Finite-wave-vector electromagnetic response of fractional quantized Hall states

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A fractional quantized Hall state with filling fraction $\nu = p/(2mp+1)$ can be modeled as an integer quantized Hall state of transformed fermions, interacting with a Chern-Simons field. The electromagnetic response function for these states at arbitrary frequency and wave vector can be calculated using a semiclassical approximation or the random-phase approximation. However, such calculations do not properly take into account the large effective-mass renormalization which is present in the Chern-Simons theory. We show how the mass renormalization can be incorporated in a calculation of the response function within a Landau-Fermi-liquid theory approach such that Kohn's theorem and the f-sum rules are properly satisfied. We present results of such calculations.

I. INTRODUCTION

Although the ground state for fractional quantized Hall systems is reasonably well understood at the Laughlin filling fractions $\nu = 1/(2m+1)$ where m is an integer, we have only a qualitative understanding of the elementary excitations of the system.[1] Furthermore, theories of experimentally observed fractional quantized Hall states at other filling fractions remain controversial. In order to understand the more general series of states at filling fractions $\nu = p/(2mp+1)$ where m and p are integers, Jain has constructed trial wave functions based on a picture of "composite fermions," which may be described loosely as electrons bound to an even number of magnetic flux quanta.[2] The fractional quantized states correspond to integer quantized Hall states for the composite fermions in Jain's description. Lopez and Fradkin[3] showed how one can formally transform the electron system at $\nu = p/(2mp+1)$ into a system of fermions interacting with a Chern-Simons gauge field, such that in the mean-field approximation the ground state is indeed a system of p filled Landau levels for the transformed fermions, in accord with Jain's analysis. Moreover, Lopez and Fradkin proposed that, going beyond mean-field theory, one could employ the random-phase approximation (RPA) or time-dependent Hartree approximation to calculate the linear response functions to an external electromagnetic field at wave vector $\mathbf{q}$ and frequency ω. This calculation was carried out[4] to obtain an optical excitation spectrum for these quantized Hall states in the limit of $q \to 0$. The fermion–Chern-Simons picture of Lopez and Fradkin was further developed by Halperin, Lee, and Read [henceforth referred to as HLR (Ref. 5)] who used it to study even-denominator filling fractions such as $\nu = 1/2$, where no quantized Hall effect is observed. Their analysis also had implications for the excitation spectra of Jain's quantized Hall states, especially in the limit $p \to \infty$, where the value of ν approaches an even fraction.

An important correction to both the mean-field theory and the RPA, noted by HLR, is that fluctuations in the Chern-Simons gauge field lead to a large correction to the effective mass m^* that describes low-energy excitations. The RPA, in its standard form, as used by Lopez and Fradkin, assumes an effective mass which is equal to the bare electron band mass m_b. If one arbitrarily changes the value of the mass in the RPA in order to get reasonable energies for the lowest branch of the excitation spectrum for the fractional quantized Hall states, then one violates both the f-sum rule and Kohn's theorem, which says that in the limit $q \to 0$ a mode at the bare cyclotron frequency $\omega_c = |eB|/(m_b c)$ has all the weight of the f-sum rule. In the present paper we propose a modification of the RPA which we believe gives a good representation of the low-energy branches of the spectrum, while at the same time preserving the f-sum rule when $m^* \neq m_b$. Our modified RPA may be obtained as a natural extension of Landau-Silin Fermi-liquid theory if one includes, in addition to the direct Coulomb potential and the self-consistent Chern-Simons field, a nonzero value of the Fermi-liquid coefficient A_1, chosen to satisfy the constraint imposed by Galilean invariance. We present numerical results of this approximation for a wide range of q values for three representative quantized Hall states ($\nu = \frac{1}{3}, \frac{3}{7}, \frac{10}{21}$), comparing the results with those of the unrenormalized RPA and a semiclassical approximation suggested by HLR.

Although the electromagnetic response of the quantized Hall state is trivial in the absence of impurities at zero wave vector ($q = 0$) because of Kohn's theorem, the finite wave-vector excitation spectrum can display a very rich structure. Theoretical calculations of such excitation spectra have been accurately performed in a controlled perturbation theory only for integer quantized Hall states.[6] The spectra for fractional quantized Hall states have been much harder to calculate. To this end, Girvin, MacDonald, and Platzman[7] used a single-mode approximation in analogy with the Feynman theory of superfluid helium to determine the dispersion relation of the lowest-energy branch of the excitation spectrum of the Laughlin states $\nu = 1/(2m+1)$. In this approximation, it was shown that there is a gap at zero wave

vector, and a minimum in the dispersion curve at finite wave vector. This minimum was called the "magnetoroton" in analogy with superfluid helium.

Excitation spectra have also been calculated exactly by numerical diagonalization of small spherical systems restricted to the first Landau level.[8,9] The results obtained in the present paper are in at least reasonable qualitative agreement with available exact calculations of the density response function for these finite systems.[8] (A detailed comparison will be given elsewhere).[10] A particularly interesting feature of our results is that for large p, the lowest branch in the excitation spectrum acquires a series of deep minima, similar to the magnetoroton minimum at $\nu = 1/3$, at wave vectors given approximately by $q_n \approx k_F(n+\frac{1}{4})\pi/(2|p|)$ where n is an integer, and k_F is related to the electron density n_e by $k_F = (4\pi n_e)^{\frac{1}{2}}$.

It should also be possible to experimentally observe the finite wave vector excitation spectrum via resonant inelastic light scattering with an angle of incidence far from the normal. In fact, such measurements have been performed recently on integer quantized Hall states.[11] It may also be possible to use a grating near the surface of the quantized Hall system to measure the electromagnetic response at the wave vector of the grating. Comparison of the results of these experiments to our present calculations should provide an excellent test of our current understanding of the fractional quantized Hall effect.

The outline of this paper is as follows. In Sec. II we review the model used by HLR.[5] We describe the RPA and the semiclassical approximations. In these models, an unperturbed response function is calculated and the interactions as well as the gauge fluctuations are accounted for in perturbation theory (essentially by summing bubble diagrams). In the RPA, the unperturbed response function is simply the response function for the mean-field system, whereas for the semiclassical approximation the unperturbed response function is determined from a semiclassical approximation of the quasiparticle conductivity. These unperturbed response functions are calculated in Sec. III. The results of these calculations are put in more usable form in Appendix A and Appendix B. In Sec. IV we discuss the issue of the renormalization of the quasiparticle mass. A Fermi-liquid theory approach is used to account for the renormalized mass in the calculation of the electromagnetic response. Using this approach, we construct what we call the "modified semiclassical" approximation and the "modified RPA." A more general Fermi-liquid theory calculation that can be generalized to include the effects of additional nonzero Fermi-liquid coefficients is performed in Appendix C and agrees with the results of Sec. IV. In Sec. V we display electromagnetic response spectra for the quantized Hall states $\nu = \frac{1}{3}, \frac{3}{7}, \frac{10}{21}$ calculated in the semiclassical approximation, the RPA, and the modified RPA. Finally, we summarize our findings in Sec. VI.

II. MODEL

We consider a two-dimensional system of spinless electrons of band mass m_b and charge $-e$, with interactions given by a potential $v(r)$, in a uniform magnetic field B perpendicular to the plane of the system (in the $\hat{\mathbf{z}}$ direction). We will generally take the interaction potential to be the physically interesting Coulomb interaction given by

$$v(r) = \frac{e^2}{\epsilon|r|} , \tag{1}$$

where ϵ is the background dielectric constant.

Following HLR,[5] we make a singular gauge transformation to write the Hamiltonian for this system in terms of the composite fermion quasiparticle creation operator $\psi^\dagger(\mathbf{r})$ that creates an electron at point $\mathbf{r}$ bound to $\tilde{\phi}$ quanta of Chern-Simons flux. In terms of these quasiparticle operators, the Hamiltonian for this system can be written exactly as

$$H = K + V , \tag{2}$$

where K is the kinetic energy given by

$$K = \frac{\hbar^2}{2m_b}\int d^2r\ \psi^\dagger(\mathbf{r})\left[-i\boldsymbol{\nabla} + \left\{\frac{e}{c}\mathbf{A}(\mathbf{r}) - \mathbf{a}(\mathbf{r})\right\}\right]^2\psi(\mathbf{r}). \tag{3}$$

and V is the potential energy

$$V = \frac{1}{2}\int d^2r \int d^2r'\ v(\mathbf{r}-\mathbf{r}') : \rho(\mathbf{r})\rho(\mathbf{r}') : . \tag{4}$$

Here, the colons represent normal ordering of the creation and annihilation operators, $\mathbf{A}(\mathbf{r})$ is the vector potential due to the magnetic field B such that $\boldsymbol{\nabla}\times\mathbf{A} = B$, and $\mathbf{a}(\mathbf{r})$ is the vector potential associated with the Chern-Simons flux which can be written as

$$\mathbf{a}(\mathbf{r}) = \tilde{\phi}\int d^2r'\mathbf{g}(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r}') , \tag{5}$$

$$\mathbf{g}(\mathbf{r}) = (\hat{\mathbf{z}}\times\mathbf{r})/r^2 , \tag{6}$$

where $\hat{\mathbf{z}}$ is the unit vector perpendicular to the plane of the system and $\tilde{\phi} = 2m$ is the even number of flux quanta bound to each electron. The point $\mathbf{r} = \mathbf{r}'$ should be excluded from the Green's function $\mathbf{g}(\mathbf{r}-\mathbf{r}')$.

If we use a mean-field description and average the effect of the fluctuating gauge field $\mathbf{a}$, the Hamiltonian (2) simply represents quasiparticle fermions in a magnetic field

$$\Delta B = B - B_{(1/2m)} = B - \frac{4m\pi\hbar c n_e}{e} , \tag{7}$$

where n_e is the electron number density. With the idea of perturbing around this mean-field description to account for gauge fluctuations and interactions, we write the mean-field reference Hamiltonian as

$$H_0 = \frac{\hbar^2}{2m_b}\int d^2r\ \psi^\dagger(\mathbf{r})\left[-i\boldsymbol{\nabla} + \frac{e}{c}\Delta\mathbf{A}(\mathbf{r})\right]^2\psi(\mathbf{r}) , \tag{8}$$

where the mean-field vector potential $\Delta\mathbf{A}(\mathbf{r})$ satisfies $\boldsymbol{\nabla}\times(\Delta\mathbf{A}) = \Delta B$.

Since the mean-field Hamiltonian describes fermions in a magnetic field ΔB, the energy levels are simply the usual Landau levels, but they are now the energy levels

of the quasiparticle wave functions. If we add a small amount of disorder to the system, we expect to see the integer quantized Hall effect for the quasiparticles where the steps are centered around integer quasiparticle filling fractions. Since the integer quantized Hall effect for electrons occurs when the filling fraction $\nu = n_e 2\pi\hbar c/(eB)$ is an integer, we now have the integer quantized Hall effect for *quasiparticles* when $p = n_e 2\pi\hbar c/(e\Delta B)$ is an integer. Substituting in the definition (7) of ΔB now yields stable states at

$$B = B_{(1/2m)} + \frac{2\pi\hbar c n_e}{ep} \tag{9}$$

$$= \frac{2\pi\hbar c n_e}{e}\left[2m + \frac{1}{p}\right] \tag{10}$$

which corresponds to the Jain states $\nu = p/(2mp+1)$.

We now define the electromagnetic linear response function $K_{\mu\nu}(\mathbf{q},\omega)$ where μ and ν take on the values $(0,x,y)$ by the relation

$$j_\mu(\mathbf{q},\omega) = \frac{e}{c} K_{\mu\nu}(\mathbf{q},\omega) A_\nu^{\text{ext}}(\mathbf{q},\omega)\ , \tag{11}$$

where A_ν^{ext} is an external perturbing scalar ($\nu = 0$) or vector ($\nu = x,y$) potential with frequency ω and wave vector $\mathbf{q}$, and j_μ is the induced change in the particle density ($\mu = 0$) or current ($\mu = x,y$). Following HLR (Ref. 5) again, we choose $\mathbf{q} \| \hat{\mathbf{x}}$, and we work in the Coulomb gauge so that the longitudinal part of $\mathbf{A}$ is zero (i.e., so that $A_x = 0$). With these choices, the longitudinal part of $\mathbf{j}$ is simply $(\omega/q)j_0$. Thus we can consider $K_{\mu\nu}$ as a 2×2 matrix in which the indices take on the values 0 and 1 where the index 1 indicates the transverse or $\hat{\mathbf{y}}$ direction.

The response function can be calculated within the RPA or time-dependent Hartree approximation in analogy with recent work on anyon superconductivity.[12–14] In this work, the RPA equations are derived through a Hamiltonian formalism. Alternatively one can derive the same relations from a more field-theoretic Lagrangian approach.[3,5] It should be noted that the RPA formalism of the above mentioned works on anyon superconductivity[12–14] differs slightly from the formalism of HLR (Ref. 5) that we have chosen to follow. In particular the HLR formalism is simplified because the diamagnetic term is included in the bare response. Nonetheless, both approaches give the same final results. It should also be noted that the above mentioned formalisms of Refs. 12–14, as well as that of Lopez and Fradkin,[3,4] involve the calculation of a 3×3 response matrix, whereas the HLR approach uses a convenient gauge to reduce the problem to the calculation of a 2×2 matrix.

The RPA equation for the electromagnetic response function is given by

$$K = K^0[1 + UK^0]^{-1}\ , \tag{12}$$

where $K^0_{\mu\nu}$ is the response function for the noninteracting system of quasiparticles governed by the Hamiltonian H_0, and the interaction matrix U is given by

$$U = V + C^{-1}\ , \tag{13}$$

where C^{-1} is the Chern-Simons interaction, with

$$C = \frac{1}{2\pi\hbar\tilde{\phi}}\begin{bmatrix} 0 & iq \\ -iq & 0 \end{bmatrix} \tag{14}$$

and

$$V = \begin{bmatrix} v(q) & 0 \\ 0 & 0 \end{bmatrix} \tag{15}$$

represents the interaction of the quasiparticles through the potential $v(q)$ which is just the Fourier transform of the potential $v(r)$. For the physically relevant case of the Coulomb potential [Eq. (1)] we have

$$v(q) = \frac{2\pi e^2}{\epsilon q}. \tag{16}$$

Note that the potential V couples the density of particles to the scalar potential, whereas the Chern-Simons interaction C—like a magnetic field—couples to the current also.

It is sometimes more useful to think in terms of the conductivity rather than the electromagnetic response.[5] The conductivity σ is defined as the response to the total electromagnetic field A_μ whereas the electromagnetic response K is the response to the external electromagnetic field A_μ^{ext}. The magnetic field generated by the quantum Hall system is small, so there is essentially no difference between $\mathbf{A}$ and $\mathbf{A}^{\text{ext}}$. On the other hand, the scalar potentials in Fourier space $eA_0(q)$ and $eA_0^{\text{ext}}(q)$ differ by the Coulomb potential $v(q)j_0(q)$ generated by the density fluctuations. Thus, we will define a 2×2 matrix $\Pi(q,\omega)$, which is more closely related to the conductivity, to be the electromagnetic response without this Coulomb contribution:

$$K^{-1} = \Pi^{-1} + V. \tag{17}$$

Similarly, it is convenient to define a 2×2 matrix $\tilde{K}(q,\omega)$ to be the electromagnetic response without the Coulomb contribution or the Chern-Simons contribution:

$$K^{-1} = \tilde{K}^{-1} + U. \tag{18}$$

In other words, Π is the contribution from all Feynman diagrams for K that are irreducible with respect to V and $\tilde{K}$ is the sum of all diagrams for K that are irreducible with respect to U. Note that the RPA equation (12) is obtained by simply approximating $\tilde{K}$ as the mean-field noninteracting quasiparticle response function K^0. The perturbation U—which includes the Coulomb and Chern-Simons interactions—is then incorporated in Eq. (18) to give the full response function. Similarly, in our semiclassical approximation we will directly try to approximate the unperturbed response function $\tilde{K}$ for the quasiparticles.

Maintaining our convention that $\mathbf{q} \| \hat{\mathbf{x}}$ we can now follow HLR (Ref. 5) to define the conductivity tensor $\sigma_{ij}(q,\omega)$ as

$$\sigma_{xx}^{-1}(q,\omega) = \frac{iq^2}{\omega e^2}\left[\Pi_{00}^{-1}(q,\omega) - \Pi_{00}^{-1}(q,0)\right] , \tag{19}$$

$$\sigma_{yy}(q,\omega) = \frac{-ie^2}{\omega}\left[\Pi_{11}(q,\omega) - \Pi_{11}(q,0)\right] , \tag{20}$$

$$\sigma_{xy}(q,\omega) = -\sigma_{yx}(q,\omega) = \frac{ie^2}{q}\Pi_{01}(q,\omega) . \tag{21}$$

Similarly, we can define the "quasiparticle conductivity tensor" $\tilde{\sigma}_{ij}(q,\omega)$ as

$$\tilde{\sigma}_{xx}^{-1}(q,\omega) = \frac{iq^2}{\omega e^2}[\tilde{K}_{00}^{-1}(q,\omega) - \tilde{K}_{00}^{-1}(q,0)] , \tag{22}$$

$$\tilde{\sigma}_{yy}(q,\omega) = \frac{-ie^2}{\omega}[\tilde{K}_{11}(q,\omega) - \tilde{K}_{11}(q,0)] , \tag{23}$$

$$\tilde{\sigma}_{xy}(q,\omega) = -\tilde{\sigma}_{yx}(q,\omega) = \frac{ie^2}{q}\tilde{K}_{01}(q,\omega). \tag{24}$$

These definitions have been chosen so that the conductivities are finite in the $\omega \to 0$ limit for any fixed value of q. Although these are not necessarily the only such definitions that are possible, we will not be overly concerned with the low-frequency limit in this paper. In fact the contributions from the zero-frequency parts of these relationships are suspected to be negligibly small in all cases that we will consider. Thus, we can approximate these relationships by dropping the additive zero-frequency pieces to write the results in a convenient matrix form as

$$\sigma = T\Pi T , \tag{25}$$

$$\tilde{\sigma} = T\tilde{K}T , \tag{26}$$

where T is the conversion matrix

$$T = e\begin{bmatrix} \frac{i\sqrt{i\omega}}{q} & 0 \\ 0 & \frac{1}{\sqrt{i\omega}} \end{bmatrix} . \tag{27}$$

Now from these relations and the definitions of Π, $\tilde{K}$, and U we can derive

$$K^{-1} = T\rho T + V, \tag{28}$$

$$\rho = \tilde{\rho} + \rho_{\mathrm{CS}}, \tag{29}$$

$$\rho_{\mathrm{CS}} \equiv T^{-1}C^{-1}T^{-1} = \frac{2\pi\hbar\tilde{\phi}}{e^2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{30}$$

where ρ_{CS} is the contribution from the Chern-Simons interaction, and ρ and $\tilde{\rho}$ are the associated resistivity matrices defined as

$$\rho = \sigma^{-1} , \tag{31}$$

$$\tilde{\rho} = \tilde{\sigma}^{-1}. \tag{32}$$

Note that if we approximate $\tilde{\sigma}$ by the mean-field noninteracting result TK^0T, then Eqs. (28) and (29) become equivalent to the RPA prescription [Eq. (12)].

III. UNPERTURBED RESPONSE

In order to calculate the response function in the RPA approximation we must first find the response K^0 of the unperturbed mean-field Hamiltonian H_0. In the RPA, the mean-field response K^0 is used as an approximation for the U-irreducible diagrams $\tilde{K}$. Similarly, for our semiclassical approximation, the semiclassical quasiparticle conductivity $\tilde{\sigma}$ is used as an approximation for $T\tilde{K}T$. For a noninteracting two-dimensional system of spinless fermions of mass m_b and charge $-e$ at density n_e in a perpendicular magnetic field ΔB, the magnetic length l_Δ is defined by

$$l_\Delta = \sqrt{\frac{c\hbar}{e(\Delta B)}} \tag{33}$$

and the cyclotron frequency $\Delta\omega_c$ is given by

$$\Delta\omega_c = \frac{e(\Delta B)}{m_b c} . \tag{34}$$

The number of "effective" quasiparticle Landau levels filled is given by

$$p = \frac{2\pi n_e \hbar c}{e(\Delta B).} \tag{35}$$

It is also convenient to define the dimensionless reduced wave vector

$$X = qR_\Delta = \frac{2qp}{k_F} , \tag{36}$$

where

$$R_\Delta = \frac{\hbar k_F c}{e(\Delta B)} \tag{37}$$

is the effective semiclassical cyclotron radius and $k_F = (4\pi n_e)^{\frac{1}{2}}$ is the Fermi momentum. We will find that X is a very natural parameter in terms of which to express our results. In particular, we will find that our results approximately scale in terms of X in the semiclassical (large p) limit. For simplicity of notation, we will assume from now on that $\Delta B \geq 0$, and hence $p \geq 0$.

A. K^0 for the RPA

For the simple case where p is an integer, the unperturbed electromagnetic response function K^0 has been derived in connection with the theory of anyon superconductivity by Fetter, Hanna, and Laughlin[12] for the $p = 2$ case, and then for general p by Chen *et al.*[13] (and later by Dai *et al.*[14]) The calculation is performed by realizing that the response function can be related to the ground-state expectation value of the time-ordered product of current operators. This quantity is then calculated by inserting complete sets of states for free electrons in a magnetic field. The final result in matrix form is

$$K^0 = T^{-1}\tilde{s}T^{-1} , \tag{38}$$

where the conductivity matrix $\tilde{s}$ is given by[13,14]

$$\tilde{s} = \frac{pe^2}{2\pi\hbar}\begin{bmatrix} i\left(\frac{\omega}{\Delta\omega_c}\right)\Sigma_0 & -\Sigma_1 \\ \Sigma_1 & i\left(\frac{\Delta\omega_c}{\omega}\right)(\Sigma_2+1) \end{bmatrix} \tag{39}$$

and

$$\Sigma_j = \frac{e^{-Y}}{p} \sum_{l=0}^{p-1} \sum_{m=p}^{\infty} \left\{ \frac{l!}{m!} \frac{(m-l) Y^{m-l-1} [L_l^{m-l}(Y)]^{2-j}}{\left(\frac{\omega+i0^+}{\Delta\omega_c}\right)^2 - (m-l)^2} \left[(m-l-Y) L_l^{m-l}(Y) + 2Y \frac{dL_l^{m-l}(Y)}{dY} \right]^j \right\} , \tag{40}$$

where L_l^m is a Laguerre polynomial, and the expansion parameter Y is given by

$$Y = \frac{1}{2}(q l_\Delta)^2 = \frac{p}{4} X^2 . \tag{41}$$

It should be noted that the (l,m)th term in the sum (40) represents a particle in the lth Landau level making a virtual transition up to the mth level and back. In Appendix A we perform some of the above summations explicitly such that each Σ_j is written as a single sum. This simplification has proven to be quite useful for both analytic and numerical analyses.

For noninteger values of p one can interpolate to find the residues of a given pole in K^0 at nonzero frequency. In terms of the imaginary part of the response function, we can write

$$\begin{aligned} \mathrm{Im}\left[K_p^0(\omega \neq 0)\right] &= ([p] + 1 - p)\ \mathrm{Im}[K_{[p]}^0(\omega)] \\ &\quad + (p - [p])\ \mathrm{Im}[K_{[p]+1}^0(\omega)] , \end{aligned} \tag{42}$$

where $[p]$ is the greatest integer less than or equal to p. To find the weight of the pole in the response function at zero frequency, one must use an f-sum rule (see also Sec. IV). We will, however, limit our attention to the integer values of p.

B. Semiclassical $\tilde{\sigma}$

The other approach we will use is to semiclassically approximate the quasiparticle conductivity $\tilde{\sigma}$ and hence $\tilde{K}$. The semiclassical regime is the region where the energy levels are closely spaced with respect to the other energy scales of the problem. This regime occurs at low effective fields (large p) and long wavelength (small q) and when $\hbar\omega$ is much less than the Fermi energy. If we are in this semiclassical regime we can consider the quasiparticles as localized wave packets moving under the influence of the magnetic field ΔB and (as described in Appendix C) we can approximate the quasiparticle conductivity $\tilde{\sigma}$ as[5,15]

$$\tilde{\sigma}_{ij} = \frac{2 i n_e e c}{\Delta B} \sum_{n=-\infty}^{\infty} \frac{V_i^{(n)*} V_j^{(n)}}{\frac{\omega}{\Delta\omega_c} - n + \frac{i}{\tau}} , \tag{43}$$

where τ is the quasiparticle scattering time, and the velocity coefficients $V_i^{(n)}$, whose meaning is further elucidated in Appendix C, are defined as

$$V_x^{(n)} = \frac{n}{X} J_n(X) , \tag{44}$$

$$V_y^{(n)} = -i \frac{dJ_n(X)}{dX} , \tag{45}$$

where J_n is the nth Bessel function. Note that unlike the RPA, the semiclassical approximation gives no special significance to integer values of p.

In this semiclassical approximation the quasiparticle scattering time τ is left as a free parameter whereas the above quantum-mechanical calculation of K^0 is inherently in the no-scattering ($\tau \to \infty$) limit since no mechanism has been included to account for scattering. Although we have the freedom to perform our semiclassical calculations for finite τ, it is actually more useful to think of the no-scattering limit. In this limit poles will appear in the density-density response function K_{00} at exactly the frequencies corresponding to the collective modes of the system. Furthermore, by taking the $\tau \to \infty$ limit we can compare our results with the RPA. Making this simplification we can sum the series of Bessel functions exactly to yield a closed form expression for the quasiparticle conductivity $\tilde{\sigma}$. This sum is performed explicitly in Appendix B.

IV. EFFECTIVE-MASS RENORMALIZATION

A. General considerations

Within the theory considered so far, the quasiparticle effective mass m^* is just the bare band mass m_b. In this theory we perturb around a reference Hamiltonian H_0 that describes particles of this unrenormalized mass [Eq. (8)]. We expect, however, that the effective mass should be renormalized by interactions. In order to estimate the importance of this mass renormalization, we follow HLR (Ref. 5) to make a crude estimate of the value of the effective mass. Assuming that the electron interaction energy is much less than the spacing between Landau levels, the Landau-level mixing can be neglected and all energies of interaction must then be proportional to the electron-electron interaction energy scale $e^2(4\pi n_e)^{1/2}/\epsilon$. Thus, dimensional analysis tells us that the effective mass should have the form (if it is in fact finite)

$$m^* = \frac{\hbar^2 (4\pi n_e)^{1/2} \epsilon}{e^2 C} , \tag{46}$$

where C is a dimensionless constant. HLR (Ref. 5) use results from the exact diagonalization of small spherical systems restricted to the lowest Landau level to estimate that $C = 0.3$. Using the experimentally relevant dielectric constant $\epsilon = 12.6$ appropriate for GaAs, a field of $B = 10$ T and a filling fraction of $\nu = \frac{1}{2}$ yields the result

$$m^* \approx 4 m_b . \tag{47}$$

Using a self-consistent analysis of a selected set of diagrams for the self-energy of the transformed fermions which describes the interaction with long-wavelength fluctuations in the Chern-Simons vector potential, HLR

(Ref. 5) conclude that for the case of the Coulomb interaction between the electrons, the effective mass m^* should actually exhibit a logarithmic divergence for energies near the Fermi energy, and for $p \to \infty$ (i.e., for $\nu \to \frac{1}{2}$). The coefficient in front of the logarithm obtained by HLR is relatively small, however, and the resulting values of the effective mass, in practice, will not be very different from those given by Eq. (46).

The important thing to note here is that the mass is renormalized considerably. Thus, perturbing around an unrenormalized Hamiltonian is likely to give very poor results. The first naive thing one could do to try to correct this problem is simply to use this renormalized effective mass in the reference Hamiltonian H_0. In fact, in Sec. V we will see that this approach can sometimes give reasonable results for the dispersion relation of the lowest excitation mode. However, this approach will give an incorrect value for the cyclotron frequency, thereby violating Kohn's theorem and the f-sum rule. The focus of this section is the construction of a method of repairing our naive approach so that these rules are satisfied.

First we stop to think about the properties we want our result to have. To begin with, we recall Kohn's theorem (a result of Galilean invariance) requires that the $q \to 0$ behavior of our system be determined by the band mass m_b rather than any renormalized mass. One can imagine all of the electrons in the system oscillating in unison so that electron-electron interactions have no effect. Similarly the f-sum rule simply says that the behavior of our system in the $\omega \to \infty$ limit is also determined by the band mass m_b. This is easily imagined since at high frequency one can think of the electrons oscillating very quickly with very small magnitude so that these oscillations do not appreciably change the positions of the electrons or couple to the electron-electron interaction. Often this rule is stated in terms of the conductivity such that by using a Kramers-Krönig relation it can be written as an integral over frequency (an f sum).

In the long-wavelength or high-frequency limit the free electron result is written most easily in terms of the resistivity

$$\rho \sim \frac{m_b}{e^2 n_e} \begin{bmatrix} -i\omega & \omega_c \\ -\omega_c & -i\omega \end{bmatrix} + O(q^2/\omega). \tag{48}$$

If the resistivity of our system indeed has this high-frequency, long-wavelength limit, then Kohn's theorem and the f-sum rules are satisfied. We now want to turn this condition on the resistivity (or equivalently the response function) into a condition on the properties of the quasiparticle system. Consider the resistivity $\tilde{\rho}$ for quasiparticles with the band mass m_b in the effective field ΔB. The analogous free quasiparticle high-frequency, long-wavelength limit is

$$\tilde{\rho} \sim \frac{m_b}{e^2 n_e} \begin{bmatrix} -i\omega & \Delta\omega_c \\ -\Delta\omega_c & -i\omega \end{bmatrix} + O(q^2/\omega) , \tag{49}$$

where $\Delta\omega_c = e\Delta B/(m_b c)$ is the cyclotron frequency associated with the effective magnetic field and the band mass. Now if we convert this into the associated resistivity for the original electron system using Eq. (29) we find that

$$\rho \sim \frac{m_b}{e^2 n_e} \begin{bmatrix} -i\omega & \Delta\omega_c \\ -\Delta\omega_c & -i\omega \end{bmatrix} + \frac{2\pi\hbar\tilde{\phi}}{e^2} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{50}$$

$$\sim \frac{m_b}{e^2 n_e} \begin{bmatrix} -i\omega & \omega_c \\ -\omega_c & -i\omega \end{bmatrix} , \tag{51}$$

where we have made use of the fact [Eq. (7)] that $\Delta B = B - 2\pi\hbar c\tilde{\phi} n_e/e$. We conclude that if our quasiparticle system satisfies Kohn's theorem and the f-sum rules with respect to the effective magnetic field ΔB and the band mass m_b, then our original electron system satisfies the same rules with respect to the full field B.

We must now arrange for our *quasiparticle* system to satisfy Kohn's theorem and the f-sum rule when we renormalize the quasiparticle mass (which we have so far taken to be equal to the bare band mass m_b) to some new value m^*. If we naively try to calculate $\tilde{\rho}$ with the new renormalized mass, by simply replacing the band mass m_b by the (phenomenological) effective mass m^* everywhere it occurs, we must clearly end up with the high-frequency, long-wavelength behavior

$$\tilde{\rho}^{\rm n} \sim \frac{m^*}{e^2 n_e} \begin{bmatrix} -i\omega & \Delta\omega_c^* \\ -\Delta\omega_c^* & -i\omega \end{bmatrix} + O(q^2) , \tag{52}$$

where the effective-mass renormalized cyclotron frequency is defined as

$$\Delta\omega_c^* = \frac{e\Delta B}{m^* c} . \tag{53}$$

This limiting behavior [Eq. (52)] is clearly different from the desired limit given in Eq. (49). Note that the off-diagonal terms are independent of the value of m^*, so only the diagonal terms are in violation of the sum rules. We must now find a way to "fix" this result so that the resistance takes the proper form. In other words, we must find a way to calculate the quasiparticle conductivity that more properly incorporates the effective mass and corrects for the fact that the effective mass is in general not equal to the band mass.

This problem is now almost exactly the same as the well-studied problem of Fermi-liquid effects on magnetoplasma modes in metals[16–18] where one considers the excitation modes of electrons in a strong magnetic field. Theoretically, one can use an approach similar to our semiclassical calculation of the quasiparticle conductivity to predict the spectrum of such a system.[15] Once again the electron mass is renormalized due to interactions, and a naive semiclassical approach will either not account for this mass renormalization or will violate the sum rules. One solution that has been used is to account for the mass renormalization within a formal Landau-Silin Fermi-liquid theory.[16] Such a Fermi-liquid approach should be valid at long wavelengths and when $\hbar\omega$ and $\hbar\omega_c$ are much less than the Fermi energy. Even more analogous to our problem, Lee and Quinn have used such a theory to study two-dimensional electron systems.[17] They show within this approach that (within the semi-

classical regime where the cyclotron energy as well as $\hbar\omega$ are much less than the Fermi energy) in the $q \to 0$ limit the frequency of the nth excitation mode is given by

$$\omega = (1 + A_n)n\omega_c^* + O(q^2) , \tag{54}$$

where A_n is the nth Fermi-liquid coefficient. Note that since the effective mass is controlled by the first Fermi-liquid coefficient (a result of Galilean invariance of Fermi-liquid theory) via[19]

$$m^* = (1 + A_1)m_b \tag{55}$$

the location of the first ($n = 1$) excitation mode—the cyclotron frequency—is unchanged when the mass is renormalized. This is exactly the type of result we want. Unfortunately, we will need to know the full conductivity, not just the frequency of the excitation modes, so we will be unable to use the results of Lee and Quinn directly. Nonetheless, we will be able to use this type of theory to calculate the conductivity for quasiparticles with renormalized mass. One additional advantage of using this type of Landau-Fermi-liquid theory is that we do not need to know exactly how or even why the electron mass is renormalized, since all of the relevant details of the electron-electron interaction are included within the single mass renormalization coefficient A_1.

B. Modified semiclassical approximation for $\tilde{\rho}$

We begin by discussing the Fermi-liquid corrections to the quasiparticle resistivity tensor $\tilde{\rho}$ in the semiclassical approximation. To do this, we first ignore the direct Chern-Simons and Coulomb interactions, and consider the current induced in the Fermi liquid by a specified electromagnetic vector potential $\mathbf{A}(\mathbf{r}, t)$. Eventually we shall use the result for $\tilde{\rho}$ in Eq. (29), which will be equivalent to replacing the electromagnetic field by the sum of the self-consistent Chern-Simons and electromagnetic fields.

In performing this Fermi-liquid theory calculation we make several simplifying assumptions. To begin with, as explained above, we consider only the low scattering $\tau \to \infty$ limit. (The effects of impurity scattering are considered as the more general case in Ref. 17). We also assume that higher Fermi-liquid coefficients (A_l for $l > 1$) are progressively less important and we can set $A_0 = 0$ since we can include the effects of the density-density interaction in the RPA treatment by modifying the interaction $v(r)$ at short distances. Thus, we assume that all of the Fermi-liquid coefficients are zero except for the coefficient A_1 that controls the mass renormalization via Eq. (55). In Appendix C it is shown how to include the effects of other nonzero Fermi-liquid coefficients. Finally, in order to use a Fermi-liquid theory, we must assume that we are in the semiclassical regime where the quasiparticles can be treated as localized wave packets. In other words, we should have the wave vector q much less than the Fermi wave vector k_F while $\hbar\omega$ and the spacing between effective Landau levels $\hbar\Delta\omega_c^*$ must both be much less than the Fermi energy E_F. The last condition, in particular, is well satisfied for large p, but only marginally satisfied for small p such as at $\nu = \frac{1}{3}$.

With these assumptions we follow the usual Fermi-liquid approach[19] and write the energy functional for a two-dimensional system of spinless quasiparticles of effective mass m^* at density n_e, in an electromagnetic field defined by the vector potential $\mathbf{A}$, as

$$\begin{aligned} E[n(\mathbf{p},\mathbf{r})] = {} & E_0[n(\mathbf{p},\mathbf{r})] \\ & + \frac{A_1}{2n_e m^*} \sum_{\mathbf{p},\mathbf{p}'} (\mathbf{p} + e\mathbf{A}) \cdot (\mathbf{p}' + e\mathbf{A}) \\ & \times n(\mathbf{p},\mathbf{r}) n(\mathbf{p}',\mathbf{r}) , \end{aligned} \tag{56}$$

where

$$E_0[n(\mathbf{p},\mathbf{r})] = \sum_{\mathbf{p}} \frac{(\mathbf{p} + e\mathbf{A})^2}{2m^*} n(\mathbf{p},\mathbf{r}) \tag{57}$$

and $n(\mathbf{p},\mathbf{r})$ is the phase-space density at momentum $\mathbf{p}$ and position $\mathbf{r}$. More generally, as discussed in Appendix C the energy functional will have additional interaction terms with other nonzero Fermi-liquid coefficients. Note that in this section we have set $c = 1$ for simplicity. Equation (56) assumes implicitly that $n(\mathbf{p},\mathbf{r})$ is a slowly varying function of $\mathbf{r}$. We also assume in this section that we have chosen a gauge where the scalar potential is zero.

We can now calculate the local current by differentiating E with respect to the vector potential

$$\mathbf{J} = \frac{-\delta E[n(\mathbf{p},\mathbf{r})]}{\delta \mathbf{A}} = \sum_{\mathbf{p}} \frac{-e(\mathbf{p} + e\mathbf{A})}{m^*} (1 + A_1) n(\mathbf{p},\mathbf{r}) , \tag{58}$$

where we have used the fact that

$$\sum_{\mathbf{p}} n(\mathbf{p},\mathbf{r}) = n(\mathbf{r}) \tag{59}$$

is the local particle density which to lowest order we have taken to be equal to the average particle density n_e. On the other hand, by Galilean invariance, we expect that

$$\mathbf{J} = \sum_{p} \frac{-e(\mathbf{p} + e\mathbf{A})}{m_b} n(\mathbf{p},\mathbf{r}). \tag{60}$$

By equating these two expressions for the current we easily derive the relation (55) between bare mass m_b and effective mass m^*.

We are now faced with actually trying to self-consistently compute the time dependence of the phase-space density $n(\mathbf{p},\mathbf{r})$ when we apply a perturbing electromagnetic field. In order to do this, we begin by constructing an effective single-particle Hamiltonian

$$H_{\text{eff}}(\mathbf{p},\mathbf{r}) = \frac{\delta E[n(\mathbf{p},\mathbf{r})]}{\delta n(\mathbf{p},\mathbf{r})} ; \tag{61}$$

we then have Hamilton's equations of motion

$$\frac{d\mathbf{p}}{dt} = -\nabla_{\mathbf{r}} H_{\text{eff}}, \tag{62}$$

$$\frac{d\mathbf{r}}{dt} = \nabla_{\mathbf{p}} H_{\text{eff}} \, . \tag{63}$$

These then are used to construct the Boltzman (Fokker-Plank) equation of motion for $n(\mathbf{p},\mathbf{r})$

$$\frac{\partial n(\mathbf{p},\mathbf{r})}{\partial t} = -\left[\nabla_{\mathbf{p}} \cdot \frac{d\mathbf{p}}{dt} + \nabla_{\mathbf{r}} \cdot \frac{d\mathbf{r}}{dt}\right] n(\mathbf{p},\mathbf{r}) \tag{64}$$

$$= [(\nabla_{\mathbf{r}} H_{\text{eff}}) \cdot \nabla_{\mathbf{p}} - (\nabla_{\mathbf{p}} H_{\text{eff}}) \cdot \nabla_{\mathbf{r}}] n(\mathbf{p},\mathbf{r}). \tag{65}$$

The general self-consistent solution to this equation is nontrivial, and is outlined in Appendix C. However, we know the solution of this equation for the simple case where all of the Fermi-liquid interaction terms are set to zero (i.e., there is no mass renormalization $m^* = m_b$). In this case, the trivial effective Hamiltonian is just

$$H^0_{\text{eff}} = \frac{\delta E_0}{\delta n(\mathbf{p},\mathbf{r})} = \frac{[\mathbf{p} + e\mathbf{A}(\mathbf{r})]^2}{2m_b} \tag{66}$$

and yields the conductivity $\tilde{\sigma}$ as given in Eq. (43) in the $\tau \to \infty$ limit as usual. Now, if we try to naively account for the mass renormalization by replacing the band mass m_b by the effective mass m^* everywhere (as well as replacing $\Delta\omega_c$ by $\Delta\omega_c^*$), we call the result the naive semiclassical conductivity $\tilde{\sigma}^{\text{n}}$. As mentioned before, this naive approach violates the f-sum rule and Kohn's theorem. Nonetheless the naive conductivity will provide the starting point for our modified semiclassical calculation.

We now calculate the effective Hamiltonian (61) from our energy functional [Eq. (56)]. We find

$$H_{\text{eff}} = \frac{[\mathbf{p} + e\mathbf{A}(\mathbf{r}) - \frac{A_1 m_b}{n_e e}\mathbf{J}(\mathbf{r})]^2}{2m^*} + O(\mathbf{J}^2) \tag{67}$$

which to lowest order in the perturbing electromagnetic field is exactly the above trivial Hamiltonian [Eq. (66)] but with a renormalized mass m^* and a renormalized vector potential

$$\mathbf{A}_{\text{re}} = \mathbf{A} - \frac{A_1 m_b}{n_e e^2}\mathbf{J} \tag{68}$$

$$= \mathbf{A} - \frac{(m^* - m_b)}{n_e e^2}\mathbf{J}. \tag{69}$$

Equation (68) is equivalent to using a renormalized electric field

$$\mathbf{E}_{\text{re}} = \mathbf{E} + \frac{(m^* - m_b)}{n_e e^2}\frac{\partial \mathbf{J}}{\partial t} \tag{70}$$

$$= \mathbf{E} - i\omega\frac{(m^* - m_b)}{n_e e^2}\mathbf{J}. \tag{71}$$

We can neglect the associated magnetic field renormalization to first order.

Since the effective Hamiltonian H_{eff} looks like the trivial Hamiltonian H^0_{eff}, we can calculate the current by using the naive conductivity and the renormalized electric field

$$\mathbf{J} = \tilde{\sigma}^{\text{n}}\mathbf{E}_{\text{re}} \tag{72}$$

$$= \tilde{\sigma}^{\text{n}}\left[\mathbf{E} - i\omega\frac{(m^* - m_b)}{n_e e^2}\mathbf{J}\right] \, . \tag{73}$$

We then solve for the self-consistent current

$$\mathbf{J} = \left[1 - i\omega\frac{(m_b - m^*)}{n_e e^2}\tilde{\sigma}^{\text{n}}\right]^{-1}\tilde{\sigma}^{\text{n}}\mathbf{E} \tag{74}$$

and thus extract the conductivity $\tilde{\sigma}$ for the system of quasiparticles of effective mass m^* in a magnetic field ΔB,

$$\tilde{\sigma} = \left[1 - i\omega\frac{(m_b - m^*)}{n_e e^2}\tilde{\sigma}^{\text{n}}\right]^{-1}\tilde{\sigma}^{\text{n}} \, . \tag{75}$$

In terms of the resistivity $\tilde{\rho}$ (which is the inverse of $\tilde{\sigma}$) this can be written simply as

$$\tilde{\rho} = \tilde{\rho}^{\text{n}} - \frac{i\omega(m_b - m^*)}{n_e e^2}1 \, , \tag{76}$$

where 1 is the identity matrix and $\tilde{\rho}^{\text{n}} = (\tilde{\sigma}^{\text{n}})^{-1}$. It should be noted that as long as $\tilde{\rho}^{\text{n}}$ satisfies the sum rule (52) then $\tilde{\rho}$ satisfies the desired sum rule (49).

The full prescription for calculating the resistivity (and hence the response) of the fractional quantized Hall state in this modified semiclassical formalism is to calculate the naive quasiparticle conductivity $\tilde{\sigma}^{\text{n}}$ using Eq. (43) in the $\tau \to \infty$ limit, where we replace all occurrences of the cyclotron frequency $\Delta\omega_c$ by the mass renormalized cyclotron frequency $\Delta\omega_c^*$. (The infinite sum in this equation is performed explicitly in Appendix B.) Next we invert to get the associated resistivity $\tilde{\rho}^{\text{n}} = (\tilde{\sigma}^{\text{n}})^{-1}$. We then add the diagonal effective-mass correction term [Eq. (76)] to get the quasiparticle resistivity and the off-diagonal Chern-Simons correction term [Eq. (29)] to get the true resistivity ρ. Altogether

$$\rho = \tilde{\rho}^{\text{n}} - \frac{i\omega(m_b - m^*)}{n_e e^2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{2\pi\hbar\tilde{\phi}}{e^2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \, . \tag{77}$$

The resistivity ρ can then be converted to an electromagnetic response K using Eq. (28).

C. Modified RPA

The above semiclassical prescription (76), which accounts for mass renormalization by adding a constant resistivity, looks very much like the RPA prescription (29) for taking into account the effect of the Chern-Simons field by simply adding a constant to the resistivity tensor. This encourages us to try to account for the mass renormalization in the RPA calculation by the following analogous method. We write a quantum-mechanical energy functional Eq. (56) where we now think of the phase space distribution $n(\mathbf{r},\mathbf{p})$ as its quantum-mechanical analog, the Wigner function. As above, we can differentiate to get the effective single-particle Hamiltonian (61) except now we should think of this as a quantum-

mechanical operator. As above in Eq. (66), if we neglect the mass renormalization by setting the Fermi-liquid coefficient A_1 to zero, our effective Hamiltonian is the same trivial free particle Hamiltonian except that now $\mathbf{p}$ and $\mathbf{r}$ must be treated as quantum-mechanical operators. Using this as a single-particle Hamiltonian, we know how to calculate the electromagnetic response K^0 given in Eq. (38) which we write in terms of the quantum-mechanical conductivity matrix $\tilde{s}$ in Eq. (39). Again, if we naively substitute the effective mass m^* and the mass-renormalized cyclotron frequency $\Delta\omega_c^*$ for the band mass m_b and the cyclotron frequency $\Delta\omega_c$ in Eqs. (39) and (40) we obtain the naive quantum-mechanical conductivity $\tilde{s}^{\rm n}$. Once again, we know that this expression violates the f-sum rule and Kohn's theorem.

Now if we add in the interaction term in the energy functional, we get the effective Hamiltonian Eq. (67) except that now $\mathbf{p}$ and $\mathbf{r}$ are operators, and $\mathbf{J}$ is a current expectation value. Again, this Hamiltonian is just the trivial Hamiltonian with a renormalized mass and a renormalized vector potential. We can thus follow the rest of the modified semiclassical prescription exactly.

Thus, the complete prescription for the modified RPA is to first calculate the naive quantum-mechanical conductivity $\tilde{s}^{\rm n}$ using Eqs. (39) and (40) where we substitute the mass-renormalized cyclotron frequency $\Delta\omega_c^*$ for the cyclotron frequency $\Delta\omega_c$. The sums that occur in this equation are simplified in Appendix A. We then invert this conductivity matrix to get the naive resistivity $\tilde{\rho}^{\rm n} = (\tilde{s}^{\rm n})^{-1}$. Finally we include the mass-renormalization and Chern-Simons terms exactly as we did for the modified semiclassical case by using Eq. (77) to get the resistivity ρ. Again the resistivity can be converted to an electromagnetic response using Eq. (28).

Although this Fermi-liquid approach is certainly appropriate in the semiclassical regime, it is not as clear that it is appropriate for correcting our RPA calculation. Formally one should probably use a diagrammatic expansion in the electron-electron interaction to calculate both the value of the effective mass m^* and the correction to the conductivity. The problem with this approach is that, as shown by HLR,[5] these types of calculations are plagued with divergences (although the coefficients of the diverging terms may be small so that they are easier to ignore in practice). Furthermore, since the mass is so greatly renormalized, such a perturbative calculation might converge only very slowly. Nonetheless, we believe that our approach is at least reasonably accurate as well as being the simplest approach that still satisfies all of the sum rules. Furthermore, it should be noted that the form of our approximation (76) for the quasiparticle resistivity $\tilde{\rho}$ coincides with an approximation proposed by Ando in 1976 for the total resistivity of a two-dimensional electron system.[18] Ando derived this approximation using a particular short-range form of the electron-electron interaction which he treated in lowest-order perturbation theory, corresponding to a Hartree-Fock approximation for the electron self-energy and a ladder approximation for the vertex correction to the polarization bubble. Although Ando also considers the possible effects of impurity scattering, his analysis is restricted to the $q = 0$ limit, and of course he does not include a Chern-Simons contribution in his model.

V. NUMERICAL RESULTS

We begin by limiting our attention to the series of quantum Hall states given by $\nu = p/(2p+1)$. This is the most stable experimentally observed series of states, and is thus the most interesting. We focus on these states by setting the flux attached to each quasiparticle to be exactly two quanta ($\tilde{\phi} = 2m = 2$). The results we would find for the more general case $\nu = p/(2mp+1)$ are quantitatively very similar to the results for $\nu = p/(2p+1)$ except the poles in the density-density response function have much smaller weight in general.

A. Semiclassical

We first examine the case where there is no mass renormalization ($m^* = m_b$) and where we turn off the direct Coulomb interaction by taking the limit of large dielectric constant ($\epsilon \to \infty$), or by setting $V = 0$ in Eq. (28). In the semiclassical approximation, we have used Eq. (43) in the low scattering ($\tau \to \infty$) limit to calculate the quasiparticle conductivity $\tilde{\sigma}$. An equivalent, but more convenient form of this equation is given in Appendix B. We then convert this quasiparticle conductivity to a response function K by using Eqs. (28)–(32).

We are most interested in the poles in the density-density response function K_{00}. A pole in K_{00} with respect to frequency indicates the existence of a collective mode, and the weight of the pole indicates the strength of the coupling of this mode to a fluctuation in density. In Fig. 1 we show the location (heavy solid) of these poles in K_{00} as a function of reduced frequency ($\omega/\Delta\omega_c$) and reduced wave vector ($X = qR_\Delta = 2qp/k_F$). The width of the striped bands around the lines of poles indicates q^{-2} times the relative weight of the poles. In accordance with Kohn's theorem, we see that the cyclotron mode [the mode at $\omega = (2p+1)\Delta\omega_c = \omega_c$] has all of the weight at long wavelength and moreover that this weight scales as q^2. It appears as though some of the lines of poles get very thin and disappear at certain wave vectors. What is actually happening here is just that the residue of the line of poles has become too small to see on the scale of the graph shown.

We have shown results for filling fractions $\nu = \frac{1}{3}$, $\frac{3}{7}$, and $\frac{10}{21}$, corresponding to effective Landau-level fillings of $p = 1$, 3, and 10 where $m = 1$. By examining the $p = 10$ case we see that the semiclassical approximation has a very simple large p, large X/R limit (where $R = \omega/\Delta\omega_c$ as usual). In this limit we see the series of crossing straight lines with equal slopes but with opposite signs. More precisely, we have

$$X_{\rm pole} \sim \pm\frac{\pi}{2}\frac{\omega}{\Delta\omega_c} + \left(n + \frac{1}{4}\right)\pi\,, \tag{78}$$

where n is an integer. This behavior can be derived most easily by using the analytic form for $\tilde{\sigma}$ described in Ap-

pendix B and expanding the Bessel functions for large argument using Eq. 9.2.1 of Ref. 22. The above linear relation then follows after converting $\tilde{\sigma}$ to a response function.

We can gain some insight into the physics behind this linear relation with the following heuristic argument. In the present approximation, we have ignored the Coulomb interaction V in Eq. (28). Therefore, in order to have a pole in the electromagnetic response K, we must have a zero of the determinant of the resistivity ρ. But $\rho = \tilde{\rho} + \rho_{CS}$ [see Eq. (29)] can only have a zero if the quasiparticle resistivity $\tilde{\rho}$ is large enough to cancel the Chern-Simons resistivity ρ_{CS}. Since ρ_{CS} is large, we can only have this cancellation very near to a pole in $\tilde{\rho}$ which occurs when there is a zero eigenvalue of the quasiparticle conductivity $\tilde{\sigma}$. Thus, the poles in the electromagnetic response K occur at approximately the same wave vector as the zero eigenvalues of the quasiparticle conductivity tensor $\tilde{\sigma}$.

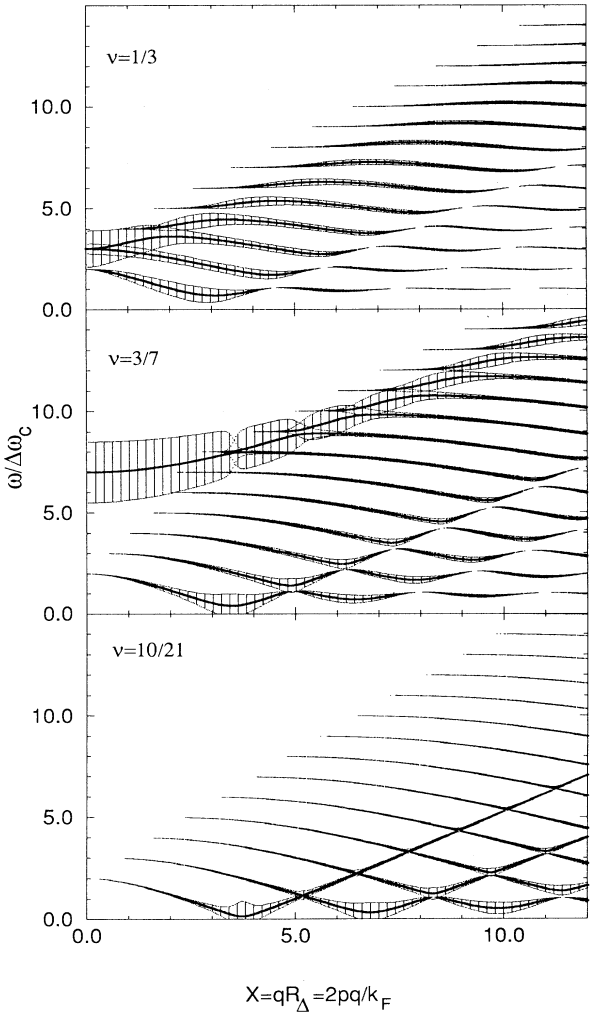

FIG. 1. The location and weights of the poles in the response function $K_{00}(q,\omega)$ calculated in the semiclassical approximation [Eqs. (43) and (28)–(32)] and using the bare band mass m_b, a vanishing Coulomb potential $\epsilon \to \infty$, and no scattering ($\tau \to \infty$). Results are shown for filling fractions $\nu = \frac{1}{3}, \frac{3}{7}$, and $\frac{10}{21}$ corresponding to effective Landau level fillings $p = 1, 3$, and 10, respectively, where $m = 1$. Solid curves show the locations of the poles; the width of the striped band around each pole is q^{-2} times the weight of the pole in K_{00}. Note that the cyclotron mode [$\omega = (2p+1)\Delta\omega_c = \omega_c$] has all of the weight at $q \to 0$ in accordance with Kohn's theorem.

We now imagine a semiclassical quasiparticle orbiting in the magnetic field ΔB with a semiclassical cyclotron radius of R_Δ in the presence of a perturbing electric field $\mathbf{E}_{\text{eff}}$ which is the sum of the Chern-Simons and actual electric fields. If the perturbation is applied at $\omega \approx 0$ and at a wave vector $\mathbf{q}$ such that the wavelength is less than the cyclotron radius ($2\pi/q \ll R_\Delta$), then during the course of one orbit the quasiparticle experiences $\mathbf{E}_{\text{eff}}$ in oppposite directions as shown in Fig. 2. When the quasiparticle is moving essentially parallel to the wave vector $\mathbf{q}$ (the $\hat{\mathbf{x}}$ direction in our convention) as shown by the dotted lines of the orbit in Fig. 2, it experiences a quickly oscillating field. Thus there is no net energy loss or gain from moving through this part of the orbit. On the other hand, when the quasiparticle is moving perpendicular to the wave vector (the $\hat{\mathbf{y}}$ direction) as shown by the solid lines in Fig. 2, it experiences the same force for an extended period of time. This is the part of the orbit where the quasiparticle can gain or lose energy due to its interaction with the field.

We now use the fact that the semiclassical quasiparticle conductivity is just the Fourier transform of the velocity autocorrelation function:[19]

$$\tilde{\sigma}_{ij} \propto \int_0^\infty d(t-t') e^{-i\omega(t-t')} \langle v_i(t) v_j(t') e^{i\mathbf{q}\cdot[\mathbf{r}(t)-\mathbf{r}(t')]} \rangle \ . \tag{79}$$

Since the quasiparticle is undergoing cyclotron motion,

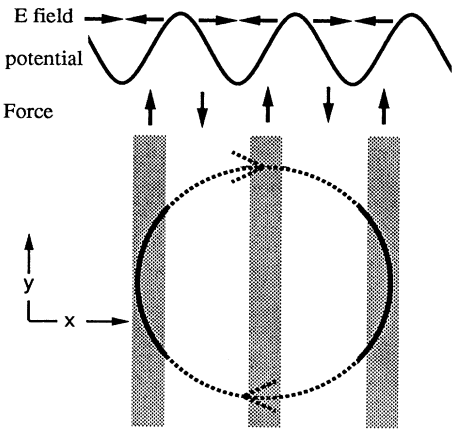

FIG. 2. The semiclassical orbiting mode of a quasiparticle in a perturbing field $\mathbf{E}_{\text{eff}}$ that represents the sum of the Chern-Simons and actual electromagnetic fields. Here we show the effect of applying a zero-frequency wave with wavelength less than the effective cyclotron diameter to an orbiting quasiparticle. As described in the text, when the quasiparticle is moving essentially parallel to the wave vector (shown dotted) it experiences an oscillating field. However, when the quasiparticle is moving perpendicular to the wave vector (shown solid) then it feels the same force for a long period of time.

$\exp\{i\mathbf{q}\cdot[\mathbf{r}(t)-\mathbf{r}(t')]\}$ oscillates very quickly when the particle is moving in the $\hat{\mathbf{x}}$ direction, and stays constant when the particle is moving in the $\hat{\mathbf{y}}$ direction (see Fig. 2). Thus, when we integrate to obtain the low-frequency conductivity, we expect that $\tilde{\sigma}_{yy}$ will, in general, be the largest component of the conductivity tensor. This assertion is verified by examining Eqs. (B15), (B17), and (B19) of Appendix B where we see that $\tilde{\sigma}_{yy}$ is of highest order in q, and thus dominates in the large q limit. We conclude that the zero eigenvalue of the quasiparticle conductivity tensor (and hence the pole in the electromagnetic response) must occur very close to the point where $\tilde{\sigma}_{yy}=0$.

We now set t' to be the time when the quasiparticle is at the extreme right of its orbit such that $v_y(t')$ is large and negative. Clearly the exponential factor $\exp\{i\mathbf{q}\cdot[\mathbf{r}(t)-\mathbf{r}(t')]\}$ is unity whenever the quasiparticle returns to the extreme right of the orbit. Furthermore, if the diameter of the orbit is approximately an integer number of wavelengths, then the exponential factor is approximately unity when the quasiparticle is at the extreme left of the orbit also. Now since $v_y(t)$ oscillates (with frequency $\Delta\omega_c$) and is a maximum at the far left and a minimum at the far right, we see that these two pieces will approximately cancel in the integral in Eq. (79). Thus if the diameter of the orbit is approximately an integer number of wavelengths, $\tilde{\sigma}_{yy}$ will be zero, and hence there will be a pole in the response function. More careful analysis shows that the condition for having a pole in the response function at zero frequency is

$$2R_\Delta=(n+\tfrac{1}{4})\lambda=\frac{2\pi}{q}(n+\tfrac{1}{4}) \tag{80}$$

which is exactly the $\omega\to 0$ limit of Eq. (78). The "$+\frac{1}{4}$" is included because the average separation of the two transverse parts of the orbit is somewhat less than the the full diameter of the orbit (see Fig. 2). These poles at zero frequency were first predicted by HLR,[5] and are somewhat analogous to the "geometric resonances" found in the propagation of acoustic waves in a direction perpendicular to an applied magnetic field in three-dimensional metals.[15]

Now we consider the effect of nonzero frequency. When the wave is in motion, we want to arrange that the phase of the wave when the quasiparticle is at one transverse part of the orbit is the same as the phase of the wave when the quasiparticle reaches the other transverse part of the orbit such that $\exp\{i\mathbf{q}\cdot[\mathbf{r}(t)-\mathbf{r}(t')]-i\omega t\}$ is equal at the extreme left and extreme right of the orbit. This is most easily visualized by considering only the coordinate of the quasiparticle which is parallel to the wave vector (the $\hat{\mathbf{x}}$ coordinate in our previous convention). Now consider the linear world lines of the crests of the wave and the sinusoidal world line of the quasiparticle as shown in Fig. 3. There are two possible ways to have exactly no net contibution to the integral (79). The first possibility is that every time the particle moves to the right it begins and ends at the same phase of the wave (case I in Fig. 3). Alternately, the particle can begin and end at the same phase of the wave every time it moves to the left (case II in Fig. 3). Note that these two cases are not equivalent since the phase of the wave is different at the beginning of each orbit. It is not too hard to see that these two possible conditions are exactly the conditions written above in Eq. (78). If either of these conditions are met, then the corresponding contributions to the integral (79) from the extreme left and extreme right of the orbit cancel, and we should have a zero of $\tilde{\sigma}_{yy}$ and hence a pole in the response.

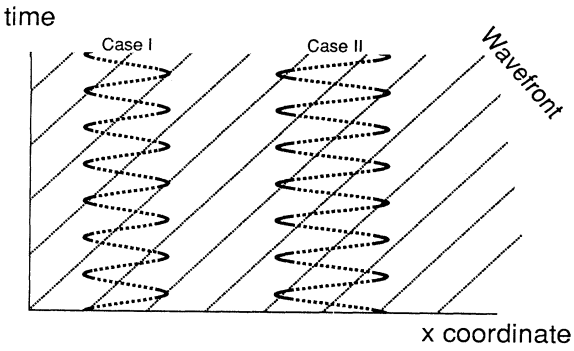

FIG. 3. Semiclassical orbiting modes of a quasiparticle in an electromagnetic wave at nonzero frequency. Here we show the world lines of oscillating particles (sinusoidal) and the world lines (parallel lines) of the wave crests. As described in the text, in order to have a zero in the quasiparticle conductivity and hence a pole in the electromagnetic response, we must arrange so that the phase of the wave is the same on the extreme left as it is on the extreme right so that the largest contributions to the integral (79) cancel. In case I, each time the quasiparticle moves to the right it begins and ends at the same phase of the wave. In case II, each time the quasiparticle moves left it begins and ends at the same phase of the wave. These two cases are not, in general, equivalent since the phase of the wave is not the same at the beginning and the end of an orbit.

Finally we consider the special case when $\omega=n\Delta\omega_c$. Since ω and $\Delta\omega_c$ are commensurate, we can have energy absorbed and reemitted at the applied frequency, and hence a pole in the quasiparticle conductivity. It is easy to see that in this conditon ($\omega=n\Delta\omega_c$) both above cases I and II [both signs of Eq. (78)] are satisfied simultaneously. This would correspond to the "crossing" of the lines of poles in the above spectrum (Fig. 1) at multiples of the effective cyclotron frequency $\Delta\omega_c$ as predicted by Eq. (78). Note, however, that the lines of poles in Fig. 1 do not actually cross at these points. The fact that the quasiparticle conductivity has a pole rather than a zero at these special frequencies creates a "level-repulsion" keeping the lines of poles from crossing. In terms of the integral (79), the pole in the conductivity occurs because the phase of $\exp\{i\mathbf{q}\cdot[\mathbf{r}(t)-\mathbf{r}(t')]-i\omega t\}$ is the same at the beginning of each orbit. Thus, any small noncancellation of the contributions to the integral will occur identically for each orbit, and thus these terms will add and cause a diverging conductivity.

Although the effects of this semiclassical orbiting behavior are most obvious in the large X and large p limits, the same general behavior is seen even for $p=1$ (although the validity of the semiclassical approximation at low p and low frequency is questionable). One notes

that the semiclassical theory predicts a series of magnetorotonlike minima, i.e., minima in the dispersion curve of the magnetoexciton (the lowest neutral excited mode). The first of these minima occurs for any given p approximately where the first zero-frequency mode would occur in the large p limit ($X \approx \frac{5}{4}\pi$).

B. Unrenormalized RPA

We now consider the RPA in the same limit where the mass is unrenormalized ($m^* = m_b$). This is the calculation considered by Lopez and Fradkin in the $q \to 0$ limit.[4] However, unlike Lopez and Fradkin, we will begin by considering the case when the Coulomb interaction is turned off ($\epsilon \to \infty$). In this case we calculate the mean-field unperturbed response function K^0 using Eqs. (38)–(40) which are given as a simplified single sum in Appendix A. This mean-field result is then converted into the full RPA response function using the RPA equation (12). The results of such a calculation are shown in Fig. 4. Once again we have shown the location of the pole (heavy solid) in the density-density response function K_{00} as a function of reduced wave vector and reduced frequency, and the width of the striped bands indicate q^{-2} times the relative weights of the poles. As before we have shown results for filling fractions $\nu = \frac{1}{3}$, $\frac{3}{7}$, and $\frac{10}{21}$ corresponding to effective Landau level fillings of $p = 1$, $p = 3$, and $p = 10$, respectively, where $m = 1$.

In the large p limit, the observed behavior looks very much like the above semiclassical picture of particles undergoing cyclotronlike oscillations while interacting with the Chern-Simons force. This is to be expected since the semiclassical picture is thought to be accurate in this limit. We also note that in all cases except $p = 1$, we have additional modes of small residue in the RPA calculation that did not show up in the semiclassical case. The difference between the RPA and the semiclassical calculations is that for the semiclassical quasiparticle conductivity, the residue at the pole $n\Delta\omega_c$ is proportional to projection matrix $[V_i^n]^* V_j^n$ [see Eq. (43)]. Such projection matrices have a zero eigenvalue. This singular situation causes the disappearance of additional solutions of the equation for the poles of the system. Similarly, for the $p = 1$ case of the RPA when we calculate K^0 (see Appendix A) there is only one way for a virtual transition to take place with a given energy difference $n\Delta\omega_c^*$ (a jump from the single filled level to the nth empty level and back). When only a single process contributes to a given pole, then the residue takes the projectionlike form $\langle i|j_i|f\rangle\langle f|j_j|i\rangle$ where $|i\rangle$ is the initial state and $|f\rangle$ is the final state. In the $p > 1$ case for the RPA several processes can contribute to each pole, except the lowest one, and thus interfere to ruin the perfect projection form of the residues in the quasiparticle response function, and so we see additional poles. However, to the extent that the semiclassical calculation approximates the RPA, we expect that the additional poles will have a very small residue.

If we were to add a Coulomb interaction by taking ϵ finite (for typical experimental parameters the density of electrons is $n_e = 10^{-11}$ cm^{-2} and the dielectric constant $\epsilon = 12.6$) the resulting response function would be quite similar to Fig. 4, so we have not shown it here. The main effect of the Coulomb interaction is to push the weight of the poles up to higher-frequency modes. In addition, very small shifts in the frequency of the modes are also seen. Since the Coulomb interaction couples to density fluctuations, it is mainly those modes corresponding to large residues of the density-density response function that are affected. At small q, the cyclotron mode has all the weight, so it is shifted the most. However, exactly at $q = 0$, Kohn's theorem must be satisfied so the total weight of the cyclotron pole (which is the sum of two modes in the $p = 1$ case) must stay the same.

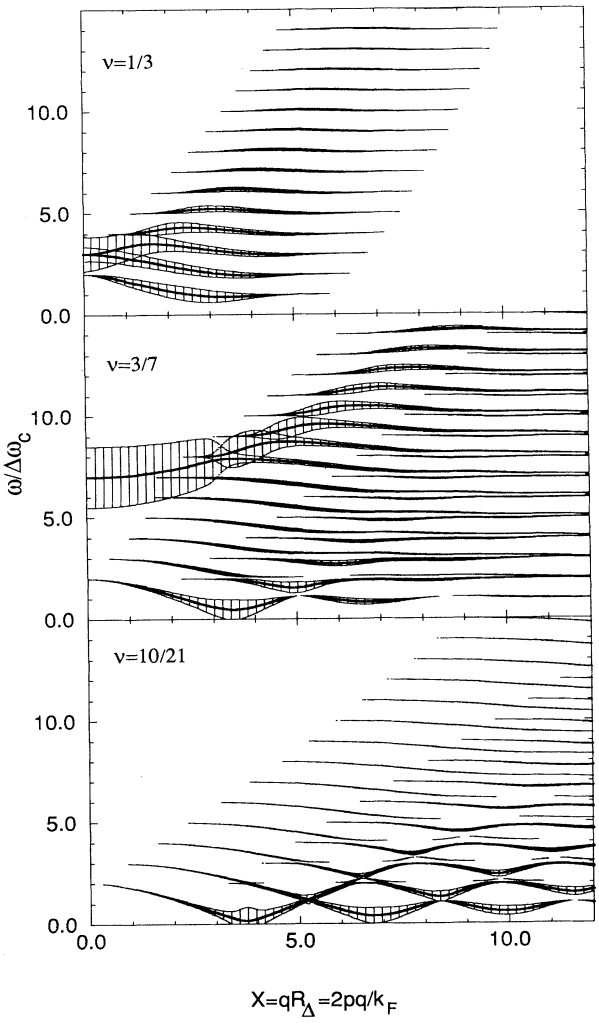

FIG. 4. The location and weights of the poles in the response function $K_{00}(q,\omega)$ calculated in the RPA [Eqs. (38)–(40)] using the bare band mass m_b and a vanishing Coulomb potential ($\epsilon \to \infty$). Results are shown for filling fractions $\nu = \frac{1}{3}, \frac{3}{7}$, and $\frac{10}{21}$ corresponding to effective Landau level fillings $p = 1, 3$, and 10, respectively, where $m = 1$. The width of the striped band around each pole (solid) is q^{-2} times the weight of the pole in K_{00}. In the large p limit our results look very much like the semiclassical results of Fig. 1.

The Coulomb interaction also seems to slightly reduce the depth of the magnetoexciton minima.

C. Modified RPA

Here we consider the effect of using the renormalized mass. The method of calculating the response function—which we outline here—was described in Sec. IV C. We begin by using Eq. (39) to calculate the naive quasiparticle conductivity $\tilde{s}^{\rm n}$ in a field ΔB where here we have replaced $\Delta\omega_c$ by $\Delta\omega_c^*$ in Eqs. (39) and (40). Again, the necessary sums are given in a simplified form in Appendix A. We then set $\tilde{\rho}^{\rm n} = (\tilde{s}^{\rm n})^{-1}$ and use Eq. (77) to include the mass-renormalization and Chern-Simons terms to yield the resistivity ρ. Finally, the resistivity is converted to an electromagnetic response K using Eq. (28). Results of these calculations are shown in Fig. 5.

Figure 5 should be compared to Fig. 4 where we have not renormalized the mass or included the Coulomb interaction. First, however, we note that if we were to make the naive RPA approximation (simply inserting m^* into the Hamiltonian in place of m_b to calculate $\tilde{\rho}^{\rm n}$, then including the Chern-Simons term and converting this into a response via $K^{-1} = T[\tilde{\rho}^{\rm n} + \rho_{\rm CS}]T$) the end result would be exactly like Fig. 4 except that the vertical scale would now be $\omega/\Delta\omega_c^*$ as it is in Fig. 5 and all the weights would be scaled by the same factor. Although this approach gives the correct energy scale for the low-energy excitations, it fails to satisfy the f-sum rule, and the cyclotron frequency is incorrect.

On the other hand, if we include the mass renormalization properly by using the modified RPA [Eqs. (39), (40), (77), and (28) as described above] but still ignore the Coulomb interaction, we find that the cyclotron frequency is pushed up to its correct value [$\omega_c = m^*(2p+1)\Delta\omega_c^*/m_b$] and the weights of the poles satisfy the f-sum rule. At the same time, the low-lying modes in such a calculation are virtually unchanged from the above mentioned Fig. 4 rescaled.

Finally, when we include the Coulomb interaction as well as the mass renormalization in the modified RPA, we obtain Fig. 5. Here we show the results of such a calculation for filling fractions $\nu = \frac{1}{3}, \frac{3}{7}$, and $\frac{10}{21}$ corresponding to effective Landau-level fillings $p = 1, 3$, and 10, respectively, where $m = 1$. Once again the width of the striped bands indicates q^{-2} times the weight of the pole (solid). We have used the experimentally relevant parameters $n_e = 10^{-11}$ cm^{-2}, $\epsilon = 12.6$, and an effective mass of $m^* = 3.9m_b$ for illustrative purposes. As discussed in Sec. IV, the value $m^* = 3.9m_b$ is thought to be approximately correct for certain relevant experimental conditions. ($m^* = 4m_b$ was avoided simply because there is no reason to assume that the effective mass should be an integer multiple of the band mass.) Although, as we discussed in Sec. IV, the effective mass is in general a function of the magnetic field, we will treat it as a constant here. The results given in Fig. 5 are the complete theory including Coulomb interaction and the mass renormalization due to the Fermi-liquid interaction A_1.

We note that the general structures of Fig. 4 and Fig. 5 are similar, except that the cyclotron frequency has been pushed up from the effective cyclotron frequency $\omega_c^* = (2p+1)\Delta\omega_c^*$ in Fig. 4 (where we read the vertical scale as $\omega/\Delta\omega_c^*$) to the bare cyclotron frequency $\omega_c = m^*(2p+1)\Delta\omega_c^*/m_b$ in Fig. 5. In the cases of $p = 3$ and $p = 10$, the cyclotron frequency is pushed off of the top of the graph shown. As mentioned above, Fig. 4 rescaled provides a good approximation of the low-lying excitations when the Coulomb interaction is ignored. By comparing Fig. 4 and Fig. 5, we see that the inclu-

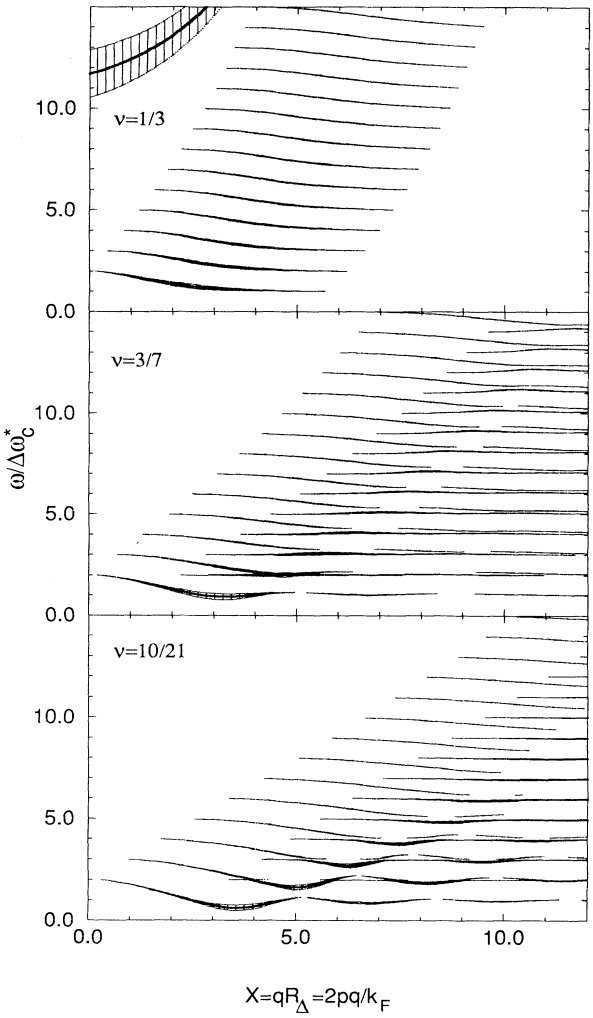

FIG. 5. The location and weights of the poles in the response function $K_{00}(q,\omega)$ calculated in the modified RPA as described in Sec. IV C using a renormalized mass $m^* = 3.9m_b$ and a Coulomb potential for an electron density $n_e = 10^{-11}$ cm^{-2} and a dielectric constant $\epsilon = 12.6$. Results are shown for filling fractions $\nu = \frac{1}{3}, \frac{3}{7}$, and $\frac{10}{21}$ corresponding to effective Landau level fillings $p = 1, 3$, and 10, respectively, where $m = 1$. The width of the striped band around each pole (solid) is q^{-2} times the weight of the pole in K_{00}. This figure is the complete theory including both Coulomb interaction and mass renormalization from the Fermi-liquid interaction term A_1. The quantity $\Delta\omega_c^*$ is defined as $e|\Delta B|/(m^*c)$, where $\Delta B = B/(2p+1)$.

sion of the direct Coulomb interaction causes a significant change in the shape of the lowest magnetoexciton curve. In particular the magnetoexciton minima are much less pronounced. The Coulomb interaction has a much larger effect in the modified RPA than it did in the simple RPA primarily because the overall energy scale is smaller.

In the case of $p = 1$ ($\nu = \frac{1}{3}$), these results may be compared directly with results of the single mode approximation[7] which is believed to be quite accurate near the magnetoroton minimum. One finds that the actual minimum is significantly deeper than that found in either the RPA, the modified RPA, or the semiclassical approximation. A relatively deep magnetoroton minimum has also been found in numerical work on finite systems[8,9] and in the analytic approach of Zhang, Hanson, and Kivelson.[20]

We speculate that the inclusion of Coulomb ladder diagrams (i.e., the attraction between the quasihole and quasiparticle of the exciton) would enhance the size of the magnetoroton minimum relative to that found in the RPA or modified RPA, and would perhaps bring the perturbative Chern-Simons calculation into better agreement with the other calculations in this regime. We also speculate that the Coulomb ladder diagrams may be relatively less important in the case of large p, where the charges of the quasiparticle and quasihole are small, so that the modified RPA may give an accurate description of the dispersion of the lowest excitation mode in this case. We find that for very large p, the dispersion curves show a series of deep minima which are respresented in the RPA, modified RPA, or semiclassical approximation. However, at $p = 10$ ($\nu = 10/21$) there are still significant differences in the depths of the exciton minima according to Figures 1, 4, and 5. Again we note that the depths of the magnetoexciton minima are somewhat suppressed by the Coulomb interaction in the modified RPA. The other main contribution of the Coulomb interaction is simply to push the weight of the poles to higher-frequency modes.

VI. SUMMARY

In this paper we have reviewed the Chern-Simons construction that allows us to think of certain fractional quantized Hall states as integer quantized Hall states of fermionic quasiparticles bound to an even number of flux quanta. The electromagnetic response function was first calculated in a semiclassical approximation and within the RPA. If one uses the bare electron band mass m_b in these calculations, one obtains an incorrect frequency scale for the low-energy excitations; if one simply replaces m_b by an effective mass m^* which is chosen to give the correct scale for the low-energy excitations, then one obtains an incorrect value for the cyclotron energy, in violation of Kohn's theorem, and one obtains intensities that fail to satisfy the f-sum rule. A modified RPA was then constructed that accounts for the effective-mass renormalization by using a Fermi-liquid theory approach. The results of the modified RPA calculation properly satisfy the f-sum rule and Kohn's theorem and also have the low-energy excitation spectrum approximately correct. In the semiclassical regime (ν approaching an even denominator fraction and large wavelength compared to the magnetic length) we clearly see the orbiting behavior that results in geometric resonances including a series of magnetoexciton minima at increasing wave vector. At $\nu = \frac{1}{3}$, the magnetoroton minimum is not as deep in our approximation as previous works predict. (We speculate that Coulomb ladder diagrams which have not been included within the RPA may increase the depth.) Finally, we note that within the RPA the main effect of the direct interquasiparticle Coulomb interaction is to slightly reduce the depth of the magnetoexciton minima and to push some of the weight of the poles of the response function up to modes of higher frequencies.

All of the approximations discussed in this paper omit the possible effects of quasiparticle scattering. When such effects are taken into account, we expect that, in general, the higher excitation modes will acquire a finite energy width, as they can generally decay into two or more modes of lower energy, while conserving momentum and energy. If the decay rate becomes larger than the spacing between modes for some regions of the k, ω plane, then the energy spectra indicated in Fig. 5 will cease to be meaningful in that region. By contrast, we expect that the lowest-energy branch will remain perfectly sharp, in the absence of impurity scattering, at least near the magnetoexciton minima because there are no lower-lying excitations to decay to with conservation of energy and momentum.

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APPENDIX A: QUANTUM-MECHANICAL SUM

To reduce the sums in the definition of the Σ_j [Eq. (40)] to single sums, we start by reparametrizing our dummy variables using $n = m - l$ such that

$$\Sigma_j = \frac{e^{-Y}}{p} \sum_{n=1}^{\infty} \frac{n Y^{n-1}}{R^2 - n^2} S_j(n, Y) , \tag{A1}$$

$$S_j(n, Y) = \sum_{l=\max(0, p-n)}^{p-1} G_j(n, l, Y) , \tag{A2}$$

where

$$G_j(n,l,Y) = \frac{l!}{(n+l)!}\left[L_l^n(Y)\right]^{2-j} \times \left[(n-Y)L_l^n(Y) + 2Y\frac{dL_l^n(Y)}{dY}\right]^j , \quad \text{(A3)}$$

and $L_l^n(Y)$ is the associated Laguerre polynomial. In the calculation of $\tilde{s}$ to get K^0 for the RPA in Sec. III A, we use $R = \omega/\Delta\omega_c$ whereas in the calculation of $\tilde{s}^n$ for the modified RPA we use $R = \omega/\Delta\omega_c^*$. In this notation, the (n,l)th term in the S_j sum represents a particle making a virtual transition from the lth Landau-level up n levels and back. This form for Σ_j is also quite appealing physically since it groups terms of the sum in terms of which pole they contribute to. It will be convenient to think of the sum over l as the difference of two sums both of whose lower limit is zero. In other words,

$$S_j(n,Y) = \left[\sum_{l=0}^{p-1} G_j(n,l,Y)\right] - \Theta(p-n)\left[\sum_{l=0}^{p-n-1} G_j(n,l,Y)\right] , \quad \text{(A4)}$$

where Θ is the step function

$$\Theta(x) = \begin{cases} 0, & x \le 0 \\ 1, & x > 0. \end{cases} \quad \text{(A5)}$$

The point of this appendix is to perform the sums over l to yield closed form expressions for $S_j(n,Y)$.

To perform our sums it will be necessary to evaluate the three quantities

$$T_n^\alpha(Y) = \sum_{m=0}^{n} \frac{m!}{(m+\alpha)!}\left[L_m^\alpha(Y)\right]^2 , \quad \text{(A6)}$$

$$U_n^\alpha(Y) = \sum_{m=0}^{n} \frac{m!}{(m+\alpha)!} L_m^\alpha(Y)\frac{d}{dY}L_m^\alpha(Y) , \quad \text{(A7)}$$

$$V_n^\alpha(Y) = \sum_{m=0}^{n} \frac{m!}{(m+\alpha)!}\left[\frac{dL_m^\alpha(Y)}{dY}\right]^2 . \quad \text{(A8)}$$

These can be calculated by first considering the Christoffel-Darboux formula[21]

$$\sum_{m=0}^{n} \frac{m!}{(m+\alpha)!} L_m^\alpha(x)L_m^\alpha(y) = \frac{(n+1)!}{(n+\alpha)!(x-y)}\left[L_n^\alpha(x)L_{n+1}^\alpha(y) - L_{n+1}^\alpha(x)L_n^\alpha(y)\right] . \quad \text{(A9)}$$

By setting $x = Y+\delta$ and $y = Y$ and differentiating with respect to δ at $\delta = 0$, and then using the identity[21]

$$\frac{dL_n^\alpha(Y)}{dY} = -L_{n-1}^{\alpha+1}(Y) , \quad \text{(A10)}$$

where a Laguerre polynomial of negative lower index is defined here to be zero, one easily derives

$$T_n^\alpha(Y) = \frac{(n+1)!}{(n+\alpha)!}\left[L_n^\alpha(Y)L_n^{\alpha+1}(Y) - L_{n-1}^{\alpha+1}(Y)L_{n+1}^\alpha(Y)\right] . \quad \text{(A11)}$$

Differentiating this result yields

$$U_n^\alpha(Y) = \frac{1}{2}\frac{(n+1)!}{(n+\alpha)!}\left[L_{n-2}^{\alpha+2}(Y)L_{n+1}^\alpha(Y) - L_n^\alpha(Y)L_{n-1}^{\alpha+2}(Y)\right] . \quad \text{(A12)}$$

The easiest way to find $V_n^\alpha(Y)$ without running into divisions by zero is to write the derivatives as limits

$$V_n^\alpha(Y) = \lim_{\gamma,\epsilon,\delta\to 0}\sum_{m=0}^{n} \frac{m!}{(m+\alpha)!}\left[\frac{L_m^\alpha(Y+\gamma+\delta) - L_m(Y+\gamma)}{\delta}\right]\left[\frac{L_m^\alpha(Y+\epsilon) - L_m^\alpha(Y)}{\epsilon}\right] \quad \text{(A13)}$$

such that we end up with four terms in the Christoffel-Darboux form. The sums are then performed using Eq. (A9). At this point one must be very careful in taking the limits. The easiest way to do this is to expand each resulting Laguerre polynomial in a Taylor series around Y to *third order* such that the parameters γ, ϵ, and δ no longer occur inside the arguments of the polynomials. (At the end of the calculation it is easy to see that higher-order terms are irrelevant since we will take the small parameters γ, ϵ, and δ to zero anyway.) Finally one can take the limits and find that the first- and second-order terms of the expansion vanish leaving the result

$$V_n^\alpha(Y) = \frac{(n+1)!}{(n+\alpha)!}\left[\tfrac{1}{6}\{L_{n-2}^{\alpha+3}(Y)L_n^\alpha(Y) - L_{n-3}^{\alpha+3}(Y)L_{n+1}^\alpha(Y)\} + \tfrac{1}{2}\{L_{n-1}^{\alpha+2}(Y)L_{n-1}^{\alpha+1}(Y) - L_{n-2}^{\alpha+2}(Y)L_n^{\alpha+1}(Y)\}\right] , \quad \text{(A14)}$$

where we have used the above Laguerre polynomial identity (A10) several times.

We can now use the three derived sums (T, U, and V) to perform the l sums in $S_j(n,Y)$ Using Eq. (A11) and the definition of $T_\alpha^n(Y)$ we have immediately

$$S_0(n,Y) = T^n_{p-1}(Y) - \Theta(p-n)T^n_{p-n-1}(Y) \quad \text{(A15)}$$

and similarly we use the sums (A11) and (A12) to yield

$$\begin{aligned} S_1(n,Y) = &[(n-X)T^n_{p-1}(Y) + 2YU^n_{p-1}(Y)] \\ &-\Theta(p-n)[(n-Y)T^n_{p-n-1}(Y) \\ &+2YU^n_{p-n-1}(Y)] \,. \end{aligned} \quad \text{(A16)}$$

Now by using the Laguerre polynomial identities 8.97.4 and 8.97.5 from Ref. 21 this can be reduced to

$$\begin{aligned} S_1(n,Y) = &\frac{p!}{(p+n-1)!}L^{n+1}_{p-1}(Y)L^{n-1}_p(Y) \\ &-\Theta(p-n)\frac{(p-n)!}{(p-1)!}L^{n+1}_{p-n-1}(Y)L^{n-1}_{p-n}(Y). \end{aligned} \quad \text{(A17)}$$

And finally we have

$$\begin{aligned} S_2(n,Y) = &\left[(n-Y)^2T^n_{p-1}(Y) + 4Y(n-Y)U^n_{p-1}(Y) + 4Y^2V^n_{p-1}(Y)\right] \\ &-\Theta(p-n)\left[(n-Y)^2T^n_{p-n-1}(Y) + 4Y(n-Y)U^n_{p-n-1}(Y) + 4Y^2V^n_{p-n-1}(Y)\right] . \end{aligned} \quad \text{(A18)}$$

Although these results look somewhat messy, they eliminate one infinite sum which is beneficial for both analytic and numerical work.

APPENDIX B: SEMICLASSICAL SUM

When we take the $\tau \to 0$ limit we can rewrite the quasiparticle conductivity given in Eq. (43) as

$$\tilde{\sigma}_{ij} = \frac{ipe^2}{\pi\hbar}\sum_{n=-\infty}^{\infty}\frac{V_i^{(n)}V_j^{(n)*}}{R-n} \,, \quad \text{(B1)}$$

where p is the number of effective Landau-levels filled which need not be an integer. We use $R = \omega/\Delta\omega_c$ for the semiclassical calculation and $R = \omega/\Delta\omega_c^*$ for the modified semiclassical approximation (which then yields $\tilde{\sigma}^n$ as a result).

Using symmetry relations of the Bessel functions (Eq. 9.1.5 from Ref. 22) the quasiparticle conductivity matrix can be rewritten as

$$\tilde{\sigma}_{xx} = \frac{ipe^2}{\pi\hbar}\sum_{n=1}^{\infty}\frac{n^2J_n^2(X)}{X^2}\frac{2R}{R^2-n^2} \,, \quad \text{(B2)}$$

$$\tilde{\sigma}_{xy} = \frac{pe^2}{\pi\hbar}\sum_{n=1}^{\infty}\frac{n^2J_n(X)J_n'(X)}{X}\frac{2}{R^2-n^2} \,, \quad \text{(B3)}$$

$$\tilde{\sigma}_{yy} = \frac{ipe^2}{\pi\hbar}\left[\frac{[J_0'(X)]^2}{R} + \sum_{n=1}^{\infty}[J_n'(X)]^2\frac{2R}{R^2-n^2}\right] . \quad \text{(B4)}$$

In order to evaluate these sums, we first consider the general quantity

$$W_\alpha = \sum_{n=1}^{\infty}\frac{J_{n+\alpha}(X)J_{n-\alpha}(X)}{R^2-n^2} . \quad \text{(B5)}$$

Using the integral identities 3.715.19 and 6.681.1 from Ref. 21 we can rewrite W_α as a sum over a double integral of cosines,

$$\begin{aligned} W_\alpha = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{R^2-n^2}\int_0^{\frac{\pi}{2}}dt\int_0^{\frac{\pi}{2}}dt'&\cos(2X\,\cos t\cos t') \\ &\times\cos(2\alpha t)\cos(2nt') \,. \end{aligned} \quad \text{(B6)}$$

The sum is then performed by using sum 1.445.8 from Ref. 21 leaving us with

$$W_\alpha = Q_\alpha + P_\alpha \,, \quad \text{(B7)}$$

where

$$Q_\alpha = \frac{-2}{\pi^2R^2}\int_0^{\pi/2}dt\int_0^{\pi/2}dt'F_\alpha[t,t',X] \,, \quad \text{(B8)}$$

$$P_\alpha = \frac{2}{\pi R\,\sin(R\pi)}\int_0^{\pi/2}dt\int_0^{\pi/2}dt'F_\alpha[t,t',X]\cos(2Rt') \,, \quad \text{(B9)}$$

and

$$F_\alpha[t,t',X] = \cos(2X\cos t\cos t')\cos(2\alpha t). \quad \text{(B10)}$$

For both of these terms the integral over t can be performed using 3.715.19 from Ref. 21, and the integral over t' can then be performed using 6.681.1 from Ref. 21. The end result is the desired quantity

$$W_\alpha = (-1)^\alpha\left[-\frac{J_\alpha^2(X)}{2R^2} + \frac{\pi J_{\alpha+R}(X)J_{\alpha-R}(X)}{2R\,\sin(R\pi)}\right] . \quad \text{(B11)}$$

Using this partial result we can calculate the more relevant quantity

$$\tilde{W}_\alpha = \sum_{n=1}^{\infty}\frac{n^2}{R^2-n^2}J_{n+\alpha}(X)J_{n-\alpha}(X) \quad \text{(B12)}$$

$$= \sum_{n=1}^{\infty}\left[-1 + \frac{R^2}{R^2-n^2}\right]J_{n+\alpha}(X)J_{n-\alpha}(X) . \quad \text{(B13)}$$

The first term can be evaluated using the orthogonality relation of Bessel functions (Eq. 9.1.75 of Ref. 22) and the second term is just $R^2 W_\alpha$. Thus we have

$$\tilde{W}_\alpha = -\frac{1}{2}\delta_{\alpha,0} + (-1)^\alpha \frac{\pi R}{2\sin(R\pi)} J_{\alpha+R}(X) J_{\alpha-R}(X) \ . \tag{B14}$$

Now using the $\alpha = 0$ case of this result immediately allows us to perform one of the desired sums giving us the result

$$\tilde{\sigma}_{xx} = \frac{ipe^2}{\pi\hbar}\frac{2R}{X^2}\left[-\frac{1}{2} + \frac{\pi R}{2\sin(R\pi)} J_R(X) J_{-R}(X)\right] \ . \tag{B15}$$

Furthermore, by differentiating $\tilde{W}_\alpha$ with respect to X, we can derive

$$\tilde{\sigma}_{xy} = \frac{pe^2}{\pi\hbar}\frac{\pi R}{2X\sin(R\pi)}\left[J_R(X) J_{-R}(X)\right]' \ . \tag{B16}$$

By using the Bessel function identities 9.1.27 from Ref. 22 as well as the Wronskian identity 9.1.15 from Ref. 22 we can rewrite the off-diagonal conductance as

$$\tilde{\sigma}_{xy} = -i\tilde{\sigma}_{xx} - \frac{pe^2}{\pi\hbar}\frac{R\pi}{X\sin(R\pi)} J_{R+1}(X) J_{-R}(X) \ . \tag{B17}$$

The evaluation of $\tilde{\sigma}_{yy}$ is achieved by using the same Bessel function identities 9.1.27 from Ref. 22 to derive

$$[J_n'(X)]^2 = -J_{n-1}(X) J_{n+1}(X) + \frac{n^2}{X^2} J_n^2(X). \tag{B18}$$

The sum over the first term is just W_1 whereas the sum over the second term is $X^{-2}\tilde{W}_0$. The result is easily simplified to

$$\tilde{\sigma}_{yy} = \tilde{\sigma}_{xx} + \frac{ipe^2}{\pi\hbar}\frac{\pi}{\sin(R\pi)} J_{1+R}(X) J_{1-R}(X). \tag{B19}$$

We can evaluate some of the limits of this expression for the conductivity by expanding the Bessel functions in their defining series (Eq. 9.1.10 of Ref. 22). The condition for this series expansion to be a good approximation is that X^2/R be much less than one. If we insert this expansion into the above expressions (and using Eq. 6.1.17 of Ref. 22) to simplify the result, we find the expected result

$$\tilde{\sigma} = \frac{e^2 n_e}{m_b}\frac{1}{(\Delta\omega_c)^2 - \omega^2}\begin{bmatrix} -i\omega & -\Delta\omega_c \\ \Delta\omega_c & -i\omega \end{bmatrix} + O(X^2/R) \ . \tag{B20}$$

The important thing to realize here is that within the semiclassical approximation, a low q expansion and a large ω expansion are equivalent. This is not obvious from the original expression for the conductivity [Eq. (43)], but becomes clear once we have this closed form expression.

APPENDIX C: FERMI-LIQUID THEORY

In this appendix we use the Landau-Silin Fermi-liquid theory[16,17,19] to determine the effect of mass renormalization on the conductivity of a system in a magnetic field. We assume here a two-dimensional system of spinless Fermions in a magnetic field B. (Note that in the text we consider a system in a field ΔB. We have dropped the delta for simplicity of notation.) For this system, the linearized semiclassical transport equation (which should be accurate in the semiclassical regime as discussed in Sec. III) is written as[16,19]

$$\frac{\partial \delta n}{\partial t} + \left[\mathbf{v_k}\cdot\boldsymbol{\nabla}_\mathbf{r} - \frac{e}{c}(\mathbf{v_k}\times\mathbf{B})\cdot\boldsymbol{\nabla}_\mathbf{k}\right]\left[\delta n + \delta\epsilon_1\frac{\partial n_0}{\partial\epsilon_0}\right] = -e\mathbf{E}\cdot\mathbf{v_k}\frac{\partial n_0}{\partial\epsilon_0} + I \ , \tag{C1}$$

where $\mathbf{k} = \mathbf{p} + \frac{e}{c}\mathbf{A}$ is the kinetic momentum, $\epsilon_0 = k^2/(2m^*)$ is the kinetic energy of a noninteracting quasiparticle, m^* is the effective mass of a quasiparticle at the Fermi surface, $\delta n(\mathbf{k},\mathbf{r})$ is the local deviation from the equilibrium distribution $n_0(\mathbf{k})$, the local quasiparticle velocity is given by $\mathbf{v_k} = \boldsymbol{\nabla}_\mathbf{k}(\epsilon_0 + \delta\epsilon_1)$, the effects of scattering are included in the collision integral $I(n_0+\delta n)$, and the local change in quasiparticle energy $\delta\epsilon_1$ due to interaction is given by

$$\delta\epsilon_1(\mathbf{k},\mathbf{r}) = \frac{1}{(2\pi)^2}\int d^2k'\Phi(\mathbf{k},\mathbf{k}')\delta n(\mathbf{k}',\mathbf{r}) \ , \tag{C2}$$

where $\Phi(\mathbf{k},\mathbf{k}')$ is the (unknown) Landau interaction function. The above transport equation (C1) is derived (as described loosely in Sec. IV B) by using a single-particle effective Hamiltonian derived from a local energy functional and then using Hamilton's equations of motion for this effective Hamiltonian.[16,19] The equations of motion must be expanded to linear order in the effects of the perturbing electromagnetic field to yield Eq. (C1).

Although the inclusion of a nonzero scattering integral is straightforward,[17] we will assume that $I \to 0$ ($\tau \to \infty$) for simplicity. Keeping with our convention that $\mathbf{q}\|\hat{\mathbf{x}}$, we apply an electromagnetic perturbation proportional to $e^{iqx-i\omega t}$. Following Lee and Quinn[17] we rewrite the linearized transport equation (C1) as[16,17]

$$-i\omega f(\phi) + \left(iqv_x(\phi) + \omega_c^*\frac{\partial}{\partial\phi}\right)[f(\phi) + \delta\epsilon_1(\phi)] = -e\mathbf{E}\cdot\mathbf{v}(\phi) \ , \tag{C3}$$

where $\omega_c^* = eB/(m^*c)$ is the mass renormalized cyclotron frequency, $f(\phi)$ is defined by

$$\delta n(\mathbf{k}) = f(\phi)\frac{-\partial n_0}{\partial\epsilon_0(\mathbf{k})} \tag{C4}$$

with ϕ the angle defining the direction of the kinetic momentum $\mathbf{k}$, and the velocity vector is given by

$$\mathbf{v}(\phi) = v_F(\sin\phi, -\cos\phi). \tag{C5}$$

with $v_F = k_F/m^*$. Since f is periodic in ϕ we can expand it in a Fourier series,

$$f_l = \frac{1}{2\pi}\int_0^{2\pi} d\phi\, f(\phi) e^{il\phi} , \tag{C6}$$

$$f(\phi) = \sum_l e^{-il\phi} f_l . \tag{C7}$$

Similarly, $\Phi(\mathbf{k},\mathbf{k}')$ is periodic in $\phi - \phi'$ so we can write

$$\Phi(\mathbf{k},\mathbf{k}') = \frac{2\pi}{m^*}\sum_l A_l e^{-il(\phi-\phi')} . \tag{C8}$$

Furthermore, since Φ is symmetric and real, we have $A_l = A_{-l}$ and A_l is real.

We can define a displacement vector for the quasiparticles on the Fermi surface

$$\mathbf{R}(\phi) = -\frac{1}{\omega_c^*}\int_{\pi/2}^{\phi} \mathbf{v}(\phi')d\phi' \tag{C9}$$

so that[21]

$$e^{i\mathbf{q}\cdot\mathbf{R}(\phi)} = e^{iX\cos\phi} = \sum_n i^n J_n(X) e^{in\phi} \tag{C10}$$

for $X = qR_c^* = qv_F/\omega_c^*$ and J_n is the Bessel function. We now expand the following periodic functions of ϕ into Fourier series

$$f(\phi)e^{-i\mathbf{q}\cdot\mathbf{R}(\phi)} = \sum_l e^{-il\phi} F_l , \tag{C11}$$

$$\delta\epsilon_1(\phi)e^{-i\mathbf{q}\cdot\mathbf{R}(\phi)} = \sum_l e^{-il\phi}\epsilon_l , \tag{C12}$$

$$\mathbf{v}(\phi)e^{-i\mathbf{q}\cdot\mathbf{R}(\phi)} = \sum_l e^{-il\phi}\mathbf{v}_l . \tag{C13}$$

By inverting these Fourier series and inserting into above definitions we derive the following relations:

$$\epsilon_l = \sum_m A_m f_m i^{m-l} J_{l-m}(X) , \tag{C14}$$

$$f_l = \sum_m i^{m-l} J_{m-l}(X) F_m , \tag{C15}$$

$$\mathbf{v}_l = -v_F i^{-l}\begin{pmatrix} lJ_l(X)/X \\ -iJ_l'(X)\end{pmatrix} , \tag{C16}$$

where the Bessel function identities 9.1.27 from Ref. 22 have been used to derive $\mathbf{v}_l$. Note that the coefficients $\mathbf{v}_l$ are the velocity coefficients used in Eq. (43) to calculate the quasiparticle conductivity up to multiplicative constants. By using the Bessel function orthogonality relation 9.1.75 from Ref. 22 we can also derive the inverse relation

$$F_m = \sum_l i^{l-m} J_{m-l}(X) f_l . \tag{C17}$$

Now multiplying our kinetic equation (C3) by $(2\pi)^{-1}\exp[il\phi - i\mathbf{q}\cdot\mathbf{R}(\phi)]$ and integrating over ϕ yields the kinetic equation in terms of our new variables (previously derived by Lee and Quinn[17])

$$i\omega F_l + il\omega_c^*[F_l + \epsilon_l] = e\mathbf{E}\cdot\mathbf{v}_l , \tag{C18}$$

where we have used the fact that

$$iqv_x e^{-i\mathbf{q}\cdot\mathbf{R}(\phi)} = \omega_c^*\frac{d}{d\phi}e^{-i\mathbf{q}\cdot\mathbf{R}(\phi)} \tag{C19}$$

in our evaluation of the integral.

We now want to express the current in terms of the motion of quasiparticles. This can be done using the standard result of Fermi-liquid theory[16,19]

$$\mathbf{j} = \frac{-e}{(2\pi)^2}\int d^2k\,\delta n(\mathbf{k}) \times\left[\mathbf{v}(\mathbf{k}) + \frac{1}{(2\pi)^2}\int d^2k'\Phi(\mathbf{k},\mathbf{k}') \times\mathbf{v}(\mathbf{k}')\left(\frac{-\partial n_0}{\partial\epsilon(\mathbf{k}')}\right)\right] , \tag{C20}$$

where the second term represents the backflow current due to interactions. By interchanging the order of integration we can rewrite this as

$$\mathbf{j} = \frac{-e}{(2\pi)^2}\int d^2k\,\mathbf{v}(\mathbf{k})\left(\frac{-\partial n_0}{\partial\epsilon(\mathbf{k})}\right)[f(\mathbf{k}) + \delta\epsilon_1(\mathbf{k})] , \tag{C21}$$

which can be expressed in terms of our new variables as

$$\mathbf{j} = \frac{-em^*}{2\pi}\sum_l \mathbf{v}_l^*(F_l + \epsilon_l). \tag{C22}$$

Combining this with Eq. (C18) yields the result

$$\mathbf{j} = \frac{-em^*}{2\pi}\sum_l \mathbf{v}_l^*\left[\frac{e\mathbf{E}\cdot\mathbf{v}_l - i\omega F_l}{il\omega_c^*}\right] . \tag{C23}$$

At this point let us consider what happens in a noninteracting system. In this case, all the Fermi-liquid coefficients A_l and hence ϵ_l are zero. The kinetic equation (C18) is solved by

$$F_l = \frac{e\mathbf{E}\cdot\mathbf{v}_l}{i\omega + il\omega_c^*} \tag{C24}$$

so that the noninteracting current is given by

$$\mathbf{j}^{\mathrm{n}} = \frac{-e^2m^*}{2\pi}\sum_l\left[\frac{1}{i\omega + il\omega_c^*}\right]\mathbf{v}_l^*\ \mathbf{v}_l\cdot\mathbf{E} \tag{C25}$$

and thus the conductivity is

$$\sigma_{ij}^{\mathrm{n}} = \frac{-e^2m^*}{2\pi}\sum_l\left[\frac{1}{i\omega + il\omega_c^*}\right](\mathbf{v}_l^*)_j(\mathbf{v}_l)_i \tag{C26}$$

which is exactly the semiclassical expression[5,15] for the conductivity of a system of noninteracting quasiparticles in the $\tau\to\infty$ limit given in Eq. (43).

We now want to analyze this system when the interaction coefficients are nonzero. To do this, we must be able to solve the kinetic equation (C18). We express the kinetic equation in terms of the unknown variables f_l by using Eqs. (C14) and (C15) to yield the matrix equation[17]

$$c_n = \sum_m (a_n^m - \delta_{nm}) f_m \ , \tag{C27}$$

where

$$c_n = -\sum_l i^{n-l} \frac{e\mathbf{E}\cdot\mathbf{v}_l J_{l-n}}{i\omega + il\omega_c^*} \ , \tag{C28}$$

$$a_n^m = i^{n-m} A_m \left[-\delta_{nm} + \sum_l \frac{i\omega J_{l-m} J_{l-n}}{i\omega + il\omega_c^*} \right] \ , \tag{C29}$$

where the Bessel functions and the velocity coefficients $\mathbf{v}_l$ are evaluated at X. Although this system of equations is infinite dimensional, if we assume that A_i is zero for i greater than some number $i_{\max}$, then we have a_j^i also zero for $i > i_{\max}$. In this case the equations with $-i_{\max} \le n \le i_{\max}$ form a closed system of equations with variables $\{f_{(-i_{\max})} \cdots f_{(i_{\max})}\}$ where f_0 is real and all other f_n are complex. Once this smaller system is solved, the remaining f_n are defined trivially since they only depend on the already determined values. Then one can solve for the F_n using Eq. (C17) and then find the current using Eq. (C23) and hence extract the conductivity.

As an illustrative example we consider the case where A_1 is the only nonzero Fermi-liquid coefficient and using this approximation (whose validity is discussed in Sec. IV B) we derive the same result [Eq. (76)] as in Sec. IV B. Note that in Sec. IV B we use a trick to perform this same calculation that cannot be generalized to account for an arbitrary number of nonzero Fermi-liquid coefficients. The method shown below is more difficult, but more generalizable (in principal one could also generalize this method to include the effects of impurity scattering also).

With the simplification that A_1 is the only nonzero Fermi-liquid coefficient, we now have the decoupled system of two equations

$$\begin{aligned} c_1 &= (a_1^1 - 1) f_1 + a_1^{-1} f_{-1} \ , \\ c_{-1} &= (a_{-1}^{-1} - 1) f_{-1} + a_{-1}^1 f_1 . \end{aligned} \tag{C30}$$

Solving this system yields the result

$$\begin{aligned} f_1 &= D^{-1}[(a_{-1}^{-1} - 1)c_1 - a_1^{-1} c_{-1}] \ , \\ f_{-1} &= D^{-1}[(a_1^1 - 1)c_{-1} - a_{-1}^1 c_1] \ , \\ D &= (a_{-1}^{-1} - 1)(a_1^1 - 1) - a_1^{-1} a_{-1}^1 . \end{aligned} \tag{C31}$$

Once f_1 and f_{-1} are determined, all of the f_n are then given by the Eq. (C27) which now takes the form

$$f_n = a_n^1 f_1 + a_n^{-1} f_{-1} - c_n . \tag{C32}$$

Using this result in Eq. (C17), inserting the definition of c_n, and simplifying by using the Bessel function orthogonality equation (Eq. 9.1.75 Ref. 22) yields

$$F_l = \frac{e\mathbf{E}\cdot\mathbf{v}_l}{i\omega + il\omega_c^*} + \frac{A_1 i^l l\omega_c^*}{i\omega + il\omega_c^*}[f_{-1} J_{m+1} - f_1 J_{m-1}]. \tag{C33}$$

Notice that the first term is just the noninteracting result given in Eq. (C24), whereas the second term is clearly an interaction term. Substituting this expression into Eq. (C18) and using Bessel function identities (9.1.27 from Ref. 22) yields the current

$$\mathbf{j} = \mathbf{j}^{\mathrm{n}} + \delta\mathbf{j} \ , \tag{C34}$$

where $\mathbf{j}^{\mathrm{n}}$ is the previous noninteracting current defined in Eq. (C25) and

$$\delta\mathbf{j} = \frac{-\omega e m^* A_1}{2\pi} \sum_l \frac{i^l \mathbf{v}_l^*}{i\omega + i\omega_c^*} \left[f_1 \left(\frac{l J_l}{X} + J_l' \right) - f_{-1} \left(\frac{l J_l}{X} - J_l' \right) \right] . \tag{C35}$$

By using the definition of $\mathbf{v}_l$ in terms of Bessel functions as given in Eq. (C16), we can put this in the simple form

$$\delta\mathbf{j} = \frac{\omega A_1}{v_F e}[f_{-1}\sigma^{\mathrm{n}}\hat{\mathbf{r}}_- - f_1 \sigma^{\mathrm{n}}\hat{\mathbf{r}}_+] \ , \tag{C36}$$

where $\hat{\mathbf{r}}_+ = \hat{\mathbf{r}}_-^* = \hat{\mathbf{x}} + i\hat{\mathbf{y}}$ and σ^{n} is the previous noninteracting conductivity defined in Eq. (C26). It should be noted that the coefficients f_i are linear in the c_i's which in turn are linear in $\mathbf{E}$ as can be seen from Eqs. (C31) and (C28). Hence $\delta\mathbf{j}$ and $\mathbf{j}$ will be linear in the field $\mathbf{E}$ such that a linear conductivity can be defined properly.

At this point a great deal of very tedious algebra (along with clever use of the definition of σ^{n} in terms of Bessel functions) can be used to simplify the result into the form given in Eq. (76).

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