Two-dimensional Bose liquid with strong gauge-field interaction

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Two unrelated problems can be reduced to a model of a Bose gas interacting with a gauge field: (i) the effect of thermal fluctuations on a system of vortices in bulk superconductors in fields $H_{c1} \ll H \ll H_{c2}$, and (ii) charged, spinless excitations in two-dimensional (2D) strongly correlated electron systems. Both problems are important for the theory of high-temperature superconductors. We study this model in three regimes: at finite temperatures, assuming that the gauge field is purely transverse; at $T = 0$, for the purely static (2D Coulomb) interaction; and at $T = 0$, for a weak Coulomb interaction and a strong transverse one. Transverse interactions suppress the temperature of the superfluid transition significantly. A sufficiently strong transverse interaction is shown to produce a phase separation as the temperature decreases (in the absence of Coulomb repulsion). If there is Coulomb repulsion, the ground state does not have off-diagonal long-range order but the superfluid density is not zero unless the Coulomb constant exceeds a critical value. Sufficiently strong coupling to the transverse field destroys superfluidity as well. In the normal state formed at large couplings, the translational invariance is intact. We propose a bosonic ground state that is not superfluid at $T = 0$. We discuss the implications of these results both for vortex liquids and strongly correlated electron systems.

I. INTRODUCTION

Two seemingly unrelated problems in the theory of superconductivity can both be reduced to the general problem of the two-dimensional Bose liquid strongly interacting with an Abelian gauge field. The first of these problems is the nature of the intermediate state between the conventional Abrikosov flux lattice and the conventional normal state in high-T_c superconductors (HTSC's).[1] This state can be visualized as a liquid of Abrikosov vortices. The second problem is the Bose condensation of the holons that is predicted by the resonating valence bond (RVB) theory[2] of the doped Mott insulator.[3]

We start, in the Introduction, with a brief review of these physical problems and their reduction to the Bose liquid model. Then, in the bulk of the paper we consider the Bose liquid model itself in different regimes. Finally, we apply the results obtained in the bulk of the paper to these two physical problems.

A. Liquid of flux lines in superconductors

Consider a usual three-dimensional (3D) type-II superconductor in a magnetic field H: $H_{c1} < H < H_{c2}$. The conventional wisdom is that in such a system the normal metal undergoes a second-order phase transition into the Abrikosov state where the flux lines form an ordered triangular lattice. However, it has been recently suggested[1] that thermal fluctuations are much more effective melting the flux lattice than completely destroying the superconductive order parameter. In this case an intermediate phase, in which the vortices form a liquid rather than a solid, becomes possible.

Although there is no small parameter that ensures the existence of this intermediate phase (as there is no small parameter in the theory of conventional liquids), the estimates of the numerical factors indicate that this intermediate liquid state exists in a relatively wide temperature range. The temperature scale of this range is governed by the strength of the thermal fluctuations.

Certainly, in conventional superconductors the thermal fluctuations become strong only in the neighborhood of the transition temperature: In weak magnetic fields the width of this region is $\tau_{\text{fl}} = (T_c - T)/T_c \approx \mathcal{G}$, where $\mathcal{G}$ is the Ginzburg parameter:

$$\mathcal{G} = \frac{1}{2}(T_c/H_c^2\xi_\parallel^2\xi_\perp)^2. \tag{1.1}$$

For a clean isotropic superconductor $\mathcal{G} \sim (T_c/\epsilon_F)^4$. The width of the fluctuating region is increased in the magnetic field: $\tau_{\text{fl}} \approx [H_{c2}(T)/H_{c2}(0)]^{2/3}\mathcal{G}^{1/3}$, but it is still very narrow for conventional superconductors. Within this temperature range the thermal fluctuations are sufficiently strong to decrease the amplitude of the superconducting order parameter substantially. The properties of this region deserve a special study; we shall not consider them in this paper. The intermediate liquid vortex state exists in a wider temperature range $\tau_{\text{melt}} > \tau > \tau_{\text{fl}}$, but even this wider range remains narrow for conventional superconductors.

The situation changes in the family of high-T_c superconductors (see Ref. 4 for an extended discussion of that problem). So in $Bi_2Sr_2CaCu_2O_8$ (2:2:1:2) the Ginzburg parameter becomes as large as $\mathcal{G} \sim 0.5$ and is less but still significant in other high-T_c superconductors, $\mathcal{G} \sim 0.01$ in $YBa_2Cu_3O_7$ material, and is still less in the $La_{2-x}Sr_xCu_2O_4$ family. To avoid confusion, note that in strongly layered materials the value of $\mathcal{G}$ defined by (1.1) might be different from an actual width of the fluctuational region around T_c, e.g., in 2:2:1:2 $\tau_{\text{fl}}^{2\text{D}} \approx 0.1$.

For such a large value of the Ginzburg parameter, as in 2:2:1:2 material, the phase diagram looks qualitatively different from that of a conventional superconductor. We discuss the phase diagram of a more conventional superconductor (with $\mathcal{G} \leq 0.01$) first and then turn to an exceptional case of a large $\mathcal{G}$.

The difference between τ_{melt} and τ_{fl} is due to the numerical factors only: $\tau_{\text{melt}}/\tau_{\text{fl}} = 0.32 c_L^{-4/3} \approx 3\text{–}7$, where c_L is a Lindemann number.[4,5] The estimate of the τ_{melt} is based on the Lindemann melting criterion; i.e., the melting of the lattice is believed to happen when the average square of the atomic thermal displacement $\langle u^2 \rangle_T$ exceeds $(c_L a_0)^2$ where a_0 is the lattice constant. As is well known in the theory of conventional liquids, the Lindemann number is small: $c_L = 0.1\text{–}0.2$. The value of the Lindemann number for the quantum melting is known from Monte Carlo simulations:[6] $c_L = 0.2\text{–}0.3$; the same number is also given by Monte Carlo simulations of the model vortex lattice.[7] Two effects combine to make the ratio $\tau_{\text{melt}}/\tau_{\text{fl}}$ large: (i) The shear modulus of the triangular lattice is small, and so the fluctuations are large, and (ii) the Lindemann number c_L is also small. The latter effect is well known in the theory of liquids and is, actually, responsible for their very existence, whereas the former is specific for the 2D triangular lattice.

The effect of this large numerical factor is even more pronounced in a weak magnetic field, where the melting temperature is close to $T_c(0)$. Namely, for a sufficiently small $\mathcal{G}$, there is a range of magnetic fields

$$B < B_0 \approx \beta H_{c2}, \qquad \beta = 0.2 c_L^{-4} \mathcal{G}, \tag{1.2}$$

where τ_{melt} is even larger than $[T_c(B) - T_c(0)]/T_c$ (Fig. 1). Because of a large numerical factor in front of $\mathcal{G}$, the condition (1.2) is compatible with $B > B_{\text{fl}} = \mathcal{G} H_{c2}$, which ensures that we are outside of the scaling region. In these fields even the temperature dependence of $B_{\text{melt}}(T)$ becomes different from that of $H_{c2}(T)$:

$$\frac{B_{\text{melt}}(T)}{H_{c2}(0)} \approx \frac{(T_c - T)^2}{\beta T_c^2}. \tag{1.3}$$

The melting of the flux lattice occurs in a field $B_{\text{melt}}(T) \ll H_{c2}(T)$ in this region of the phase diagram.[1,4,5,8,9] This field range becomes broad in high-T_c materials. We show a typical phase diagram (at T close to T_c) for the material with moderately low $\mathcal{G}$ (such as 1:2:3 compounds) in Fig. 1(a). The melting line was calculated with the use of general expression[5] using the parameters $\mathcal{G} = 5 \times 10^{-3}$ and $c_L = 0.25$. This value of $\mathcal{G}$ was extracted from the width of the transition as observed in the resistivity measurements in $B = 0$, which is about 5 times less than $\mathcal{G}$; the value of c_L was chosen to give the correct position of the melting line as observed in Refs. 8 and 9. Note that, because of the extremely high value of $\kappa = \lambda/\xi \sim 100$ in these materials, the field scale of H_{c1} is not visible in Fig. 1.

For 2:2:1:2 material the Ginzburg parameter is so large that coefficient β in front of H_{c2} in (1.2) exceeds unity, so the melting of the flux lattice happens in much lower magnetic fields than H_{c2} even at $T \sim T_c/2$. Only at much lower temperatures does B_{melt} become comparable to H_{c2}. For such materials the temperature depen-

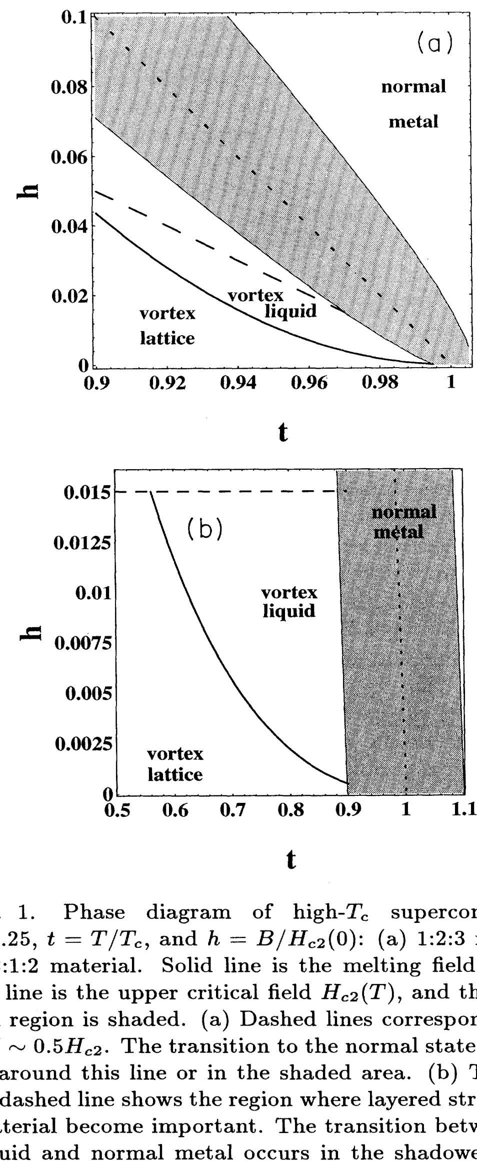

FIG. 1. Phase diagram of high-T_c superconductors. $c_L = 0.25$, $t = T/T_c$, and $h = B/H_{c2}(0)$: (a) 1:2:3 material, (b) 2:2:1:2 material. Solid line is the melting field $B_M(T)$, dotted line is the upper critical field $H_{c2}(T)$, and the fluctuational region is shaded. (a) Dashed lines correspond to the field $H \sim 0.5 H_{c2}$. The transition to the normal state happens either around this line or in the shaded area. (b) The horizontal dashed line shows the region where layered structure of the material become important. The transition between vortex liquid and normal metal occurs in the shadowed region and around the dashed line.

dence of the melting field (1.3) holds in the field range $B \ll \min(H_{c2}/\beta, B')$, where B' is the field range where the layered structure of 2:2:1:2 becomes important (see below). We show a typical phase diagram for the material with a large Ginzburg parameter in Fig. 1(b). In this diagram we used the parameters of the 2:2:1:2 material: $\lambda(T=0) = 200$ nm, $\xi(T=0) = 3$ nm, $d = 1.5$ nm, $M/m = 2500$, $H_{c2}(0) = 65$ T, which gives $\mathcal{G} \approx 0.5$ and $\tau_{\text{fl}}^{2\text{D}} \approx 0.1$.

For $B \sim B_{\text{melt}}(T) \ll H_{c2}(T)$ the distance between the vortices is much larger than their core size which simplifies the problem considerably. In this paper we shall consider this field range only.

Undoubtedly, at $T_{\text{melt}}(H)$ a genuine phase transition takes place and the long-range translational order is broken by the flux lattice. The phase formed right above $T_{\text{melt}}(H)$ is a vortex liquid. We shall show that the vortex liquid has zero resistance in the direction parallel to the magnetic field and is separated from the normal metal by a genuine phase transition.[10]

The phase diagram of real high-T_c superconductors, such as 2:2:1:2 is additionally complicated by weak coupling between layers. In low magnetic fields $B < B' = \pi\Phi_0 m/(d^2 M)$ (where Φ_0 is the flux quantum, d interlayer spacing, m/M is the mass anisotropy) the layered nature of these superconductors is not important and the problem can be reduced to that of an anisotropic superconductor [Fig. 1(b)].

The opposite limit $B > B'$ was considered in Ref. 11. In these fields one expects two phase transitions as well, but the reason for this is very different: The melting of the 2D Abrikosov lattice leads directly to the normal metal, whereas the second phase transition occurs at lower temperatures, in the state in which the translational invariance has been already broken. These two phase transitions should not be confused with the two phase transitions at lower fields which we discuss in this paper.

B. Vortex liquid: Reduction to the boson model

We use the London approximation to find the free energy of the vortex system. This approximation is very good for a type-II superconductor with large $\kappa = \lambda/\xi \ll 1$ (where λ is London penetration depth, ξ is the coherence length) if the magnetic field $B \ll H_{c2}(T)$. The free energy of the arbitrary curved vortices becomes

$$F = \frac{\Phi_0^2}{32\pi^2\lambda^2} \sum_{i,j} \int \frac{\exp(-|\mathbf{r}_i - \mathbf{r}_j|/\lambda)}{|\mathbf{r}_i - \mathbf{r}_j|} d\mathbf{r}_i d\mathbf{r}_j - \frac{\mathbf{H}_{\text{ext}} \cdot \mathbf{B}}{4\pi}, \tag{1.4}$$

where the sum runs over all flux lines and $d\mathbf{r}_i$ is the tangent vector element of the ith vortex.

Far from the fluctuating region ($\tau \gg \tau_{\text{fl}}$) the vortices are curved smoothly and have no overhangs. The number of the thermally activated vortex loops is also exponentially small at these temperatures. All this makes it possible to view the flux lines as world lines of Bose particles. In this representation the integral over all possible configurations of these lines that yields the partition function becomes the path integral of a nonrelativistic quantum Bose system in imaginary time. The direction of the "time" coincides with the direction of the external magnetic field in the original superconductor. The temperature T of the superconductor becomes a "Planck constant" $\hbar_B$ of the Bose system (below we will put $\hbar_B = 1$ in the bulk of our discussion). The inverse temperature of the Bose system is then given by $T^B = T/L_z$, where L_z is the length of the superconductive sample in the direction of the magnetic field. This useful representation was invented by Nelson.[12]

The statistics of these particles is determined by the condition that all configurations of the flux lines result in positive contributions to the partition function. Unfortunately, the conventional partition function of the Bose system implies periodic boundary conditions in imaginary time. Translated into the language of the original problem such boundary conditions mean that the sample is a torus. Free boundary conditions would be much more appropriate for a physical superconductor of finite width. However, the properties of the superconductors should have the same thermodynamic limit at $L_z \to \infty$ regardless of their boundary conditions, and so the results for the Bose system at zero temperature can be translated directly into the bulk properties of superconductors.

The effective action of the quantum Bose system follows from (1.4). It is useful to view the interaction between these bosons as being mediated by an auxiliary gauge field $(a_0, \mathbf{a})$:[10]

$$S = \sum_j \int dz \left[\frac{m}{2} \left(\frac{d\mathbf{r}_j}{dz} \right)^2 - \mu \right] + \int d^3x \left[i \left(j_\alpha - \frac{1}{\Phi_0} \epsilon_{\alpha\beta\gamma} \nabla_\beta A_\gamma \right) a_\alpha + \frac{1}{4g^2} f_{\alpha\beta}^2 + \frac{1}{8\pi} (\boldsymbol{\nabla} \times \mathbf{A})^2 \right], \tag{1.5}$$

where $\mathbf{A}$ is the electromagnetic vector potential ($\boldsymbol{\nabla} \times \mathbf{A} = \mathbf{B}$), $f_{\alpha\beta} = \partial_\alpha a_\beta - \partial_\beta a_\alpha$, the coupling constant $g = \Phi_0/(\lambda\sqrt{4\pi})$, the bare vortex line tension $m = g^2/4\pi$, $\mu = (\Phi_0 H_{\text{ext}}/4\pi - m \ln \kappa)$ is the "chemical potential" for vortex lines, and $j_\alpha = (n, \mathbf{j})$ is a three-component vector defined by a vortex line configuration:

$$n(\mathbf{r}, z) = \sum_j \delta[\mathbf{r} - \mathbf{r}_j(z)], \qquad \mathbf{j}(\mathbf{r}, z) = \sum_j \tfrac{d\mathbf{r}}{dz} \delta[\mathbf{r} - \mathbf{r}_j(z)]. \tag{1.6}$$

Actually, n and $\mathbf{j}$ represent the density and current of the 2D Bose liquid we are going to discuss. To reproduce the interaction energy (1.4) from the action (1.7) one has to integrate out the fluctuations of the electromagnetic field first. This yields the gauge-invariant mass λ^2 of the field a. Then the Gaussian integral over the remaining field a results in the interaction (1.4).

The action (1.5) is written in terms of vortex configurations or, in other words, in terms of world lines of Bose particles. Instead of doing a Feynman path integral over trajectories, we go over to a coherent-state formulation (see, e.g., Refs. 13 and 14) where the integration is over complex Bose fields ψ and $\psi^\dagger$:

$$S = \int [L_B\{\psi, a\} + L_G\{A, a\}] d^2r d\tau, \quad (1.7)$$

$$L_B\{\psi, a\} = \psi^* \left[\frac{\partial}{\partial \tau} - \frac{1}{2m}(\nabla - i\mathbf{a})^2 + i a_0 - \mu \right] \psi + V_{\rm sr}(\psi^*\psi) + \frac{1}{4g^2} f_{\alpha\beta}^2,$$

$$L_G\{A, a\} = \frac{i}{4\sqrt{\pi}\lambda g} \epsilon_{\alpha\beta\gamma} A_\alpha f_{\beta\gamma} + \frac{1}{8\pi}[\nabla \times \mathbf{A}]^2,$$

where $V_{\rm sr}(\psi^*\psi)$ represents the short-range repulsion between the bosons cores. In the new representation, μ is the chemical potential of the bosons, whereas m is their mass. Its bare value corresponds to a small-scale deformation of the flux line and equals $m = (\Phi_0/4\pi\lambda)^2 = g^2/4\pi$. At larger scales it renormalizes significantly due to the self-interaction of the flux lines. It reaches $m \ln(\lambda/\xi)$ at scales λ for a single-line problem. The characteristic scale of the many-body problem (1.7) is set by the distance l_0 between the flux lines (or bosons) which is governed by the average external magnetic field B: $l_0 = n^{-1/2}$, where $n = B/\Phi_0$ is the flux line density. We shall consider only strong fields $B \gg H_{c1}$, so that the distance between the lines $l_0 \ll \lambda$. In these fields the Bose mass becomes $m \ln(l_0/\xi)$ at scales l_0.

In such fields one may neglect the screening of the interaction (1.4) which occurs at scales λ. This approximation should have almost no effect on the liquid-state properties governed mainly by the short scales. In the formalism of a Bose model (1.7) it amounts to neglecting the electromagnetic field fluctuations. We shall use this approximation in the main part of the paper and treat the electromagnetic field $\mathbf{A}$ in (1.7) as a source field. The problem is reduced then to the problem of bosons interacting with a gauge field a as described by the Lagrangian $L_B\{\psi, a\}$ alone.

Below we shall discuss the generalized version of this model described by Lagrangian (which we now rewrite in real time):

$$L_B\{\psi, a\} = \psi^* \left[i\left(\frac{\partial}{\partial \tau} - i a_0 \right) + \frac{1}{2m}(\nabla - i\mathbf{a})^2 \right] \psi - V_{\rm sr}(\psi^*\psi) + \frac{1}{2g^2} f_{0\alpha}^2 - \frac{c^2}{2g^2} f_{12}^2. \quad (1.8)$$

Lagrangian (1.8) simplifies to (1.7) at $c = 1$; at $c \neq 1$ it describes a more general situation when the strength of the scalar interaction is different from that of the transverse one. This situation is realized in another physical context, e.g., in strongly correlated electron systems.

C. RVB theory of the strongly correlated electron systems

The nature of the ground state of a lightly doped Mott insulator remains an unsolved problem. Even the ground states of the simplest models of it such as the t-J or Hubbard model are subjects of controversy. Quite a few ground states were proposed.[2,15–17]

Phenomenologically, the uniform RVB state proposed in Ref. 15 seems to be the most likely candidate for the description of high-T_c oxides at moderate temperatures (see Refs. 18–20 and references therein). In this state the spin and charge degrees of freedom are decoupled. Formally, the operator of the physical electron (c_α^+) in this state becomes the product of the fermion (spinon, f_α^+) and the boson (holon, b) operators: $c_\alpha^+ = b f_\alpha^+$. This decoupling implies gauge invariance: $b \to \exp(i\theta) b$, $f_\alpha \to \exp(i\theta) f_\alpha$. This is why the gauge field inevitably appears in the spectrum of the low-energy excitations as a consequence of spin charge separation.

Most physical responses in this state (such as electrical and thermal conductivities, Hall coefficient, etc.) are governed by the Bose subsystem.[18,20] In other words, the effective theory of these properties contains the Bose particles and the gauge field only, but no spin degrees of freedom which are integrated out of the effective action:

$$S_h(\psi, a) = \int \left\{ \psi^* \left[\frac{\partial}{\partial \tau} - i a_0 + \frac{1}{2m}(\nabla - i\mathbf{a})^2 \right] \psi \right\} d^2r d\tau + \frac{1}{2} \sum_{\omega, k} [\Pi_\parallel(\omega, k) a_0^2(\omega, k) + \Pi_\perp(\omega, k) \mathbf{a}^2(\omega, k)]. \quad (1.9)$$

Instead of the usual term $f_{\alpha\beta}^2$ describing the action of a free gauge field this action contains the polarization operators $\Pi_\parallel(\omega, k)$ and $\Pi_\perp(\omega, k)$ of longitudinal and transverse photons. Formally, these terms appear after the spinons are integrated out of the effective theory.[3] The behavior of these terms at small ω, k depends crucially on the spectrum of the spin excitations which were integrated out or, in other words, on the state of the spinon subsystem.

In the uniform RVB state the spinons form a Fermi sea with a large Fermi surface. In this state the spinons screen the fluctuations of the scalar part of the gauge field very effectively: longitudinal polarization operator $\Pi_\parallel(\omega, k) = \nu$ at $\omega, k \to 0$, where ν is the density of states of the spinons at their Fermi surface. The vector part of the gauge field remains unscreened but its dynamics is overdamped: $\Pi_\perp(\omega, k) = \gamma|\omega|/k + \chi k^2$, where γ is the Landau damping coefficient and χ is the orbital susceptibility of the spinon fluid. A very soft overdamped spectrum $\omega \propto k^3$ of the transverse gauge field means that these fluctuations are very slow. If their dynamics is ignored completely, the model (1.9) is reduced to the model (1.8) with $c \to 0$ but finite ratio $c^2/g^2 = \chi$.

In the straightforward RVB scenario nothing happens to the spinon subsystem at low temperatures until a Bose condensation of the holons occurs, resulting in a Fermi liquid. Another scenario seems to be more likely to describe most of the high-T_c materials.[21] In this scenario the gap (or the pseudogap) is formed in the spinon spectrum at moderate temperatures ($T \sim 200$ K), far above the superconductive transition temperature. Significant experimental evidence seems to indicate that some pseudogap is really formed above T_c in a lot of materials, such as the 60 K 1:2:3 material, 2:2:1:2 compound, and La family of high-T_c superconductors.[21] In this state the spinons can no longer screen the scalar part of the gauge field, and so the longitudinal polarization operator becomes $\Pi_{\|}(\omega,k) = k^2/\Delta$ (where Δ is the formed gap) at small frequencies and wave vectors. Landau damping caused by the gapless spinons on the Fermi surface also disappears and the transverse polarization operator becomes $\Pi_{\perp}(\omega,k) = \omega^2/\Delta + \chi k^2$. Thus, in this case the model (1.8) is reproduced identically with $g^2 = \Delta$, $c^2/g^2 = \chi$.

D. Formulation of the problem

Now we return to the basic model (1.8). The qualitative properties of the model are governed by three dimensionless parameters. The most important are the two dimensionless coupling constants of the gauge field:

$$\alpha_C = \frac{g^2 m}{16\pi^2 n}, \qquad \alpha_g = \frac{g^2}{8\pi mc^2}. \tag{1.10}$$

The third dimensionless parameter is related to the strength of the hard core interaction [described by $V_{\rm sr}(\psi^*\psi)$ in (1.8)]: $\delta = nv_0$, where n is the density of the bosons and v_0 is the volume of their hard core. At small $\delta \ll 1$ (and in the absence of other interactions) bosons form a gas; at larger $\delta \sim 1$ they constitute a liquid, but the properties of these states are similar at low temperatures. To simplify the problem we shall consider only the dilute gas limit $\delta \ll 1$ from now on.

Even in this limit, quite a few qualitatively different phases are possible depending on the values of the other two dimensionless parameters. In the following we consider them one by one.

II. WEAK COULOMB AND TRANSVERSE INTERACTIONS

We start with the simplest case: a vanishing Coulomb interaction and a weak transverse one. Quantitatively it means that $g^2m/n \to 0$ and $g^2/mc^2 \ll 1$. As we shall see below we need even more bounding condition on the transverse interaction to ensure that its effect on the Bose system is small (however, even this very small interaction changes some properties qualitatively):

$$\frac{g^2 \ln(1/\delta)}{16\pi mc^2} \ll 1, \tag{2.1}$$

where we retained a large numerical factor which invariably appears in all estimates. The easiest way to estimate the effect of the transverse interaction on the Bose system is to compare the characteristic temperature at which this effect would become important in the ideal Bose gas with the Bose condensation temperature. To avoid confusion, we emphasize again that the temperature of the Bose gas corresponds to the thickness of the sample in the vortex problem. It is this "temperature" that we shall mean by this word and denote by T in this and the following sections. To obtain the Bose condensation temperature we find the susceptibility of the ideal Bose gas:

$$\chi = \frac{N_B(0)}{24\pi m} = \frac{e^{T_0/T}-1}{24\pi m}, \qquad T_0 = \frac{2\pi n}{m}, \tag{2.2}$$

where $N_B(\epsilon) = \exp[(\epsilon-\mu)/T - 1]^{-1}$ is the Bose distribution function, and we use the chemical potential for the ideal Bose gas of fixed density: $\mu_0 = T\ln[1-\exp(-T_0/T)]$. The susceptibility (2.2) is small compared to the "vacuum" one (c^2/g^2) at high temperatures, but grows rapidly below T_0 and reaches the "vacuum" value at

$$\tilde{T}_g = \frac{T_0}{\ln[24\pi mc^2/(g^2)]}, \tag{2.3}$$

which sets the temperature scale. If condition (2.1) is satisfied, this temperature is lower than the superfluid transition temperature of the dilute Bose gas:[13,22]

$$T_{\rm sf} = \frac{T_0}{\ln\ln(1/\delta)}. \tag{2.4}$$

Thus, the Bose condensation happens at the temperatures at which the effects of the gauge field are unimportant. Below the Bose condensation temperature the usual superfluid order parameter is formed in the Bose liquid at short scales. However, the effect of the gauge field changes the qualitative behavior of this state at larger scales. The reason for this is that the gauge field screens the interaction between the vortices in the superfluid, and so the energy of a single vortex is no longer infinite, but

$$E_v = \frac{\pi n_s}{m}\ln\lambda_B/\xi_B, \tag{2.5}$$

where n_s is a superfluid density, $\lambda_B = [g^2 n_s/(mc^2)]^{-1/2}$ is the penetration depth of the gauge field in the Bose liquid, and ξ_B is the coherence length of the superfluid which is conveniently expressed through the scattering amplitude $\Gamma(\mu)$ of the Bose particles at energy μ:[22]

$$\xi_B^{-2} = 4mn_s\Gamma(\mu), \qquad \Gamma(\mu) \approx \frac{4\pi}{m\ln(1/\delta)}. \tag{2.6}$$

Certainly, the formula (2.5) is valid (and the vortices themselves are stable) only if $\lambda_B/\xi_B \geq 1$. This ratio is actually Ginzburg-Landau parameter of this Bose liquid:

$$\kappa_B^2 = \frac{16\pi mc^2}{g^2\ln(1/\delta)}, \tag{2.7}$$

The condition (2.1) ensures that $\kappa \gg 1$ and the vortices are stable and well-defined excitations in this state.

A finite value of the vortex energy (2.5) means that at any finite temperature the density n_v of these vortices is finite:

$$n_v \simeq \xi_B^{-2}\exp(-E_v/T) \simeq \frac{16\pi n_s}{\ln(1/\delta)}\exp\left(-\frac{\pi n_s\ln(\kappa_B)}{mT}\right). \tag{2.8}$$

Thus, at large scales $L \gg n_v^{-1/2}$ the correlations of the order parameter decay exponentially and all properties at these scales are the same as the ones of the normal liquid at all nonzero temperatures. The density of vortices goes to zero exponentially at $T \to 0$, and so the ground state is a superfluid Bose liquid.

The crucial changes are induced in the ground state by the transverse gauge field if the interaction constant is larger than (2.1).

III. STRONG COUPLING TO THE TRANSVERSE FIELD, THERMAL FLUCTUATIONS

Now we turn to the case of a stronger transverse interaction (but vanishing Coulomb repulsion):

$$\frac{g^2 \ln(1/\delta)}{16\pi mc^2} \gg 1. \tag{3.1}$$

In this case the interaction with a gauge field becomes important for the Bose system below T_g [Eq. (2.3)]. At temperatures slightly below T_g the superfluid correlations are still weak, and so in this temperature range one may neglect them and simplify the problem to the problem of an ideal Bose gas with a transverse interaction.

We shall argue that in this system the phase separation is likely to happen. This phase separation is reminiscent of the intermediate state of type-I superconductors in a magnetic field. Qualitatively it is due the fact that magnetic fluctuations are suppressed in the superfluid phase, and so the energy of zero point fluctuations is smaller in the vacuum than in the Bose liquid. Thus, Bose particles are repelled from the regions where the magnetic field fluctuates.

In order to describe this phase separation quantitatively we find the equation of state for this gas, namely, the dependence of its chemical potential on the gas density $\mu(n)$, and check it for the stability of the homogeneous solution. We restrict ourselves to a one-loop approximation in the gauge-field fluctuations. This approximation is justified if the transverse interaction is not very strong: $g^2/(16\pi mc^2) \ll 1$. If it is stronger, the results of a one-loop approximation should be regarded as an estimate of the effect only. Then, the chemical potential of the Bose gas becomes

$$\mu = \mu_0 + \frac{\partial F_g}{\partial n},$$

$$F_g = T\frac{1}{2}\sum_\omega \int \frac{d^2k}{(2\pi)^2} \ln\left[\frac{\omega^2}{g^2} + \frac{c^2k^2}{g^2} + \Pi_\perp(\omega,k)\right], \tag{3.2}$$

where μ_0 is the chemical potential of the Bose gas without gauge interactions; above T_{sf} it is given by $\mu_0 = T\ln[1-\exp(-T_0/T)]$, and $\Pi_\perp(\omega,k)$ is the polarization operator of the Bose gas.

A small value of the charge g, but finite g^2/c^2, implies that the spectrum of the gauge-field fluctuations $\omega = ck$ is very soft since c is small. Thus, we may neglect their dynamics in this limit. Formally, it means that we leave only the term with $\omega = 0$ in the sum (3.2). This is justified if in the next term of this sum (i.e., at $\omega = 2\pi T$) $\omega^2/g^2 \gg c^2k^2/g^2 + \Pi_\perp(\omega,k)$ for all important wave vectors k. The important k^2 are no larger than $\max(mT, n)$ which sets the smallest scale of the Bose gas. Inserting these scales in the above inequality we get a sufficient condition for the static approximation at $T \sim T_0$:

$$\frac{g^2}{mc^2} \gg \frac{g^2 m}{n}, \tag{3.3}$$

which is satisfied if the Coulomb interaction is relatively weak. In this approximation the free energy (3.2) becomes

$$F_g = \frac{1}{2}T\int \ln\left[1 + \frac{g^2}{c^2k^2}\Pi_\perp(k)\right]\frac{d^2k}{(2\pi)^2}, \tag{3.4}$$

where we have subtracted the constant part of the free energy that does not depend on the Bose system. To evaluate it we need the form of polarization operator. In the one-loop approximation we find

$$\Pi_\perp(k) = \frac{1}{m}\int\left[1 - \frac{2[p^2 - (\mathbf{p}\cdot\mathbf{k})^2/k^2]}{m(\epsilon_{p+k} - \epsilon_p)}\right] N_B(\epsilon_p)\frac{d^2p}{(2\pi)^2}, \tag{3.5}$$

where $\epsilon_p = p^2/(2m)$ is the energy spectrum of bosons. Integrating over the angles formed by $\mathbf{p}$ and $\mathbf{k}$ and introducing the dimensionless variable $z = 4p^2/k^2$ we simplify (3.5):

$$\Pi_\perp(k) = \frac{k^2}{16\pi m}\int \sqrt{1-z}\, N_B\left(\frac{zk^2}{8m}\right) dz. \tag{3.6}$$

The one-loop approximation employed in the derivation of (3.6) is justified only if the interaction is not too strong: $\alpha_g = g^2/(8\pi mc^2) \ll 1$. Together with (3.1) this condition determines the range of the interaction which we shall consider in this section.

For such interaction the corrections to the thermodynamical properties of the Bose system caused by the gauge field are small at T_0, but increase at lower temperatures. We find them at $T \ll T_0$.

At these temperatures the integral in (3.6) can be evaluated analytically in two overlapping regions of k:

$$\Pi_\perp(k) = \begin{cases} \frac{T}{\pi}\left[\frac{\sqrt{1+(k\xi_T)^2}}{k\xi_T}\ln\left[\sqrt{1+(k\xi_T)^2} + k\xi_T\right] - 1\right], & k^2 \ll 8mT, \\ \frac{n_B}{m} - \frac{T}{2\pi}f(\frac{k^2}{8mT}), & k^2 \gg \xi_T^{-2}, \end{cases} \tag{3.7}$$

$$f(y) = -\ln[1-\exp(-y)] + y\int_0^1 \frac{1-\sqrt{1-z}}{\exp(zy)-1}dz,$$

where $\xi_T^{-2} = -8m\mu_0$.

For a moderate transverse interaction the second term in the brackets in the expression (3.4) is small at $k^2 \geq 8mT_0$: $g^2\Pi_\perp(k)/(c^2k^2) \leq g^2/(16\pi mc^2) \ll 1$. At these k the ln in (3.4) can be expanded. Using (3.5) for the polarization operator we see that the integral (3.4) over this region of k leads to the terms in the free energy that are constant or linear in the boson density. These terms do not affect the equation of state for bosons since they shift their chemical potential by a constant value only. We neglect these terms and consider only the region of small $k^2 \ll 8mT$ in the integral (3.4) which yields a nontrivial correction to the chemical potential of bosons. In this region we use small k asymptotics in (3.7):

$$F_g = \frac{g^2T^2}{8\pi^2c^2}\int_0^\infty \ln\left\{1+\frac{1}{\zeta}\left[\frac{\sqrt{1+G\zeta}}{\sqrt{G\zeta}}\ln\left(\sqrt{G\zeta+1}+\sqrt{G\zeta}\right)-1\right]\right\}d\zeta, \tag{3.8}$$

where we introduced the dimensionless parameter $G = g^2\exp(T_0/T)/(8\pi mc^2)$ which measures the effective strength of the transverse interaction. In (3.8) we extended the integral over $k^2 = g^2T\zeta/(\pi c^2)$ to the whole real axis including the region of large k where the large k asymptotics in (3.7) should have been used. This changes the free energy by irrelevant terms which are constant or linear in the boson density.

At $T \gg T_0$ the interaction parameter G is small ($G \ll 1$), the argument of the outer ln is close to unity, and the correction to the free energy does not depend on the state of the Bose system. At lower temperature G increases rapidly and becomes large at $T \ll \tilde{T}_g$ [cf. (2.3)]. At these temperatures we evaluate the integral (3.8) with logarithmic accuracy and find the shift of the chemical potential induced by the gauge field:

$$\mu_B = \mu_0 - \frac{g^2T}{8\pi mc^2}\ln\left[\ln G(T_0)\right]. \tag{3.9}$$

In (3.9) we emphasize by $G(T_0,T)$ that the shift of the chemical potential depends on the T_0 and therefore on the concentration of bosons. As a result of this shift the compressibility of the Bose system becomes

$$\frac{\partial\mu_B}{\partial n} = \frac{2\pi}{m}\left[e^{\frac{-T_0}{T}} - \frac{g^2}{8\pi mc^2}\frac{T}{T_0+T\ln[g^2/(8\pi mc^2)]}\right]. \tag{3.10}$$

At low temperatures, $T \leq T_g \approx T_0/\ln[(8\pi mc^2)/g^2]$, the compressibility (3.10) becomes negative, signifying the instability of this phase with respect to a phase separation. The phases which are formed have smaller and larger concentrations of bosons than a homogeneous phase. To find these states we must find the stable solutions of the equation of state with lower and higher densities than the original one. The existence of a stable solution with a lower concentration is clear from (3.10): At a concentration $n_B \leq n_{\min}$,

$$n_{\min} \approx \frac{mT}{2\pi\ln(8\pi mc^2/g^2)}, \tag{3.11}$$

the compressibility becomes positive and the phase is stable.

The phase with a higher concentration does not become stable until the concentration becomes so high that a Bose condensation happens. To find the compressibility in this phase we repeat the calculations resulting in (3.10) for a phase with Bose condensation. To simplify the problem we consider only very large concentrations, so that the formed Bose system turns out to be at effectively low $T \ll T_{\text{sf}}(n)$. The chemical potential μ_0 of the Bose gas without gauge interaction at these concentration changes sign and becomes

$$\mu_0 = \frac{4\pi n}{m\ln(1/\delta)}. \tag{3.12}$$

The polarization operator $\Pi_\perp(k) = n/m$ depends weakly on the wave vector k, and so the free energy of the gauge field is

$$F_g = \frac{T}{2}\int \ln\left[1+\frac{g^2n}{mc^2k^2}\right](dk). \tag{3.13}$$

This free energy results in the correction to the chemical potential of the superfluid (3.12) similar to (3.9):

$$\mu = \frac{4\pi n}{m\ln(1/\delta)} + \frac{g^2T\ln[mc^2/(g^2n)]}{8\pi mc^2}. \tag{3.14}$$

In this superfluid phase the compressibility becomes positive at sufficiently large densities $n \geq n_{\max}$:

$$n_{\max} = \frac{g^2\ln(1/\delta)}{16\pi mc^2}\frac{mT}{2\pi}. \tag{3.15}$$

The dependence $\mu(n)$ in the whole region of the densities is given by the combination (3.9,3.14). The Bose liquid with the density $n_{\min} \leq n \leq n_{\max}$ (3.11,3.15) is absolutely unstable. The ratio

$$n_{\max}/n_{\min} = \frac{g^2}{16\pi mc^2}\frac{\ln(1/\delta)}{\ln(8\pi mc^2/g^2)} \tag{3.16}$$

is large if $g^2\ln(1/\delta)/(16\pi mc^2) \gg 1$. In this case the dilute phase can be named a normal gas and a dense phase a superfluid liquid of bosons. The densities of these phases become comparable close to a critical point where $g^2\ln(1/\delta)/(8\pi mc^2) \approx 1$ and $T \approx T_g = T_{\text{sf}}$.

The density of the normal gas n_n is less than $n_{\min}$ and the density of the superfluid liquid $n_{\rm sf}$ is greater than $n_{\rm max}$. To find their equilibrium values at the temperatures at which these phases coexist we have to find the chemical potential from the equilibrium condition:

$$\int_{n_n}^{n_{\rm sf}} \mu(n)dn = 0. \quad (3.17)$$

Thus, we have shown that a moderate interaction with a gauge field makes the Bose system unstable with respect to the phase separation below T_g. Two phases are formed: one with a lower concentration of bosons and another one with a higher concentration. The phase with a higher concentration is superfluid, since the density in this state is large enough to suppress the fluctuations of the gauge field. Such phase separation is suppressed even by weak Coulomb repulsion. A weak Coulomb repulsion leads to a formation of finite size droplets. Strong Coulomb repulsion leads to a homogeneous phase.

IV. STRONG COULOMB REPULSION

In this section we consider the Bose gas with Coulomb interaction only, i.e., model (1.8) in the limit $c \to \infty$. The strength of the interaction is measured by the dimensionless parameter α_C defined in (1.10).

In the limit of very large interaction the Bose gas condenses into the Wigner crystal breaking spontaneously the translational invariance of the original model. We shall argue below that, for intermediate values of the interaction, the Bose gas forms a normal liquid at zero temperature. This transition is associated with a spontaneous breaking of a Galilean invariance.

This statement needs elaboration. If the Galilean invariance is *exact*—i.e., it is not broken by any boundaries or arbitrary weak impurity potential—the superfluid density is $\rho_s = n/m$ at $T = 0$. However, it is possible that an arbitrarily weak pinning potential or other mechanism violating the Galilean invariance leads to a complete loss of superfluidity. We call this state the normal liquid. Below we discuss this definition for the special case of annular geometry.

Unfortunately we cannot prove this conclusion rigorously; we can present only qualitative arguments supporting it. Since each of these arguments has loopholes, we shall present few of them. These arguments are based on the long-range behavior of the off-diagonal order parameter.

The phase sequence superfluid–normal liquid–crystal taking place in a charged Bose gas should be contrasted with two phases (superfluid-crystal) that are realized in a charged Bose gas on a commensurate lattice.[23,24] The difference between the models is due to the fact that in the lattice model the Galilean invariance is broken explicitly by the lattice.

A. Off-diagonal order in a Coulomb gas

We begin with a weak Coulomb interaction $\alpha_C \ll 1$. In this case the effects of the Coulomb interaction are small at short scales. To analyze the long-wave properties in this limit one can use a "hydrodynamic" approach and derive the effective action of the Bose gas at intermediate scales.[25] These scales should be larger than the correlation length, which in the dilute Bose gas is due to the hard core repulsion. The state of the Bose gas at these scales is characterized by two variables: phase ϕ and density $\rho = n + \delta\rho$, which are canonically conjugate to each other. At the intermediate scales the Lagrangian has the usual form[13]

$$L\{\phi,\delta\rho\} = i\frac{d\phi}{dt}\delta\rho + \frac{1}{2}\left(\nu^{-1}\delta\rho^2 + \frac{n}{m}(\nabla\phi)^2 + \frac{g^2}{2\pi}\ln(r-r')\delta\rho_r\delta\rho_{r'}\right), \quad (4.1)$$

where ν is compressibility of the Bose gas due to the hard core repulsion, and the last term describes the effects of the Coulomb interaction. At large scales the Bose gas can be described by the phase ϕ only. The effective Lagrangian of the phase is[25]

$$L_{\rm eff}\{\phi\} = \frac{1}{2}\left[\frac{d\phi_{-q}}{dt}\frac{1}{\nu^{-1}+g^2/q^2}\frac{d\phi_q}{dt} + \frac{n}{m}\phi_{-q}q^2\phi_q\right]. \quad (4.2)$$

The fluctuations of the phase described by the action (4.2) are large and lead to the decay of the off-diagonal order parameter at large scales:

$$\langle\psi_0^\dagger\psi_r\rangle = n\langle\exp[i\phi(r) - i\phi(0)]\rangle \propto \frac{1}{r^\alpha}, \quad (4.3)$$

$$\alpha = \sqrt{\alpha_C}, \quad (4.4)$$

since $\langle[\phi(r)-\phi(0)]^2\rangle = 2\alpha\ln r$.

The power law decay of the off-diagonal long-range order does not necessarily imply the absence of the superfluid density, as demonstrated by the Berezinsky phase in a 2D Bose liquid at finite temperature.[26] In a 2D Bose liquid the exponent of the power law increases with temperature, and at $T = T_{\rm BKT}$ the transition into the normal state occurs. At the transition the exponent is $1/4$; the superfluid density does not change much between 0 and $T_{\rm BKT}$, but jumps to zero at $T_{\rm BKT}$.[26,27]

At zero temperature the exponent increases with increase of Coulomb repulsion. Below we argue that when this exponent exceeds a critical value $\alpha^{\rm cr} \sim 1$ the transition into the normal state takes place.

However, before that we shall show that Eq. (4.3) relating the exponent α to the strength of the Coulomb interaction derived in the hydrodynamic approximation is actually an exact result in the superfluid state. There are two ways to prove it.

Galilean invariance ensures that there are no corrections to the superfluid density in perturbation theory. So the coefficient n/m in the action (4.1) is not renormalized until the transition into the normal state happens. In this respect the Bose problem with Coulomb interaction at zero temperature is different from the dilute Bose gas at finite temperature. The Coulomb interaction between total densities is not renormalized as well. Thus, the coefficients of the action (4.2) are not renormalized.

In a different approach one considers a Hamiltonian that contains a Coulomb long-range interaction and a specific short-range interaction in which the ground state is known exactly.[28] In this ground state the off-diagonal correlator obeys (4.3). Then one may prove that the effect of the short-range interaction at large scales is negligible.[28]

Apart from the Coulomb repulsion this Hamiltonian contains a short-range three-body interaction

$$H_{\rm int} = \frac{g^2}{4\pi}\int \ln(r_1-r_2)\,\delta\rho(r_1)\,\delta\rho(r_2)d^2r_1d^2r_2 + H_{\rm sr},$$
$$H_{\rm sr} = \frac{4\pi\alpha}{m}\int[\delta\rho(r)]^2d^2r + \int\delta\rho(r_1)\delta\rho(r_2)\delta\rho(r_3) \times V(r_1-r_2,r_1-r_3)d^2r_1d^2r_2d^2r_3, \quad (4.5)$$
$$V(r,r') = \frac{2\alpha^2}{m}\frac{\mathbf{r}\cdot\mathbf{r}'}{r^2r'^2}.$$

The ground state of it is given by the Jastrow form wave function

$$\Psi(r_1,\ldots,r_n) = \prod_{i>j}(r_i-r_j)^{2\alpha}\exp\left(-\alpha\pi n\sum_i r_i^2\right). \quad (4.6)$$

The energy of this ground state is exactly zero.

The simple form of the wave function (4.6) allows one to evaluate equal time correlators. Given the wave function (4.6) Girvin and MacDonald have shown[29] that the off-diagonal correlator obeys the power law (4.3). Such correlators are given by the integrals over all coordinates of $|\Psi|^2$. The particular form of the integral depends on the correlator considered, but all of them are similar to the correlators of the 2D classical problem with energy

$$\beta E = -\frac{1}{2}\sum_{i,j}4\alpha\ln(r_i-r_j). \quad (4.7)$$

B. States of Bose gas with strong Coulomb interaction

Let us consider the Coulomb gas with a three-body short-range interaction (4.5), the ground state of which is described by the wave function (4.6). The explicit form of the ground-state wave function makes the discussion of the three-body case easier; however, all following arguments can be also applied to the Coulomb problem without a three-body interaction with only minor modifications.

This wave function looks deceptively simple, but it describes both the liquid state (at small α) and the crystal (at large α). The energy (4.7) describes a Coulomb plasma with effective charge 4α. At large α it undergoes the transition into the solid Wigner crystal. The charge at which it happens turns out to be numerically large: $\alpha_W \approx 30$.[30] Thus, the Wigner crystal state is realized only for $\alpha_C \geq \alpha_W^2 \approx 900$ in the model (4.5) which contains, apart from Coulomb interaction, also a short-range potential. This value of $\alpha_C^{\rm crit}$ would correspond to the Lindemann number $c_L \approx 0.08$ which is less than usual. This discrepancy is not surprising since the short-range part of the Hamiltonian, (4.5), affects the value of shear modulus of the crystal and, therefore, the melting criteria.

In the crystal phase superfluidity disappears. Below we shall argue that it disappears at $\alpha \sim 1$ long before the transition into the Wigner crystal happens. To determine the superfluid density it is not sufficient to know the wave function of the ground state. Many equivalent definitions of the superfluid density exist. We shall use the one that relates the superfluid density to the difference between the energy of the ground state with periodic and antiperiodic boundary conditions. The following discussion is similar in spirit to the old arguments of Leggett.[31] However, because of the presence of the long-range interaction, the conclusion that we reach is different.

To simplify the discussion we consider a Bose gas that fills an annulus of outer radius L and an inner radius $L/2$. In this system periodic boundary conditions correspond to zero flux of the magnetic field through the hole, the antiperiodic to a half quantum of a flux. The energy of the state with a flux does not depend on L in the limit $L \to \infty$ in the superfluid state and goes to zero in the normal state.

In the absence of any impurity potential the Hamiltonian commutes with rotations around the center of the annulus. It will be convenient to describe each particle by axial coordinates (θ_i, r_i). The phase of any wave function should be linear in $\Theta = \sum_i \theta_i$, and so the wave function that satisfies the antiperiodic boundary conditions contains an additional factor $\exp(i\Theta/2)$, which costs finite energy. Thus, in the absence of any pinning perturbation, superfluidity is preserved for any Galilean invariant interaction.

The wave function (4.6) obeys periodic boundary conditions. We shall try to construct a trial wave function which obeys antiperiodic boundary conditions. We shall look for the trial wave functions which have the form

$$\Psi_A(r_1,\ldots,r_N) = \exp[i\phi(r_1,\ldots,r_N)]\Psi(r_1,\ldots,r_N).$$

The energy of this wave function is

$$\Delta E = \frac{\int[\nabla\phi(r_1,\ldots,r_N)]^2|\Psi(r_1,\ldots,r_N)|^2dr_1\cdots dr_N}{\int|\Psi(r_1,\ldots,r_N)|^2dr_1\cdots dr_N}. \quad (4.8)$$

The phase $\phi(r_1,\ldots,r_N)$ should obey the condition that it changes by π when the particle is moved around the hole.

Qualitatively, if the wave function of each particle is localized, its phase can be changed only in the region where the wave function is small. Such a change would not cost a considerable kinetic energy. For the particle to be localized, its wave function should decay faster than $1/r$ [then the integral $\int |\Psi(r)|^2$ is governed by short scales]. The analog of the one-particle wave function for the many-body problem is the equal time Green's function (4.3) which measures how the wave function decays as a function of one variable with all others being fixed.

In an attempt to formalize these arguments for a many-body problem we consider a path in $2N$-dimensional space that connects the points

$$A = [(\theta, r), (\theta_2, r_2), \ldots, (\theta_N, r_N)]$$

and

$$A' = [(\theta + 2\pi, r), (\theta_2, r_2), \ldots, (\theta_N, r_N)].$$

If the wave function satisfies antiperiodic boundary conditions, the phase changes by π along this path. In the following we shall try to construct implicitly the phase that obeys this condition.

If the amplitude of the wave function is small somewhere on this path, one can construct a trial wave function whose phase changes only in the region where the amplitude is small. Then the energy of this trial wave function is small provided that two conditions are satisfied.

(A) The region in which the phase changes does not contribute much to the average (4.8).

(B) Each path connecting the two points A and A' passes through the region where the amplitude of the wave function is small.

In a crystal state the wave function is small unless almost all particles are close to their equilibrium positions in a periodic lattice. In this case the path in configuration space that connects points A and A' can be viewed as a path in 2D space with all other coordinates being fixed. The wave function along this path decays to a value of $1/L^\alpha$ at $\theta' = \theta + \pi$. Thus, in a crystal state one may change the phase of the wave function only in the region where the wave function is small, proving that the superfluid density is zero in this state.

In the liquid state the wave function along an analogous path (in which all coordinates but one are fixed) decays as $1/r^\alpha$. Consider an integral over this variable in (4.8). The distant parts of this path do not contribute to this integral (4.8) if $\alpha \geq 1$.

This construction can be generalized to paths in which a small number ($N \ll L\sqrt{n}$) of coordinates is changed. Such a path necessarily goes through the region where the 2D "energy" is at least $E \geq \ln(L/N)$. This "energy" is attained in the configurations where the average distance between "moved" particles (and their initial positions) is $r \sim L/N$. The contribution from this region to the integral (4.8) contains a small factor $(N/L)^\alpha$ and a phase volume $(L/r)^{2N}$. Thus, although such configurations have smaller "energy" than a one-particle motion, their contribution is additionally suppressed by a small phase volume. Thus, the contribution of these regions to the variational energy is small.

We were not able to extend this construction and show that it is possible to define the phase so that it changes by π on arbitrary path and that this change happens only when the amplitude of the function is small. The fundamental difficulty of such construction can be traced to the fact that the wave function (4.6) contains large components which correspond to the creation of large but smooth variations of the density. If this variation happens on a macroscopic scale (L) and the total charge of this fluctuation is 1, the corresponding 2D "energy" is only $E \sim \alpha$. The motion of such configuration as a whole around a disk does not pass through the regions where the amplitude of the wave function is small. Instead, such configurations give small contributions only due to weak pinning by defects or by the sample boundary.

We believe that in the full problem that includes boundaries (or weak pinning by defects) it is possible to construct the phase of the wave function so that the boundary conditions are satisfied and the variational energy is small provided that $\alpha \geq 1$.

The same conclusion is reached if one introduces the auxiliary complex field h coupled to the superfluid order parameter ($H_h = \mathrm{Re} h\Psi$) and compares two field configurations.

(A) The field is nonzero and real in narrow radial regions around angles $\theta \approx 0$, $\theta \approx 2\pi/3$, and $\theta \approx 4\pi/3$: $h = h_0$.

(B) The field is nonzero in the same regions, but acquires a phase: $h = h_0 \exp\theta$.

In the second case the phase of the order parameter turns by 2π around the circle. It is natural to attribute the difference between the energies of these configurations to the rotation of the phase. In this formulation the field itself provides the pinning perturbation, and so the difficulty associated with the implicit introduction of weak pinning is removed.

Comparing the energies of these two field configurations evaluated in the perturbation expansion over h_0 ($\delta E \propto \int \langle \Psi_{\mathbf{r}} \Psi^\dagger_{\mathbf{r}'} \rangle d^2r d^2r'$, where the integral over r and r' is performed over different narrow regions), we see that the second configuration costs more energy: $\Delta E \propto \mathrm{const} + L^{2-\alpha}$. The energy cost does not grow with L if $\alpha \geq 2$. From this argument we conclude that the superfluid density is zero if $\alpha \geq 2$. We note that the distribution of Bose particles n_k has no singularity at zero momentum at $\alpha \geq 2$ [$n_k = \int \langle \Psi^\dagger(r)\Psi(0)\rangle \exp(ikr) d^2r$], and so this value of α is most likely an upper bound for $\alpha_{\rm cr}$ at which superfluidity disappears.

Finally, we explain why the Landau conclusion that superfluidity is present if the interaction is Galilean invariant and the spectrum of quasiparticles is not softer than linear does not work here. This conclusion is based on the following argument: Consider the Bose liquid in a coordinate system moving with the liquid. In this coordinate system impurities (or the boundaries) are moving with velocity v. The dissipation of the supercurrent would mean that these impurities can excite quasiparticles. Since the impurities are connected with the vessel, they are infinitely heavy. In this case, the energy transferred to the quasiparticle is $\epsilon = pv$, where p is the transferred momentum. Such a process is impossible if v is less than velocity of the quasiparticle. This argument does not work if the state has broken translational order and processes in which momentum is transferred to the crystal as a whole are allowed. This process is also possible if the interaction is long ranged (as is the case here), so that the momentum is transferred to the system as a whole even in the absence of the broken translational order.

V. COULOMB AND TRANSVERSE INTERACTIONS: GROUND-STATE PROPERTIES

In this section we study the effects of the transverse interaction on a superfluid ground state formed by a relatively weak Coulomb interaction (i.e., with $\alpha \leq 1$). The transverse coupling constant $\alpha_g = g^2/8\pi mc^2$ will also be considered to be relatively small, $\alpha_g \leq 1$. We shall use a perturbation expansion in α_g around superfluid ground state in order to calculate a depletion of the superfluid density $n - n_s$ at $T = 0$. This depletion is entirely due to the breaking of Galilean invariance caused by the retarded nature of the gauge-field-mediated interaction. Then we shall go beyond perturbation theory and use a self-consistent approach to calculate n_s when it becomes small.

A. Perturbation theory

1. Green's functions at $\alpha_g = 0$

We start with the calculation of bare Green's functions for the superfluid ground state at $\alpha_g = 0$, $\alpha \leq 1$. In this case a kind of Bogolyubov (Hartree-Fock) approach can be used for a bare Lagrangian (cf. Refs. 32–34):

$$L_0 = \int \psi^\dagger \left(i\frac{\partial}{\partial t} + \frac{1}{2m}\nabla^2 \right) \psi d^2r + \frac{g^2}{4\pi}\int (\psi^\dagger\psi - n)_{r'} (\psi^\dagger\psi - n)_r \ln(r-r') d^2r d^2r'. \tag{5.1}$$

The idea is to represent Bose field ϕ as the sum of a condensate (zero-momentum) part ψ_0 and a nonzero-momentum part ψ_1, to neglect terms higher than quadratic over $\psi_1,\psi_1^\dagger$, and to calculate normal and anomalous Green's functions $G(x-x') = -i\langle T\psi_1(x)\psi_1^\dagger(x')\rangle$, $F(x-x') = -i\langle T\psi_1(x)\psi_1(x')\rangle$. Equations for the G and F functions can be derived in a standard way[35]; in the Fourier representation we obtain

$$\left(\epsilon - \frac{p^2}{2m} - \Sigma_0^n(\epsilon,\mathbf{p})\right) G(\epsilon,\mathbf{p}) - \Sigma_0^a(\epsilon,\mathbf{p}) F(\epsilon,\mathbf{p}) = 1,$$
$$\left(-\epsilon - \frac{p^2}{2m} - \Sigma_0^n(\epsilon,\mathbf{p})\right) F(\epsilon,\mathbf{p}) - \Sigma_0^a(\epsilon,\mathbf{p}) G(\epsilon,\mathbf{p}) = 0, \tag{5.2}$$

where lowest-order expressions for a normal (Σ_0^n) and anomalous (Σ_0^a) self-energy parts are given by

$$\Sigma_0^n(\epsilon,\mathbf{p}) = \Sigma_0^a(\epsilon,\mathbf{p}) = \frac{ng^2}{p^2}. \tag{5.3}$$

Diagrams for Σ_0^n and Σ_0^a are shown in Fig. 2. In the mean-field approximation employed here we neglected the difference [of the order of $\alpha \ln L$; cf. Eq. (4.3)] between the total density n and the density of the condensate $\psi_0^\dagger\psi_0$. This is justified since this difference does not contribute to the correction to superfluid density n_s which we study.

In the time-momentum representation the solution of Eqs. (5.2) is given by

$$iG(\mathbf{p},t) = \left[\frac{1}{2\epsilon_0(p)}\left(\frac{p^2}{2m} + \frac{g^2 n}{p^2}\right) + \frac{1}{2}\mathrm{sgn}(t)\right] e^{-i\epsilon_0(p)|t|}, \tag{5.4}$$

$$iF(\mathbf{p},t) = -\frac{1}{2\epsilon_0(p)}\frac{g^2 n}{p^2} e^{-i\epsilon_0(p)|t|}, \tag{5.5}$$

where $\epsilon_0(p)$ is bare excitation spectrum in the superfluid state given by

$$\epsilon_0^2(p) = \left(\frac{p^2}{2m}\right)^2 + \frac{g^2 n}{m}. \tag{5.6}$$

This energy spectrum has a finite energy gap which corresponds to a bare plasmon frequency $\epsilon_0 = \sqrt{g^2 n/m}$, as could have been anticipated for a problem with a Coulomb interaction. Another feature which is common for a system with the Coulomb interaction is a screening of a longitudinal potential. In the lowest-order approximation we are using here the effective longitudinal interaction is

$$V_{\rm eff}(p) = \frac{1}{p^2} - 2\frac{1}{p^2}[G(0,p) + F(0,p)]\frac{g^2}{p^2} = \frac{p^2}{p^4 + 4ng^2 m}. \tag{5.7}$$

In the real-space representation the effective potential is equal to

$$V_{\rm eff}(r) = \frac{1}{2\pi}\ker(\sqrt{2}q_1 r) = \frac{1}{2\pi}\mathrm{Re}K_0[q_1 r(1+i)], \tag{5.8}$$

where $\ker(x)$ and $K_0(x)$ are Tompson and McDonald functions of the zeroth order, and $q_1 = (ng^2 m)^{1/4}$. At large r the potential $V_{\rm eff}(r)$ decreases exponentially with a characteristic Debye length $r_D = q_1^{-1}$, but also oscillates with a wave vector q_1 (cf. Refs. 33 and 34). Usually the mean-field-type approximation works well for a Coulombic problem if the number of particles within a

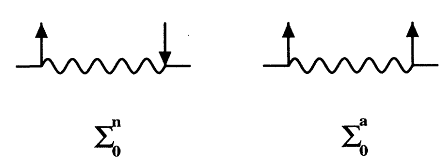

FIG. 2. Normal (Σ^n) and anomalous (Σ^a) self-energy parts within Bogolyubov-Hartree-Fock approximation for the Coulomb Bose liquid. Lines with arrows denote Bose condensate ψ_0; wavy line stands for the Coulomb potential.

sphere of the Debye radius is large; in the present case this condition reads

$$4\pi n r_D^2 = \frac{4\pi}{g}\sqrt{\frac{n}{m}} \geq 1; \tag{5.9}$$

i.e., it coincides with the inequality $\alpha \leq 1$ which we assumed to be valid.

2. Perturbative corrections to n_s

Now we are in a position to find corrections to a superfluid density $n_s = m\Pi_\perp(q=0)$. To do that, we take into account a coupling between transverse bosonic current and the gauge field **a** perburbatively, using, for the bare Green's functions, the functions determined above in the context of purely Coulombic interaction [cf. Eqs. (5.4,5.5)]. Then the transverse current-current correlator is given by $\Pi_\perp = \Pi_{\perp 0} + \Pi_{\perp 1}$, where $\Pi_{\perp 0} = n/m$. The diagrams giving the lowest-order contributions to $\Pi_{\perp 1}$ are shown in Fig. 3; the corresponding analytical expression becomes simpler in the $x = (\mathbf{r}, t)$ representation:

$$\Pi_{\perp 1}(x) = i\frac{2n}{m^2}D(x)[G(x) + F(x)], \tag{5.10}$$

where $D(x)$ is a bare photon propagator in a superfluid state,

$$D(\mathbf{q}, t) = -i\frac{g^2}{2\omega_0(q)}e^{-i\omega_0(q)|t|}, \tag{5.11}$$

where bare photon spectrum is

$$\omega_0^2(q) = c^2q^2 + \frac{ng^2}{m}. \tag{5.12}$$

Combining (5.10,5.11), we get

$$\Pi_{\perp 1}(q=0) = -\frac{ng^2}{m^2}\int\frac{d^2p}{(2\pi)^2}\frac{p^2/2m}{\epsilon_0(p)\omega_0(p)\left[\epsilon_0(p) + \omega_0(p)\right]}. \tag{5.13}$$

Finally, using Eq. (5.13) and the definition of n_s, one gets

$$1 - \frac{n_s}{n} = 2\alpha_g u J(u), \tag{5.14}$$

where $u = \alpha/\alpha_g$ and the dimensionless integral $J(u)$ is given by

$$J(u) = \int_0^\infty \frac{x\,dx}{\sqrt{x^2+1}\sqrt{ux+1}(\sqrt{x^2+1}+\sqrt{ux+1})}. \tag{5.15}$$

At small u, $J(u) \approx \ln(2/u)$, whereas at $u \to \infty$, $J(u) \approx (1/2u)\ln(2u^2)$. The function $uJ(u)$ is plotted in Fig. 4. We use Eqs. (5.14,5.15) to estimate the density range, n, in which $n \gg n_s$. To do that, we solve numerically (5.14) with $n_s = 0$ in the left-hand side for u at moderately small values of α_g and then find the value of the parameter $\alpha = \alpha_g u$. In the case of $\alpha_g = 1/2$ (which corresponds to the vortex liquid problem) one finds that the vanishing of the superfluid density takes place at $\alpha = \alpha^{\rm cr} \approx 0.3$ [i.e., at $\alpha_C^{\rm cr} = (\alpha^{\rm cr})^2 \approx 0.1$]. The $\alpha^{\rm cr}(\alpha_g)$ dependence is shown in Fig. 4(b). This curve is reliable in the range $0.3 \leq \alpha_g \leq 0.6$ where both α_g and α are considerably less than unity, so that low-α, α_g perturbative expansion can be trusted.

3. Corrections to the spectrum of longitudinal excitations

It is also instructive to check what is the effect of the coupling to a gauge field on the excitation spectrum (5.6). To do that, we calculate self-energy parts Σ^n and Σ^a in the lowest order in α_g and add them to the bare functions given by the (5.3). There are two types of these corrections: Σ_1 containing a vector vertex part and Σ_2 with a scalar vertex part; the corresponding diagrams are shown in Fig. 5. The renormalized energy spectrum is

FIG. 3. Lowest-order perturbative corrections to the transverse polarization function $\Pi_\perp$. Straight line stands for the bosonic Green's function, the line with arrows in opposite directions represents the anomalous Green's function F, and the dashed line stands for the photon propagator D.

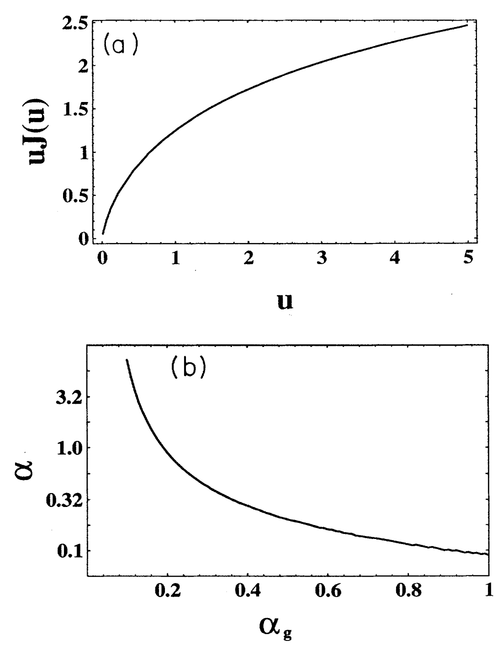

FIG. 4. (a) The function $uJ(u)$, (b) critical line $\alpha(\alpha_g)$ determined within first order in a perturbative expansion.

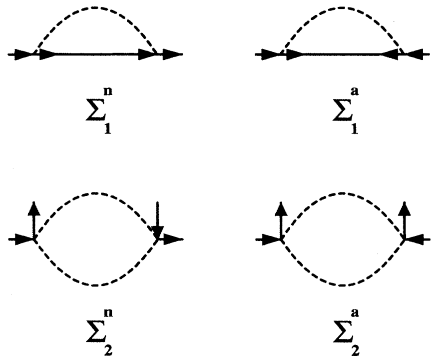

FIG. 5. Self-energy parts giving lowest-order corrections to the plasmon spectrum $\epsilon_0(p)$.

given by

$$\epsilon_p^2 = \epsilon_0^2 + \frac{p^2}{m}[\Sigma_1^n(\epsilon_p, p) + \Sigma_2^n(\epsilon_p, p)] + 2\frac{ng^2}{p^2}[\Sigma_1^n(\epsilon_p, p) + \Sigma_2^n(\epsilon_p, p) - \Sigma_1^a(\epsilon_p, p) - \Sigma_2^a(\epsilon_p, p)]. \quad (5.16)$$

In the following we shall need only two features of the renormalized energy spectrum: the renormalized value of the plasmon frequency $\epsilon(p = 0)$ and the dispersion of the spectrum at small $p \ll q_1 = (ng^2m)^{1/4}$. The calculations simplify noting that at low p, $\Sigma_1^{n,a} \propto p^2$, whereas $\Sigma_2^n = \Sigma_2^a \sim$ const. We get

$$\epsilon_{\rm pl}^2 = \epsilon_0^2 + 2ng^2 \left.\frac{\Sigma_1^n(\epsilon_{\rm pl}, p) - \Sigma_1^a(\epsilon_{\rm pl}, p)}{p^2}\right|_{p=0} \quad (5.17)$$

for the plasmon frequency and

$$\epsilon^2(p) - \epsilon_{\rm pl}^2 = \frac{p^2}{m}\Sigma_2(\epsilon_{\rm pl}, p = 0) \quad (5.18)$$

for the dispersion of $\epsilon(p)$ at low momenta. The diagrams for $\Sigma_1^{n,a}/p^2$ are the same as the ones for $\Pi_{\perp 1}$ [cf. Eq. (5.13)]. As a result,

$$\epsilon_{\rm pl}^2 = \epsilon_0^2 [1 - \alpha_g u J(u)] , \quad (5.19)$$

where $J(u)$ is defined in (5.15). From (5.19) we see that the relative reduction of the plasmon frequency is 2 times smaller than the reduction of n_s. This conclusion does not change if we take into account a weak frequency dependence of $\Sigma_{1,2}$ which makes the reduction of plasmon frequency even smaller. The calculation of $\Sigma_2(0)$ gives the dispersion

$$\epsilon^2(p) = \epsilon_{\rm pl}^2 - \alpha_g \epsilon_0 \frac{p^2}{2m}; \quad (5.20)$$

i.e., $\epsilon(p)$ decreases at small p and thus has a minimum at some finite value of p. This minima never falls as low as $\epsilon = 0$ (if it happens it would indicate the instability of the charge-density-wave type). To prove it we consider the "radiative" correction to a functional $E\{\delta\rho\}$ determining an energy of a static inhomogeneous density distribution:

$$E\{\delta\rho\} = \frac{1}{2}\sum_{\mathbf{q}} \delta\rho_{\mathbf{q}}\delta\rho_{-\mathbf{q}} \left[\frac{g^2}{q^2} + \frac{q^2}{4mn} - \alpha_g \frac{g}{2\sqrt{mn}}\right] , \quad (5.21)$$

where the radiative correction is given by the last term in the square brackets. Obviously, the minimum (over q's) of the expression Eq. (5.21) never touches zero in the range $\alpha_g \le 1$.

Thus, we conclude that the corrections to a Bose particle excitation spectrum are noncritical in the region of parameters where n_s vanishes.

B. Self-consistent calculation of n_s

Clearly, the above arguments for the vanishing of n_s are based on the first-order perturbation expansion over α_g and are not quite conclusive. We improve the approximation deriving a self-consistent equation for n_s and looking for the value of α where the solution of this equation with positive n_s ceases to exist. This is analogous to the Pokrovsky-Uimin[36] approach to the problem of phase transition in a classical 2D XY model. To use a similar procedure in the present problem, we prefer an approach based on the functional integral representation over the diagram expansion that is used above.

To use this representation we note that the corrections to n_s calculated above come from the following term in the Lagrangian Eq. (1.8):

$$L_{\psi,\mathbf{a}}^{(2)} = -\frac{1}{2m}\mathbf{a}^2\psi^\dagger\psi = -\frac{1}{2m}\mathbf{a}^2(n + \delta\rho), \quad (5.22)$$

where we introduced a density fluctuation field $\delta\rho$. Here and below we mean by **a** *transverse* (i.e., gauge-invariant) part of the **a** field. We need to evaluate the functional integral

$$\int D\psi D\psi^\dagger D\mathbf{a}\exp(iS).$$

We shall keep only term (5.22) in the interaction part of the action S. Then we simplify the functional integral passing from the integration over ψ to the integration over density fluctuations $\delta\rho$. The integral over density fluctuations should be performed with the Gaussian weight

$$\exp\left(\frac{i}{2}\int \delta\rho_x\delta\rho_{x'}C^{-1}(x - x')dxdx'\right) , \quad (5.23)$$

where $C(x)$ is the correlation function of the density fluctuations. $C(x)$ can be obtained from the hydrodynamic Lagrangian (4.1):

$$C(\omega, \mathbf{q}) = \frac{nq^2/m}{\omega^2 - \frac{ng^2}{m} - (q^2/2m)^2 + i0} = 2n[G(\omega, \mathbf{q}) + F(\omega, \mathbf{q})] , \quad (5.24)$$

where the functions G and F are defined in Eqs. (5.4, 5.5). Certainly, the transformation from the functional integral over ψ to the one over $\delta\rho$ is not exact: It is just the same kind of mean-field approximation valid at $\alpha \ll 1$ that we have used in Sec. V A. After the integration over $\delta\rho$ the effective action acquires an additional term originating from the second term in (5.22):

$$S_{\text{eff}}\{\mathbf{a}\} = -\frac{n}{2m}\int d^3x\mathbf{a}_x^2 - \frac{i}{8m^2}\int d^3xd^3x'\mathbf{a}_x^2\mathbf{a}_{x'}^2C(x-x') + S_0\{\mathbf{f}\}, \tag{5.25}$$

where $S_0\{\mathbf{f}\}$ is a bare part of the gauge-field action. Then we use mean-field decoupling to treat the quartic in the $\mathbf{a}$ term of the action (5.25). We represent the gauge field as a sum of the slow $\mathbf{a}_0$ and fast parts $\mathbf{a}_1$ and integrate over $\mathbf{a}_1$ in order to obtain an effective Lagrangian

$$L\{\mathbf{a}_0\} = -\frac{n_s}{2m}\mathbf{a}_0^2 - \frac{c^2}{2g^2}f_{12}^2 + \frac{1}{2g^2}f_{0\alpha}^2 \tag{5.26}$$

for the slowly varying field $\mathbf{a}_0$, where n_s is given by

$$n_s = n + \frac{1}{m}\int d^3xC(x)\langle\mathbf{a}_1(0)\mathbf{a}_1(x)\rangle. \tag{5.27}$$

The idea of the self-consistent calculation is that the propagator of the "fast" field $\mathbf{a}_1$ and the propagator of the slow field $\mathbf{a}_0$ are determined by the same Lagrangian (5.26), and so the correction term $\Pi_{\perp 1}$ is given by (5.13) with the renormalized photon spectrum

$$\omega^2(q) = c^2q^2 + \frac{n_sg^2}{m}. \tag{5.28}$$

Finally, the self-consistent equation for n_s is

$$1 - w = 2\alpha_g uJ(w,u), \tag{5.29}$$

where $w = n_s/n$ and

$$J(w,u) = \int_0^\infty \frac{xdx}{\sqrt{x^2+1}\sqrt{ux+w}(\sqrt{x^2+1}+\sqrt{ux+w})}. \tag{5.30}$$

Equation (5.29) can be solved numerically for $n_s = wn$ as function of α at a given α_g. Unlike the analogous equations in Ref. 36 Eqs. (5.29,5.30) have no singularity at $n_s \to +0$. Therefore n_s may go to zero with a finite slope at some critical value $\alpha_{\text{sc}}^{\text{cr}}$ with a finite slope [$n_s \propto (\alpha_{\text{sc}}^{\text{cr}} - \alpha)$]. The other possibility is a hard-type instability (disappearance of the solution with nonzero n_s), like the one obtained in Ref. 36. Actually both of these scenarios realize at different values of α_g, as is seen from the results of numerical solution of Eq. (5.29) shown in Fig. 6 for $\alpha_g = 0.35$, 0.5, and 0.65.

In the small-α_g region the self-consistent critical value $\alpha_{\text{sc}}^{\text{cr}}$ can be determined as a function of α_g via the solution of the equation $1 = 2\alpha_g uJ(0,u)$ for $u = \alpha_{\text{sc}}^{\text{cr}}/\alpha_g$, whereas at larger α_g the critical value $\alpha_{\text{sc}}^{\text{cr}}$ corresponds to the point where the solution of Eq. (5.29) disappears. The corresponding "phase diagram" in the α_g-α plane is shown in Fig. 7. In the region of hard instability ($\alpha_g \geq 0.38$) the dashed line corresponds to the instability of the superfluid state whereas a true phase transition line should be determined as a point where the ground-state energies of both phases are equal. Thus, the true first-order transition line may fall below the dashed line in Fig. 7. Actual calculation of the position of the first-order transition line requires the knowledge of the ground-state energy of the normal state formed above this transition. At present we can only guess the nature of this state but are unable to calculate its energy.

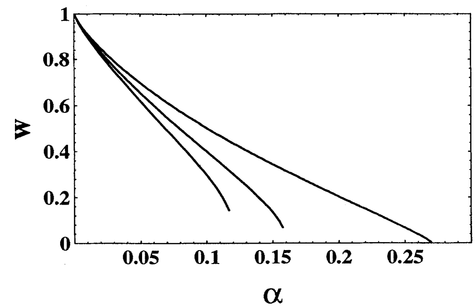

FIG. 6. The solution of the self-consistent equation. At $\alpha_g \leq 0.38$ the ratio $w = n_s/n$ goes to zero continuously; at larger α_g the solution with positive w disappears abruptly.

In the self-consistent calculations that lead to the phase diagram in Fig. 7 we implicitly assumed that the velocity of the photons, c, is not renormalized so that the only effect of the renormalization on the photon spectrum is the change in the superfluid density n_s. To check this assumption we use the self-consistent approach that results in Eq. (5.29) to derive the renormalization of the photon velocity. We get

$$\nu = (c_r/c)^2 = 1 - \frac{1}{2}\alpha Q(n_s/n, \alpha/\alpha_g),$$
$$Q(w,u) = \int_0^\infty \frac{xdx}{b}\left[ux\frac{8a^2+9ab+3b^2}{a^5(a+b)^3} - \frac{4a+2b}{a^3(a+b)^2}\right],$$
$$a = \sqrt{w+ux}, \qquad b = \sqrt{1+x^2}. \tag{5.31}$$

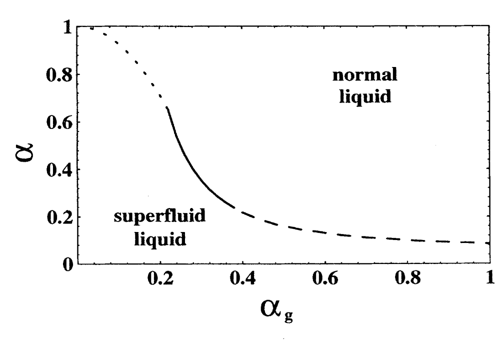

FIG. 7. The phase diagram of the Bose liquid at $T = 0$ in the plane of the coupling constants α and α_g. Solid line designates a second-order transition line in which n_s varies continuously. The solution with $n_s \neq 0$ becomes completely unstable at the dashed line. Actual line of the first-order phase transition is somewhat below the dashed line. Dotted line is a conjectured transition line matching the results of Secs. IV and V.

We plot the renormalization of the velocity at the transition line $\alpha(\alpha_g)$ in Fig. 8. As is evident from this plot the renormalization of the velocity is small everywhere outside of the vicinity of the tricritical point $\alpha_g = 0.38$ where the order of the transition changes.

C. Nature of the normal Bose liquid at $T = 0$

Summarizing results of the previous subsection, we have found that above a critical line shown in Fig. 7 the superfluid ($n_s > 0$) ground state is unstable. This conclusion is in qualitative agreement with the arguments of Wheatley and Hong[37] that strong interaction with gapless degrees of fredom may lead to the normal Bose liquid at $T = 0$.

In the vicinity of this line the density-density correlation function (corresponding to a plasmon branch) does not show any anomalies; instead, a vanishing of n_s (which occurs at $\alpha_g \leq 0.38$) indicates that this transition is due to the instability of the photon (transverse) spectrum. In other words, a formation of a nonsuperfluid ground state is driven by the changes in the fluctuation spectrum of the gauge field.

What is the state which is formed above this line? Two classes of normal states are possible: states with a new order parameter and states that do not break any symmetry of the original Hamiltonian.

If a new order parameter is formed, the instability of the photon spectrum which we found in the vicinity of the transition indicates that this order parameter is a component of the gauge field. This component may have nonzero wave vector. Clearly, in a state with the staggered magnetic field the Bose condensation is suppressed.

If the new state does not break any symmetry, a completely different state may form in which large fluctuations of the gauge field change the properties of the Bose quasiparticles so drastically that they would resemble fermions more than bosons (statistics transmutation).

Thus, we have the following candidates for the normal state: a state with a uniform magnetic field (orbital ferromagnet), a state with staggered magnetic field (orbital antiferromagnet), a toroidal magnet, and a state where the fluctuations of the gauge field lead to the statistic transmutation. Now we discuss these states.

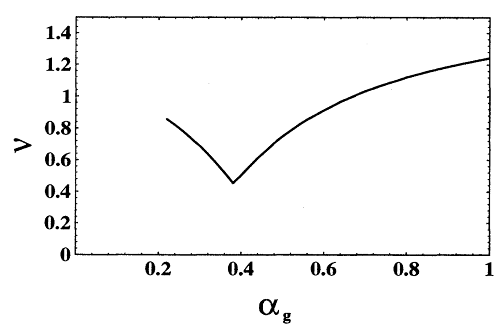

FIG. 8. The ratio of the renormalized squared light velocity c_r^2 to its bare value c^2 at the transtion line shown in Fig. 7, as a function of α_g.

Orbital ferromagnet. In this state a uniform magnetic field violates time reversal symmetry. Formally, this state may be realized only if the coefficient in the effective gauge-field Lagrangian (5.26) obeys a set of conditions: First, the $\mathbf{a}^2$ term is absent; second, the coefficient in front of the f_{12}^2 term is negative. It is unlikely that both of these conditions are satisfied for general values of the coupling constants. We have found no signature that these two conditions are satisfied for some particular values of the coupling constants.

Orbital antiferromagnet (OAF). In this state the staggered magnetic field preserves the combined symmetry under time reversal and translation but violates each of them separately. This state was originally proposed[38] and then extensively studied (see, e.g., Refs. 39 and 40) in order to describe the materials which have very high diamagnetic susceptibility but are not superconductors. This property makes this state a natural candidate for our intermediate (between crystalline and superfluid) state.

If the OAF is formed in a second-order phase transition, the spectrum of the gauge field should acquire a minimum at nonzero q in the vicinity of this transition. The transition happens when the energy of this minimum touches zero. As explained in the end of Sec. V B we have checked that within the self-consistent scheme the photon energy increases with q at small momenta ($c_r^2 > 0$; see Fig. 8). It indicates that a deep minimum of the photon spectrum at large q is very unlikely (though it does not prove it, of course). This makes an orbital antiferromagnet an unlikely candidate for the normal state of the Bose liquid, at least in the region of parameters α, α_g where the transition is of the second order.

Toroidal magnet (TM). This state was also expected[40] to produce a large (but finite) diamagnetic susceptibility like OAF. Unlike OAF it does not break the translational invariance. It is characterized by order parameter $\mathbf{T}$ which has the same symmetry properties as the current $\mathbf{j}$. However, a simple proportionality between $\mathbf{T}$ and $\mathbf{j}$ is forbidden by gauge invariance;[40] instead $\mathbf{j} \propto \nabla^2\mathbf{T}$. In the case of a two-dimensional system the TM state can be visualized as the state with an equal number of (+) and (−) magnetic fluxes bonded in "dipolelike" pairs. In this picture the toroidal order parameter $\langle\mathbf{T}\rangle$ is proportional to an average "dipolar polarization" produced by those pairs. We have no indication in favor of this state, but we cannot exclude it as candidate for a normal ground state of the Bose liquid.

State with statistics transmutation. In this state the strong fluctuations of the gauge field change the properties of the Bose particles entirely. In this scenario the instability of the superfluid solution found in the previous subsection is interpreted as a signature of a quantum phase transition to a state characterized by strong non-Gaussian fluctuations of the gauge field $\mathbf{a}$. These strong fluctuations can be viewed as flux tubes carrying "magnetic" flux 2π (in dimensionless units) within a space scale of the order of the interparticle distance $1/\sqrt{n}$.

Such flux tubes, if bound to bosons, change their statistics.[41,42] Indeed, when a composite particle (bosons

+ flux tube) moves half a circle around another composite particle, it acquires a phase $\phi = \pi$. Since this process is equivalent to the interchange of particles, this interchange leads to the additional phase $\phi = \pi$ of the wave function. Thus, this binding converts the bosons into the fermions. If all bosons are bound to fluxes, the Bose liquid becomes a Fermi liquid which remains a normal liquid even at zero temperature.

If this mapping of the Bose liquid with strong gauge interactions onto the Fermi liquid is correct, the properties of this state can be deduced from the known properties of the Fermi liquid at zero temperatures. Note, however, that these properties are different from a conventional Fermi liquid due to the presence of the long-range gauge field.[43,44]

This mapping implies that the excitations in the Bose liquid become Fermi particles. Such statistics transmutations are known in one-dimensional physics where the particle bound to a soliton changes its statistics. In the Bose liquid the role of the solitons is played by the flux tubes of the gauge field which carry flux 2π. As well as in 1D solitons these configurations of the gauge field are very special even in the superfluid state: They carry nonzero flux, but their energy is finite. Such flux tubes can move; they can be characterized by their energy and the mass.

Let us provide qualitative arguments which show that a charged Bose excitation may form a bound state with the flux tube and change its statistics thereby. Consider one Bose particle interacting with one flux tube. Certainly, a particle is repelled from the static magnetic field. Consider, however, a state in which the flux tube is rotating around the center of mass of the pair formed of a boson and a tube. According to Ampère's law, this rotation produces a radial electrical field, which (for the correct sign of the rotation) may bind the charged Bose particle. Likewise, the magnetic field created by the current of the Bose particle may bind the flux.

In a different approach we consider a charged 2D particle in a spatially localized state with nonzero angular momentum l. The orbital motion of a charged particle produces an electric current and, therefore, a magnetic field. It is easy to show that the total magnetic flux produced by such a localized state is not sensitive to the form of the wave function; it can be expressed only through the orbital moment of the particle:

$$\Phi(l) = 4\pi\alpha_g l. \tag{5.32}$$

Particularly, for the case $\alpha_g = 1/2$, it means that the state with unit ($l = \pm 1$) orbital momentum produces a flux quantum $\Phi = \pm 2\pi$ (note, to avoid confusion, that in our units the flux quantum is simply 2π — as distinct from the flux quantum Φ_0 of 3D electrodynamics). Now let us consider two quasiparticles, each of them in a quasilocalized state with angular momentum $l = \pm 1$ (by "quasilocalized" we mean that at a given position of the center of packet the wave function decays rapidly away from that center, but the position of the center itself is not fixed and can drift). Then the phase the combined two-particle wave function acquires after the particles are exchanged (via slow winding of one around another) is exactly π, as if they were fermions.

Two types of flux tubes are possible: $+2\pi$ and -2π. Because of the physical effects discussed above, both types are attracted to bosons and may form a bound state. If only one type is used to form a composite particle, the resulting state would have a nonzero average magnetic field. Such a state has a larger energy than the state in which both types of flux tubes are used so that the average "magnetic" field is zero.

The state with two types of fluxes can be viewed as an orbital antiferromagnet "melted" by strong quantum fluctuations. The external magnetic field changes the balance between positive and negative fluxes (like it does with $\pm$ spins in the case of usual Fermi liquid). Thus, in this state there is an additional "paramagnetic" contribution to a magnetic susceptibility.

Obviously, these arguments in favor of the statistics transmutation are very far from a proof that such a state is indeed formed in the strongly interacting Bose gas. A proof of this conjecture would involve the construction of those composite fermions and the calculation of the ground-state energy of the Fermi liquid. Moreover, one should check that no pairing instability happens in this Fermi liquid, which would lead to a small but nonzero n_s.

VI. APPLICATIONS

In this section we discuss implications of the nonsuperfluid Bose liquid ground state in two physical contexts described in the Introduction: the classical statistical mechanics of the vortex liquid in HTSC's and the gauge theory of the Mott dielectric.

A. Vortex liquid as an intermediate phase of a type-II superconductor in a magnetic field

1. Superconductor–Bose-liquid duality: Formal derivation

Now we return to the problem of a superconductor in a mixed state and derive a duality relation between the superfluidity of the 2D boson system and the superconductivity of the orginal 3D metal in a magnetic field.[10] From now on we will relax the condition $\lambda^{-1} = 0$ which we used in the bulk of the paper. We will characterize ground state of the 2D liquid by its response to a slowly varying external gauge field $\mathbf{a}$:

$$\mathbf{j}(\mathbf{q}) = -\Pi_\perp(\mathbf{q})\mathbf{a}_\perp(\mathbf{q}). \tag{6.1}$$

With the definition (6.1) an effective Lagrangian for the slowly varying gauge field $a = (a_0, \mathbf{a})$ and the electromagnetic vector potential $\mathbf{A}$ becomes [see Eq. (1.5)]

$$L_{\text{eff}}\{A, a\} = \frac{i}{4\sqrt{\pi}\lambda g}\epsilon_{\alpha\beta\gamma}A_\alpha f_{\beta\gamma} + \frac{1}{8\pi}[\boldsymbol{\nabla}\times\mathbf{A}]^2 + \frac{1}{4g^2}f_{\alpha\beta}^2 + \frac{1}{2}\Pi_\perp\mathbf{a}_\perp^2 + L_\parallel\{a_0\}, \tag{6.2}$$

where the last term in Eq. (6.2) accounts for the Bose liquid response to the scalar potential a_0. The Lagrangian (6.2) allows us to express the correlation function $\langle A_z(\mathbf{q})A_z(-\mathbf{q})\rangle$ of the electromagnetic vector potential component along the background magnetic field in terms of the Bose liquid polarization function $\Pi_\perp$ (note that in this section we use the imaginary-time representation as it corresponds directly to the vortex liquid problem):

$$\mathcal{D}_{zz}(\mathbf{q}) = \langle A_z(\mathbf{q})A_z(-\mathbf{q})\rangle = \frac{4\pi T}{q^2+\mathcal{P}(q)},$$
$$\mathcal{P}(q) = \frac{1}{\lambda^2}\frac{q^2}{q^2+g^2\Pi_\perp(q)}. \qquad (6.3)$$

Here and below in this subsection T means the temperature of the 3D superconductor. The function $\mathcal{P}(q)$ defined in Eq. (6.3) is nothing but the irreducible correlator of superconductive currents $J_z^{(s)}$ along the direction of the external magnetic field $\mathbf{H}_{\text{ext}}$. The relation between $\mathcal{P}$ and $\Pi_\perp$ [Eq. (6.3)] is just the duality relation that we need. In the following analysis we will assume that q_z component of the wave vector $\mathbf{q}$ is set to zero *before* the limit $\mathbf{q}_\perp \to 0$ is taken. This order of limits is mapped to the limit $\Pi_\perp(\omega=0,\mathbf{q})$ in the Bose liquid representation. Suppose, first, that the Bose liquid is in a superfluid ground state; then $\Pi_\perp(q\to 0) = n_s/m > 0$, which leads immediately to a correlation function of the electromagnetic field,

$$\mathcal{D}_{zz}(q) = 4\pi T\mu_d/q^2, \qquad \mu_d = \frac{1}{1+1/(4\pi\lambda^2 n_s)}; \quad (6.4)$$

i.e., we get to a phase with a finite permeability μ_d which does not differ qualitatively from a normal state. If, on the other hand, the Bose liquid is in a "normal" state with a finite diamagnetic susceptibility χ^B, then $\Pi_\perp(q) \approx (4\pi/g^2)\chi^B q^2$ and, with Eq. (6.3), we get

$$\mathcal{D}_{zz}(q) = \frac{4\pi T}{q^2+\lambda_{\text{eff}}^{-2}}, \qquad (6.5)$$

where $\lambda_{\text{eff}} = \lambda\sqrt{1+4\pi\chi^B}$; i.e., the "normal" Bose liquid ground state corresponds to an anisotropic superconductive phase with a London relation between the current and vector potentials when both are in the direction of the background magnetic field. For the vector potential which is transverse to the magnetic field $\mathbf{H}_{\text{ext}}$ the superconductive response is never possible in this problem since, in the absence of pinning, vortices can move under the action of the Lorentz force $\mathbf{F}_L = \frac{1}{c}[\mathbf{J}\times\mathbf{B}]$.

Qualitatively, the duality between superconductivity in the mixed state and superfluidity of 2D bosons can be understood as follows: These bosons represent Abrikosov vortices which are topological defects of the superconductive ground state; therefore off-diagonal long-range order in terms of bosons should appear as the *disorder* parameter for the original superconductivity.

Thus, the existence of a nonsuperfluid Bose liquid ground state implies that there is an intermediate vortex liquid phase of a superconductor in a magnetic field where A_z and J_z^s obey the London equation

$$J_z^s = -\rho_s^{zz}A_z, \qquad \rho_s^{zz} = \frac{1}{4\pi\lambda_{\text{eff}}^2}. \qquad (6.6)$$

Relation (6.6) holds also for the Abrikosov lattice state (mapped to a Wigner crystal in the 2D boson representation), as can be easily derived from the expression for the free energy of this state given in Ref. 45 in the mean-field approximation (MFA). Within the MFA an effective penetration depth λ_{eff} coincides with a bare London length λ. We also calculated a fluctuational correction to it [it is mapped to the diamagnetic susceptibility χ^B in the crystalline ground state of the Bose problem; cf. (6.5)] and found that it is small even at the melting line: $\lambda_{\text{eff}}^{\text{Abr}}/\lambda - 1 \le \pi c_L^2/2$. Recently relation (6.6) was shown also to hold[46] in a vortex liquid model with a wave-vector-dependent shear modulus $C_{66}(\mathbf{q}\neq 0)\neq 0$. The existence of an intermediate vortex liquid phase with a nonzero ρ_s^{zz} was also confirmed by recent Monte Carlo simulations.[47]

The first indications that this phase was observed experimentally were reported recently in Ref. 48. In this experiment the resistivity ρ_{zz} in the artificially layered MoGe/Ge materials was observed to vanish below some temperature whereas the resistivity perpendicular to the field varied smoothly in this temperature range.

The relations (6.3)–(6.5) are quite general and do not depend on the approximation $\lambda\to\infty$ which we made when discussing the Bose liquid ground states.

For intermediate values of $n\lambda^2$ one should take into account the finite range of the interaction mediated by the **a** field when calculating the boson polarization function $\Pi_\perp$. For example, at very low flux-line densities, $n\lambda^2 \ll 1$, the Bose ground state is a superfluid,[12] and with Eq. (6.4) we get an effective diamagnetic permeability of the vortex liquid, $\mu_d \approx n\lambda^2 \ll 1$; i.e., in this limit one deals with an "almost superconductive" system. On the other hand, the high-density Bose superfluid corresponds to a weakly diamagnetic state, $1-\mu_d \ll 1$, that resembles fluctuational diamagnetism known to exist above H_{c2}.[49]

The above discussion pertains only to the description of equilibrium thermodynamic properties of the superconductive mixed state; kinetic quantities such as resistivity have no direct analog in the 2D boson picture, and so we are unable to extract any quantitative information for resistivity from the study of the 2D Bose liquid ground state. However, it is still possible to estimate the rate of the dynamic processes which govern the resistivity ρ_{zz}. (The resistivity in other directions is not sensitive to the state of the flux lattice and is determined by the flux flow.)

2. Qualitative picture and estimates for resistivity

A useful description of the superfluidity in terms of the Bose particle's world lines was given by Ceperley and Pollock.[50] They introduce a winding number W as a quantitative measure for the role of multiparticle cooperative-ring-exchange processes which are known to be important in the superfluid phase.[51] The superfluid density in the 2D Bose liquid can be expressed through the mean-squared fluctuations of the winding number: $n_s = mT^B\langle W^2\rangle/2\hbar_B^2$. In the limit $T^B \to 0$ (mapped to a bulk superconductor with thickness $L_z \to \infty$) the existence of finite n_s means that $\langle W^2\rangle \propto 1/T^B \propto L_z$. Qualitatively, it implies the existence of arbitrary large planar vortex loops in the vortex liquid phase (these loops are the projections of entangled vortex lines onto the xy plane). An equivalence between vortex line entanglement and superfluidity of the 2D Bose liquid was first proposed by Nelson.[12]

We should differentiate between weak entanglement, which is present even in the Abrikosov lattice state due to elementary pairwise exchange processes existing in any bosonic ground state, and strong entanglement, in which planar loops of arbitrary length appear. These loops are responsible for a finite value of $\langle W^2\rangle/L_z$. If only short (like pairwise) vortex exchanges are present, the main contributions to $\langle W^2\rangle$ from the exchanging "partners" cancel each other. Then $n_s = 0$, despite that each vortex line, if traced, becomes arbitrary far from its original position in the xy plane. In other words, the ever present diffusion of the Bose particles does not lead to superfluidity.

The presence or absence of arbitrary large planar vortex loops is directly related to the behavior of the resistivity ρ_{zz}. Dissipation of the longitudinal current J_z is due to the growth of planar vortex loops of appropriate (for a given direction of current) vorticity. In a weakly entangled vortex liquid (no superfluidity in the 2D Bose liquid representation) the free energy of a large planar loop is proportional to its size R: $F = 2\pi R\epsilon_1$. The line tension ϵ_1 is some fraction of the vortex line tension: $\epsilon_1 \lesssim \epsilon_0 = m = (\Phi_0/4\pi\lambda)^2$. In the presence of current the free energy acquires an additional term proportional to the current:

$$F(R, J_z) \approx 2\pi R\epsilon_1 - \frac{1}{c}J_z\Phi_0\pi R^2. \tag{6.7}$$

For the critical size of the loop the energy $F \approx \pi R_c\epsilon_1$. The size of the critical loop and its free energy are inversely proportional to the value of current, $F_c(J_z) \propto 1/J_z$. Therefore, an electric field $E_z(J_z)$ produced by this current is exponentially weak at low J_z:

$$E_z \propto e^{-J_T/J_z}, \qquad J_T \approx \frac{\pi c\epsilon_1^2}{\Phi_0 T}. \tag{6.8}$$

Thus, the resistivity $\rho_{zz} = E_z/J_z$ goes to zero exponentially as the current decreases.

In the strongly entangled vortex phase (superfluid Bose liquid), arbitrary long planar loops are present in thermodynamic equilibrium even in the absense of current. In the presence of current the equilibrium between loops with a positive and negative vorticity is biased. In this case, an electric field is proportional to current and a linear (at $J_z \to 0$) resistivity $\rho_{zz}^{\rm lin}$ appears. Its value is determined by an energy barrier controlling the growth of the planar loops. Since these loops are projections of the entangled vortex lines onto the xy plane, the growth of loops implies cutting and reconnecting vortex lines. The energy barrier for such process governs the growth of large loops and, thus, the value of the resistivity:

$$\rho_{zz}^{\rm lin} \propto e^{-F_{\rm cr}/T}, \quad F_{\rm cr} = A_{\rm cr}\frac{\Phi_0^2}{(4\pi\lambda)^2\sqrt{n}}, \tag{6.9}$$

where a numerical factor $A_{\rm cr} \sim 1$.

At sufficiently low temperatures the energy barrier $F_{\rm cr}$ becomes significantly higher than temperature. At these temperatures the decay of a metastable state takes a long time, and so the equilibrium is never reached. Such situation was observed recently by Li and Teitel[52] in extensive Monte Carlo simulations of the vortex line system. In this case the behavior of a vortex liquid on finite time scales can differ significantly from the results obtained using equilibrium statistical mechanics. In particular, this equilibration time problem may invalidate the direct use of the results obtained from the Bose liquid mapping of the vortex problem. From the viewpoint of the 2D bosons, this metastability does not correspond to any physical reality, and so care should be exercised in any quantum Monte Carlo simulation of the 2D Bose gas.

3. *H-T* phase diagram

The results obtained above show that the vortex liquid state is a genuine thermodynamic state, intermediate between vortex lattice and normal metal. This phase is squeezed between vortex lattice on one side and normal metal on another. The position of the melting line is determined by the Lindeman melting criteria (Sec. I A). We shall discuss here the position of the critical-line vortex-liquid–normal-metal state.

We consider first the 3D superconductors with moderate $\mathcal{G}$, such as 1:2:3 materials. Formally, one can use the relations between parameters of the Bose liquid and parameters of superconductor,

$$\frac{B}{H_{c2}} = \frac{(T-T_c)^2}{16\alpha^2\mathcal{G}\,T_c^2} \tag{6.10}$$

in order to map the line of the superfluid–normal-ground state transition, Fig. 7, into the H-T phase diagram of a superconductor. Unfortunately, such a mapping would lead to the wrong conclusion that the transition line is *above* $H_{c2}(T)$ line. It means that in the whole region where the mapping is valid, the vortex system exists either in a solid or in an intermediate liquid phase. The transition to the normal metal occurs after this mapping breaks down which happens at $H \sim 0.5H_{c2}$ when the vortex cores start to overlap strongly. Thus, the line $H \sim 0.5H_{c2}$ provides a lower bound for the position of the critical line. It is also possible that the transition to the normal state does not occur until one approaches the region around $H_{c2}(T)$ line where the fluctuations of the amplitude of the order parameter are large. This region serves as an upper bound for the critical line. At present we are unable to find the exact location of the critical line. In Fig. 1(a) we have shown both upper and lower bounds for the position of the critical line.

The situation is different for strongly anisotropic materials with large $\mathcal{G}$, such as 2:2:1:2. To be specific, we use the parameters of 2:2:1:2 ($\mathcal{G} \sim 0.5$; see Sec. I B). The actual width of the critical region in this layered material is less than $\mathcal{G}$: $\tau_{\rm fl}^{2\rm D} \approx 0.1 \ll \mathcal{G}$. The mapping to the Bose system is valid everywhere outside of the critical region; the transition to the normal state occurs inside this region. At higher fields ($B > 1$ T [we use $H_{c2}(T=0) \approx 65$ T]), the layered structure of the material becomes important. At these fields vortices in different planes decouple and the transition to the normal metal follows. The phase diagram in this range of fields is shown in Fig. 1(b).

B. Normal liquid of bosons in the strongly correlated electron systems

Here we discuss possible implications for strongly correlated electron systems with spin charge separation, in particular, for the theory of the normal state of high-T_c oxides (Sec. I C). In the metallic state of these systems the charged excitations are gapless. The charge carriers are Bose quasiparticles. As discussed in Refs. 18 and 20 such a theory provides a semiquantitative explanation of the properties of the normal state, provided that no Bose condensation occurs. Below we shall investigate the validity of this assumption.

The important parameters of the theory are the mass and the density of the bosons in the low-energy limit. The density of the bosons coincides with the hole density. It is difficult to extract the mass from the experimental data, because in this problem we expect a significant renormalization of the mass, whereas conventional methods determine the mass of the quasiparticles at large energies. For instance, the width of the Drude peak determines the effective mass of the charge carrier at $E \sim 1000$ K; the penetration depth determines the mass of the carrier in the superconducting state where all fluctuations with the energies below the superconducting gap ($2\Delta \sim 700$ K) are completely suppressed. Thus, both types of measurements determine the effective mass of the charge carrier at $E \sim 1000$ K. The high-energy estimates based on these measurements give the mass $m_B \approx 3m_{\rm el}$. As we shall see below it is not possible to reconcile the experimental data with the theory if the mass is so light. However, if the renormalization increases the mass to $m_B \approx 15m_{\rm el}$, the agreement is much better. This value of the Bose mass agrees with the value extracted[19] from the diamagnetic susceptibility measured in Ref. 53.

The spin excitations may be either gapless or have a gap. We consider these two cases separately.

1. *Gapless spin excitations*

In this case, the low-energy excitations involve chargeless fermions with spin 1/2, charged bosons, and a gauge field. Neglecting the direct interaction between bosons and spinons we conclude that charge transport in this system is governed by bosons interacting with a slowly propagating gauge field (Sec. I C). The properties of the Bose system in this regime were considered in Secs. II and III. We now apply the estimates of these sections to 90 K $YBa_2Cu_3O_7$. In this material the density of holes is large ($n_h \approx 0.3$), and so $\ln(1/\delta) \gtrsim 1$.

If the interaction constant $\alpha_g \ll 1$, the superfluid phase transition is replaced by a crossover to a phase with a finite concentration of vortices, n_s [Eq. (2.8)]. At larger interaction constants, $\alpha_g \gtrsim 1$, phase separation happens. The value of the interaction constant for the high-T_c cuprates can be estimated using the values of the resistivity slope to determine $\chi = 500Ka^2$[18] [here a is the lattice constant within (a,b) plane]. Using for the Bose mass $m_B \approx 3.0m_{\rm el}$ we get the interaction constant $\alpha_g \approx 0.15$. At such interaction strength the phase separation is unlikely to happen, but the phase transition is smeared significantly. The effects of the superfluidity do not show up until the density of vortices becomes less than the density of Bose particles. Using (2.8) we get an upper bound on the temperature at which the crossover to the normal state begins: $T_{\rm cr} \lesssim 0.15T_0$. Estimating the transition temperature in the ideal Bose gas by (2.4) we conclude that the effects of superfluidity are suppressed by gauge-field fluctuations down to $T_{\rm cr} \lesssim 600$ K. Below this temperature the conventional Fermi liquid is gradually formed. This conclusion is in obvious contradiction to the experiments in which the anomalous behavior persists to the transition temperature $T_c \sim 100$ K. Using for the Bose mass $m_B \approx 15m_{\rm el}$, we find a lower interaction constant $\alpha_g \approx 0.04$. Still the phase transition is smeared and the effects of superfluidity are suppressed down to a temperature $T_{\rm cr} \lesssim 200$.

2. *Spin gap*

The results of the NMR measurements show that, in some high-T_c cuprates, spin excitations acquire some sort of gap well above the superconducting transition. This effect is most pronounced in 60 K $YBa_2Cu_3O_{6.5}$.[54] We believe that this spin gap is due to pairing between the spinons in different planes.[21] Such pairing gives a mass to the symmetric combination of gauge fields in these planes. The low-energy degrees of freedom involve bosons in these planes and gauge field which is an antisymmetric combination of the gauge fields in each plane. Bosons on different planes have opposite charges with respect to this gauge field.

The scalar potential corresponding to the antisymmetric combination is not screened and the action of the gauge field acquires the general form (1.8). The dimensionless parameters of this model are

$$\alpha_g = \frac{1}{8\pi m\chi}, \qquad \alpha_C = \frac{3}{32\pi}\frac{\Delta^2 m_B}{\epsilon_F n_h}.$$

For the estimate of the spin gap we use the value $2\Delta \approx 700$ K which was observed as a threshold in the infrared reflectivity.[55] This threshold persists well above T_c, especially in 60 K material, but it disappears at higher temperatures. We associate this threshold with a spin gap formed in the spinon subsystem.[21]

Using the Bose mass $m_B \approx 3m_{\rm el}$, we get $\alpha_g \approx 0.1$, $\alpha \approx 0.06$. According to the phase diagram in Fig. 7 and the estimates of the Sec. II, the Bose liquid with such an interaction has a superfluid ground state at $T = 0$, but the effects of superfluidity are suppressed down to the temperatures $T_{cr} \lesssim 400$ K. Below this temperature a conventional Fermi liquid is formed. This conclusion is in contradiction to experiment.

The agreement improves for $m_B \approx 15m_{el}$. For this mass the interaction constants become $\alpha_g \approx 0.02$, $\alpha \approx 0.15$. Taking into account only static thermal fluctuations and using (2.8) we estimate that the superfluidity of the bosons is suppressed down to $T_{cr} \lesssim 150$ K. The dynamic fluctuations suppress the crossover further down, but are unable to destroy the superfluidity completely at $T = 0$ according the phase diagram on Fig. 7.

The Bose condensation discussed so far was the one-particle Bose condensation. If bosons on different planes have different charges, as happens in the model of Ref. 21, pair condensation of the bosons becomes more likely. Pairs of bosons have zero total charge with respect to the gauge field, and so the vortices in their order parameter are not screened by the gauge field. If the attraction between the bosons is weak, the size of the pair is large and such a transition is analogous to the Kosterlitz-Thouless (KT) transition in the superconductive film. Unlike the one-particle Bose condensation in a gapless spin situation, this is a real thermodynamic transition which leads to a superconductive state. Combining KT estimates with the value of the pair mass $m_p = 30m_{el}$, we estimate this transition temperature: $T_c \sim 80$ K.

Thus, the spin charge separation provides an explanation of the experiments in both spin gapless and gapped materials, if we assume that the mass of the boson is renormalized to $m_B \approx 15m_{el}$.

Note added in proof

Recently the results of the transport multiprobe measurements on Y-Ba-Cu-O single crystals in a mixed state were reported in Ref. 56 in which a sharp onset of the vortex rigidity in the c direction was observed at a temperature significantly above the vortex glass transition. Both position and nature of the new transition line are in agreement with the prediction of the vortex liquid–normal metal transition [Fig. 1(a)] if we assume $H_{c2} = 100$ T for this material.

A separate set of recent experimental results[57–59] seem to indicate that the previously accepted value for the anisotropy in 2:2:1:2 material was underestimated by a factor of 10: $M/m \approx 25\,000$; in this case the vertical axes of the predicted phase diagram [Fig. 1(b)] of the 2:2:1:2 should be multiplied by an additional factor 0.1 suppressing the region of the existence of a new phase to low fields $H \leq 500$ G. Measurements of the resistivity in the c direction in such fields on this material would give the evidence for the existence of a vortex liquid phase.

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