Coupled dynamics of fast spins and slow interactions: An alternative perspective on replicas

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It is shown that an Ising spin system in which both spins and interactions evolve in time according to dynamical laws suggested by neural processes, but with widely separated timescales, leads to a thermodynamic equilibrium corresponding to a system of averaged replicas, where the replica number can take any real value determined by the ratio of characteristic temperatures. The resultant phase structure has interesting features which are explored.

In this paper we show that a simple model of coupled dynamics of fast spins and slow interactions, stimulated by considerations of simultaneous learning and retrieval in recurrent neural networks, leads naturally to an effective statistical mechanics characterized by a partition function which is an average over a replicated system. This latter is reminiscent of the replica trick used to consider disordered systems, such as spin glasses, but with the important difference that the number of replicas has a *physical* meaning as the ratio of two characteristic temperatures and can be varied throughout the whole range of real values. We further demonstrate that the model has interesting phase consequences as a function of varying this ratio, and external stimuli, and that it can be extended to a range of other models.

As the basic archetypal model we consider a system of Ising spins $\sigma_i \in \{-1,1\}$, $i \in \{1,\ldots,N\}$, interacting via continuous-valued symmetric exchange interactions J_{ij}, which themselves evolve in response to the states of the spins.

The spins are taken to have a stochastic field-alignment dynamics which is fast compared with the evolution rate of the interactions J_{ij}, such that on the timescale of J_{ij} dynamics, the spins are effectively in equilibrium according to a Boltzmann distribution,

$$P_{\{J_{ij}\}}(\{\sigma_i\}) \propto \exp[-\beta H_{\{J_{ij}\}}(\{\sigma_i\})] , \quad (1)$$

where

$$H_{\{J_{ij}\}}(\{\sigma_i\}) = -\sum_{i<j} J_{ij}\sigma_i\sigma_j , \quad (2)$$

and the subscript $\{J_{ij}\}$ indicates that the $\{J_{ij}\}$ are to be considered as quenched variables. In practice, several specific types of dynamics which obey detailed balance lead to the equilibrium distribution (1), such as a Markov process with single-spin-flip Glauber dynamics.[1] The quantity β is an inverse temperature characterizing the stochastic gain.

For the J_{ij} dynamics we choose the form,

$$\tau\frac{d}{dt}J_{ij} = \frac{1}{N}\langle\sigma_i\sigma_j\rangle_{\{J_{ij}\}} - \mu J_{ij} + \frac{1}{\sqrt{N}}\eta_{ij}(t) \quad (i<j) , \quad (3)$$

where $\langle\cdots\rangle_{\{J_{ij}\}}$ refers to a thermodynamic average over the distribution (1) with the effectively instantaneous $\{J_{ij}\}$, and $\eta_{ij}(t)$ is a stochastic Gaussian white noise of zero mean and correlation

$$\langle \eta_{ij}(t)\eta_{kl}(t')\rangle = 2\tau\tilde{\beta}^{-1}\delta_{(ij),(kl)}\delta(t-t') .$$

The first term on the right-hand side of (3) is inspired by the Hebbian process in neural tissue in which synaptic efficacies are believed to grow locally in response to the simultaneous activity of presynaptic and postsynaptic neurons.[2] The second term acts to limit the magnitude of J_{ij}; $\tilde{\beta}$ is the characteristic inverse temperature of the interaction system. (A related interaction dynamics without the noise term, equivalent to $\tilde{\beta}=\infty$, was introduced by Shinomoto.[3]) Generalizing spin systems by considering the interactions to be slowly time dependent was also proposed by Horner.[4] There are, however, important differences with our approach: we define explicit stochastic dynamical laws for the interactions and arrive at a direct physical interpretation for the replica dimension n of a corresponding spin-glass model in terms of parameters controlling the two dynamical processes. Furthermore, our dynamical laws provide a clear link to neural network theory.

Substituting for $\langle\sigma_i\sigma_j\rangle$ in terms of the distribution (1) enables us to rewrite (3) as

$$N\tau\frac{d}{dt}J_{ij} = -\frac{\partial}{\partial J_{ij}}\mathcal{H}(\{J_{ij}\}) + \sqrt{N}\,\eta_{ij}(t) , \quad (4)$$

where the effective Hamiltonian $\mathcal{H}(\{J_{ij}\})$ is given by

$$\mathcal{H}(\{J_{ij}\}) = -\frac{1}{\beta}\ln Z_\beta(\{J_{ij}\}) + \frac{1}{2}\mu N\sum_{i<j}J_{ij}^2 , \quad (5)$$

where $Z_\beta(\{J_{ij}\})$ is the partition function associated with (2):

$$Z_\beta(\{J_{ij}\}) = \mathop{\mathrm{Tr}}_{\{\sigma_i\}} \exp[-\beta H_{\{J_{ij}\}}(\{\sigma_i\})] .$$

We now recognize (4) as having the form of a Langevin equation, so that the equilibrium distribution of the interaction system is given by a Boltzmann form. Henceforth, we concentrate on this equilibrium state which we can characterize by a partition function $\tilde{Z}_{\tilde{\beta}}$ and an associated free energy $\tilde{F}_{\tilde{\beta}}$:

$$\tilde{Z}_{\tilde\beta} \equiv \int \prod_{i<j} dJ_{ij} [Z_\beta(\{J_{ij}\})]^n \exp\left[-\frac{1}{2}\tilde\beta\mu N \sum_{i<j} J_{ij}^2\right],$$
$$\tilde{F}_{\tilde\beta} \equiv -\tilde\beta^{-1} \ln \tilde{Z}_{\tilde\beta}, \qquad (6)$$

where $n \equiv \tilde\beta/\beta$. We may use $\tilde{Z}_{\tilde\beta}$ as a generating functional to produce thermodynamic averages of state variables $\Phi(\{\sigma_i\};\{J_{ij}\})$ in the combined system by adding suitable infinitesimal source terms to the spin Hamiltonian (2):

$$H_{\{J_{ij}\}}(\{\sigma_i\}) \rightarrow H_{\{J_{ij}\}}(\{\sigma_i\}) + \lambda\Phi(\{\sigma_i\};\{J_{ij}\}),$$
$$\lim_{\lambda\to 0} \frac{\partial \tilde{F}_{\tilde\beta}}{\partial\lambda} = \overline{\langle \Phi(\{\sigma_i\};\{J_{ij}\})\rangle_{\{J_{ij}\}}},$$
$$\equiv \frac{\int \prod_{i<j} dJ_{ij} \langle \Phi(\{\sigma_i\};\{J_{ij}\})\rangle_{\{J_{ij}\}} e^{-\tilde\beta\mathcal{H}(\{J_{ij}\})}}{\int \prod_{i<j} dJ_{ij} e^{-\tilde\beta\mathcal{H}(\{J_{ij}\})}}, \qquad (7)$$

where the bar refers to an average over the asymptotic $\{J_{ij}\}$ dynamics.

The form (6) with $n \to 0$ is immediately reminiscent of the effective partition function which results from the application of the replica trick to replace $\ln Z$ by $\lim_{n\to 0}(1/n)(Z^n - 1)$ in dealing with a quenched average for the infinite-ranged spin glass,[5] while $n=1$ relates to the corresponding annealed average, although we note that in the present model the timescales for spin and interaction dynamics remain completely disparate. These observations correlate with the identification of n with $\tilde\beta/\beta$, which implies that $n \to 0$ corresponds to a situation in which the interaction dynamics is dominated by the stochastic term $\eta_{ij}(t)$, rather than by the behavior of the spins, while for $n=1$ the two characteristic temperatures are the same. For $n \to \infty$, the influence of the spins on the interaction dynamics dominates. In fact, any real n is possible by tuning the ratio between the two β's. In the formulation presented in this paper n is always nonnegative, but negative values are possible if the Hebbian rule of (3) is replaced by an anti-Hebbian form with $\langle\sigma_i\sigma_j\rangle$ replaced by $-\langle\sigma_i\sigma_j\rangle$ (the case of negative n is being studied by Mézard and co-workers[6]).

The model discussed above is range-free (infinite-ranged) and can, therefore, be analyzed in the thermodynamic limit $N \to \infty$ by the replica mean-field theory as devised for the Sherrington-Kirkpatrick (SK) spin glass.[5,7,8] This can be developed precisely for integer n[5,7–9] and analytically continued. In the usual manner there enters a spin-glass order parameter,

$$q^{\gamma\delta} = \overline{\langle \sigma_i^\gamma \sigma_i^\delta \rangle_{\{J_{ij}\}}} \quad (\gamma \neq \delta),$$

where the superscripts are replica labels. $q^{\gamma\delta}$ is given by the extremum of

$$F\left[\{q^{\gamma\delta}\}\right] = -\frac{\tilde\beta}{2\mu n^2}\sum_{\gamma<\delta}[q^{\gamma\delta}]^2 + \ln \mathop{\mathrm{Tr}}_{\{\sigma^\gamma\}} \exp\left[\frac{\tilde\beta}{\mu n^2}\sum_{\gamma<\delta}\sigma^\gamma q^{\gamma\delta}\sigma^\delta\right],$$

while $\tilde{Z}_{\tilde\beta}$ is proportional to $\exp[N \mathrm{extr} F(\{q^{\gamma\delta}\})]$. In the replica-symmetric region (or ansatz) one assumes $q^{\gamma\delta} = q$.

We will choose as the independent variables n and β, taking $\tilde{J} \equiv (\tilde\beta_\mu)^{-1/2} = 1$, and briefly discuss the phase picture of our model (fuller details may be found elsewhere[10]). The system exhibits a paramagnetic ($q=0$) to spin-glass ($q>0$) transition at a critical $\beta_c(n)$. For $n \le 2$ this transition is second order at $\beta_c = 1$, down to the SK spin-glass limit, $n \to 0$, but for $n > 2$ the coupled dynamics leads to a qualitative, as well as quantitative, change to first order. The transition temperature and the corresponding value of the order parameter q are shown in Fig. 1 as a function of n. Replica symmetry is stable above a critical value $n_c(\beta)$, at which there is a de Almeida-Thouless (AT) transition (cf. Kondor[11]). As expected from spin-glass studies, $n_c(\beta)$ goes to zero as $\beta \downarrow 1$ but rises for larger β, having a maximum of order 0.3 at β of order 2. Thus, for $n > n_c(\max) \approx 0.3$ there is no instability against small replica-symmetry-breaking fluctuations, while for smaller n there is reentrance in this stability. The onset of local-replica-symmetric (RS) instability for various temperatures is shown in Fig. 2.

Several simple modifications of the above model are possible. One consists of adding external fields to the spin dynamics and/or to the interaction dynamics, by making the substitutions,

$$H_{\{J_{ij}\}}(\{\sigma_i\}) \rightarrow H_{\{J_{ij}\}}(\{\sigma_i\}) - \sum_i \theta_i\sigma_i,$$
$$\mathcal{H}(\{J_{ij}\}) \rightarrow \mathcal{H}(\{J_{ij}\}) - \sum_{i<j} J_{ij}K_{ij},$$

in (2) and (5), respectively. These external fields may be viewed as generating fields in the sense of (7); for example,

$$-\frac{\partial\tilde{F}}{\partial\theta_i} = \overline{\langle\sigma_i\rangle}, \quad \frac{\partial^2\tilde{F}}{\partial\theta_i\partial\theta_j} = \tilde\beta[\overline{\langle\sigma_i\rangle\langle\sigma_j\rangle} - \overline{\langle\sigma_i\rangle}\,\overline{\langle\sigma_j\rangle}] + \beta[\overline{\langle\sigma_i\sigma_j\rangle} - \overline{\langle\sigma_i\rangle\langle\sigma_j\rangle}],$$
$$-\frac{\partial\tilde{F}}{\partial K_{ij}} = \overline{J_{ij}}, \quad -\frac{\partial^2\tilde{F}}{\partial K_{ij}\partial K_{kl}} = \tilde\beta[\overline{J_{ij}J_{kl}} - \overline{J_{ij}}\,\overline{J_{kl}}].$$

For neural network models, a natural first choice for

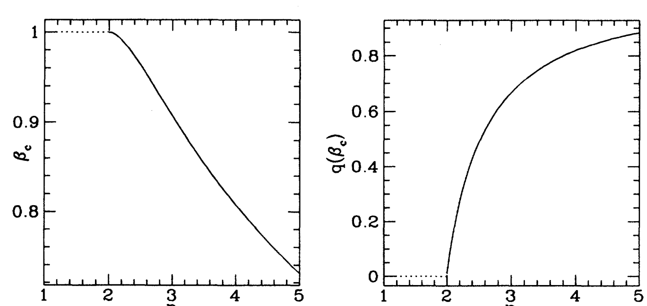

FIG. 1. The P→SG transition for various n, at $J_0=0$, $\tilde{J}=1$. For $n>2$ the transition is first order, but is second order elsewhere. The inverse spin temperature β at the transition is shown on the left; the corresponding value of q (in the SG phase) is indicated on the right.

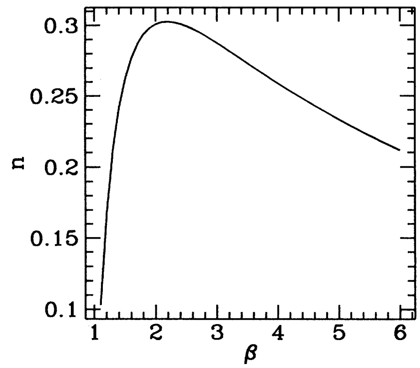

FIG. 2. The AT line as a function of the inverse spin temperature β for $J_0=0$, $\tilde{J}=1$.

the external fields would be $\theta_i \equiv h\xi_i$ and $K_{ij} \equiv K\xi_i\xi_j$, $\xi_i \in \{-1,1\}$, where the ξ_i are quenched random variables corresponding to an imposed pattern. Without loss of generality all the ξ_i can be taken as $+1$, via the gauge transformation $\sigma_i \to \sigma_i\xi_i$, $J_{ij} \to J_{ij}\xi_i\xi_j$. Henceforth, we shall make this choice. The spin-perturbation field h induces a finite magnetization characterized by a new order parameter

$$m^\alpha = \overline{\langle \sigma_i^\alpha \rangle} ,$$

which is independent of α in the replica-symmetric assumption (which turns out to be stable with respect to variation in this parameter). As in the case of the spin glass, there is now a critical surface in (h,n,β) space characterizing the onset of replica-symmetry breaking. In introducing the interaction perturbation field K we find that K/μ is the analog of the mean exchange J_0 in the SK spin-glass model, $\tilde{J}^2 \equiv (\beta n \mu)^{-1}$ being the analog of the variance. If large enough, this field leads to a spontaneous "ferromagnetic" order. Again we find further examples of both second- and first-order transitions (details can be found in Ref 10). For the paramagnetic (P; $m=0$, $q=0$) to ferromagnetic (F; $m\neq 0$, $q\neq 0$) case, the transition is second order at the SK value $\beta J_0=1$, so long as $(\beta\tilde{J})^{-2} \geq 3n-2$. Only when $(\beta\tilde{J})^{-2} < 3n-2$ do the interaction dynamics influence the transition, changing it to first order at a lower temperature. Regarding the ferromagnetic to spin-glass (SG; $m=0$, $q\neq 0$) transition, this exhibits both second-order (lower J_0) and first-order (higher J_0) sections separated by a tricritical point for n less than a critical value of the order of 3.3. This tricritical point exhibits reentrance as a function of n.

A different type of generalization is to consider the whole system as of lower connectivity with only pairs of connected sites being available for interaction upgrade. For example, the system could be on a lattice, in which case the corresponding coupled partition function will have the usual greater complication of a finite-dimensional system, or randomly connected with each bond present with a probability C/N, in which case there results an analog of the Viana-Bray[12] spin glass. In each of these cases the explicit factors involving N in the $\{J_{ij}\}$ dynamics (3) should be removed (their presence or absence being determined by the need for statistical relevance and physical scaling).

Yet another generalization is to higher-order interactions; for example, to p-spin ones:

$$H_{\{J\}}(\{\sigma_i\}) = -\sum_{i_1,\ldots,i_p} J_{i_1,\ldots,i_p}\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_p} ,$$

with corresponding interaction dynamics,

$$\tau\frac{d}{dt}J_{i_1,\ldots,i_p} = \frac{1}{N}\langle \sigma_{i_1}\cdots\sigma_{i_p}\rangle_{\{J\}} - \mu J_{i_1,\ldots,i_p} + \frac{1}{\sqrt{N}}\eta_{i_1,\ldots,i_p}(t) ,$$

or to more complex spin types.

Thus we see that a coupled dynamics of fast spins and slow interactions, in which the evolution of interactions is defined by the chosen stochastic laws, present the replica method from a novel perspective, providing a direct interpretation of the replica dimension n in terms of parameters controlling dynamical processes and leads to new phase transition characters. As a model for neural learning, the example presented here is, however, only a first step, with h and K as introduced corresponding to only a single pattern. Its adaptation to treat many patterns is the next challenge.

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