

Asymptotic correlation functions in the one-dimensional Hubbard model with applications to high- T_c superconductivity

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The one-dimensional Hubbard model with repulsive interaction is studied. We calculate the asymptotic correlation functions in the less-than-half-filled band case. In particular, our approach centers on the large- U wave function and emphasizes the finite phase shifts on the Fermi surface. This is important since the concept of finite phase shifts can be easily applied to higher dimensions. We then discuss the non-Fermi-liquid behavior of this model and the relevance to the understanding of the two-dimensional Hubbard model, which is believed to give a good description of the high- T_c superconductors.

I. INTRODUCTION

Although it is one of the simplest models in statistical physics, many important and interesting questions about the Hubbard model still remain unanswered. It has recently attracted a lot of attention because of the belief that it is the right model for high- T_c superconducting materials. So far, very little knowledge has been gained about the two-dimensional Hubbard model, although progress is being made. On the other hand, following the recent excellent numerical work by Ogata and Shiba,¹¹ and Sorella *et al.*,¹³ the low energy behavior of the one-dimensional (1D) model has been clarified. Here we discuss this development and the implications on high- T_c superconductivity.

II. LIEB-WU SOLUTION

Following Yang's solution¹ of the one-dimensional Fermi gas with δ -function interactions, Lieb and Wu² succeeded in solving the one-dimensional Hubbard model

$$H = -t \sum_{i,j,\sigma} c_{i\sigma}^\dagger c_{j\sigma} + U \sum_i n_{i\uparrow} n_{i\downarrow} \quad (1)$$

using the Bethe-ansatz wave function

$$\psi = \sum_P [Q, P] \exp \left(i \sum_{j=1}^N k_{P_j} x_{Q_j} \right), \quad (2)$$

where $P = (P_1, P_2, \dots, P_N)$ and $Q = (Q_1, Q_2, \dots, Q_N)$ are two permutations of the numbers $(1, 2, \dots, N)$ for a system with N electrons and M down spins. It is supposed that $N \leq N_a$ and $2M \leq N$.

The equations determining the $N! \times N!$ coefficients $[Q, P]$ lead to coupled equations for quantum numbers k_j and Λ_α :

$$\begin{aligned} N_a k_j &= 2\pi I_j + \sum_{\beta=1}^M \theta(2 \sin k_j - \Lambda_\beta), \\ j &= 1, 2, \dots, N, \\ -\sum_{j=1}^N \theta(2\Lambda_\alpha - 2 \sin k_j) &= 2\pi J_\alpha - \sum_{\beta=1}^M \theta(\Lambda_\alpha - \Lambda_\beta), \quad (3) \\ \alpha &= 1, 2, \dots, M, \\ \theta(p) &= -2 \tan^{-1}(2p/U), \quad -\pi \leq \theta \leq \pi, \end{aligned}$$

I_j 's are integers (half-odd integers) for M even (odd), and J_α 's are integers (half-odd integers) for $N - M$ odd (even). N_a is the length of the system. The momentum and energy of the state are given by

$$p = \sum_{j=1}^N k_j, \quad (4)$$

$$E = -2 \sum_{j=1}^N \cos k_j.$$

For a half-filled band, Lieb and Wu calculated the ground state energy

$$E_{\text{gs}} = -4N_a \int_0^\infty \frac{J_0(\omega) J_1(\omega)}{\omega [1 + \exp(\frac{1}{2}\omega U)]} \quad (5)$$

and the Mott-Hubbard gap

$$\Delta = U - 4 + 8 \sum_{n=1}^\infty (-1)^n [(1 + n^2 U^2/4)^{1/2} - \frac{1}{2} n U]. \quad (6)$$

They showed that for $U > 0$, the ground state energy and wave function are analytic in U , and concluded there is no Mott transition at finite U . The low lying excitation spectrum has been studied by Ovchinnikov³ and Coll.⁴ They calculated the velocities of the spin wave and the charge density wave in various limits. More recently, Woynarovich⁵ studied the excited states of a half-filled

band with electron pairs occupying the same sites. These correspond to states in the upper Hubbard band.

III. THE g -OLOGY THEORY

The bosonization technique has been successfully applied to the one-dimensional electron models.⁶ For spinless fermions with the Hamiltonian

$$H = v_F \sum_k |k| a_k^\dagger a_k + \frac{1}{2L} \sum_k V_k \rho(k) \rho(-k), \quad (7)$$

the transformation to the bosonic representations

$$\begin{aligned} \rho(k) &= \rho_1(k) + \rho_2(k), \\ \rho_1(k) &= \sum_{p>0} a_{p-k/2}^\dagger a_{p+k/2} = b_k \left| \frac{kL}{\pi} \right|^{1/2}, \\ \rho_2(k) &= \sum_{p<0} a_{p-k/2}^\dagger a_{p+k/2} = b_{-k} \left| \frac{kL}{\pi} \right|^{1/2} \end{aligned} \quad (8)$$

makes both the kinetic term and the interactions quadratic in boson operators,

$$H = v_F \sum_k |k| b_k^\dagger b_k + \frac{1}{2\pi} \sum_k |k| V_k (b_k + b_{-k}^\dagger)(b_k^\dagger + b_{-k}) \quad (9)$$

and the Hamiltonian is easily diagonalized. For spin 1/2 fermions, it is convenient to introduce the charge and spin operators

$$\begin{aligned} c_p^\dagger &= (b_{p\uparrow}^\dagger + b_{p\downarrow}^\dagger)/\sqrt{2}, \\ s_p^\dagger &= (b_{p\uparrow}^\dagger - b_{p\downarrow}^\dagger)/\sqrt{2}. \end{aligned} \quad (10)$$

If the original Hamiltonian is invariant under spin reversal, there will be no terms in the product of one charge and one spin creation (annihilation) operators and we may say the charge and spin degrees of freedom are separated. This is a consequence of the fact that in one dimension the only possible excitations are density fluctuations:

$$\begin{aligned} H &= H_c + H_s \\ &= v_c \sum_k |k| c_k^\dagger c_k + \frac{1}{2\pi} \sum_{k>0} (g_1 - 2g_2) |k| (c_k^\dagger c_{-k}^\dagger + c_{-k} c_k) \\ &\quad + v_s \sum_k |k| s_k^\dagger s_k + \frac{1}{2\pi} \sum_{k>0} g_1 |k| (s_k^\dagger s_{-k}^\dagger + s_{-k} s_k). \end{aligned} \quad (11)$$

However, the existence of the spin degrees of freedom allow more possible terms to appear. There are the backward scattering

$$H_{BS} = g_1 \int dx \psi_{R\uparrow}^\dagger \psi_{L\downarrow}^\dagger \psi_{R\downarrow} \psi_{L\uparrow} + \text{H.c.} \quad (12)$$

in which electrons of opposite spin cross the Fermi surface in opposite directions and the umklapp scattering

$$H_{um} = g_3 \int dx e^{i(4k_F - G)x} \psi_{L\uparrow}^\dagger \psi_{L\downarrow}^\dagger \psi_{R\downarrow} \psi_{R\uparrow} + \text{H.c.} \quad (13)$$

The Hamiltonian including the backward and umklapp scattering cannot be diagonalized exactly. Luther and Emery showed, at particular values of g 's, it is possible to reduce H_c and H_s to a free fermion Hamiltonian by a canonical transformation, and the resulting Hamiltonian can again be diagonalized. The result combined with the renormalization group analysis determines the ground state properties in different regions of interaction parameters. In the case of the Hubbard model, for repulsive U the backward scattering Hamiltonian⁷ has the wrong sign for producing a gap in the spin wave spectrum while the umklapp scattering responsible for a gap in the charge density wave spectrum is only important when the momentum transfer to the lattice is $4k_F$, i.e., at half filling. This agrees with the Lieb-Wu result.

The renormalization group analysis for the model been done by Solyom.⁸ The charge renormalization is only relevant at half filling. The spin renormalization flow has a fixed point at $g_{1\parallel} = g_{1\perp} = 0$.

IV. THE MODEL IN THE LARGE- U LIMIT

To get a better understanding of the ground state wave function given by the Lieb-Wu solution, we rewrite the wave function in the infinite- U limit, after Ogata and Shiba,¹¹ as

$$\psi(x_1, \dots, x_N) = (-1)^Q \det \left[\exp(ik_i x_{Q_j}) \right] \phi(y_1, \dots, y_M). \quad (14)$$

The $\phi(y_1, \dots, y_M)$ is the Bethe solution to a 1D Heisenberg model on the "squeezed" lattice x_i (i.e., all hole sites are squeezed out). Here y_1, \dots, y_M are positions of the down (or up) spins on this squeezed lattice of the spins. The wave function now is an explicit product of a charge part and spin part and this kind of separation in the ground state wave function only happens in the large- U limit. The form of the wave function immediately tells us that the charge part is renormalized to a spinless fermion system, and most properties are easily understood in this picture. Naively, the correlation functions are just simple products of the spinless fermion charge part and the Heisenberg spin part, but there is an important constraint to be taken into account. While the quantum numbers Λ_α depend only on J_α 's, the k_j 's depend on the sum of all J_α 's through

$$N_\alpha k_j = 2\pi I_j + \frac{2\pi}{N} \sum_\alpha J_\alpha. \quad (15)$$

When only charge or spin density excitation exists; i.e., the I_j 's and J_α 's do not differ from their value in the ground state value simultaneously, and the coupling be-

tween charge and spin may be neglected. However, when an electron excitation is created, the last term in the last equation generates a phase shift for the charge part. Let us say that we remove an electron of momentum k_F from the system. The way that costs the least energy will be removing the I_j at the “charge Fermi surface” $2k_F$ and removing the J_α at the “spin Fermi surface” k_F . This in turn will shift all the rest of the k_j 's, in terms of phase shift, by

$$\delta = N_a \delta k_j = \frac{2\pi M}{N} \frac{M}{2} = \frac{M\pi}{N}, \quad (16)$$

which is equal to $\pi/2$ for equal numbers of up and down spins. Let us consider the change in n_k when we add one particle to the system at k_F . This is given by

$$\Delta n_k = |\langle \psi_{N+1,k} | c_k^\dagger | \psi_N \rangle|^2. \quad (17)$$

The overlap above, that of the state of $N+1$ with total momentum k_F with the ground state of N particles plus a free particle of momentum k_F , can be calculated using the technique developed by one of the authors⁹ in the treatment of the Kondo problem. The method is for finding overlaps between wave functions of determinantal forms in terms of the single particle phase shifts. For two single particle wave functions differing by a phase shift of δ , the overlap is

$$\langle \psi_k, \psi'_k \rangle = \frac{\sin \delta}{\pi(n_k - n'_k) + \delta_k}; \quad (18)$$

thus the overlap integral between the determinants made up of states ψ_k and ψ'_k is

$$\begin{aligned} \text{Det}|\langle \psi_k, \psi'_k \rangle| &= \text{Det} \left| \frac{\sin \delta}{\pi(n_k - n'_k) + \delta_k} \right| \\ &= \prod_n^N \frac{\sin \pi \delta_n}{\pi} \prod_{m < n}^N (n - m + \delta_n - \delta_m)(m - n) / \prod_{m,n}^N (n - m + \delta_n) \\ &\propto \exp \left[-\frac{1}{2} \left(\frac{1}{2\pi^2} \sum_{n=1}^N \delta_n^2 (n^{-1} + (N - n + 1)^{-1}) + \frac{1}{2} \sum_{n=1}^N \sum_{m=1}^{N-n} \frac{(\delta_{n+m} - \delta_n)^2}{m^2} \right) \right] \\ &= \exp \left[-\frac{1}{4} \left(\frac{\delta}{\pi} \right)^2 \ln N \right]. \end{aligned} \quad (19)$$

The extra factor $1/2$ in the formula is due to the fact that we should only pair particles with the same spin direction while we have a determinant of both up and down spins. If the Bethe solution were at the exact free fermion fixed point described by Sutherland,¹⁰ Haldane,¹⁰ and Shastry,¹⁰ the overlap used here turns out to be a constant. As it is, there may be at least logarithmic corrections to correlation functions due to spin function overlaps. The annihilation operator for the spin commutes with P_G ; thus the overlap between the spin part of the wave functions before and after taking one spin out is a finite constant independent of N . We have

$$\Delta n_k = n^{-1/8} \propto (\Delta k)^{-1/8}; \quad (20)$$

this implies near the Fermi surface,

$$n_k = n_{k_F} - \text{const} |k - k_F|^{1/8} \text{sgn}(k - k_F). \quad (21)$$

V. MORE CORRELATION FUNCTIONS IN THE LARGE- U LIMIT

We want now to show that by further formalizing the phase shift idea, more correlation functions can be “visualized” in the large- U limit. For a 1D fermion system with linear low energy excitations, we can write for the

wave functions of the left and right moving branch

$$\psi_R(x) = e^{i\phi_R(x)}, \quad \psi_L(x) = e^{i\phi_L(x)}, \quad (22)$$

with

$$\phi_R(x) = \sum_{k>0} \left(\frac{2\pi}{|k|L} \right)^{1/2} [b_k^\dagger e^{-ikx} + b_k e^{ikx}] e^{-\alpha|k|/2}, \quad (23)$$

$$\phi_L(x) = \sum_{k<0} \left(\frac{2\pi}{|k|L} \right)^{1/2} [b_k^\dagger e^{-ikx} + b_k e^{ikx}] e^{-\alpha|k|/2},$$

where α is introduced for convergence. The ϕ fields are connected to the density field $\rho(x)$ by

$$\nabla \phi_{R,L}(x) = 2\pi \rho_{R,L}(x). \quad (24)$$

Each time a fermion is added, $\phi(x)$ increases by 2π , and so $\phi(x)$ is the properly normalized “phase shift” field. The effect of interaction is to change the phase shift coefficients, i.e., $e^{i\phi_R} \rightarrow e^{i\lambda\phi_R}$, for example, by mixing the left and right branch through a Bogoliubov transformation as we usually observe in the Tomonaga-Luttinger model:

$$\begin{aligned} \phi_R(x) &\rightarrow \cosh \varphi \phi_R(x) + \sinh \varphi \phi_L(x), \\ \phi_L(x) &\rightarrow \cosh \varphi \phi_L(x) + \sinh \varphi \phi_R(x), \end{aligned} \quad (25)$$

For our purpose here it enough to write down the expression for calculating the correlation function

$$\langle e^{i\lambda\phi_R(x,t)} e^{-i\lambda\phi_R(0,0)} \rangle = [\langle e^{i\phi_R(x,t)} e^{-i\phi_R(0,0)} \rangle]^{\lambda^2}. \quad (26)$$

Let us define the "holon" as the excitation associated with adding or removing one I_j , which are called type-II, type-III excitations in Lieb and Wu. The Green's function for the holon is easy to find. When we add one I_j to the system, the Λ_α 's do not change. We are simply adding one k_j to the Slater determinant, which behaves like a noninteracting spinless fermion system, filled up to momentum $2k_F$. So we have

$$G^c(x, t) \sim \frac{e^{2ik_F x}}{x - v_c t + i\delta t}, \quad (27)$$

where v_c is the velocity of the charge excitations in Hubbard model, which together with the velocity of spin excitations v_s have been calculated by, e.g., Coll.⁴ The $i\delta$ term is put in in such a way to make the analyticity correct.

We next consider the excitation associated with removing one of the J_α 's, i.e., type-I in Lieb and Wu. When we remove a J_α at $-k_F$, we get a spinon—a spin half excitation with momentum k_F , basically an electron removed from a Heisenberg system,

$$\langle \psi(x, t) \psi^\dagger(0, 0) \rangle \sim \frac{e^{ik_F x}}{(x - v_s t)^{1/2}}. \quad (28)$$

[The $2k_F$ contribution in the spin-spin correlation function of the Heisenberg model consists of one left (right) moving particle and a right (left) moving hole, and so

$$\langle \mathbf{S}(x, t) \cdot \mathbf{S}(0, 0) \rangle \sim \frac{\cos 2k_F x}{(x^2 - v_s^2 t^2)^{1/2}}; \quad (29)$$

see, e.g., Luther and Peschel.¹² Moreover, each $k_j L$ is shifted by $\frac{\pi}{2}$, compared to the interspacing 2π . This leads to $1/4$ for the phase shift coefficient we mentioned earlier, and since there appear a right moving particle and a left moving hole, we have

$$\begin{aligned} G^s(x, t) &\sim \frac{e^{ik_F x}}{(x - v_s t + i\delta t)^{\frac{1}{2}} (x - v_c t + i\delta t)^{\frac{1}{4}} (x + v_c t - i\delta t)^{\frac{1}{2}}^2} \\ &= \frac{e^{ik_F x}}{(x - v_s t + i\delta t)^{\frac{1}{2}} (x - v_c t + i\delta t)^{\frac{1}{16}} (x + v_c t - i\delta t)^{\frac{1}{16}}}. \end{aligned} \quad (30)$$

The spin-spin correlation is now given by the joint spinon-(spinon-hole) correlation. For the $2k_F$ contribution, the phase shifts are in the same direction, which determines the relative signs when we perform the calculations in the bosonization formalism. We easily get

$$\begin{aligned} \langle \mathbf{S}(x, t) \cdot \mathbf{S}(0, 0) \rangle &\sim \frac{\cos 2k_F x}{(x - v_s t)^{\frac{1}{2}} (x + v_s t)^{\frac{1}{2}} (x - v_c t)^{(\frac{1}{4} + \frac{1}{4})^2} (x + v_c t)^{(\frac{1}{4} + \frac{1}{4})^2}} \\ &= \frac{\cos 2k_F x}{(x - v_s t)^{\frac{1}{2}} (x + v_s t)^{\frac{1}{2}} (x - v_c t)^{\frac{1}{4}} (x + v_c t)^{\frac{1}{4}}}. \end{aligned} \quad (31)$$

This gives

$$\langle \mathbf{S}(x) \cdot \mathbf{S}(0) \rangle \sim x^{-\frac{3}{2}} \cos 2k_F x \quad (32)$$

(except, again, for logarithmic corrections). This is confirmed by numerical calculations by Ogata and Shiba¹¹ and Sorella *et al.*¹³ On the other hand, when the calculation of the susceptibility involves only the zero momentum contribution of the spin-spin correlation function, we will get exactly the same result as in the Heisenberg model. The $T = 0$ susceptibility has been calculated by Shiba.¹⁴

Another correlation function of interest is the density-

density correlation. The $4k_F$ contribution is rather simple:

$$\langle \rho(x, t) \rho(0, 0) \rangle \sim \frac{\cos 4k_F x}{(x - v_c t)(x + v_c t)}, \quad (33)$$

also confirmed by Sorella *et al.*

Finally we calculate the electron Green's function and the momentum distribution. To make an electron hole, we have to remove a I_j at $2k_F$ and a J_α at $-k_F$, but then the shifting of k_j 's adds a quarter k_j at $2k_F$ and removes a quarter k_j at $-2k_F$. Thus at $2k_F$ we have two opposing phase shifts. Noting this, we immediately have

$$\begin{aligned} G^e(x, t) &\sim \frac{e^{ik_F x}}{(x - v_c t + i\delta t)^{(1 - \frac{1}{4})^2} (x + v_c t - i\delta t)^{(\frac{1}{4})^2} (x - v_s t + i\delta t)^{\frac{1}{2}}} \\ &= \frac{e^{ik_F x}}{(x - v_c t + i\delta t)^{\frac{9}{16}} (x + v_c t - i\delta t)^{\frac{1}{16}} (x - v_s t + i\delta t)^{\frac{1}{2}}}, \end{aligned} \quad (34)$$

which also leads to the momentum distribution

$$n(k) = -i \int_{-\infty}^{\infty} dx e^{-ikx} G(x, 0^-) = \text{const} - \text{const} |k - k_F|^{\frac{1}{8}} \text{sgn}(k - k_F). \quad (35)$$

But there are other ways of making an electron excitation; we can remove a I_j at $2k_F$, and remove a J_α at k_F . This gives an electron excitation at momentum $3k_F$, with the Green's function

$$G_e^{3k_F} \sim \frac{e^{3ik_F x}}{(x - v_c t + i\delta t)^{(1+\frac{1}{4})^2} (x - v_c t + i\delta t)^{(\frac{1}{4})^2} (x - v_s t + i\delta t)^{\frac{1}{2}}}. \quad (36)$$

The momentum distribution singularity is given by $|k - 3k_F|^{9/8}$. We will have more discussions on the Green's functions later.

VI. HALDANE'S LUTTINGER LIQUID CONCEPT

A better framework for formulating the low energy structure of one-dimensional models has been developed by Haldane in a series of papers. The basic idea is that the low energy effective Hamiltonian of 1D quantum models could be mapped onto the spectrum of the Tomonaga-Luttinger model. In particular, Haldane demonstrated the applicability of the Luttinger liquid formalism to several Bethe-ansatz-type systems.

The ‘‘Luttinger liquid’’ is a universality class of 1D fermion systems with a gapless linear low energy spectrum. The structure of such a model is determined by a single parameter $e^{2\varphi}$. In the long wavelength approximation, a fermion field can be represented as

$$\psi_F(x) \sim |\rho(x)|^{1/2} e^{i\theta(x)} e^{i\phi(x)}; \quad (37)$$

here we define $\phi(x)$ and $\theta(x)$ as

$$[\rho(x), \phi(x')] = i\delta(x - x'), \quad (38)$$

$$\nabla\theta(x) = \pi\rho(x).$$

$\rho(x)$ is the density so $\phi(x)$ is the phase field and $\theta(x)$ is the ‘‘phase shift’’ field. The factor $e^{i\theta(x)}$ comes from the ‘‘Jordan-Wigner’’ transformation for 1D fermions. The Fourier transforms are

$$\begin{aligned} \rho(x) &= \rho + \frac{1}{\sqrt{L}} \sum_k e^{-ikx} \left| \frac{k}{2\pi} \right|^{1/2} (b_k + b_{-k}^\dagger), \\ \theta(x) &= \bar{\theta} + \frac{\pi Nx}{L} + i \sum_{k \neq 0} \frac{1}{k} \left| \frac{\pi k}{2L} \right|^{1/2} e^{-ikx} (b_k + b_{-k}^\dagger), \\ \phi(x) &= \bar{\phi} + \frac{\pi Jx}{L} + i \sum_{k \neq 0} \left| \frac{\pi}{2kL} \right|^{1/2} e^{ikx} (b_k^\dagger - b_{-k}). \end{aligned} \quad (39)$$

The Hamiltonian

$$H = \frac{\hbar^2}{2m} \int dx |\nabla\psi|^2 + \frac{1}{2} \int \int dx dy V(x - y) \rho(x) \rho(y) \quad (40)$$

can be rewritten in the low energy regime as

$$H = \frac{\hbar}{2\pi} \int dx [v_J (\nabla\phi)^2 + v_N (\nabla\theta - \pi\rho_0)^2], \quad (41)$$

where $v_J = \pi\hbar\rho_0/m$ and $v_N = \kappa/\pi\hbar\rho_0^2$ are the current and number velocity, and κ is the compressibility per unit length. The Hamiltonian can be diagonalized by introducing a Bogoliubov transformation parameter e^φ ,

$$\theta(x) = \bar{\theta} + \frac{\pi Nx}{L} + i \sum_{k \neq 0} \frac{1}{k} \left| \frac{\pi k}{2L} \right|^{1/2} e^{-ikx} e^\varphi (b_k + b_{-k}^\dagger), \quad (42)$$

$$\phi(x) = \bar{\phi} + \frac{\pi Jx}{L} + i \sum_{k \neq 0} \left| \frac{\pi}{2kL} \right|^{1/2} e^{ikx} e^{-\varphi} (b_k^\dagger - b_{-k}),$$

and choosing

$$v_J e^{-2\varphi} = v_N e^{2\varphi} = v_S, \quad (43)$$

and so

$$H = \frac{\pi}{2L} [v_J J^2 + v_N (N - N_0)^2] + \hbar v_S \sum_k |k| b_k^\dagger b_k, \quad (44)$$

$$P = \frac{\pi NJ}{L} + \sum_k k b_k^\dagger b_k,$$

with the selection rule

$$(-1)^{\Delta J} = (-1)^{\Delta N}. \quad (45)$$

$v_S = (\kappa/m\rho_0)^{1/2}$ is the velocity of the density oscillation, or ‘‘sound velocity.’’ For free fermions $e^{-2\varphi} = 1$ and for free bosons $e^{-2\varphi} = 0$.

For the Hubbard model, we have

$$\begin{aligned} H &= \frac{\hbar}{2\pi} \int dx \{ v_c [e^{2\varphi} (\nabla\phi_c)^2 + e^{-2\varphi} (\nabla\theta_c - \pi N_0/L)^2] \\ &\quad + v_\sigma [2(\nabla\phi_\sigma)^2 + \frac{1}{2}(\nabla\theta_\sigma)^2] \}. \end{aligned} \quad (46)$$

Here, we have taken into account the fact that the charge part does not get renormalized from the umklapp term when the band is not half-filled, and that the marginal backward scattering term for the spin part renormalizes logarithmically to zero. In the limit of large U , the charge density velocity v_c and spin wave velocity v_σ are given by¹⁴

$$v_c = 2t \left(\sin \frac{\pi N}{N_a} \right) \frac{L}{N_a}, \quad (47)$$

$$v_\sigma = \frac{1}{2} \pi \left(\frac{4t^2}{U} \right) \left(1 - \frac{\sin 2\pi N/N_a}{2\pi N/N_a} \right) \frac{L}{N_a}.$$

The parameter e^φ is decided by the ratio of the two velocities v_N and v_c . It is easy to determine e^φ in both the large- and small- U limits. For large U ,

$$v_N = \frac{L}{\pi} \frac{\partial \mu}{\partial N}$$

$$= 2t \frac{L}{N_a} \sin Q$$

$$= v_c; \quad (48)$$

i.e., $e^{-2\varphi} = 1$. This is expected since in the large- U limit the charge part renormalizes to free fermions. In the small- U limit,

$$v_N = \frac{L}{\pi} \frac{\partial \mu}{\partial N}$$

$$= 2t \frac{L}{2N_a} \sin Q$$

$$= \frac{1}{2} v_c; \quad (49)$$

so $e^{-2\varphi} = 1/2$. For general U , one can obtain e^φ using the method similar to that given by, e.g., Haldane for a Bethe-ansatz system without internal degrees of freedom.¹⁵ The results of Bethe-ansatz-soluble models are given in terms of a linear inhomogeneous Fredholm integral equation of the second kind:

$$2\pi\rho(k) = 1 + \int_{-Q}^Q dk' \frac{\partial \vartheta(k, k')}{\partial k} \rho(k'). \quad (50)$$

If we define $\sigma(k)$ by

$$2\pi\sigma(k) = \vartheta(k, Q) - \int_{-Q}^Q dk' \frac{\partial \vartheta(k, k')}{\partial k'} \sigma(k'), \quad (51)$$

then

$$e^{-\varphi} = 1 - \sigma(Q) + \sigma(-Q). \quad (52)$$

In the large- U limit,

$$\frac{\partial \vartheta(k, k')}{\partial k} = \cos k \frac{\ln 2}{U} \quad (53)$$

or

$$\vartheta(k, k') = \frac{\ln 2}{U} (\sin k - \sin k'); \quad (54)$$

this gives

$$\sigma(Q) - \sigma(-Q) = \frac{\ln 2}{\pi U} \sin Q, \quad (55)$$

and so we have

$$e^{-\varphi} \rightarrow 1 - \frac{\ln 2}{\pi U} \sin Q. \quad (56)$$

For convenience in the following formulas, we redefine

$$e^{-2\varphi} = 2[e^{-2\varphi}]_{\text{old}}. \quad (57)$$

Now we can write down all kinds of correlation functions by using the bosonization technique for the Tomonaga-Luttinger model. We have

$$\theta_c(x) = \bar{\theta}_c + \frac{\pi N_c x}{L} + i \sum_{k \neq 0} \frac{1}{k} \left| \frac{\pi k}{L} \right|^{1/2} e^{-ikx} e^\varphi (c_k + c_{-k}^\dagger),$$

$$\phi_c(x) = \bar{\phi}_c + \frac{\pi J_c x}{L} + i \sum_{k \neq 0} \left| \frac{\pi}{4kL} \right|^{1/2} e^{ikx} e^{-\varphi} (c_k^\dagger - c_{-k}), \quad (58)$$

$$\theta_s(x) = \bar{\theta}_s + \frac{\pi N_s x}{L} + i \sum_{k \neq 0} \frac{1}{k} \left| \frac{\pi k}{L} \right|^{1/2} e^{-ikx} (s_k + s_{-k}^\dagger),$$

$$\phi_s(x) = \bar{\phi}_s + \frac{\pi J_s x}{L} + i \sum_{k \neq 0} \left| \frac{\pi}{4kL} \right|^{1/2} e^{ikx} (s_k^\dagger - s_{-k}),$$

and

$$\theta_c = \theta_\uparrow + \theta_\downarrow, \quad N_c = N_\uparrow + N_\downarrow,$$

$$\theta_s = \theta_\uparrow - \theta_\downarrow, \quad N_s = N_\uparrow - N_\downarrow, \quad (59)$$

$$\phi_c = (\phi_\uparrow + \phi_\downarrow)/2, \quad J_c = (J_\uparrow + J_\downarrow)/2,$$

$$\phi_s = (\phi_\uparrow - \phi_\downarrow)/2, \quad J_s = (J_\uparrow - J_\downarrow)/2,$$

Note that according to this definition, at large U $e^{2\varphi} = 1/2$, and for $U = 0$, $e^{2\varphi} = 1$.

In calculating the correlation functions, we have to follow the selection rules

$$(-1)^{\Delta N_\uparrow} = (-1)^{\Delta J_\uparrow},$$

$$(-1)^{\Delta N_\downarrow} = (-1)^{\Delta J_\downarrow}, \quad (60)$$

and the term to enter will be

$$\psi_c(x) = e^{i\Delta N_c \phi_c(x)} e^{\Delta J_c \theta_c(x)},$$

$$\psi_s(x) = e^{i\Delta N_s \phi_s(x)} e^{\Delta J_s \theta_s(x)}, \quad (61)$$

which will have $\Delta J_c \times 2k_F$ oscillations. The leading term in the electron Green's function is given by a $\Delta N_\uparrow = 1$, $\Delta J_\uparrow = 1$, $\Delta N_\downarrow = 0$, $\Delta J_\downarrow = 0$ (k_F) piece and a $\Delta N_\uparrow = 1$, $\Delta J_\uparrow = 3$, $\Delta N_\downarrow = 0$, $\Delta J_\downarrow = 0$ (k_F) piece:

$$G(x, t) = \langle \psi_\uparrow(x, t) \psi_\uparrow^\dagger(0, 0) \rangle \sim e^{ik_F x} (x - v_c t)^{-1/2} (x - v_\sigma t)^{-1/2} |(x - v_c t)(x + v_c t)|^{-\alpha}$$

$$+ e^{3ik_F x} (x - v_c t)^{-(3e^\varphi + e^{-\varphi})^2/8} (x + v_c t)^{-(3e^\varphi - e^{-\varphi})^2/8} (x - v_\sigma t)^{-1/2}, \quad (62)$$

with

$$\alpha = (e^{2\varphi} + e^{-2\phi} - 2)/8. \quad (63)$$

This gives

$$\begin{aligned} n_k &\sim -\text{sgn}(k - k_F)|k - k_F|^{(e^{-2\varphi} + e^{2\varphi} - 2)/4} \\ &\sim -\text{sgn}(k - 3k_F)|k - 3k_F|^{(9e^{2\varphi} + e^{-2\varphi} - 2)/4}. \end{aligned} \quad (64)$$

The first few terms in the density-density correlation function are

$$\begin{aligned} n(x, t) &= \psi_{\uparrow}^{\dagger}\psi_{\uparrow} + \psi_{\downarrow}^{\dagger}\psi_{\downarrow}, \\ \langle n(x, t)n(0, 0) \rangle &= n_0^2 + \frac{e^{2\varphi}}{x^2 - v_c^2 t^2} + \\ &C_1 \frac{\cos 2k_F x}{(x^2 - v_c^2 t^2)^{e^{2\varphi}/2}(x^2 - v_{\sigma}^2 t^2)^{1/2}} + C_2 \frac{\cos 4k_F x}{(x^2 - v_c^2 t^2)^{2e^{2\varphi}}}. \end{aligned} \quad (65)$$

$$\hat{O}_{\text{SP}}(x) = \psi_{\sigma}^{\dagger}(x)\psi_{-\sigma}^{\dagger}(x),$$

(67)

$$\begin{aligned} \langle \hat{O}_{\text{SP}}^{\dagger} \hat{O}_{\text{SP}} \rangle &\propto \frac{1}{|x^2 - v_c^2 t^2|^{e^{-2\varphi}/2}|x^2 - v_{\sigma}^2 t^2|^{1/2}} + E_0 \frac{e^{i2k_F x}}{(x - v_c t)^{(e^{\varphi} + e^{-\varphi})^2/2}(x + v_c t)^{(e^{\varphi} - e^{-\varphi})^2/2}} \\ &+ E_0 \frac{e^{-i2k_F x}}{(x + v_c t)^{(e^{\varphi} + e^{-\varphi})^2/2}(x - v_c t)^{(e^{\varphi} - e^{-\varphi})^2/2}}. \end{aligned}$$

The triplet pairing $\Delta N_{\uparrow} = 2$, $\Delta N_{\downarrow} = 0$ has leading terms $\Delta J_c = \Delta J_s = 0$:

$$\hat{O}_{\text{TP}}(x) = \psi_{\sigma}^{\dagger}(x)\psi_{\sigma}^{\dagger}(x),$$

$$\begin{aligned} \langle \hat{O}_{\text{TP}}^{\dagger} \hat{O}_{\text{TP}} \rangle &\propto \frac{1}{|x^2 - v_c^2 t^2|^{e^{-2\varphi}/2}|x^2 - v_{\sigma}^2 t^2|^{1/2}} \\ &+ \text{terms} \propto e^{i2k_F x}/x^{3+8\alpha} + \text{etc.} \end{aligned} \quad (68)$$

In the Luttinger liquid language, $e^{-\varphi}$ and e^{φ} are related to v_N and v_J respectively. Actually, Haldane has shown that $e^{-\varphi}$ is proportional to the phase shift on the Fermi surface when we add equal number of electrons on both sides of the Fermi surface; similarly, e^{φ} is proportional to the phase shift when we add some number of electrons on one side and the same number of holes on the other. What we did in Sec. V is to consider what happens if we add electrons only to one side of the Fermi surface. This turns out to be proportional to $\frac{1}{2}(e^{\varphi} \pm e^{-\varphi})$ and gives exactly the same result obtained in this section.

It is very straightforward to show that the correlation functions are directly given by the phase shifts in the more general Bethe-ansatz soluble models. This is of course trivial since the phase shifts approach will be correct even if a model is not solved by Bethe ansatz.

The central result of the Bethe-ansatz solution is the equation relating various pseudomomenta k_i 's to quantum number I_i 's:

The first two terms come from the $k = 0$ piece in $\nabla\theta_c(x)\nabla\theta_c(0)$. The last two terms correspond to $\Delta J_c = 1$, $\Delta J_s = 1$ and $\Delta J_c = 2$, $\Delta J_s = 0$ respectively. For large U , we expect the $4k_F$ term to be dominant. The leading terms in the spin-spin correlation are

$$\begin{aligned} \langle S_z(x, t)S_z(0, 0) \rangle &= \text{const} + D_1 \frac{1}{x^2 - v_{\sigma}^2 t^2} \\ &+ D_2 \frac{\cos 2k_F x}{(x^2 - v_c^2 t^2)^{e^{2\varphi}/2}(x^2 - v_{\sigma}^2 t^2)^{1/2}}. \end{aligned} \quad (66)$$

The singlet pairing $\Delta N_{\uparrow} = \Delta N_{\downarrow} = 1$ has leading terms $\Delta J_{\uparrow} = \pm 1$, $\Delta J_{\downarrow} = \pm 1$:

$$k_i L = 2\pi I_i - \sum_{j=1}^N \vartheta(k_i, k_j), \quad k \in [-Q, Q]. \quad (69)$$

Consider the single particle Green's function

$$G \sim \frac{1}{(x - vt)^{(e^{\varphi} + e^{-\varphi})^2/4}(x + vt)^{(e^{\varphi} - e^{-\varphi})^2/4}}. \quad (70)$$

Haldane has demonstrated that the Luttinger liquid parameter e^{φ} can be derived from the kernel $\vartheta(k, k')$ in the Bethe-ansatz equation

$$e^{-\varphi} = 1 - \sigma(Q) + \sigma(-Q), \quad (71)$$

$$e^{\varphi} = 1 - \sigma(Q) - \sigma(-Q), \quad (72)$$

where

$$2\pi\sigma(k) = \vartheta(k, Q) - \int_{-Q}^Q dk' \frac{\partial}{\partial k'} \vartheta(k, k')\sigma(k') \quad (73)$$

or

$$G \sim \frac{1}{(x - vt)^{[1 - \sigma(Q)]^2}(x + vt)^{[\sigma(-Q)]^2}}. \quad (74)$$

We prove here that $1 - \sigma(Q)$ and $\sigma(-Q)$ are nothing but the phase shifts on the two Fermi points.

Consider adding a particle to the right Fermi point and the shift in every pseudomomenta Δk ,

$$\begin{aligned}
k_i L &= 2\pi I_i - \sum_{j=1}^N \vartheta(k_i, k_j), \\
k'_i L &= 2\pi I_i - \sum_{j=1}^N \vartheta(k'_i, k'_j) - \theta(k'_i, Q), \\
(k'_i - k_i)L &= -\vartheta(k'_i, Q) - \sum_{j=1}^N \left[\vartheta(k'_i, k'_j) - \vartheta(k_i, k_j) \right].
\end{aligned} \tag{75}$$

We have

$$\begin{aligned}
\Delta k L &= (k'_i - k_i)L \\
&= -\theta(k, Q) - \int dk' \rho(k') \frac{\partial}{\partial k} \vartheta(k, k') \Delta k L \\
&\quad - \int dk' \rho(k') \frac{\partial}{\partial k'} \vartheta(k, k') \Delta k' L.
\end{aligned} \tag{76}$$

The density of the pseudomomenta k 's is defined by the

function

$$\rho(k_i) = \frac{1}{(k_{i+1} - k_i)L}, \tag{77}$$

$$2\pi\rho(k) = 1 + \int_{-Q}^Q dk' \frac{\partial}{\partial k} \vartheta(k, k') \rho(k'). \tag{78}$$

We can then rewrite the equation for Δk ,

$$2\pi\rho(k)\Delta k L = -\vartheta(k, Q) - \int dk' \frac{\partial}{\partial k'} \vartheta(k, k') \rho(k') \Delta k' L. \tag{79}$$

Comparing this with the equation for $\sigma(k)$, we see

$$\sigma(k) = -\rho(k)\Delta k L \tag{80}$$

$$= -\frac{k'_i - k_i}{k_{i+1} - k_i} \tag{81}$$

This is nothing but the definition of phase shifts.

VII. GREEN'S FUNCTION IN MOMENTUM SPACE

Following Dzyaloshinskii, we write the k_F Green function in the form

$$G(x, t) = \frac{1}{2\pi} \frac{1}{x - t + i\delta \operatorname{sgn}(t)} \left[\frac{x - t + i/\Lambda \operatorname{sgn}(t)}{x - vt + i/\Lambda \operatorname{sgn}(t)} \right]^{1/2} \left[\Lambda^2 \{x - vt + i/\Lambda \operatorname{sgn}(t)\} \{x + vt - i/\Lambda \operatorname{sgn}(t)\} \right]^{-\alpha}, \tag{82}$$

where $v_\sigma = 1$, $v_c = v$, and Λ is a cutoff. The signs of the δ and $1/\Lambda$ term are so chosen to satisfy the analytic continuation condition. For example, to see that it has the correct analyticity, let us calculate the momentum distribution

$$\begin{aligned}
n(k) &= -i \int_{-\infty}^{\infty} dx e^{-i(k-k_F)x} G(x, 0^-) \\
&= -\frac{i}{2\pi} \int_{-\infty}^{\infty} dx e^{-i(k-k_F)x} \frac{1}{x - i\delta} (1 + \Lambda^2 x^2)^{-\alpha} \\
&= -\frac{i}{2\pi} \int_{-\infty}^{\infty} dx e^{-i(k-k_F)x} i\pi \delta(x) (1 + \Lambda^2 x^2)^{-\alpha} - \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \frac{\sin(k - k_F)x}{x} (1 + \Lambda^2 x^2)^{-\alpha} \\
&= \frac{1}{2} - \operatorname{const} |k - k_F|^{2\alpha} \operatorname{sgn}(k - k_F).
\end{aligned} \tag{83}$$

For Green's function in momentum space, a naive Fourier transform gives

$$\begin{aligned}
G(p, \omega) &= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dt e^{-ipx} e^{i\omega t} G(x, t) \\
&= \frac{1}{\Lambda^{2\alpha} (\omega + p)^{-\alpha} (\omega - p)^{1-\alpha}}.
\end{aligned} \tag{84}$$

To find the behavior of $G(p, \omega)$ for various regions, we write $G(x, t)$ in two parts:

$$\begin{aligned}
G(x, t) &= \frac{1}{2\pi} \frac{1}{x - t} \left[\frac{x - t + i \operatorname{sgn}(t)/\Lambda}{x - vt + i \operatorname{sgn}(t)/\Lambda} \right]^{1/2} \left[\Lambda^2 (x - t + i \operatorname{sgn}(t)/\Lambda) (x + t - i \operatorname{sgn}(t)/\Lambda) \right]^{-\alpha} \\
&\quad + \frac{1}{2\pi} (-i\pi) \operatorname{sgn}(t) \delta(x - t) \left[\frac{x - t + i \operatorname{sgn}(t)/\Lambda}{x - vt + i \operatorname{sgn}(t)/\Lambda} \right]^{1/2} \left[\Lambda^2 (x - t + i \operatorname{sgn}(t)/\Lambda) (x + t - i \operatorname{sgn}(t)/\Lambda) \right]^{-\alpha}.
\end{aligned} \tag{85}$$

The second term is

$$G_2(p, \omega) = -\frac{i}{2} \left[\int_0^\infty dt e^{-ipt + i\omega t} \frac{2^{-\alpha} e^{-i\pi\alpha/2 - i\pi/4}}{\Lambda^{\frac{1}{2} + \alpha} (v - 1)^{1/2} t^{\alpha + \frac{1}{2}}} - \int_{-\infty}^0 dt e^{-ipt + i\omega t} \frac{2^{-\alpha} e^{-i\pi\alpha/2 - i\pi/4}}{\Lambda^{\frac{1}{2} + \alpha} (v - 1)^{1/2} |t|^{\alpha + \frac{1}{2}}} \right], \tag{86}$$

when $p, \omega \ll \Lambda$, the main contribution comes from $t \gg 1/\Lambda$, so

$$G_2(p, \omega) = \frac{1}{2}(2\Lambda)^{-\alpha} \frac{1}{\sqrt{\Lambda(v-1)}} \text{sgn}(\omega - p) \left(e^{-i\pi\alpha} - \frac{i}{2} \right) \frac{\Gamma(\frac{1}{2} - \alpha)}{|\omega - p|^{\frac{1}{2} - \alpha}}. \quad (87)$$

In particular

$$\text{Im} G_2(p, \omega) = -\frac{1}{2}(2\Lambda)^{-\alpha} \frac{1}{\sqrt{\Lambda(v-1)}} \text{sgn}(\omega - p), \quad \left(\sin \pi\alpha + \frac{1}{2} \right) \frac{\Gamma(\frac{1}{2} - \alpha)}{|\omega - p|^{\frac{1}{2} - \alpha}}. \quad (88)$$

For small U , the main contribution from the first term in (85) can be obtained by approximating v to be 1 and omitting $1/\Lambda$ terms. Define $x^+ = (x + t)/2$, $x^- = (x - t)/2$; then

$$\begin{aligned} G_1(p, \omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dt e^{-ip(x^+ + x^-) + i\omega(x^+ - x^-)} 2^{-\alpha} \frac{1}{2x^-} [\Lambda^2(x^- + i/2\Lambda)(x^+ - i/2\Lambda)]^{-\alpha} \\ &= \frac{1}{2\pi} 2^{-\alpha} \Lambda^{-2\alpha} \int_{-\infty}^{\infty} dx^- \frac{1}{x^-} e^{-i(p+\omega)x^-} (x^-)^{-\alpha} \int_{-\infty}^{\infty} dx^+ e^{-i(p-\omega)x^+} (x^+)^{-\alpha} \\ &= \frac{1}{2\pi} (2^{-\alpha}) \Lambda^{-2\alpha} \frac{\Gamma(-\alpha)\Gamma(1-\alpha)}{|\omega + p|^{-\alpha} |\omega - p|^{1-\alpha}} \begin{cases} i(1 - e^{-2i\pi\alpha}), & \omega - p > 0, \omega + p > 0 \\ -2 \sin \pi\alpha, & \omega - p > 0, \omega + p < 0 \\ 2 \sin \pi\alpha, & \omega - p < 0, \omega + p > 0 \\ -i(1 - e^{-2i\pi\alpha}), & \omega - p < 0, \omega + p < 0. \end{cases} \end{aligned} \quad (89)$$

That is,

$$\text{Im } G_1(p, \omega) = -\frac{1}{\pi} \theta(\omega^2 - p^2) \text{sgn}(\omega - p) (2^{-\alpha}) \Lambda^{-2\alpha} \frac{\Gamma^2(1-\alpha)}{\alpha} \sin^2 \pi\alpha \frac{1}{|\omega + p|^{-\alpha} |\omega - p|^{1-\alpha}}, \quad (90)$$

$$\text{Re } G_1(p, \omega) = -\frac{2^{-\alpha}}{2\pi} \Lambda^{-2\alpha} \Gamma(-\alpha)\Gamma(1-\alpha) \frac{\text{sgn}(\omega - p)}{|\omega + p|^{-\alpha} |\omega - p|^{1-\alpha}} (\theta(|\omega| - |p|) \sin 2\pi\alpha + \theta(|p| - |\omega|) 2 \sin \pi\alpha).$$

For $p, \omega \ll \Lambda$, $G_2(p, \omega) \ll G_1(p, \omega)$, and so in what follows, we only consider the contribution from $G_1(p, \omega)$.

For small but finite α , $\text{Im } G$ does not contain a part of the form $Z\delta(\omega - \epsilon)$; this means the quasiparticle picture fails.

For general interaction, since three branch cuts exist in $G(x, t)$, the Fourier transform is rather involved. If we throw away all the $i\delta$'s first, we can show

$$\begin{aligned} G(\omega, p) &\propto \frac{1}{(\omega - v_c p)^{1/2-\alpha} (\omega - v_\sigma p)^{1/2-\alpha}} \\ &\times F\left(\alpha, \frac{1}{2} - \alpha; \frac{1}{2} + \alpha; \mu \frac{\omega - v_c p}{\omega - v_\sigma p}\right), \end{aligned} \quad (91)$$

where $\mu = (v_\sigma + v_c)/2v_c$ and $F(\alpha, \beta; \gamma; z)$ is the hypergeometric function. The function has branch points at the $z = 0, 1, \infty$ and corresponds to $\omega = v_c p$, $\omega = -v_c p$, and $\omega = v_\sigma p$ respectively. The singularities at the three points are

$$\frac{1}{(\omega - v_c p)^{\frac{1}{2}-\alpha}}, \quad \frac{1}{(\omega - v_\sigma p)^{\frac{1}{2}-2\alpha}}, \quad \frac{1}{(\omega + v_c p)^{-\alpha}}. \quad (92)$$

Since $F(\alpha, \beta; \gamma; x)$ is convergent for $0 < x < 1$, and apparently real, the Fourier transform is indeed real along the $\omega = 0$ axis:

$$\begin{aligned} G(\omega, p) &\sim \frac{1}{(\omega - v_c p)^{1-\alpha} (\omega + v_c p)^{-\alpha}} \\ &\times F\left(\frac{1}{2}, 1 - \alpha; 1 + \alpha; \mu' \frac{v_c p + \omega}{v_c p - \omega}\right) \end{aligned} \quad (93)$$

($\mu' = \frac{v_c - v_\sigma}{v_c + v_\sigma}$) and has nonzero imaginary part in other ω, p regions (Fig. 1).

VIII. A 2D LUTTINGER LIQUID?

The Luttinger liquids in 1D are alternative fixed points to the Landau Fermi liquid. It is evident from our approach that the non-Fermi-liquidness only comes from the finite phase shifts on the Fermi surface and does not really depends on any special properties of the one-dimensional systems as is commonly believed. It is proposed that the normal state of high- T_c superconductors may well be described by a two-dimensional Luttinger

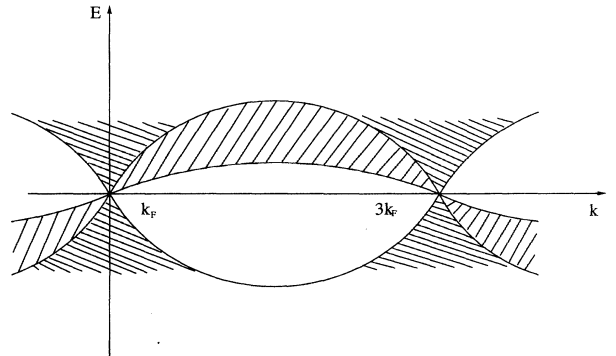


FIG. 1. Regions where $\text{Im} G$ is nonzero.

liquid.²³ As such, the correlation functions in such a system will be similar to that of 1D Hubbard model. The most immediate consequences are manifested by properties that are given directly by the correlation functions.

One of the notable features of the Luttinger-Tomonaga models is the phenomenon of conductivity enhancement first pointed by Luther and Peschel.¹⁶ They found that in the weak interaction regime, the dc conductivity is modified according to

$$\sigma(T) = \pi\sigma_0(T) \frac{[\pi k_B T / 2W]^g}{2\Gamma(2g) \sin(\pi g)}, \quad (94)$$

where $g = V\rho/(1 + 2V\rho)$ and V, ρ, W are the interaction strength, the density of states, and the half bandwidth, respectively. $\sigma_0(T)$ is the Born approximation conductivity for the corresponding noninteracting electron system. In the strong interacting limit, the above formula is essentially correct except that the value of the exponent g will be modified. In the resistivity of the normal state of high- T_c superconductors, the expected T^2 electron-electron scattering contribution is conspicuously absent and replaced by a linear T term apparently caused by neither impurity nor phonon scattering, yet the scattering rate T^2 is still observed, e.g., in the measurement of the Hall angle.¹⁷ It is interesting to ask if one equivalent way of understanding this anomalous resistivity is that the linear T resistivity is the enhancement of T^2 scattering rate of electron-electron scattering (or spinon-spinon scattering). On the other hand, the enhancement of the impurity resistivity would normally lead to the onset of localization. Such a contribution is not observed in the resistivity measurement. We suspect that such a localization is actually absent in the 2D Luttinger liquid, because of the separation of charge and spin degrees of freedom.

There is strong evidence of the existence of an incommensurate magnetic order in the normal state of high- T_c materials. Nuclear magnetic relaxation is one of the most anomalous experiments, and also one that is easily and uniquely explained by the Luttinger liquid theory. Experimentally, it is found that there are two different kinds of behavior when the temperature is above T_c . The planar Cu(2) nuclear relaxation differs substantially in its temperature dependence from that of the O(2) relaxation. Below T_c , the two relaxation rates are similar, suggesting there is a single species of fluctuating electronic spins relaxing the two neighboring nuclei. In YBa₂CuO₃, the relaxation rate on ¹⁷O in the plane and ⁸⁹Y is conventional Korringa-like. The Cu(2) in the plane has a very unconventional and large relaxation rate.²⁵ This indicates that there exists strong antiferromagnetic correlations which get stronger at lower T .

To explain simultaneously the two different kinds of NMR behavior, one has to understand the crystal structure of the CuO₂ plane. The Cu atoms alone form a cubic lattice and O atoms are located at the middle of every Cu-Cu bond.

The effective hyperfine coupling Hamiltonian¹⁸⁻²⁰ is

$$H'(\text{Cu}) = \frac{8\pi}{3} \gamma_{\text{Cu}} \gamma_e \hbar^2 \mathbf{I}_{r+\delta} \cdot \mathbf{S}_r, \quad (95)$$

$$H'(\text{O}) = \frac{8\pi}{3} \gamma_{\text{Cu}} \gamma_e \hbar^2 \mathbf{J}_{r+\hat{x}/2} \cdot (\mathbf{S}_r + \mathbf{S}_{r+\hat{x}}). \quad (96)$$

According to standard arguments, this will lead to

$$\frac{1}{T_1 T} \Big|_{\text{Cu}(2)} \propto \sum_q \frac{\text{Im}\{\chi^{-+}(q, \omega_0)\}}{\omega_0} (\cos q_x + \cos q_y)^2 \quad (97)$$

and

$$\frac{1}{T_1 T} \Big|_{\text{O}(2)} \propto \sum_q \frac{\text{Im}\{\chi^{-+}(q, \omega_0)\}}{\omega_0} (1 + \cos q_x), \quad (98)$$

where $\chi^{-+}(q, \omega_0)$ is the spin susceptibility at nuclear frequency ω_0 :

$$\chi^{-+}(q, \omega_0) \sim i \int_0^\infty dt e^{i\omega_0 t} \langle \delta S_q^-(t) \delta S_{-q}^+(0) \rangle. \quad (99)$$

There are two momenta that contribute the most; one is the zero momentum term that is proportional to the susceptibility, and proportional to $1/(x^2 - v_s^2 t^2)$. And there is the $2k_F$ term which behaves like $1/x^{3/2}$ in the large- U limit. What we have learned in one dimension about the $2k_F$ spin-spin correlation is that because of spin-charge separation, the $2k_F$ spin-spin correlation has two parts; one is “half” the free Fermi gas correlation, with a power law $1/x$, and only depends on spin wave velocity. In the Heisenberg model, where the charge motion is frozen, this is the only term. The other part is the disruption by the hole motion, which has a power of $1/x$ at small U and $1/\sqrt{x}$ at large U .

The temperature dependence due to the $2k_F$ spin-spin correlation is then given by

$$\frac{1}{T_1} \propto T^{e^{2\varphi}}. \quad (100)$$

But because of the form factor that is present in the oxygen $1/T_1 T$ the $2k_F$ term will not be seen in the oxygen relaxation rate, this explains the rather puzzling result of two different relaxation rates.

The Knight shift is decided by the static susceptibility at $q = 0$ and thus is still Korringa-like, as the experiments have shown.

For the relaxation rate of the copper atoms, the deviation from the normal Korringa behavior is expected. For large U , we would expect

$$\frac{1}{T_1 T} \propto \frac{1}{\sqrt{T}}. \quad (101)$$

We have plotted the NMR data of Barrett *et al.*; the fit is remarkably good (Fig. 2). A more systematic interpretation of the NMR data will be discussed elsewhere.

What we have learned in one dimension about the spin-spin correlation is that because of spin-charge separation, the spin-spin correlation has two parts; one is “half” the free Fermi gas correlation, with a power law $1/r$, and only depends on spin wave velocity, in the Heisenberg model, where the charge motion is frozen, this is the only term. The other part is the disruption by the hole motion, which has a power of $1/r$ at small U and $1/\sqrt{r}$ at large U . If we still believe in the spin-charge separation in

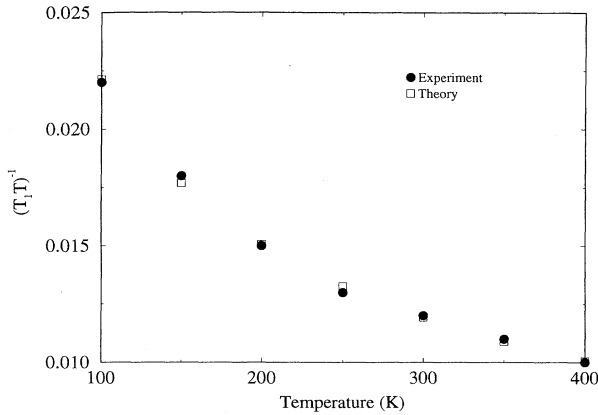


FIG. 2. Cu NMR relaxation rate $1/T_1T$ (Barrett *et al.*), fitted to power law $1/\sqrt{T}$.

2D, we should still get some anomalous power law for the spin-spin correlation. After all, finite phase shifts gives finite power laws, and Anderson has recently proved that in 2D the phase shift on the Fermi surface is finite.²¹

IX. CONCLUSION

There are many experimental properties of the high- T_c materials which show deviations from the normal Fermi-

liquid behavior that can be understood rather well by the one-dimensional large- U Hubbard model. Among them the photoemission data clearly indicate there are no Fermi-liquid-like quasielectron poles in the imaginary part of the single particle Green's function.²² The resistivity and NMR data also provide strong indication that there might be something in common in the 1D and 2D Hubbard models. As has been pointed out,²³ the breakdown of the Fermi-liquid behavior in the 1D and 2D strongly repulsive Hubbard models is caused by similar mechanisms. In Ref. 23 we have proposed a "Luttinger liquid" ground state for the 2D Hubbard model, based on the idea that as in 1D charge and spin degrees of freedom separate and the only true Fermi surface is that for the spinon gas. The charge degrees of freedom are then reduced to collective modes of motion of this Fermi sea, as in one dimension.

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