# Kondo crossover in the self-consistent one-loop approximation

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The free energy and magnetization for the general SU($N$) one-impurity Kondo model in the magnetic field $h$ are calculated by extending the previous $1/N$ expansion technique: the saddle point is determined self-consistently to the $1/N$ order. The obtained universal field-dependent magnetization $M(h/T_K)$ by this simple method is shown analytically to be asymptotically exact at both $h \ll T_K$ and $h \gg T_K$ limits. For general "$f$-electron" fillings, except half filling, the $M(h/T_K)$ curves cross continuously from the weak to strong coupling limit, but overestimate the curvature in the crossover region for moderate $N$. The magnetic Wilson crossover numbers are calculated approximately. Our results explicitly verify that the $1/N$ parameter is nonsingular under the adiabatic continuation.

## I. INTRODUCTION

The flowing of an effective interaction from weak coupling at high energy to strong coupling at low energy is an important and frequently encountered phenomenon in various physical systems. A well-known condensed matter example is the Kondo effect.[1] Usually, it is only possible to construct perturbative solutions in the weak- and strong-coupling limits. Since the Kondo problem admits an exact solution, it provides a useful testbed for new ideas and methods. Among various methods applied to the problem, the numerical renormalization group (NRG),[2] Bethe ansatz,[3] and noncrossing approximation,[4] nicely and accurately produce the crossover. Unfortunately, these methods either are very complicated or heavily rely on numerical calculations. A simple and elementary method describing the crossover is desirable and may give us new insight.

Recently, motivated by the NRG results on the two-impurity Kondo problem,[5,6] which claim that there is a line of Fermi-liquid fixed points continuously modified by the Ruderman-Kittel-Kasuya-Yosida (RKKY) interaction between the two impurity spins, we have developed an "Eliashberg equation" approach to build the magnetic correlation between the two impurity spins nonperturbatively into the ground state.[7] Naturally, we want to test our method for the one-impurity Kondo problem. In this simple case, our approach amounts to the self-consistent one-loop approximation. For the general SU($N$) impurity spin model[8] with the orbital degeneracy $N$, we expand the free energy in $1/N$ and determine the saddle point self-consistently using the free energy including one-loop ($1/N$) fluctuation contributions. We shall see that $1/N$ is a nonsingular parameter under the adiabatic continuation,[9] at least outside a narrow crossover region. The effect of high-order terms is to smooth out the crossover. Technically, $1/N$ fluctuations always involve cutoff-dependent contributions. In order to obtain the universal free energy and magnetization, all the cutoff-dependent terms have to be absorbed into the Kondo temperature $T_K$. In the following, we first sketch the procedure, then give the details in the next two sections so that whoever is not interested in details can skip from the end of the Introduction directly to the Results.

The Kondo problem describes an impurity spin antiferromagnetically coupled with strength $J$ to a wide conduction band with density of states $\rho(\epsilon)$. The Hamiltonian for the general SU($N$) model[8] in the magnetic field is

$$H = \sum_{\mathbf{k},\sigma}(\epsilon_{\mathbf{k}} + \sigma h)c^{\dagger}_{\mathbf{k}\sigma}c_{\mathbf{k}\sigma} - \frac{J}{N}\sum_{\mathbf{k},\mathbf{k}',\sigma,\sigma'}(c^{\dagger}_{\mathbf{k}\sigma}f_{\sigma})(f^{\dagger}_{\sigma'}c_{\mathbf{k}'\sigma'}) + h\sum_{\sigma=-S}^{S}\sigma f^{\dagger}_{\sigma}f_{\sigma}. \quad (1)$$

The impurity spin is represented by $N = 2S+1$ localized degenerate levels partially filled with "$f$-electrons." Their creation and annihilation operators are subject to the constraint

$$\hat{n}_f = \sum_{\sigma} f^{\dagger}_{\sigma}f_{\sigma} = q_0 N. \quad (2)$$

We have set the gyromagnetic ratio and Bohr magneton equal to one so that the magnetic field strength $h$ has the energy scale. For Ce, the lower spin-orbit split multiplet is usually $N = 6$. The coefficient $q_0$ is treated as a constant of order one[9] in the expansion and will be given any value at the end of calculation. We shall present results for $q_0 = 1/2$ and $q_0 = 1/N$.

There are two physical parameters in the Kondo problem, the bandwidth $D$ and the dimensionless coupling constant $g = J\rho(0)$. In the scaling regime, $h \ll D$ and $T_K \ll D$, physical quantities depend on $D$ and $g$ only through the Kondo temperature $T_K = T_K(D,g)$. If the initial bare $g \ll 1$, we can find $T_K$ in the $D/T_K \to \infty$ limit. This is equivalent to the ultraviolet renormalization. The renormalizability of the Kondo problem was stated long time ago[10,11] and can be proved without difficulty. After absorbing the bare parameters into $T_K$, physical quantities such as the magnetization must be a one-variable function: $M = M(h/T_K)$, since $M$ is dimensionless. Usually, there could be many different scaling

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TABLE I. Definition of symbols and notations.

| Symbol | Definition (Eq. No.) | Symbol | Definition (Eq. No.) |
|---|---|---|---|
| $D$ | Bandwidth | $\Gamma$ | (26) |
| $\rho(\epsilon)$ | Density of states | $\Gamma_1$ | (27) |
| $h$ | (1) | $\Gamma_2$ | (28) |
| $q_0$ | (2) | $\eta_1$ | (29) |
| $\epsilon_f$, $r_0$ | (15) | $\eta_2$ | (30) |
| $\Gamma_\lambda$ | (18) | $\Lambda_1$ | (32), (A8) |
| $\Gamma_{\lambda r}$ | (19) | $\Lambda_2$ | (33), (34), (A6) |
| $\Gamma_r$ | (20) | $\nu_0$ | (31) |
| $g$, $T_K^{(0)}$ | (21) | $T_K$ | (38) |
| $\epsilon_{f\sigma}$, $\Delta$ | (22) | $\widetilde{F}$ | (4), (37) |
| $F_{\rm MF}$ | (3), (23) | $F_{\rm reg}$ | (4), (39) |
| $F_{1/N}$ | (3), (24) | $F_{1/N}^{\rm reg}$ | (40) |

functions $M(x)$ with $x = h/T_K$, depending on the band structures $\rho(\epsilon)$. However, $M(x)$ for the Kondo problem is universal because changing band structure only adds in irrelevant perturbations that quickly die out under scaling if initial $g \ll 1$.[12] The only possible exception is particle-hole symmetry-breaking perturbation, which is marginal and may lead to a modified $M(x)$. Thus, the obtained scaling solution for the magnetization in our calculation is directly comparable with any previous result up to a proportionality constant between different definitions of the Kondo temperature.[13]

It has been known from the phenomenology of dilute alloys[14] that the nature of the strong-coupling fixed point of the Kondo problem is a local resonant level. The two parameters of the resonant level, its position $\epsilon_f$ and width $\Delta$, are precisely the saddle-point parameters in the $1/N$ expansion.[15] Including $1/N$ fluctuations, the free energy in the magnetic field can be written as

$$F(h,\epsilon_f,\Delta,g,D) = NF_{\rm MF}(h,\epsilon_f,\Delta,g,D) + F_{1/N}(h,\epsilon_f,\Delta,g,D), \tag{3}$$

where the mean field and $1/N$ contributions, $F_{\rm MF}$ and $F_{1/N}$, have no explicit dependence on $N$. The two parameters $\epsilon_f$ and $\Delta$ are determined by the stationary condition of the free energy. To find the Kondo temperature $T_K$, we separate out from the free energy all terms depending on the bare parameters $g$ and $D$,

$$F(h,\epsilon_f,\Delta,g,D) = \widetilde{F}(h,\epsilon_f,\Delta,g,D) + F_{\rm reg}(h,\epsilon_f,\Delta,T_K). \tag{4}$$

The regularized free energy, $F_{\rm reg}$, depends on $g$ and $D$ only through $T_K$. With a proper definition of $T_K$, $\widetilde{F}$ becomes a constant depending *only* on $g$ and $D$, representing the correction to the ground-state energy. The thermodynamics is contained in $F_{\rm reg}$ from which we obtain the field-dependent magnetization.

The paper is organized as follows. In the next section, we briefly recapture the large-$N$ approach in the magnetic field to define our notations. The renormalization procedure is described in the third section. In the fourth section, we present the field-dependent magnetization from $h \ll T_K$ to $h \gg T_K$ for several values of $N$. The magnetic Wilson crossover numbers are calculated approximately. The proof that the magnetization calculated from $F_{\rm reg}$ has the correct $h \gg T_K$ asymptotics and the integral expressions of some functions appearing in the regularization are included in the appendixes for completeness. To alleviate cross referencing, we list the frequently occurring symbols together with their defining equation numbers in Table I.

## II. LARGE-$N$ FORMALISM

Following previous treatments,[15,16] we introduce a Lagrange multiplier $\lambda$ to enforce the constraint (2). By using the fact that the constraint commutes with the Hamiltonian, we write the partition function in the magnetic field $h$ as

$$\begin{aligned}\mathcal{Z} &= {\rm Tr}\,\delta(\hat{n}_f - q_0N)\exp[-\beta H] \\ &= \int\frac{\beta\,d\lambda}{2\pi}\,{\rm Tr}\,\exp\{-\beta[H + i\lambda(\hat{n}_f - q_0N)]\} \\ &= \int\frac{\beta\,d\lambda}{2\pi}\int\mathcal{D}[c,\bar{c},f,\bar{f}]\exp\left[-\int_0^\beta d\tau(\mathcal{L}_0 + H - iq_0N\lambda)\right]\end{aligned} \tag{5}$$

$$\mathcal{L}_0 = \sum_{\mathbf{k},\sigma} c_{\mathbf{k}\sigma}^\dagger\partial_\tau c_{\mathbf{k}\sigma} + \sum_\sigma f_\sigma^\dagger(\partial_\tau + i\lambda)f_\sigma. \tag{6}$$

After performing Hubbard-Stratonovich transformation to factorize the Kondo interaction, we rewrite the partition function as

$$\mathcal{Z} = \int\frac{\beta\,d\lambda}{2\pi}\int\mathcal{D}[c,\bar{c},f,\bar{f},Q,\bar{Q}] \times\exp\left[-\int_0^\beta d\tau\left(\mathcal{L}_0 + \mathcal{L}' + \frac{N|Q|^2}{J} - iq_0N\lambda\right)\right], \tag{7}$$

$$\mathcal{L}' = \sum_{\mathbf{k},\sigma}(\epsilon_{\mathbf{k}} + \sigma h)c_{\mathbf{k}\sigma}^\dagger c_{\mathbf{k}\sigma} + \sum_{\mathbf{k},\sigma}(Qc_{\mathbf{k}\sigma}^\dagger f_\sigma + \bar{Q}f_\sigma^\dagger c_{\mathbf{k}\sigma}) + h\sum_\sigma \sigma f_\sigma^\dagger f_\sigma. \tag{8}$$

The above Lagrangian possesses a U(1) gauge invariance

$$f_\sigma \to f'_\sigma = f_\sigma e^{i\phi},$$
$$Q \to Q' = Q\, e^{-i\phi}, \tag{9}$$
$$\lambda \to \lambda' = \lambda + \frac{d\phi}{d\tau}.$$

The redundant gauge degrees of freedom can be eliminated by choosing to work in the radial gauge. Separating the complex field $Q$ into an amplitude and a phase $Q = r\,e^{-i\phi}$, the phase $\phi$ can be absorbed into new variables $f'_\sigma$ and $\lambda'$: $f'_\sigma = f_\sigma e^{-i\phi}$, $\lambda' = \lambda + d\phi/d\tau$ . In terms of new variables $r$, $\lambda'$, $f'_\sigma$ and $\bar{f}'_\sigma$, the partition function can be cast in the form, after dropping the primes,

$$\mathcal{Z} = \int \mathcal{D}[c,\bar{c},f,\bar{f},\lambda,r] \prod_\tau r(\tau) \times \exp\left[-\int_0^\beta d\tau \left(\mathcal{L}''(\tau) + \frac{Nr^2}{J} - iq_0 N\lambda\right)\right], \tag{10}$$

$$\mathcal{L}'' = \sum_{\mathbf{k},\sigma} c^\dagger_{\mathbf{k}\sigma}(\partial_\tau + \epsilon_{\mathbf{k}} + \sigma h)c_{\mathbf{k}\sigma} + \sum_{\sigma=-S}^{S} f^\dagger_\sigma(\partial_\tau + i\lambda + h\sigma)f_\sigma + \sum_{\mathbf{k}\sigma} r\,(c^\dagger_{\mathbf{k}\sigma} f_\sigma + f^\dagger_\sigma c_{\mathbf{k}\sigma}). \tag{11}$$

It is possible to completely gauge away the U(1) phase $\phi$ because it does not contain dynamics. Since the last Lagrangian is bilinear in the Grassman variables $c_{\mathbf{k}\sigma}$ and $f_\sigma$, we can integrate them out to obtain an effective action,

$$\mathcal{Z} = \mathcal{Z}_0 \int \mathcal{D}[\lambda,r]\, \exp[-S_{\text{eff}}(\lambda,r) + \delta(0)\int_0^\beta d\tau \ln r(\tau)], \tag{12}$$

$$S_{\text{eff}} = -\sum_\sigma \text{Tr}\, \ln[\partial_\tau + i\lambda + h\sigma + rG_0(\tau)r] + N\int_0^\beta d\tau \left(\frac{r^2}{J} - iq_0\lambda\right), \tag{13}$$

where $\delta(0) = (1/\beta)\sum_{\nu_n} 1$ with $\nu_n = 2\pi n/\beta$, and

$$G_0(\tau) = -\sum_{\mathbf{k}} \frac{1}{\partial_\tau + \epsilon_{\mathbf{k}}}. \tag{14}$$

$\mathcal{Z}_0$ is the partition function of the noninteracting Fermi sea.

The integration over the two real variables $\lambda$ and $r$ can be expanded around a saddle point

$$i\lambda = \epsilon_f + i\widetilde{\lambda}, \qquad r = r_0 + \widetilde{r}. \tag{15}$$

Retaining only quadratic terms in $\widetilde{\lambda}$ and $\widetilde{r}$ in the expansion, the partition function, after dropping the tilde becomes,

$$\frac{\mathcal{Z}}{\mathcal{Z}_0} = e^{-S_{\text{eff}}(\epsilon_f,r_0)} \int \prod_{\nu_n} d\lambda(\nu_n)\, dr(\nu_n) \times \exp\left[-S^{(2)}_{\text{eff}} + \sum_{\nu_n} \ln r_0\right], \tag{16}$$

$$S^{(2)}_{\text{eff}} = \frac{N}{2}\sum_{\nu_n} (\lambda(-\nu_n), r(-\nu_n)) \times \begin{pmatrix} \rho(0)r_0^2\Gamma_\lambda(\nu_n) & i\rho(0)r_0\Gamma_{\lambda r}(\nu_n) \\ i\rho(0)r_0\Gamma_{\lambda r}(\nu_n) & \rho(0)\Gamma_r(\nu_n) \end{pmatrix} \times \begin{pmatrix} \lambda(\nu_n) \\ r(\nu_n) \end{pmatrix}. \tag{17}$$

The zero-temperature expressions of the matrix elements $\Gamma$'s appearing in $S^{(2)}_{\text{eff}}$ have been given by Read and Newns.[15] Their extension to include a magnetic field is straightforward. Here we have pulled out explicitly some prefactors for later convenience:

$$\Gamma_\lambda(\nu_n) = \frac{1}{N}\sum_\sigma \frac{1}{|\nu_n|(|\nu_n| + 2\Delta)} \ln\left[\frac{\epsilon^2_{f\sigma} + (|\nu_n| + \Delta)^2}{\epsilon^2_{f\sigma} + \Delta^2}\right], \tag{18}$$

$$\Gamma_{\lambda r}(\nu_n) = -\frac{2}{N|\nu_n|}\sum_\sigma \left[\tan^{-1}\left(\frac{\epsilon_{f\sigma}}{|\nu_n| + \Delta}\right) - \tan^{-1}\left(\frac{\epsilon_{f\sigma}}{\Delta}\right)\right], \tag{19}$$

$$\Gamma_r(\nu_n) = \frac{1}{N}\sum_\sigma \left\{\ln\left[\frac{\epsilon^2_{f\sigma} + (|\nu_n| + \Delta)^2}{(T_K^{(0)})^2}\right] + \frac{2\Delta}{|\nu_n|}\ln\left[\frac{\epsilon^2_{f\sigma} + (|\nu_n| + \Delta)^2}{\epsilon^2_{f\sigma} + \Delta^2}\right]\right\}, \tag{20}$$

where we have defined the mean-field Kondo temperature,

$$T_K^{(0)} = D\exp\left(-\frac{1}{g}\right), \qquad g = J\rho(0), \tag{21}$$

and the convenient notations,

$$\epsilon_{f\sigma} = \epsilon_f + \sigma h, \qquad \Delta = \pi\rho(0)r_0^2. \tag{22}$$

The contributions to the free energy (3) are given by

$$F_{\text{MF}} = \frac{1}{N}\sum_\sigma \left\{\frac{\epsilon_{f\sigma}}{\pi}\tan^{-1}\left(\frac{\epsilon_{f\sigma}}{\Delta}\right) + \frac{\Delta}{2\pi}\ln\left[\frac{\epsilon^2_{f\sigma} + \Delta^2}{(T_K^{(0)})^2}\right]\right\} - \frac{\Delta}{\pi} + \left(\frac{1}{2} - q_0\right)\epsilon_f, \tag{23}$$

$$F_{1/N} = \frac{1}{2\beta}\sum_{\nu_n} \ln[\Gamma_\lambda(\nu_n)\Gamma_r(\nu_n) + \Gamma^2_{\lambda r}(\nu_n)] + \text{const.} \tag{24}$$

In the free energy $F_{1/N}$, we note that the prefactors in the front of $\Gamma$'s in (17) exactly cancel the contribution $\sum_{\nu_n} \ln r_0$ of (16), originating from the Jacobian of transforming to the radial gauge.

## III. RENORMALIZATION

To calculate zero-temperature quantities, we can simply replace the discrete Matsubara frequency sum by an integration

$$F_{1/N} = \frac{1}{2\pi}\int_0^\infty d\nu \ln(\Gamma_\lambda\Gamma_r + \Gamma_{\lambda r}^2),$$
$$\frac{1}{\beta}\sum_{\nu_n} \to \int_{-\infty}^{\infty}\frac{d\nu}{2\pi}, \quad |\nu_n| \to \nu. \tag{25}$$

The upper integration limit is actually cut off by the conduction-electron bandwidth $D$. One can see this from the approximation we made in deriving the mean-field free energy and $1/N$ fluctuation matrix element $\Gamma$'s,

$$\sum_{\mathbf{k}}\frac{1}{i\omega_n - \epsilon_{\mathbf{k}}} = \rho(0)\int_{-D}^{D}\frac{d\epsilon}{i\omega_n - \epsilon}$$
$$= -i2\rho(0)\tan^{-1}\left(\frac{D}{\omega_n}\right)$$
$$\simeq -i\pi\rho(0)\,\mathrm{sgn}\omega_n\theta(D - |\omega_n|).$$

Obviously, the $F_{1/N}$ of (25) contains contributions linear in $D$, which become divergent in the $D \to \infty$ limit. A little investigation shows that the subleading divergent terms of $F_{1/N}$ have the form of $\ln\ln D$.

To separate out the cutoff-dependent terms of $F_{1/N}$, which diverge as $D \to \infty$, we consider the $\nu \to \infty$ asymptotic behavior of the integrand,

$$\Gamma(\nu) = \Gamma_\lambda\Gamma_r + \Gamma_{\lambda r}^2 = \frac{1}{\nu^2}\left[\Gamma_1(\ln\nu) + \frac{2}{\nu}\Gamma_2(\ln\nu) + \mathcal{O}(\nu^{-2})\right]. \tag{26}$$

The two functions $\Gamma_1$ and $\Gamma_2$ only depend on $\ln\nu$ and have the following simple forms

$$\Gamma_1(\ln\nu) = 4\left[\ln^2\frac{\nu}{T_K} - \pi\eta_2\ln\frac{\nu}{T_K} + \pi^2\left(\frac{1}{2} - q_0 - \eta_1\right)^2\right], \tag{27}$$

$$\Gamma_2(\ln\nu) = 4\Bigg[\Delta(1 - \pi\eta_2)\ln\frac{\nu}{T_K} - \pi\eta_2\Delta\left(\frac{1}{2} - \pi\eta_2\right) - \pi\epsilon_f\left(\frac{1}{2} - q_0 - \eta_1\right)\Bigg], \tag{28}$$

where we have introduced following two shorthand notations:

$$\eta_1 = \frac{\partial F_{\mathrm{MF}}(\epsilon_f,\Delta)}{\partial\epsilon_f} = \frac{1}{2} - q_0 - \frac{1}{\pi N}\sum_\sigma \tan^{-1}\left(\frac{\epsilon_{f\sigma}}{\Delta}\right), \tag{29}$$

$$\eta_2 = \frac{\partial F_{\mathrm{MF}}(\epsilon_f,\Delta)}{\partial\Delta} = \frac{1}{N\pi}\sum_\sigma \ln\left(\frac{\sqrt{\epsilon_{f\sigma}^2 + \Delta^2}}{T_K^{(0)}}\right). \tag{30}$$

They are both independent of frequency $\nu$. The $1/N$ fluctuation free energy is regularized as follows:

$$F_{1/N} = \int_0^\infty \frac{d\nu}{2\pi}\left\{\ln\Gamma(\nu) - \left[\ln\Gamma_1(\ln\nu) + \frac{2\,\Gamma_2(\ln\nu)}{\nu\,\Gamma_1(\ln\nu)}\right]\theta(\nu - \nu_0)\right\} + \int_{\nu_0}^{D}\frac{d\nu}{2\pi}\ln\Gamma_1(\ln\nu) + \int_{\ln(\nu_0/T_K)}^{\ln(D/T_K)}\frac{dx}{\pi}\frac{\Gamma_2(x)}{\Gamma_1(x)} + \text{const.} \tag{31}$$

Since the first integral is convergent, we have extended the upper integration limit to infinity. Note that $\nu_0$ is not a parameter of the theory. $F_{1/N}$ is independent of $\nu_0$. We shall choose it for computational convenience. Actually, it provides a useful consistency check for the numerical calculation. The cutoff dependence is then separated out from the last two integrals of (31),

$$\frac{1}{2\pi}\int_{\nu_0}^{D} d\nu\ln\Gamma_1(\ln\nu) = D\,\Lambda_1(D,\eta_1,\eta_2) - \nu_0\,\Lambda_1(\nu_0,\eta_1,\eta_2), \tag{32}$$

$$\frac{1}{\pi}\int_{\ln(\nu_0/T_K)}^{\ln(D/T_K)} dx\frac{\Gamma_2(x)}{\Gamma_1(x)} = \Lambda_2(D,\eta_1,\eta_2,\epsilon_f,\Delta) - \Lambda_2(\nu_0,\eta_1,\eta_2,\epsilon_f,\Delta). \tag{33}$$

The defined two functions, $\Lambda_1$ and $\Lambda_2$, are given in Appendix A.

To treat the cutoff-dependent terms $D\,\Lambda_1(D,\eta_1,\eta_2)$ and $\Lambda_2(D,\eta_1,\eta_2,\epsilon_f,\Delta)$, we first obtain explicitly

$$\Lambda_2(D,\eta_1,\eta_2,\epsilon_f,\Delta) = \frac{\Delta}{\pi}\ln\ln\frac{D}{T_K} - \eta_2\,\Delta\ln\ln\frac{D}{T_K}, \tag{34}$$

where we have neglected terms that vanish as $D \to \infty$. Using the fact that $\eta_1$ and $\eta_2$ are the derivatives of the mean-field-free energy, we can show that $\Lambda_1$ and the second term of (34) can be renormalized away from the saddle-point equations if we let the saddle-point parameters $\epsilon_f$ and $\Delta$ acquire the following $1/N$ corrections:

$$\widetilde{\epsilon}_f = \epsilon_f + \frac{D}{N}\frac{\partial}{\partial\eta_1}\Lambda_1(\eta_1^*,\eta_2^*), \tag{35}$$

$$\widetilde{\Delta} = \Delta - \frac{\Delta}{N}\ln\ln\frac{D}{T_K} + \frac{D}{N}\frac{\partial}{\partial\eta_2}\Lambda_1(\eta_1^*,\eta_2^*), \tag{36}$$

where $\eta_1^*$ and $\eta_2^*$ are the values at the point of the saddle-point solution, $\epsilon_f = \epsilon_f^*$ and $\Delta = \Delta^*$. When we rewrite the mean-field free energy in terms of the renormalized saddle-point parameters $\widetilde{\epsilon}_f$ and $\widetilde{\Delta}$, we have to include the difference $F_{\mathrm{MF}}(\epsilon_f,\Delta) - F_{\mathrm{MF}}(\widetilde{\epsilon}_f,\widetilde{\Delta})$ into the cutoff-dependent part of the free energy $\widetilde{F}$ introduced in (4). Collecting this difference term, (34), $\Lambda_1(D,\eta_1,\eta_2)$, and a term coming from replacing $T_K^{(0)}$ by $T_K$ in $F_{\mathrm{MF}}$, the total cutoff-dependent part of the free energy is

$$\widetilde{F} = -N\frac{\Delta}{\pi}\ln\frac{T_K^{(0)}}{T_K} + \frac{\Delta}{\pi}\ln\ln\frac{D}{T_K} + D\left[\Lambda_1(\eta_1,\eta_2) - \frac{\partial\Lambda_1(\eta_1^*,\eta_2^*)}{\partial\eta_1}\eta_1 - \frac{\partial\Lambda_1(\eta_1^*,\eta_2^*)}{\partial\eta_2}\eta_2\right]. \quad (37)$$

Note that the terms in the last bracket are a constant, to the order $\mathcal{O}(\eta_1) \sim \mathcal{O}(\eta_2)$. The first two terms cancel out if we define

$$T_K = T_K^{(0)}\left(\ln\frac{D}{T_K}\right)^{-1/N} = D\left(\ln\frac{D}{T_K}\right)^{-1/N}\exp\left(-\frac{1}{g}\right). \quad (38)$$

In the spirit of order by order renormalization, we replace $\epsilon_f$, $\Delta$ and $T_K^{(0)}$ appearing in $F_{1/N}$ by $\widetilde{\epsilon}_f$, $\widetilde{\Delta}$ and $T_K$, respectively. This gives us the regularized free energy as a function of $h$, $\widetilde{\epsilon}_f$, $\widetilde{\Delta}$, and $T_K$ only. Note that our expression for the Kondo temperature is consistent with the well known expression $T_K = Dg^{1/N}\exp(-1/g)$ up to $\mathcal{O}(1/N)$.

Actually, one can simply expand $\Lambda_1(\eta_1,\eta_2)$ in $1/N$ by using the fact $\eta_1 \sim \eta_2 \sim \mathcal{O}(1/N)$, a consequence of the saddle-point equations. We immediately see that the only $\mathcal{O}(1)$ contribution of $\Lambda_1(\eta_1,\eta_2)$ to the free energy is a constant. This constant is the correction to the ground-state energy and has no effect on the physical quantities. Higher-order terms in the expansion of $\Lambda_1(\eta_1,\eta_2)$ can be neglected in the order by order renormalization. The second term of (34) is also dropped, since it is of order $\mathcal{O}(1/N)$. After we renormalize away the first term of (34) by defining the $1/N$ corrected Kondo temperature $T_K$ via (38) and replace the mean-field Kondo temperature $T_K^{(0)}$ in $\Gamma_r$ by $T_K$, the resulting regularized free energy is then only a function of $\epsilon_f$, $\Delta$, $h$, and $T_K$. All these are due to the fact that the free energy is stationary with respect to $\epsilon_f$ and $\Delta$. A $\mathcal{O}(1/N)$ shift of these parameters does not induce any change in the free energy to the order $\mathcal{O}(N) + \mathcal{O}(1)$.

After completing the renormalization, the universal free energy is, from (4) and (31)–(33),

$$F_{\rm reg} = \sum_\sigma \frac{\epsilon_{f\sigma}}{\pi}\tan^{-1}\left(\frac{\epsilon_{f\sigma}}{\Delta}\right) - \frac{N\Delta}{\pi}\left[1 - \frac{1}{N}\sum_\sigma \ln\left(\frac{\sqrt{\epsilon_{f\sigma}^2 + \Delta^2}}{T_K}\right)\right] + N\left(\frac{1}{2} - q_0\right)\epsilon_f + F_{1/N}^{\rm reg}, \quad (39)$$

$$F_{1/N}^{\rm reg} = -\nu_0\,\Lambda_1(\nu_0,\eta_1,\eta_2) - \Lambda_2(\nu_0,\eta_1,\eta_2,\epsilon_f,\Delta) + \int_0^\infty \frac{d\nu}{2\pi}\left\{\ln\Gamma(\nu) - \left[\ln\Gamma_1(\ln\nu) + \frac{2\,\Gamma_2(\ln\nu)}{\nu\,\Gamma_1(\ln\nu)}\right]\theta(\nu-\nu_0)\right\}. \quad (40)$$

The parameters $\eta_1$ and $\eta_2$ only depend on $\epsilon_f$, $\Delta$. Inside $\eta_2$ and $\Gamma_r$, $T_K^{(0)}$ is replaced by $T_K$.

The saddle-point parameters, $\epsilon_f$ and $\Delta$, are determined by solving the following two saddle-point equations,

$$\frac{1}{N}\frac{\partial}{\partial\epsilon_f}F_{\rm reg}(h,\epsilon_f,\Delta,T_K) = \frac{1}{2} - q_0 - \frac{1}{\pi N}\sum_\sigma \tan^{-1}\left(\frac{\epsilon_{f\sigma}}{\Delta}\right) + \frac{1}{N}\frac{\partial}{\partial\epsilon_f}F_{1/N}^{\rm reg} = 0, \quad (41)$$

$$\frac{1}{N}\frac{\partial}{\partial\Delta}F_{\rm reg}(h,\epsilon_f,\Delta,T_K) = \frac{1}{\pi}\sum_\sigma \ln\left(\frac{\sqrt{\epsilon_{f\sigma}^2+\Delta^2}}{T_K}\right) + \frac{\partial}{\partial\Delta}F_{1/N}^{\rm reg} = 0. \quad (42)$$

Substituting the solution $\epsilon_f = \epsilon_f^*(h/T_K)$ and $\Delta = \Delta^*(h/T_K)$ back into $F_{\rm reg}$, we obtain the scaling form of the free energy depending only on $h/T_K$, up to an additive constant. The magnetization is

$$M(h/T_K) = -\frac{\partial}{\partial h}F_{\rm reg}(h,\epsilon_f^*,\Delta^*,T_K) = \frac{1}{\pi}\sum_\sigma \sigma\,\tan^{-1}\left(\frac{\epsilon_{f\sigma}}{\Delta}\right) - \frac{\partial}{\partial h}F_{1/N}^{\rm reg}. \quad (43)$$

The one-dimensional integration in the regularized $1/N$ free energy and its derivatives, as well as solving the two coupled equations (41) and (42), are carried out numerically.

We emphasize that the obtained magnetization is *not* a $1/N$ perturbative result if we solve the equations (41) and (42) self-consistently, i.e., not by expanding $\epsilon_f^*$ and $\Delta^*$ in $1/N$. The fact that we only carried out perturbative ultraviolet renormalization only implies that the Kondo temperature defined by (38) is perturbatively accurate to the $1/N$ order. In other words, our result for $F_{\rm reg}$ or $M(h/T_K)$ is perturbative at high energy but not necessarily perturbative at low energy, depending on how we solve the saddle-point equations. As we can see, the same renormalization procedure can be carried out for every physical quantity, and their calculation is a straightforward exercise.

## IV. RESULTS

The solution of the saddle-point equations, $\epsilon_f^*(h/T_K)$ and $\Delta^*(h/T_K)$, for $q_0 = 1/6$, $N = 6$ is shown in Fig. 1 as an example. Generally for $q_0 \neq 1/2$, there is more than one solution in the weak-coupling regime for a given value of $h/T_K$. Certainly, the criterion is to choose one with the lowest energy. However, since we know the asymptotics at both weak- and strong-coupling limits, we can follow the solutions continuously by varying the magnetic field slightly each time. For $q_0 = 1/6$ and $N = 6$ as an example, there are solutions other than that shown in Fig. 1 for $h/T_K > 0.52$ and give magnetizations much closer to Hewson and Rasul's exact results[17] in near crossover region compared with the results shown in Fig. 3. But, if we follow these solutions to the high magnetic field, they do not have the correct asymptotics.

The field-dependent magnetizations $M(h/T_K)$ for $q_0 =$

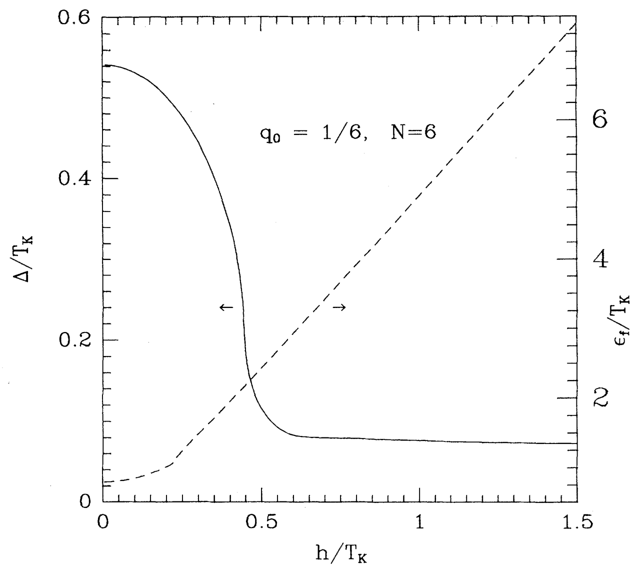


FIG. 1. The solution of the saddle-point equations in the magnetic field for $q_0 = 1/6$ and $N = 6$. $T_K$ is defined by (38). $\epsilon_f$ is the position of the resonant level and $\Delta$ is the width.

1/2 and various values of $N$ are shown in Fig. 2. Note that each curve has a window in the crossover region, where no solution is found by the present method. This happens only for $q_0 = 1/2$. The reason is the following. We try to describe the strong-coupling fixed point by a resonant level. The particle-hole symmetry presented in the $q_0 = 1/2$ case ties the position of the resonant level at the Fermi surface, $\epsilon_f^* = 0$, in the strong coupling regime. Certainly, the nature of the weak coupling is no longer

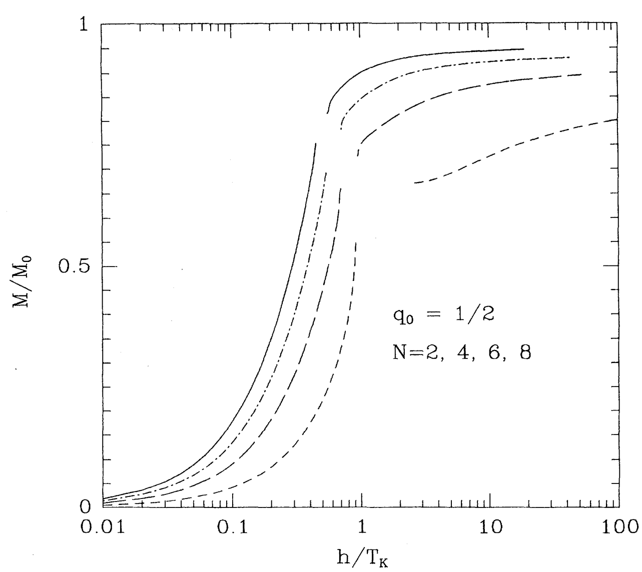


FIG. 2. The universal magnetic field dependent magnetization for $q_0 = 1/2$ and for $N = 2$ (short dashed line), $N = 4$ (long dashed line), $N = 6$ (dash-dotted line), and $N = 8$ (solid line). All curves are parameter free. Note the improving quality for larger $N$.

a resonant level, thus $\epsilon_f^* \neq 0$. A discontinuity must occur at some value of $\epsilon_f^*$ with increasing magnetic field $h$, preventing continuous crossover from one side to the other. Nevertheless, the window quickly narrows with increasing $N$. For $N = 8$, the solid line of Fig. 2, the window narrows to $0.45 < h/T_K < 0.55$. The indication is that probably we need an infinite order of terms in $1/N$ to close the window and to obtain a completely smooth crossover. The more terms we put in, the better the quality is in the crossover region. Similar features can also be seen for general values of $q_0$. In Fig. 3, we show the magnetizations for $q_0 = 1/N$, the "realistic" situation. Also shown are Hewson and Rasul's Bethe-ansatz results[17,18] for $N = 6, 8$. Although the lines can cross continuously from one side to the other, they obviously overestimate the curvature in the crossover region. With increasing $N$, the curvature is reduced.

We calculate the magnetic Wilson crossover numbers for the Coqblin-Schrieffer model,[8] $q_0 = 1/N$, although the calculation can be done for other values of $q_0$. The ambiguity in relating $T_K$ from different cutoff schemes can be eliminated by imposing the condition of a vanishing $\ln^{-2}(h/T_K)$ term in the $h/T_K \gg 1$ expansion of $M(h/T_K)$. The weak-coupling scaling form for the magnetization in terms of $T_K$ is well known,[3]

$$\frac{M}{M_0} = 1 - \frac{1}{2\ln\frac{h}{T_K}} - \frac{\ln\ln\frac{h}{T_K}}{2N\ln^2\frac{h}{T_K}} + \frac{\ln 2}{N\ln^2\frac{h}{T_K}} + \cdots, \qquad h/T_K \gg 1. \tag{44}$$

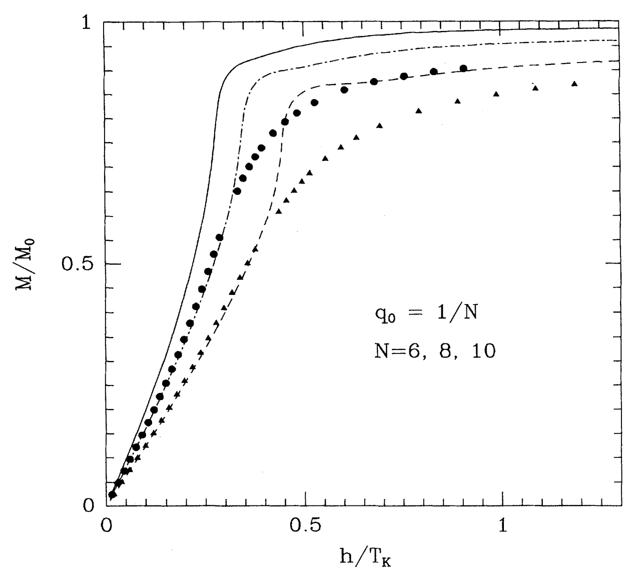


FIG. 3. The universal magnetic-field-dependent magnetization for the Coqblin-Schrieffer model, i.e., $q_0 = 1/N$, and for $N = 6$ (dashed line), $N = 8$ (dash-dotted line), and $N = 10$ (solid line). All curves are parameter free. The points are Hewson and Rasul's Bethe-ansatz results: $N = 6$ (filled triangles), and $N = 8$ (filled circles). The proportionality factor between $T_K$ defined by (38) and the $T_1$ appearing in Bethe-ansatz solution is determined for each $N$ by matching the small field gradient of the magnetization.

The last term of (44) can be removed by changing to a new energy scale

$$T_h = 2^{-2/N} T_K \simeq T_K / \left(1 + \frac{2\ln 2}{N}\right). \qquad (45)$$

Although we only explicitly prove the first log term of (44) in Appendix B, we expect that our result (43) will precisely produce all three log terms of (44), since all $1/N$ order contributions to the free energy are included in the present approach. Another direct way to see this is following. Given the second term of (44), the last two terms of (44) are determined by the second term of the weak-coupling $\beta$ function,[10]

$$\beta(g) = \frac{dg}{d\ln D} = -g^2 + \frac{g^3}{N}, \qquad (46)$$

Our expression for the Kondo temperature (38) gives exactly the same beta function. The correct asymptotic form (44) allows unambiguous determination of the energy scale $T_h$ in the present approach. In terms of the unique energy scale $T_h$, the coefficient $\alpha'$ in the strong-coupling asymptotic form of the magnetization

$$\frac{M}{M_0} = \alpha \frac{h}{T_K} = \alpha' \frac{h}{T_h}, \qquad \frac{h}{T_h} \ll 1, \qquad (47)$$

is just the magnetic Wilson crossover number. From (47) and (45), we see $\alpha' = \alpha/(1 + 2\ln 2/N)$. The slope $\alpha$ will be determined directly from $M(h/T_K)$ curve. We list the results for the general SU($N$) cases in Table II.

In summary, we calculated the universal field-deendent magnetization for the general SU($N$) one impurity Kondo model for various values of $N$ and $f$-electron fillings. At both low- and high-field limits, our results become asymptotically exact, as shown analytically in Appendix B. For other than half filling of the $f$-electrons, the magnetization curves cross continuously from one side to the other. In the crossover region, the larger the $N$, the smoother and the more accurate is the magnetization. In contrast to a continuous phase transition, the crossover involves no divergence. The other facet of the story is that one then does need high-order terms to smooth out the crossover for a given $N$.

TABLE II. The calculated magnetic Wilson crossover numbers for the Coqblin-Schrieffer model, $q_0 = 1/N$, defined as $\alpha'$ of (47). With $T_K$ defined by (38), we read off the initial gradient, $\alpha$ in (43), the magnetization curve. Then the crossover number is $\alpha' = \alpha/(1 + 2\ln 2/N)$.

| $N$ | Crossover number | Bethe ansatz |
|---|---|---|
| 2 | 0.25 | 0.342 ($=1/\sqrt{e\pi}$) |
| 4 | 0.65 | |
| 6 | 1.01 | |
| 8 | 1.36 | |
| 10 | 1.70 | |

## ACKNOWLEDGMENTS

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## APPENDIX A: INTEGRALS $\Lambda_1$ AND $\Lambda_2$

For simplicity, we set $T_K = 1$ in this section. From the definition, $\Lambda_2$ is an integral of the type,

$$\Lambda_2(D, \eta_1, \eta_2, \epsilon_f, \Delta) - \Lambda_2(\nu_0, \eta_1, \eta_2, \epsilon_f, \Delta) = \int_{\nu_0}^{D} \frac{d\nu}{\pi} \frac{w \ln \nu + v}{\nu\left(\ln^2 \nu + a \ln \nu + b\right)}, \qquad (A1)$$

where $a$, $b$, $w$, and $v$ are all independent of frequency and are given by

$$a(\epsilon_f, \Delta) = -\pi\eta_2, \qquad (A2)$$

$$b(\epsilon_f, \Delta) = \pi^2 \left(\frac{1}{2} - q_0 - \eta_1\right)^2, \qquad (A3)$$

$$w(\epsilon_f, \Delta) = \Delta(1 - \pi\eta_2), \qquad (A4)$$

$$v(\epsilon_f, \Delta) = -\pi\Delta\eta_2 \left(\frac{1}{2} - \pi\eta_2\right) - \pi\epsilon_f \left(\frac{1}{2} - q_0 - \eta_1\right). \qquad (A5)$$

By carrying out integration, we find

$$\Lambda_2(\nu_0, \eta_1, \eta_2, \epsilon_f, \Delta) = \frac{w}{2\pi} \ln\left(\ln^2 \nu_0 + a \ln \nu_0 + b\right) - \frac{2v - aw}{2\pi} \times \begin{cases} \frac{1}{\sqrt{a^2-4b}} \ln\left(\frac{2\ln\nu_0 + \sqrt{a^2-4b}}{2\ln\nu_0 - \sqrt{a^2-4b}}\right), & a^2 - 4b > 0 \\ \frac{2}{\sqrt{4b-a^2}} \tan^{-1}\left(\frac{\sqrt{4b-a^2}}{2\ln\nu_0 + a}\right), & a^2 - 4b < 0. \end{cases} \qquad (A6)$$

From the definition of $\Lambda_1$, it is an integral of the type

$$D\Lambda_1(D, \eta_1, \eta_2) - \nu_0\Lambda_1(\nu_0, \eta_1, \eta_2) = \int_{\nu_0}^{D} \frac{d\nu}{2\pi} \ln\left(\ln^2 \nu + a \ln \nu + b\right), \qquad (A7)$$

where we choose $\nu_0$ big enough so that the argument of the log function is always positive. We can see that $\Lambda_1(\nu_0, \eta_1, \eta_2)$ is analytic in $a$ and $b$ for small values of $a$ and $b$. In some cases, $\Lambda_1$ can be expressed in terms of the standard integral of exponential functions such as $Ei(x)$. In the present problem, the parameters $a$ and $b$ never get very big. A series expansion is sufficient for the practical purpose. The expression we used in the present calculation is,

$$\pi\Lambda_1(\nu_0,\eta_1,\eta_2) = \left[\frac{P_1(\ln^{-1}\nu_0)}{\ln\nu_0} - \frac{Ei(\ln\nu_0)}{\nu_0}\right]\left(e^{\sqrt{a^2/4-b}} + e^{-\sqrt{a^2/4-b}}\right)e^{-a/2} + \sum_{n=1}^{m}(-1)^{n+1}\frac{P_n(\ln^{-1}\nu_0)}{n\ln^n\nu_0}(\alpha_1^n+\alpha_2^n) + 2\left[\ln\ln\nu_0 - \frac{P_1(\ln^{-1}\nu_0)}{\ln\nu_0}\right], \tag{A8}$$

where $P_n$ are polynomials of $\ln^{-1}\nu_0$,

$$P_n(x) = 1 + nx + n(n+1)x^2 + \cdots + n(n+1)\cdots(m-1)x^{m-n}, \tag{A9}$$

and $\alpha_1$, $\alpha_2$ are related to $a$, $b$ through

$$\alpha_1 + \alpha_2 = a, \qquad \alpha_1\alpha_2 = b. \tag{A10}$$

$Ei(x)$ is the standard integral of exponential function, defined by

$$Ei(x) = \int_{-\infty}^{x}\frac{e^t}{t}dt. \tag{A11}$$

Note that $\alpha_1^n + \alpha_2^n$ are expressed as polynomials of $a$ and $b$. In the expansion (A8), $m$ is the order of expansion. The neglected terms are of the order $[\max(|\alpha_1|,|\alpha_2|)/\ln\nu_0]^{m+1}/m$. Typical values used in our calculation are $m \sim 10 - 15$ and $\ln\nu_0 \sim 5 - 8$.

## APPENDIX B: HIGH FIELD ASYMPTOTICS OF THE MAGNETIZATION

The small-field asymptotic behavior of (47) is the well-known result of the present approach.[15] Here, we prove the high-field asymptotics for $q_0 = 1/2$ and $q_0 = 1/N$. The proof for other values of $q_0$ becomes parallel. We shall set $T_K = 1$ and omit the asterisk in the notation of saddle-point solution $\epsilon_f^*(h)$ and $\Delta^*(h)$.

Let us first consider $q_0 = 1/2$ and even $N$. In the high magnetic field, the $f$-electron level is split into $N$ levels. Each of them is distant from the others. For $q_0 = 1/2$, the $f$-electrons occupy the lowest $N/2$ levels: $\sigma = -S, -S+1, \cdots, -1/2$. The $\sigma = -1/2$ level will lie close to the Fermi level. Spin exchange will result in a small resonant width. Thus, we write the solution in the form

$$\epsilon_f = \frac{h}{2} - \delta\epsilon_f, \qquad \frac{\delta\epsilon_f}{h}, \frac{\Delta}{h} \to 0, \quad \text{as } h \to \infty. \tag{B1}$$

We recall that $S$ is the spin and $N = 2S+1$. Since we are looking for $\ln^{-1}h$ asymptotic terms, we neglect all terms, which die as $h^{-1}$ or faster. Thus,

$$\epsilon_{f\sigma} = \epsilon_f + \sigma h = \begin{cases} \left(\sigma + \frac{1}{2}\right)h, & \sigma \neq -\frac{1}{2}, \\ -\delta\epsilon_f, & \sigma = -\frac{1}{2}. \end{cases} \tag{B2}$$

With this approximation, the magnetization is simplified to

$$M = M_0 - \frac{1}{2\pi}\tan^{-1}\left(\frac{\Delta}{\delta\epsilon_f}\right) - \int_0^D \frac{d\nu}{2\pi}\frac{1}{\Gamma(\nu)}\left[\Gamma_\lambda(\nu)\frac{\partial\Gamma_r}{\partial h} + \Gamma_r(\nu)\frac{\partial\Gamma_\lambda}{\partial h} + 2\Gamma_{\lambda r}(\nu)\frac{\partial\Gamma_{\lambda r}}{\partial h}\right], \tag{B3}$$

where $M_0 = \sum_{\sigma>0}\sigma$, is the saturation value of the magnetization. To shorten the notation, we use the unregularized $1/N$ fluctuation energy (25) to carry out the proof. Since the values for $\delta\epsilon_f$ and $\Delta$ are given by the saddle-point equations (41) and (42), we have to make use of them. With the simplification (B2), Eq. (41) is similarly reduced to

$$-\frac{1}{\pi}\tan^{-1}\left(\frac{\Delta}{\delta\epsilon_f}\right) + \int_0^D \frac{d\nu}{2\pi}\frac{1}{\Gamma(\nu)}\left[\Gamma_\lambda(\nu)\frac{\partial\Gamma_r}{\partial\epsilon_f} + \Gamma_r(\nu)\frac{\partial\Gamma_\lambda}{\partial\epsilon_f} + 2\Gamma_{\lambda r}(\nu)\frac{\partial\Gamma_{\lambda r}}{\partial\epsilon_f}\right] = 0\,. \tag{B4}$$

The matrix element $\Gamma$'s involve the spin component summation $\sum_\sigma$,

$$\Gamma_\lambda(\nu) = \frac{1}{N}\sum_\sigma\Gamma_\lambda^{(\sigma)}(\nu), \qquad \Gamma_r(\nu) = \frac{1}{N}\sum_\sigma\Gamma_r^{(\sigma)}(\nu), \qquad \Gamma_{\lambda r}(\nu) = \frac{1}{N}\sum_\sigma\Gamma_{\lambda r}^{(\sigma)}(\nu)\,.$$

Each spin component $\Gamma^{(\sigma)}$ of the $\Gamma$'s can be read from (18)–(20). The difference between the derivatives of the $1/N$ free energy appearing in (B3) and (B4) is that $\partial/\partial h$ in (B3) will bring down an additional factor $\sigma$ with respect to $\partial/\partial\epsilon_f$. Dividing (B4) by two and subtracting it from (B3), we find

$$M = M_0 - \frac{1}{N}\sum_\sigma\left(\sigma + \frac{1}{2}\right)\int_0^D\frac{d\nu}{2\pi\,\Gamma(\nu)}\left[\Gamma_\lambda(\nu)\frac{\partial\Gamma_r^{(\sigma)}}{\partial\epsilon_f} + \Gamma_r(\nu)\frac{\partial\Gamma_\lambda^{(\sigma)}}{\partial\epsilon_f} + 2\Gamma_{\lambda r}(\nu)\frac{\partial\Gamma_{\lambda r}^{(\sigma)}}{\partial\epsilon_f}\right]. \tag{B5}$$

Note that the $\sigma = -1/2$ component vanishes in the above $\sigma$ summation so we can replace $\epsilon_{f\sigma}$ by $(\sigma + 1/2)h$. Carrying out the derivatives, we find

$$M = M_0 - \frac{1}{\pi N} \sum_{\sigma \neq -\frac{1}{2}} \int_0^D \frac{d\nu}{\Gamma(\nu)} \frac{h(\sigma+\frac{1}{2})^2}{(\sigma+\frac{1}{2})^2 h^2 + (\nu+\Delta)^2} \left\{ \frac{2}{N} \sum_{\mu=-S}^{S} \ln \left[ \frac{\epsilon_{f\mu}^2 + (\nu+\Delta)^2}{\epsilon_{f\mu}^2 + \Delta^2} \right] \right.$$

$$\left. + \frac{\nu}{N(\nu+\Delta)} \sum_{\mu} \ln\left(\epsilon_{f\mu}^2 + \Delta^2\right) + \frac{4(\nu+\Delta)}{Nh(\sigma+\frac{1}{2})} \sum_{\mu} \left[ \tan^{-1}\left(\frac{\epsilon_{f\mu}}{\nu+\Delta}\right) - \tan^{-1}\left(\frac{\epsilon_{f\mu}}{\Delta}\right) \right] \right\}. \quad \text{(B6)}$$

By noting, from Eq. (42),

$$\frac{1}{N} \sum_{\mu=-S}^{S} \ln\left(\epsilon_{f\mu}^2 + \Delta^2\right) \sim \mathcal{O}(1/N),$$

we can expand the expression inside curly bracket of (B6) in $1/N$. We shall also expand $\Gamma(\nu)$,

$$\Gamma(\nu) = \left[ \frac{1}{N} \sum_{\mu=-S}^{S} \ln\left(\epsilon_{f\mu}^2 + (\nu+\Delta)^2\right) \right]^2 + \left\{ \frac{2}{N} \sum_{\mu=-S}^{S} \left[ \tan^{-1}\left(\frac{\epsilon_{f\mu}}{\nu+\Delta}\right) - \tan^{-1}\left(\frac{\epsilon_{f\mu}}{\nu+\Delta}\right) \right] \right\}^2 + \mathcal{O}(1/N). \quad \text{(B7)}$$

By changing the dummy variable, $\nu = h\,x$, we can make following expansion,

$$\frac{1}{N} \sum_{\mu=-S}^{S} \ln\left[\epsilon_{f\mu}^2 + (\nu+\Delta)^2\right] = 2\ln h + \frac{1}{N} \sum_{\mu=-S}^{S} \ln\left[(\tfrac{1}{2}+\mu)^2 + x^2\right]$$

$$= 2\ln h\left[1 + \mathcal{O}(\ln x/\ln h)\right],$$

where we dropped terms of order $\Delta/h$ as usual. That it is possible to make $\ln x/\ln h$ expansion in the last expression is due to the convergence of the integration in (B6). We also expand $\Gamma(\nu)$, given by (B7), in $\ln^{-1} h$ and keep the leading term. The upper integration limit in (B6) can be extended to infinity. The final result for the magnetization is, after some manipulations,

$$\frac{M}{M_0} = 1 - \frac{1}{NM_0} \sum_{\sigma \neq -\frac{1}{2}} \int_0^\infty \frac{dx}{\pi} \frac{\left(\sigma+\frac{1}{2}\right)^2}{\left(\sigma+\frac{1}{2}\right)^2 + x^2} \frac{1}{\ln h}\left[1 + \mathcal{O}(\ln x/\ln h)\right]$$

$$= 1 - \frac{1}{\ln h} \frac{1}{2NM_0} \sum_{\sigma \neq -\frac{1}{2}} |\sigma + \tfrac{1}{2}|$$

$$= 1 - \frac{1}{N\ln h} + \mathcal{O}(\ln^{-2} h). \quad \text{(B8)}$$

For $q_0 = 1/N$, strictly speaking, $1/N$ is no longer the loop expansion parameter. Nevertheless, if we repeat the above steps, we find

$$\frac{M}{S} = 1 - \frac{1}{2\ln h} + \mathcal{O}(\ln^{-2} h). \quad \text{(B9)}$$

Note that the leading log correction is independent of $N$ for $q_0 = 1/N$. It is easy to see this from the perturbation in $g$. This term comes from the linear term, $g/2$, in the $g \ll 1$ perturbation. The diagram for this term involves one conduction electron loop and one $f$-electron loop, which together contribute a factor $N^2$. The interaction vertex brings in a factor $1/N$. After normalization, i.e., dividing by $S \sim N$, it is independent of $N$.

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