Mean-field theory for underdamped Josephson-junction arrays with an offset voltage

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We study a model Hamiltonian for superconductivity in underdamped Josephson-junction arrays in the presence of an offset voltage between the array and the substrate. We develop an approximate zero-temperature ($T=0$) phase diagram as a function of Josephson coupling, charging energy, and offset voltage, using a simple Hartree-type mean-field approximation. With diagonal charging energy, the calculated phase diagram is periodic in offset voltage, in agreement with previous results. At a special value of this voltage such that states with n and $n+1$ Cooper pairs per gain are degenerate, only an infinitesimal Josephson coupling is needed to establish long-range phase coherence in this approximation. With both diagonal and nearest-neighbor charging energies, the $T=0$ phase diagram has two types of insulating lobes with different kinds of charge order, and two types of superconducting regions. One of these is a "supersolid" in which long-range phase coherence coexists with a frozen charge-density wave. We briefly discuss connections to previous calculations, and possible relevance to experiments.

I. INTRODUCTION

Josephson-junction arrays have been the subject of considerable recent research.[1] Such arrays consist of superconducting (S) grains embedded in a nonsuperconducting host and coupled together by the Josephson effect or by proximity tunneling. They exhibit a large variety of unusual behavior, both static and dynamic, in the presence of applied dc and ac currents and perpendicular magnetic fields. In general, the nonsuperconducting component can be a normal metal (N) or an insulating layer (I). While most experimental work has been carried out on the former,[1] superconductor-insulator arrays exhibit a number of new phenomena. For example, the dynamical properties of such arrays are characterized by a McCumber-Stewart parameter $\beta \gg 1$, corresponding to coupled highly underdamped Josephson junctions. Under such conditions, one observes hysteresis and resistance steps in the I-V characteristics,[2,3] which are reproduced by calculations.[4-6] Vortices in such arrays have been reported to move ballistically—that is, they may behave like massive objects which can maintain their motion, once initiated, even in an external driving current.[2,7]

When the S grains are sufficiently small, the behavior of an array is modified by quantum effects. These effects arise from the noncommutativity of the Cooper pair number operator and the phase of the superconducting wave function. Such effects were discussed by Anderson,[8] Abeles,[9] and Simanek,[10] and have since been studied by a number of authors.[11-14] For sufficiently small grains, quantum phase fluctuations lead to the suppression of superconductivity entirely.[11-19] The array is instead in an insulating state at $T=0$, in which there is no phase ordering and the average number of Cooper pairs on each grain is fixed. The interactions between the charges themselves can be logarithmic in the insulating state.[20] The phase transition has also been found to be significantly affected by dissipative tunneling of single electrons.[21-25]

In this paper, we discuss the zero-temperature behavior of an underdamped array in the presence of an additional control parameter: the offset voltage between the array and a substrate plane. Such an offset voltage behaves like a chemical potential for injection of Cooper pairs into the array. As has been discussed by several workers,[26,27] it can have a complex effect on the phase diagram of the array. Depending on the range of the charging interaction, one can even see evidence of "charge frustration" analogous to the better-known effects of a transverse magnetic field in an overdamped array.

Our contribution here is to describe a simple mean-field approximation for the phase diagram of a superconductor-insulator-superconductor array in the presence of an offset voltage. The approximation is a straightforward extension of previous approaches at zero offset voltage.[10,15-18] It is readily tractable, yet leads to a complex phase diagram with a rich variety of possible phases. In several instances, it agrees well with previous calculations based on exact statements about the Hamiltonian. In part of the phase diagram, we find a "supersolid"[27,29,30] phase in which superconducting order is found to coexist with a frozen charge-density wave. While such a supersolid phase has been reported previously in a similar model,[27] it was found only in a narrow sliver of parameter space and only with longer-range interactions than ours.

We turn now to the body of the paper. In Sec. II, we describe the model Hamiltonian, as well as the simple approximation used to treat it. Section III presents calculated phase diagrams at $T=0$ for several ranges of parameters. Finally, in Sec. IV, we compare our results to previous calculations, describe possible extensions, and discuss the relevance of the results to possible experiments on artificially synthesized arrays as well as to granular systems.

II. MODEL HAMILTONIAN AND MEAN-FIELD APPROXIMATION

We consider a two-dimensional array of N superconducting grains separated from a substrate by a thin insulating layer. The ith grain is described by a superconducting order parameter $\psi_i \equiv |\psi_i| e^{i\phi_i}$ and contains n_i Cooper pairs of charge $2e$. The ith and jth grains are coupled by a Josephson junction with critical current $I_{c;ij} \equiv 2eJ_{ij}/\hbar$.

We approximate the behavior of the array by the following Hamiltonian:

$$H = \frac{1}{2}\sum_{ij} U_{ij} n_i n_j + 2eV \sum_i n_i + \frac{1}{2}\sum_{ij} J_{ij}[1-\cos(\phi_i - \phi_j)] , \quad (1)$$

where the sums run over all grains i and j. The first summation represents the Coulomb interaction among the Cooper pairs on the various superconducting islands. The second term is the potential energy generated when the superconducting array is held at a potential V with respect to the common ground. Finally, the last term represents the energy of Josephson coupling between neighboring grains. We assume a periodic lattice of identical grains. This implies that J_{ij} and U_{ij} are functions only of the separation $|\mathbf{R}_i - \mathbf{R}_j|$ between grains i and j.

The first summation can also be written in the form

$$\frac{1}{2}\sum_{ij} n_i U_{ij} n_j = \frac{1}{2}\sum_{ij} V_i C_{ij} V_j , \quad (2)$$

where V_i is the potential on the ith superconducting island, and C_{ij} is the capacitance matrix. This makes clear that $U_{ij} = 4e^2 (C^{-1})_{ij}$, i.e., the matrix U is proportional to the inverse capacitance matrix.

To within a constant factor, the Hamiltonian (1) can also be written in the form

$$H = \frac{1}{2}\sum_{ij} (n_i - \bar{n}) U_{ij} (n_j - \bar{n}) + \frac{1}{2}\sum_{ij} J_{ij}[1-\cos(\phi_i - \phi_j)] , \quad (3)$$

where $\bar{n}$ is a constant which is determined by the offset potential V. It is readily established that forms (1) and (3) differ only by the constant term $\frac{1}{2}\sum_{ij} \bar{n} U_{ij} \bar{n}$. Henceforth, we use the form (3), which is more convenient for calculation. We also assume that the coupling energy J_{ij} vanishes except for nearest-neighbor grains, for which it has value J; and that the energy U_{ij} vanishes except for diagonal and nearest-neighbor contributions, which have values U_0 and U_1, respectively.

In order to motivate the mean-field approximation, we consider the types of order which can be expected from this model Hamiltonian. If both U_0 and U_1 are zero, the array will undergo a transition to a state of long-range phase coherence below a critical temperature T_c.[31,32] For a square array, it is known that $T_c \approx 0.95 J/k_B$, and the transition is in the Kosterlitz-Thouless universality class.[33] The low-temperature phase is superconducting but has a novel type of long-range phase coherence in which the phase correlation functions decay algebraically. When U_0 and U_1 are finite, the phase-ordering transition temperature is reduced, reaching zero at critical values of U_0 and U_1. At larger values of these parameters, one expects some kind of *charge* ordered state, the exact nature of which will depend on U_0, U_1, and $\bar{n}$.

There are various ways to develop a mean-field approximation for this Hamiltonian. One approach is to decouple the terms which involve more than one grain as follows:

$$(n_i - \bar{n})(n_j - \bar{n}) \approx \frac{1}{2}(n_i - \bar{n})(\langle n_j \rangle - \bar{n}) + \frac{1}{2}(\langle n_i \rangle - \bar{n})(n_j - \bar{n}) , \quad (4)$$

$$\cos(\phi_i - \phi_j) \approx \frac{1}{2}(\cos\phi_i \langle \cos\phi_j \rangle + \langle \cos\phi_i \rangle \cos\phi_j + \sin\phi_i \langle \sin\phi_j \rangle + \langle \sin\phi_i \rangle \sin\phi_j) . \quad (5)$$

In effect, this approximation converts the many-body Hamiltonian H into a sum of single-body Hamiltonians. The values of the canonical averages $\langle n_i \rangle$, $\langle \cos\phi_i \rangle$ and $\langle \sin\phi_i \rangle$ are then determined self-consistently by the following procedure. First, one makes an initial guess for the values of these quantities. Next, given these initial guesses, one calculates the eigenstates of H_{MF}, which is the approximation to H resulting from the substitutions (4) and (5). Since H_{MF} is a sum of single-particle operators, each such eigenstate is a product of single-particle eigenstates $\psi_i(\phi_i)$, which are solutions to the appropriate single-particle Schrödinger equation

$$\left\{ U_0 (n_i - \bar{n})^2 + U_1 (n_i - \bar{n}) \sum_j{}' (\langle n_j \rangle - \bar{n}) - \sum_j{}' J (\langle \cos\phi_j \rangle \cos\phi_i + \langle \sin\phi_j \rangle \sin\phi_i) \right\} \psi_i(\phi_i) = E_i \psi_i(\phi_i) , \quad (6)$$

where the primes indicate that sums are to be carried out over all nearest neighbors to the site i. Equation (6) can be expressed as an ordinary differential equation with the help of the representation $n_i = -i(d/d\phi_i)$. This expression follows from the canonical conjugacy of the charge operator n_i and the phase ϕ_i, which implies the commutation relation $[n_i, \phi_i] = -i$.[8-10]

In seeking self-consistent solutions to Eq. (6), we have made the assumption that $\langle \sin\phi_i \rangle = 0$, i.e., that the phase order parameters of all the grains are parallel, although we do allow for the possibility that the *amplitudes* of these order parameters are unequal on different sites.

With this assumption, and writing

$$\psi_i(\phi_i)=\exp(i\eta_i x_i)f_i(x_i)\ , \tag{7}$$

with $x_i-2\phi_i$ and $\eta_i=\bar{n}-(U_1/U_0)\sum_j'(\langle n_j\rangle-\bar{n})$, we can express the Schrödinger equation in the form of Mathieu's equation for f_i:

$$f_i''(x_i)+(e_i+2q_i\cos 2x_i)f_i(x_i)=0\ , \tag{8}$$

where $e_i=2E_i/U_0$ and $q_i=\sum_j' J\langle\cos\phi_j\rangle/U_0$. Note that since ϕ_i is the phase of the superconducting order parameter on the ith site, the solutions $\psi_i(\phi_i)$ must be 2π periodic. This means $f_i(x_i+\pi)=\exp(i\eta_i\pi)f(x_i)$.

Given the single-particle solutions to Eq. (6), the values of $\langle n_i\rangle$ and $\langle\cos\phi_i\rangle$ at temperature T can be calculated self-consistently from the relation:

$$\langle n_i\rangle=\frac{\sum_\nu \exp(-\beta E_\nu)\langle\Psi_\nu|n_i|\Psi_\nu\rangle}{\sum_\nu \exp(-\beta E_\nu)}\ ,$$
$$\langle\cos\phi_i\rangle=\frac{\sum_\nu \exp(-\beta E_\nu)\langle\Psi_\nu|\cos\phi_i|\Psi_\nu\rangle}{\sum_\nu \exp(-\beta E_\nu)}\ , \tag{9}$$

where $\beta=1/k_BT$, Ψ_ν denotes the product of the single-particle wave-function and E_ν is the corresponding energy eigenvalue, which is the sum of single-particle eigenvalues. Because of the Hartree approximation, these averages simplify to involve sums over only single-particle states on the ith grain. Once the averages are calculated, they are substituted back into the single-particle Schrödinger equations (6), and the process is repeated until convergence is obtained.

In the present paper, we will be concerned primarily with the phase diagram at $T=0$. The self-consistency procedure is then considerably simplified, since only the ground-state wave functions enter into the sums in (9). The required equations reduce to

$$\langle\cos\phi_i\rangle_o=\langle\Psi_o|\cos\phi_i|\Psi_o\rangle\ ,$$
$$\langle n_i\rangle_o=\langle\Psi_o|n_i|\Psi_o\rangle\ , \tag{10}$$

where the subscript o refers to the ground state.

III. RESULTS

We have carried out calculations for a bipartite lattice (i.e., one which can be decomposed into two sublattices), assuming only nearest-neighbor couplings J_{ij}, which we denote J. We seek self-consistent solutions in which $\langle n_i\rangle$ and $\langle\cos\phi_i\rangle$ can each have different values on the two sublattices. The resulting phase diagram depends on the parameters $\bar{n}$, U_0, U_1, and J. Since only the ratios of the last three parameters enter nontrivially, we introduce dimensionless parameters $\alpha=zJ/U_0$ and $\gamma=zU_1/U_0$, where z is the number of nearest neighbors. Thermal effects are described in terms of a scaled temperature k_BT/U_0.

We begin by considering the case of only diagonal charging energy, i.e., $\gamma=0$. In the special case $\bar{n}=0$, this problem has been previously treated by a number of workers.[10,15–18] For any given values of $\bar{n}$ and α, the self-consistently determined value of $\langle\cos\phi_i\rangle_o$ in the ground state is found to be independent of i—that is, the ground state is translationally invariant. As α decreases, $\langle\cos\phi\rangle$ also diminishes, reaching zero at a critical value $\alpha_c(\bar{n})$. At this point phase fluctuations induced by the charging energy become strong enough to destroy superconducting order. Figure 1 shows $\langle\cos\phi\rangle$ as a function of α for several values of $\bar{n}$. The dependence of α_c on $\bar{n}$ is shown in Fig. 2.

The behavior shown in Fig. 2 is readily understood. The Hamiltonian (1) is periodic in $\bar{n}$ with period unity. This implies that the phase diagram should be similarly periodic, repeating at each integer number of Cooper pairs. This periodicity has previously been noted by Bruder, Fazio, and Schön[27] on the basis of general consideration. Fisher *et al.*[26] also obtained such a periodicity in the context of a somewhat different model involving bosons with a short-range repulsive interaction on a periodic lattice. Besides this periodicity, the symmetry of the Hamiltonian implies that $\alpha_c(\bar{n})=\alpha_c(1-\bar{n})$, a symmetry which is also reflected in Fig. 2. The critical value α_c has its minimum value at $\bar{n}=\frac{1}{2}$, where $\alpha_c=0$. Thus, at $\bar{n}=\frac{1}{2}$, a superconducting ordered state can be established with even an infinitesimal Josephson coupling, at least within the mean-field approximation.

To better understand the behavior near half-integer $\bar{n}$, we note that the "interaction term," $H_{\rm int}=-2J\langle\cos\phi\rangle\cos\phi_i$, behaves like a perturbation on the "kinetic energy term" $H_0=\frac{1}{2}U_0[-(id/d\phi)-\bar{n}]^2$ in the Schrödinger equation for $\psi_i(\phi_i)$. In the absence of this perturbation, the states with n and $n+1$ pairs are degenerate at $\bar{n}=n+\frac{1}{2}$ ($n=$ integer), both representing the ground state of the unperturbed Hamiltonian (cf. Fig. 3(a), where this degeneracy is depicted for the special case $\bar{n}=\frac{1}{2}$). The interaction term breaks this degeneracy and produces phase ordering. We may estimate $\langle\cos\phi\rangle$

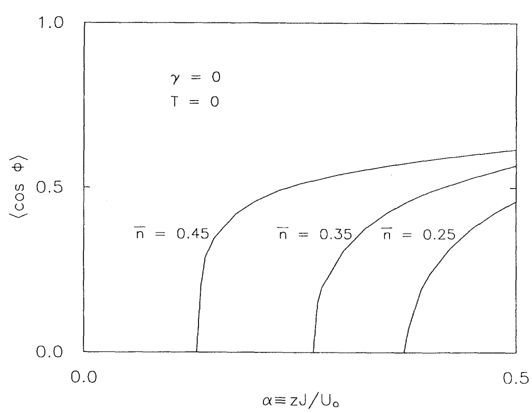

FIG. 1. Plot of the phase order parameter $\langle\cos\phi\rangle$ as a function of the parameter $\alpha\equiv zJ/U_0$ at temperature $T=0$, for several values of the average Cooper pair number $\bar{n}$, as calculated in the mean-field approximation. The nearest-neighbor charging energy $U_1=0$ in this calculation.

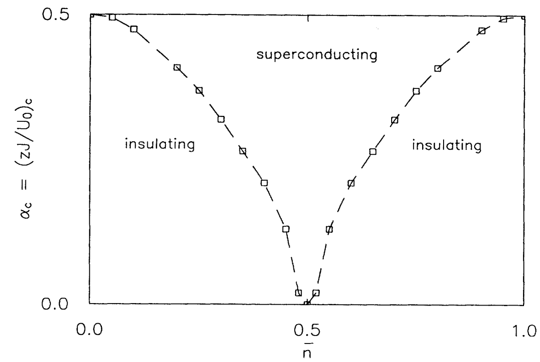

FIG. 2. Plot of the critical charging-energy parameter α_c as a function of $\bar{n}$ for the case of diagonal charging energy only, as calculated in the mean-field approximation.

in the mean-field approximation at finite temperatures, including only these two lowest states in the canonical average. The resulting self-consistent equation for $\langle\cos\phi\rangle$ at half-integer $\bar{n}$ is

$$\langle\cos\phi\rangle = \frac{2}{4-\alpha}\tanh\left[\frac{\alpha U_0}{k_B T}\langle\cos\phi\rangle\right] . \tag{11}$$

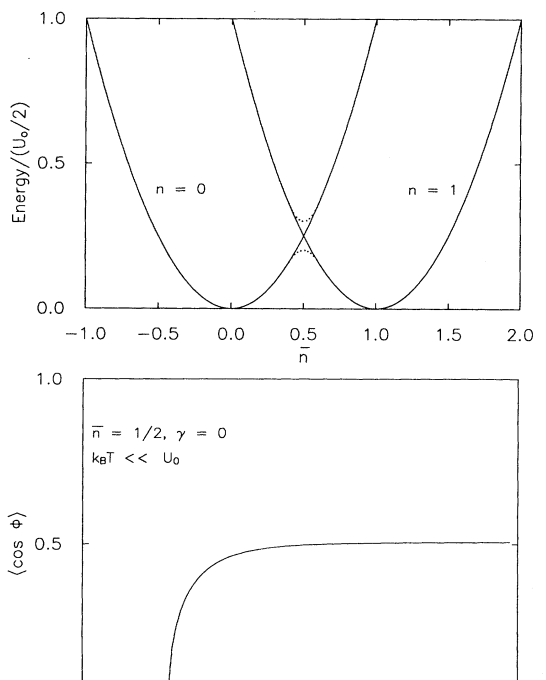

FIG. 3. (a) Plot of the ground-state energy E_0 as a function of $\bar{n}$ for the case of diagonal charging energy only, and zero Josephson-coupling energy J. The dashed lines indicate how the doubly-degenerate ground-state energy is split near $\bar{n}=\frac{1}{2}$ in the presence of a small Josephson energy J. (b) Plot of the phase order parameter $\langle\cos\phi\rangle$ as a function of temperature at $\bar{n}=\frac{1}{2}$, for small values of $\alpha=zJ/U_0$. The phase order parameter drops to zero in this limit near $2k_BT=zJ$.

Figure 3(b) shows the temperature-dependent phase order parameter $\langle\cos\phi\rangle$ resulting from (11). For $k_BT \ll U_0$, $\langle\cos\phi\rangle$ goes to zero at $\alpha=2k_BT/U_0$, or $T=zJ/(2k_B)$. Thus, for any finite J, $\langle\cos\phi\rangle$ remains finite up to this temperature, irrespective of the value of the charging energy U_0. The physics behind this behavior is easily understood. Precisely at half-integer $\bar{n}$, the two degenerate states are split by the phase-coupling perturbation into two states, in each of which the phase order parameter is nonzero. The energy splitting is of order $zJ/2$. As long as k_BT is smaller than this value, the lower state is predominantly occupied, leading to a nonzero phase order parameter. When k_BT exceeds this value, $\langle\cos\phi\rangle$ drops to zero.

We turn next to the mean-field phase diagram when the off-diagonal charging term $\gamma\neq 0$. Figures 4 and 5 show the zero-temperature mean-field phase diagram for two values of γ in the range $0<\gamma<1$.[34] The phase diagram is considerably more complex than for $\gamma=0$. At small values of α, the system is in a nonsuperconducting phase, with $\langle\cos\phi_i\rangle=0$, but we can identify two different types of charge structure. Near integer $\bar{n}$, we have a "ferromagnetic" charge ordering, such that $\langle n_i\rangle$ equals the same integer on each grain. Near half-integer $\bar{n}$, the charge ordering is "antiferromagnetic," provided the lattice is bipartite. In this case, $\langle n_i\rangle$ takes on two distinct integer values on the two sublattices, say $\langle n_i\rangle=n$ and $\langle n_i\rangle=n+1$, with n integer. These phases are "incompressible" in the sense that the total number of Cooper pairs on the lattice does not vary continuously with $\bar{n}$, but rather jumps discontinuously from one value to another at the phase boundary. These phases are thus "Mott insulators" as previously noted by Fazio and Schön. As the value of γ increases, Figs. 4 and 5 show that the antiferromagnetic lobe expands at the expense of the ferromagnetic lobes. This is to be expected, as no antiferromagnetic lobe exists when $\gamma=0$.

The mean-field approximation predicts that superconducting phase, which occurs at large α, also coexists with two different types of charge structure, dependent on the value of $\bar{n}$. Near $\bar{n}=\frac{1}{2}$ we find an antiferromagnetic

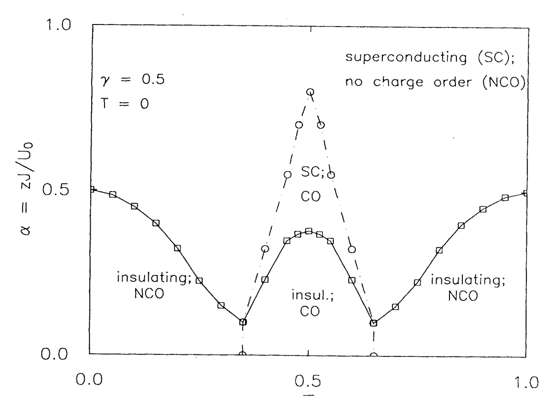

FIG. 4. Phase diagram in the α-$\bar{n}$ plane at $T=0$ for the case $\gamma\equiv zU_1/U_0=0.5$, as calculated in the mean-field approximation. The different types of order in the ground state are indicated by the legends in the figure.

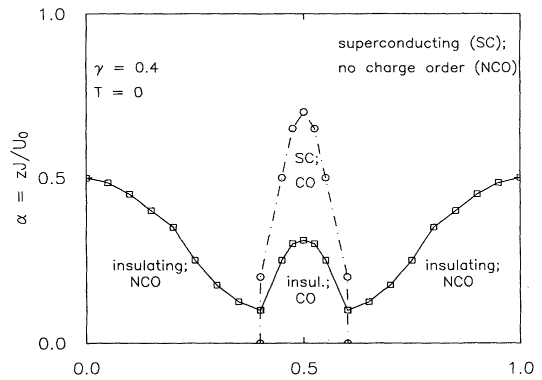

FIG. 5. Same as Fig. 4, but for $\gamma=0.4$.

charge structure coexisting with a nonzero $\langle \cos\phi_i \rangle$ (corresponding to a superconducting phase). In this phase, both $\langle n_i \rangle$ and the phase order parameter $\langle \cos\phi_i \rangle$ take on two different values on the two grain sublattices. In contrast to the corresponding Mott insulating phase, however, $\langle n_i \rangle$ is always *continuous* function of $\bar{n}$ in the superconducting phase. Because of this, we expect that the phase boundary between the charge-ordered and noncharge-ordered superconducting phases may also correspond to a continuous, rather than first-order, phase transition. As in the nonsuperconducting phases at small α, the charge-ordered superconducting lobe shrinks as γ becomes smaller.

IV. DISCUSSION

It is of interest to compare our results and approximations with those of other workers based on related models. For example, Fisher *et al.*[26] consider a model similar to ours, but with $\gamma=0$, i.e., with diagonal charging energy only. They also solve their model Hamiltonian in a mean-field approximation, which is presented as an exact solution to their model in the limit of large $\bar{n}$ and infinite-range hopping. Despite these differences, their phase diagram at $T=0$ closely resembles ours for the $\gamma=0$ case, with insulating lobes centered at each integer $\bar{n}$.

An extensive and elegant series of calculations has been carried out by the Karlsruhe group, aimed at elucidating both the static and the dynamic properties of charging-energy models. For example, Bruder *et al.*[27] consider a model Hamiltonian identical to ours but, rather than solving it in a mean-field approximation, they map it onto the so-called *XXZ* spin$=\frac{1}{2}$ Heisenberg model in two dimensions. In this model, the occupancy of a given grain is restricted to either zero or one Cooper pair, corresponding to the two states of the $S=\frac{1}{2}$ model. This correspondence is exact, however, only in the limit of "hard-core bosons," i.e., $U_0=\infty$. In order to determine the phase diagram of this *XXZ* model at $T=0$, they calculate the classical energy due to different spin configurations in this model. When only nearest-neighbor couplings are included, this approach does not produce one striking result of our calculations, namely, a charge-ordered state which coexists with superconductivity. This "supersolid phase" appears in their approximation only when next-nearest-neighbors are included, and even then only in a narrow region of the phase diagram. This sliver occurs at small α, and never includes half-integer $\bar{n}$ as does our supersolid phase. In other respects, our phase diagram closely resembles that of Bruder *et al.* for the case of diagonal and nearest-neighbor charging energies, with insulating lobes centered at half-integer and integer $\bar{n}$.

A particularly interesting locus on the phase diagram is the so-called Heisenberg point: $\bar{n}=\frac{1}{2}$, $J/U_1=1$. At this point, Bruder *et al.* observe a direct transition between a charge-ordered insulating state and a superconducting phase. This result must be exact in the limit of large U_0 (hard-core bosons). In the limit of large U_0, our results for the insulating-to-superconducting transition at half-integer $\bar{n}$ are in excellent agreement with this value. For example, when $\gamma\equiv zJ/U_0=0.3$, we observe a phase transition at $\alpha=0.26$, corresponding to $(J/U_1)_{cr}=0.26/0.3=0.87$. For $\gamma=0.2$, corresponding to a larger U_0, the transition occurs at $\alpha=0.19$ or $(J/U_1)_{cr}=0.95$, while for $\gamma=0.1$, it takes place at $\alpha=0.1$ or $(J/U_1)_{cr}=1.0$. Thus, our mean-field approach agrees well with available exact results in the appropriate limits.

It is of interest to compare these results to another recent paper by Bruder *et al.*[28] These authors use a model slightly different from that of Ref. 27 and the present work, in that they consider diagonal and nearest-neighbor *capacitances*, rather than diagonal and nearest-neighbor *charging energies*. This leads in $d=2$ to a screened logarithmic interaction between the superconducting islands, and a corresponding Kosterlitz-Thouless-like[31] charge-unbinding, rather than vortex-unbinding transition. In the presence of both an offset voltage and finite magnetic field, this model leads to a rich and fascinating phase diagram exhibiting aspects of both charge and magnetic[33,35] frustration, many of which are beyond the scope of the present work. Their phase diagram at zero magnetic field and diagonal charging energy closely resembles Ref. 27 as well as the present work. They also describe a coarse-graining approximation which differs from both the approximations of the other two papers, as well as from the present approach. However, Ref. 28 does not emphasize the charge-ordered superconducting state mentioned above.

A more recent paper[36] extends this approach (based on the model of diagonal and nearest-neighbor capacitances) to the dynamical response. These authors obtain a universal conductance at the superconductor-insulator transition, which has different values depending on the presence or absence of a small dissipative term in the dynamical Hamiltonian. Once again, they obtain a lobed phase diagram similar to that described in the other references cited above, but do not discuss the supersolid phase described here and in Ref. 27. Their approach is, in part, a mean-field-like approximation based on a coarse-graining approach which is quite different from the simple technique adopted here (which is not intended to treat the dynamical properties of the Hamiltonian).

Finally, we mention a recent discussion of Kissner and Eckern.[37] These authors, again considering the nearest-neighbor capacitance model mentioned above, propose a decoupling approximation similar to ours, but only on the Josephson-coupling part of the Hamiltonian. Since the charging-energy portion is not decoupled, it cannot be exactly evaluated, and only general statements can be made about the resulting phase diagram. It turns out that their mean-field approximation (and presumably ours also) corresponds to two different approximations for the free energy, only one of which can be derived from a variational principle. This paper emphasizes the zero or small-offset charge part of the Hamiltonian and does not discuss the supersolid phase.

The reader may be concerned that the simplicity of our approximation implies that the results are unreliable. In defense, we note that the approximation correctly gives the behavior at the Heisenberg point, as noted earlier, while giving a plausible description of the behavior at other regions of the phase diagram as well. While it is subject to the usual limitations of all mean-field approaches (for example, incorrect critical exponents at phase boundaries), it may give a qualitatively correct physical picture of a phase diagram with relatively little labor. In the limit of no Josephson coupling, our approach is equivalent to the usual mean-field treatment well known in the treatment of Ising-like Hamiltonians, which has a well-known regime of usefulness. We also note that similar approaches have proved useful in other related contexts (for example, treatment of the behavior of the phase diagram of a Josephson array in a magnetic field.[35]

Next, we briefly discuss the features of the phase diagram that emerge from our calculations. The supersolid phase, which plays a relatively prominent part in our phase diagram but a less conspicuous role in that of Ref. 27, and corresponding to the coexistence of charge and superconducting order, has a simple interpretation in the language of the "*XXZ*" model: the phase order represents ferromagnetic ordering in the *xy* plane, while charge ordering represents antiferromagnetic order in the *z* direction.[29] We speculate that the absence of the supersolid phase in the model of Ref. 27 at half-integer charging may be a consequence of the $S=\frac{1}{2}$ approximation considered by them. In that limit, at half-filling, it appears that the supersolid phase occupies a smaller and smaller length of the α line, disappearing in the limit $U_0 \rightarrow \infty$. It is not clear how this exotic state could be detected experimentally, but since it is presumably zero resistivity, the charge-ordered state could presumably be set in motion with an arbitrarily small applied voltage. This might lead to unusual effects involving oscillating currents in the superconducting state.

It would be straightforward to extend the present calculations to the case of randomness in both the capacitance and the offset potential. Fisher *et al.*[26] have considered the effects of a random external potential on their interacting Bose model. For the case of weak bounded disorder, they find that the superfluid lobes are narrowed, and a new phase emerges: an insulating, gapless Bose glass. They argue that on site randomness plays a crucial role in the phase diagram of ^{4}He adsorbed in porous media. We expect that our mean-field theory would also yield a Bose-glass-like phase in an appropriate part of the phase diagram. This can be seen by considering, e.g., the case of random diagonal capacitance energies. In the insulating regime, for a continuous bounded disorder, we expect that in the mean-field approximation, the charge number should vary continuously with offset potential in certain ranges of this potential, as successively more grains jump from one charge state to another. This would correspond to a "compressible, insulating" phase, similar to the Bose glass regime of Fisher *et al.* Presumably the Mott (incompressible) phase would disappear altogether for sufficiently strong disorder.

Conductance, such as those mentioned above, oscillations are well-established in small groups of normal junctions.[38] Presumably, it would be straightforward to carry out such measurements with an underdamped array in an appropriate geometry, seeking periodic variations in conductance with offset voltage. It would certainly be of great interest if such oscillations could be observed in superconducting arrays.

ACKNOWLEDGMENTS

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