Persistent current and voltage in a ring of Josephson junctions

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We study a ring of N Josephson junctions with gauge charges induced by external sources and in the presence of a magnetic field. At zero temperature the system is mapped into a standard two-dimensional classical XY model, thus shown to exhibit the Kosterlitz-Thouless transition between the superconducting and the insulating phases, which are characterized by the gauge-invariant phase and charge correlation functions. In particular, the persistent current and voltage in the two phases are obtained, and the duality relation between the strong-coupling limit and the weak-coupling limit is discussed.

I. INTRODUCTION

The Josephson-junction system is conveniently described by the phases of the superconducting order parameters, which represent collective degrees of freedom. In terms of these macroscopic variables, the Hamiltonian of a Josephson junction takes the form of a particle on a circular loop in a periodic potential, suggesting the possibility of macroscopic quantum phenomena.[1] In particular, externally induced gauge charge plays the role of magnetic flux, and leads to the persistent voltage across the junction,[2] which is the counterpart of the persistent current in a mesoscopic metal loop.[3] In the latter, the persistent current is a manifestation of the Aharonov-Bohm effect,[4] which reflects the nonsimply-connected (i.e., circular) geometry of the system in the real space. In contrast, the Josephson junction does not possess such nontrivial geometry in the real space; here the interference effect stems from the phase compactness, which effectively makes the topology circular. It is also of interest to note that the persistent voltage in the Josephson junction may be regarded as a manifestation of the Aharonov-Casher effect[5] which is dual to the Aharonov-Bohm effect resulting in the persistent current.

A simple Josephson-junction system of interest is the *Josephson-junction necklace*, i.e., the ring of N Josephson junctions with self-charging energies. It possesses two nontrivial topologies: the circular geometry in the *real space* in addition to the intrinsic one associated with the phase compactness. Accordingly, the system in the presence of the external magnetic field and induced gauge charge is expected to display both types of interference effects, the Aharonov-Bohm oscillation with the magnetic field and the Aharonov-Casher oscillation with the induced charge. In particular, the weak-coupling limit where the charging energy E_C is dominant over the Josephson coupling energy E_J and the opposite strong-coupling limit have been studied.[6] In both limits the system behaves like a single particle on a circular loop, suggesting the duality between the two limits.

This paper presents detailed study of the Josephson-junction necklace at zero temperature, with both the induced charge and the magnetic field. At zero temperature the system is mapped into a standard two-dimensional (2D) classical XY model, *regardless of the charge and the field.* Therefore, it exhibits the Kosterlitz-Thouless (KT) transition[7,8] between the superconducting and the insulating phases, with $\alpha \equiv E_C/E_J$ taking the role of the temperature. In particular, the critical value of α separating the two phases depends on neither the induced charge nor the magnetic field. We also calculate the gauge-invariant phase and charge correlation functions, which show the expected oscillations as the induced charge or the magnetic field is varied. The superconducting phase then displays algebraic order of phases (and charges), while the insulating phase is characterized by long-range order of charges (and phase disorder). The corresponding persistent current and the voltage carried by the system also display characteristic behaviors: The persistent current shows the periodic oscillation in the superconducting phase, while it is vanishingly small in the insulating phase. Conversely, the persistent voltage is present in the insulating phase but suppressed in the superconducting phase, decaying to zero in the strong-coupling limit ($\alpha \to 0$). Thus the duality between the two phases of the system, separated by the KT transition, is made obvious.

This paper is organized as follows: In Sec. II the Josephson-junction necklace is introduced, and the mapping onto the 2D classical XY model via the standard Feynman path integral[9,10] is briefly reviewed. In particular, it is shown that the induced gauge charge as well as the magnetic field has no effect here. Section III presents the detailed calculations of the gauge-invariant correlation functions. The phase correlation function is obtained with the help of the dual transformation,[8,11] while the charge correlation function can be conveniently obtained through the use of the results of the discrete Gaussian model.[12] Section IV is devoted to the persistent current and voltage, which are closely related to the correlation functions calculated in Sec. III. Finally, a summary and a discussion of the duality are given in Sec. V.

II. JOSEPHSON-JUNCTION NECKLACE

We consider N superconducting grains arranged into a circle, each coupled with its two neighbors via the Josephson junctions of strength E_J. On the kth grain with self-capacitance C, a gauge charge Q_k can be induced externally, e.g., by applying electric potential V_k with respect to the ground, giving $Q_k = CV_k$. In addition, a transverse magnetic field is applied, generating the vector potential along the circle. The Hamiltonian of such a Josephson-junction necklace takes the form

$$H = \frac{1}{2C}\sum_{k=1}^{N}(2ep_k + Q_k)^2 - E_J\sum_{k=1}^{N}\cos(\phi_k - \phi_{k+1} - A_k)\ , \tag{1}$$

where p_k and ϕ_k are the number of Cooper pairs and the phase of the superconducting order parameter (i.e., the macroscopic wave function of the Cooper pairs), respectively, on the kth grain. They are conjugate to each other. In Eq. (1) the intergranular capacitance has been assumed negligible, and the transverse magnetic field gives the bond angle A_k in terms of the line integral of the vector potential $\mathbf{A}$:

$$A_k \equiv \frac{2\pi}{\Phi_0}\int_k^{k+1}\mathbf{A}\cdot dl = 2\pi f\ ,$$

where f denotes the flux per junction in units of the flux quantum $\Phi_0 \equiv 2\pi\hbar c/2e$. Note that the total flux Φ scales with N: $\Phi = Nf\Phi_0$. The first term in Eq. (1) represents the charging energy of the system with the total charge $2ep_k + Q_k$ on the kth grain, while the second term corresponds to the Josephson energy with the gauge-invariant phase difference $\phi_k - \phi_{k+1} - A_k$ across the kth junction.[6,13]

For simplicity, we consider the system with uniform induced charge Q, and write the above Hamiltonian in the form

$$H = \frac{E_C}{2}\sum_k (p_k + q)^2 - E_J\sum_k \cos(\phi_k - \phi_{k+1} - 2\pi f)\ , \tag{2}$$

where $E_C \equiv 4e^2/C$ represents the charging energy scale, and $q \equiv Q/2e$ represents the uniform induced charge in units of $2e$. It is obvious that the Hamiltonian (2) is periodic both in q and in f with periods $q=1$ and $f=1$. Henceforth, the range of q will be limited on the interval $(-\frac{1}{2},\frac{1}{2})$.

We now consider the mapping of the system at zero temperature ($T \equiv \beta^{-1} = 0$) onto the 2D classical XY model, which has been done for a Josephson-junction chain in the absence of the induced charge and the magnetic field.[9,10] The generalization to the necklace with the field and charge is straightforward, and will be described briefly. Following Ref. 10, we introduce the imaginary time τ running on the interval $[0,\beta]$, and express the partition function $Z \equiv \mathrm{Tr}\, e^{-\beta H}$ in terms of the Feynman path integral:

$$Z = \left[\prod_{i,j}\int_0^{2\pi} d\phi_{ij}\right]\prod_{i=1}^{N}\prod_{j=1}^{N_\tau}\langle \phi_{i,j+1}| e^{-(E_C/2\Delta)(p_i+q)^2}|\phi_{ij}\rangle \exp\left[\frac{E_J}{\Delta}\sum_{i,j}\cos(\phi_{ij} - \phi_{i+1,j} - 2\pi f)\right]\ , \tag{3}$$

where the τ axis has been divided into N_τ steps of length Δ^{-1}, the indices i and j represents the position on the 2D x-τ lattice of size $N \times N_\tau$, and the complete sets of phase eigenstates $|\phi_{ij}\rangle$ have been inserted at each division. Here the cutoff energy Δ should be chosen sufficiently larger than both E_C and E_J. In a real system, each grain will become no more superconducting as the temperature is increased beyond T_{BCS}, which is of the order of the zero-temperature BCS gap.[14] Therefore, the Hamiltonian in Eq. (2) is valid only for the energy scales E_C and E_J sufficiently smaller than the gap, and it is natural to have the gap take the role of the cutoff Δ. The matrix element in Eq. (3) can be evaluated by inserting the complete sets of momentum eigenstates, which leads to

$$\langle \phi_{i,j+1}| e^{-(E_C/2\Delta)(p_i+q)^2}|\phi_{ij}\rangle = \sum_{\{S_{\mathbf{r},\mathbf{r}+\hat{\mathbf{y}}}\}}\exp[V_y(S_{\mathbf{r},\mathbf{r}+\hat{\mathbf{y}}} + q) + iS_{\mathbf{r},\mathbf{r}+\hat{\mathbf{y}}}(\phi_{\mathbf{r}} - \phi_{\mathbf{r}+\hat{\mathbf{y}}})]\ , \tag{4}$$

where $S_{\mathbf{r},\mathbf{r}+\hat{\mathbf{y}}}$ is an integer variable representing the momentum eigenvalue at site $\mathbf{r} \equiv (x,y) \equiv (i,j)$ of the x-τ lattice and $V_y(S) \equiv -(E_C/2\Delta)S^2$. Note that without induced charge ($q=0$), Eq. (4) would be the Villain form of the cosine action along the τ direction. In fact the induced charge cancels out in the partition function, and accordingly has no effect on the phase transition in the system. To see this, we expand the Josephson term in Eq. (3) as a Fourier series:

$$\exp\left[\frac{E_J}{\Delta}\cos(\phi_{ij} - \phi_{i+1,j} - 2\pi f)\right] = \sum_{\{S_{\mathbf{r},\mathbf{r}+\hat{\mathbf{x}}}\}}\exp[V_x(S_{\mathbf{r},\mathbf{r}+\hat{\mathbf{x}}}) + iS_{\mathbf{r},\mathbf{r}+\hat{\mathbf{x}}}(\phi_{\mathbf{r}} - \phi_{\mathbf{r}+\hat{\mathbf{x}}} - 2\pi f)]\ , \tag{5}$$

where the Fourier component $e^{V_x(S)}$ is in general given by a Bessel function of imaginary argument. The partition function in Eq. (3) then takes the form

$$Z = \left[\prod_{\mathbf{r}}\int_0^{2\pi} d\phi_{\mathbf{r}}\right]\prod_{\langle \mathbf{r}\mathbf{r}'\rangle}\sum_{\{S_{\mathbf{r}\mathbf{r}'}\}}\exp[iS_{\mathbf{r}\mathbf{r}'}(\phi_{\mathbf{r}} - \phi_{\mathbf{r}'} - 2\pi f_{\mathbf{r}\mathbf{r}'}) + V(S_{\mathbf{r}\mathbf{r}'} + q_{\mathbf{r}\mathbf{r}'})]\ , \tag{6}$$

where $f_{\mathbf{rr}'} \equiv f\delta_{\mathbf{r}',\mathbf{r}+\hat{\mathbf{x}}}$, $q_{\mathbf{rr}'} \equiv q\delta_{\mathbf{r}',\mathbf{r}+\hat{\mathbf{y}}}$, and

$$V(S_{\mathbf{rr}'}+q_{\mathbf{rr}'}) \equiv V_x(S_{\mathbf{rr}'})\delta_{\mathbf{r}',\mathbf{r}+\hat{\mathbf{x}}}+V_y(S_{\mathbf{rr}'}+q)\delta_{\mathbf{r}',\mathbf{r}+\hat{\mathbf{y}}} .$$

For convenience, site $\mathbf{r}'$ has been always chosen to be either right of or above site $\mathbf{r}$.

The integral over phases in Eq. (6) can be performed as usual to yield the zero-divergence condition for $S_{\mathbf{rr}'}$, which can be met by the dual transformation:[8] $S_{\mathbf{rr}'}=S_{\mathbf{R}}-S_{\mathbf{R}'}$ with the dual lattice site $\mathbf{R}'$ being either right of or below $\mathbf{R}$. We thus obtain the partition function in the form

$$Z=\sum_{\{S_{\mathbf{R}}\}} \exp\left\{ \sum_{\langle \mathbf{RR}'\rangle} [\,V(S_{\mathbf{R}}-S_{\mathbf{R}'}+q_{\mathbf{RR}'}) -i2\pi f_{\mathbf{RR}'}(S_{\mathbf{R}}-S_{\mathbf{R}'})]\right\} , \qquad (7)$$

with $q_{\mathbf{RR}'} \equiv q\delta_{\mathbf{R}',\mathbf{R}+\hat{\mathbf{x}}}$ and $f_{\mathbf{RR}'} \equiv f\delta_{\mathbf{R}',\mathbf{R}-\hat{\mathbf{y}}}$. In the argument of the exponential function in Eq. (7), q appears only in the terms with $\mathbf{R}'=\mathbf{R}+\hat{\mathbf{x}}$ for which

$$\begin{aligned} V(S_{\mathbf{R}}-S_{\mathbf{R}'}+q_{\mathbf{RR}'}) &= V_y(S_{\mathbf{R}}-S_{\mathbf{R}'}+q_{\mathbf{RR}'}) \\ &= -(E_C/2\Delta)(S_{\mathbf{R}}-S_{\mathbf{R}'}+q_{\mathbf{RR}'})^2 . \end{aligned}$$

Therefore we split off q-dependent terms, and express the first term in the argument as

$$V(S_{\mathbf{R}}-S_{\mathbf{R}'}+q_{\mathbf{RR}'})=V(S_{\mathbf{R}}-S_{\mathbf{R}'})-\frac{E_C}{\Delta}q_{\mathbf{RR}'}(S_{\mathbf{R}}-S_{\mathbf{R}'}) ,$$

where the q-dependent term obviously vanishes upon summing over $\mathbf{R}$ and $\mathbf{R}'$. Likewise, the (imaginary) f-dependent term in Eq. (7) also vanishes upon summing over $\mathbf{R}$ and $\mathbf{R}'$.[15] This allows us to set $q=f=0$ in Eq. (7), which then may be identified with the partition function of the 2D classical XY model with anisotropic coupling, E_J/Δ in the x direction and Δ/E_C in the τ direction. Such anisotropy may be removed by rescaling the τ axis by the factor $\sqrt{E_C E_J}$,[9,10] which leads to the partition function of an isotropic 2D XY model

$$Z=\left[\prod_{\mathbf{r}}\int_0^{2\pi} d\phi_{\mathbf{r}}\right]\exp\left[K\sum_{\langle \mathbf{r},\mathbf{r}'\rangle}\cos(\phi_{\mathbf{r}}-\phi_{\mathbf{r}'})\right] \qquad (8)$$

with the effective coupling given by $K\equiv\sqrt{E_J/E_C}$. Thus we conclude that the system at zero temperature undergoes the KT transition from the superconducting phase to the insulating phase, driven by quantum fluctuations. The critical value of $\alpha\equiv E_C/E_J$ beyond which the system is no more superconducting is given by $\alpha_c=\pi^2/4$ if the vortex core energy is sufficiently large.

The irrelevance of the induced charge as well as the magnetic field to the superconductor-insulator transition in the necklace can be understood in the following way: The uniform bond angle $2\pi f$ due to the magnetic field in the necklace leads to the same uniform bond angle in the x direction of the corresponding 2D x-τ lattice, while the uniform induced charge q gives again uniform (imaginary) bond angle.[15] Therefore, the plaquette sum of the (complex) bond angles on the x-τ lattice obviously vanishes, and there is no frustration in the system.[16] Although the induced charge and the magnetic field do not affect the phase transition, they lead to the gauge-invariant correlation functions, and generate the persistent current and voltage, which will be the subject of the next sections.

III. GAUGE-INVARIANT CORRELATION FUNCTIONS

In this section, we compute the phase and charge correlation functions, and investigate their behavior. Such correlation functions have been obtained in the absence of the magnetic field and the induced charge.[10] Here we pay attention to the gauge-invariant correlation functions in the presence of the field and charge. Unlike ordinary ("non-gauge-invariant") correlation functions, they manifest the effects of gauge-invariant quantities such as the magnetic flux and the induced charge. It is again convenient to use the known results for the isotropic 2D XY model.[8]

The phase correlation function defined by

$$g_n(\mathbf{r}-\mathbf{r}')\equiv\langle e^{in(\phi_{\mathbf{r}}-\phi_{\mathbf{r}'})}\rangle \qquad (9)$$

with $\langle\cdots\rangle$ denoting the ensemble average, may be written in the form

$$g_n(\mathbf{r}-\mathbf{r}') =\frac{1}{Z}\sum_{\{S_{\mathbf{R}}\}}\exp\left[\sum_{\langle \mathbf{RR}'\rangle}V(S_{\mathbf{R}}-S_{\mathbf{R}'}+q_{\mathbf{RR}'}-n\eta_{\mathbf{RR}'})\right] , \qquad (10)$$

where $\eta_{\mathbf{RR}'}$ is nonzero only if the path from $\mathbf{r}\equiv(x,y)$ to $\mathbf{r}'\equiv(x',y')$ cuts the dual link $\langle \mathbf{RR}'\rangle$:

$$\eta_{\mathbf{RR}'}=\delta_{\mathbf{R}',\mathbf{R}+\hat{\mathbf{x}}}+\delta_{\mathbf{R}',\mathbf{R}-\hat{\mathbf{y}}}-\delta_{\mathbf{R}',\mathbf{R}-\hat{\mathbf{x}}}-\delta_{\mathbf{R}',\mathbf{R}+\hat{\mathbf{y}}} .$$

We note the (isotropic) Villain action $V(S)=-S^2/2K$, and write

$$\sum_{\langle \mathbf{RR}'\rangle}V(S_{\mathbf{R}}-S_{\mathbf{R}'}+q_{\mathbf{RR}'}-n\eta_{\mathbf{RR}'}) =\sum_{\langle \mathbf{RR}'\rangle}V(S_{\mathbf{R}}-S_{\mathbf{R}'}-n\eta_{\mathbf{RR}'})-n\frac{q}{K}(y-y') ,$$

where the irrelevant constant term has been omitted. Therefore, we obtain

$$g_n(\mathbf{r}-\mathbf{r}')=e^{-n(q/K)(y-y')}g_n^0(\mathbf{r}-\mathbf{r}') ,$$

or

$$g_n(r)=e^{-nq\sqrt{\alpha}y}g_n^0(r) , \qquad (11)$$

where $g_n^0(r)$ is the correlation function in the absence of the induced charge.[17] Therefore the correlation function in the necklace corresponding to the "equal-time" ($y=0$) correlation function in Eq. (11) is simply given by

$$g_n(x)=g_n^0(x)\sim\begin{cases} x^{-\eta(\alpha)}, & \alpha<\alpha_c \\ e^{-x/\xi(\alpha)}, & \alpha>\alpha_c , \end{cases} \qquad (12)$$

displaying the well-known algebraic decay for $\alpha>\alpha_c$.[7,8]

The gauge-invariant phase correlation function in the presence of the magnetic field is given by

$$G_n(\mathbf{r}-\mathbf{r}')\equiv \mathrm{Re}\left\langle \exp\left[in\left[\phi_{\mathbf{r}}-\phi_{\mathbf{r}'}-\sum_{\Gamma} A_{kl}\right]\right]\right\rangle$$
$$=\cos\left[n\sum_{\Gamma} A_{kl}\right] g_n(\mathbf{r}-\mathbf{r}') , \quad (13)$$

where $\sum_\Gamma A_{kl}$ represents the sum of the bond angle over the path $\Gamma(\mathbf{r},\mathbf{r}')$ from $\mathbf{r}$ to $\mathbf{r}'$, and in general depends on the path Γ chosen.[11] In the necklace system, however, we have $A_{\mathbf{r},\mathbf{r}'}=2\pi f(\delta_{\mathbf{r}',\mathbf{r}+\hat{\mathbf{x}}}-\delta_{\mathbf{r}',\mathbf{r}-\hat{\mathbf{x}}})$, and consequently,

$$\sum_{\Gamma} A_{kl}=2\pi f(x'-x) ,$$

which is *path independent.* The gauge-invariant correlation function thus reads

$$G_n(x)=\cos(2\pi n f x) g_n^0(x) , \quad (14)$$

and exhibits periodic behavior as a function of the magnetic flux.

We next consider the charge correlation function. The gauge-invariant form is given by

$$C_n(k-l)\equiv \mathrm{Re}\langle e^{in\sum_{i<k}(p_i+q)} e^{-in\sum_{i<l}(p_i+q)}\rangle$$
$$=\mathrm{Re}[e^{inq(k-l)}c_n(k-l)] , \quad (15)$$

where $c_n(k-l)$ is the (noninvariant) correlation function. In the charge representation, $c_n(k-l)$ takes the form

$$c_n(k-l)\equiv\langle e^{in\sum_{i=l}^{k-1}p_i}\rangle$$
$$=\frac{1}{Z}\sum_{\{S_{\mathbf{R}}\}}\exp\left\{\sum_{\langle\mathbf{RR}'\rangle}V(S_{\mathbf{R}}-S_{\mathbf{R}'}+q_{\mathbf{RR}'})\right\}$$
$$\times\exp[in(S_{\mathbf{R}}-S_{\mathbf{R}+(k-l)\hat{\mathbf{x}}})] . \quad (16)$$

Since we again have

$$\sum_{\langle\mathbf{RR}'\rangle}V(S_{\mathbf{R}}-S_{\mathbf{R}'}+g_{\mathbf{RR}'})=\sum_{\langle\mathbf{RR}'\rangle}V(S_{\mathbf{R}}-S_{\mathbf{R}'}) ,$$

we conclude that $c_n(x)$ is the same as $c_n^0(x)$, the correlation function in the absence of the induced charge. The latter has been shown to be given by[10]

$$c_n^0(x)=\exp\left[-\frac{n^2}{2}G_D(x)\right] ,$$

where $G_D(x)$ is the height correlation function in the discrete Gaussian model.[12] Thus $c_n(x)$ as well as $c_n^0(x)$ decays algebraically with x for $\alpha<\alpha_c$, while it reaches a finite value for $\alpha>\alpha_c$. The gauge-invariant charge correlation function in Eq. (15) now obtains the form

$$C_n(x)=\cos(nqx)c_n^0(x) , \quad (17)$$

which displays oscillatory behavior as the induced charge is varied.

The behaviors of the correlation functions display the competition between phase ordering and charge ordering in the system. For $\alpha<\alpha_c$, we have algebraic order of phases, while charge order is suppressed. On the other hand, for $\alpha>\alpha_c$, quantum fluctuations are large enough to destroy phase order and to set up long-range order of charges. This behavior is determined solely by $g^0(x)$ and $c^0(x)$, and the induced charge as well as the magnetic field does not play a role, which reflects the absence of frustration in the system. However, the gauge-invariant correlation functions, which naturally depend on the (gauge-invariant) magnetic flux and induced charge, reveal additional features, showing characteristic oscillations. In particular, the duality in the system with the corresponding persistent current and voltage is manifested as we shall see in the next section.

IV. PERSISTENT CURRENT AND VOLTAGE

In the external magnetic field, the energy of the necklace becomes periodic in the magnetic flux f, and a persistent current is induced in the system. Similarly, the gauge charge applied externally makes the energy periodic in the charge q, thus generating a persistent voltage with respect to the ground. The persistent current and voltage, which are closely related to the correlation functions obtained in Sec. III, also show characteristic behaviors in the appropriate limits.

The current carried by the system is given by the derivative of the energy with respect to f:

$$I=\frac{e}{2\pi\hbar}\left\langle\frac{\partial H}{\partial f}\right\rangle=-\frac{2e}{\hbar}E_J\langle\sin(\phi_k-\phi_{k+1}-2\pi f)\rangle , \quad (18)$$

which is simply the supercurrent through the Josephson junctions. Thus the current in the system is given by the imaginary part of $\langle e^{i(\phi_k-\phi_{k+1}-A_{k,k+1})}\rangle$, the real part of which reduces to the gauge-invariant phase correlation function between nearest neighboring grains, $G_{n=1}(x=1)$. This leads to the desired form of the current:

$$I=\frac{2eE_J}{\hbar}g_1\sin(2\pi f) , \quad (19)$$

where $g_1\equiv g_{n=1}^0(x=1)$ is the phase correlation function between nearest-neighboring grains. For $\alpha<\alpha_c, g_1$ retains a finite value, while it becomes exponentially small as α is increased beyond α_c. The system carries a persistent current which is periodic in f in the superconducting phase ($\alpha<\alpha_c$) as expected. On the other hand, the persistent current is strongly suppressed in the insulating phase.

We next consider the potential of a grain to the ground. It is related to the time derivative of the phase (the "velocity") via the Josephson relation, and accordingly, given by the derivative of the energy with respect to q:

$$V=\frac{1}{2e}\left\langle\frac{\partial H}{\partial q}\right\rangle=\frac{E_C}{2e}\langle p_k+q\rangle , \quad (20)$$

which is simply the voltage on a capacitor C due to charge $2e(p_k+q)$. It is straightforward to calculate $\langle p_k\rangle$ in Eq. (20), which is related to the correlation function in the τ direction:

$$\langle p_k \rangle = -iK\langle \sin(\phi_{kj}-\phi_{k,j+1})\rangle = \frac{K}{2}[g(y=1)-g(y=-1)] ,$$

where $g(y=\pm 1)\equiv g_{n=1}(x=0,y=\pm 1)$ is the phase correlation function between nearest neighbors along the τ direction. Using Eq. (11) and $K\equiv\alpha^{-1/2}$, we obtain the voltage in the form

$$V=\frac{E_C}{2e}\left[q-\frac{g_1}{\sqrt{\alpha}}\sinh(q\sqrt{\alpha})\right] . \tag{21}$$

for $-\frac{1}{2}<q<\frac{1}{2}$. (Note that V as well as H is periodic in q.) In the insulating phase, the correlation function g_1 is small, and the system develops persistent voltage. In the superconducting phase, on the other hand, g_1 is finite, suppressing the voltage. As α is decreased to zero, phase order reaches its maximum ($g_1\rightarrow 1$), leaving no voltage in the system.

In summary, the system in the presence of the magnetic field and gauge charge displays the characteristic current and voltage, which are periodic in the magnetic flux and in the gauge charge, respectively. While the superconducting phase is manifested by persistent currents with the voltage strongly suppressed, persistent voltage is developed with negligible currents in the insulating phase.

V. CONCLUSIONS

The Josephson-junction necklace is simple enough to allow analytical investigation. Yet it exhibits not only a phase transition driven by quantum fluctuations but also additional interesting features associated with the two nontrivial topologies present. In particular, the system in the presence of the external magnetic field and induced gauge charge displays both types of interference effects, the Aharonov-Bohm oscillation with the magnetic field and the Aharonov-Casher oscillation with the induced charge.

We have presented detailed study of the Josephson-junction necklace at zero temperature, with both the induced charge and the magnetic field. The system has been shown to map into a two-dimensional XY model, regardless of the charge and the field. Thus a superconductor-insulator transition of the Kosterlitz-Thouless type has been concluded, which depends on neither the induced charge nor the magnetic field. We have also calculated the gauge-invariant phase and charge correlation functions, which not only characterize the superconducting and insulating phases but also display the Aharonov-Bohm and the Aharonov-Casher oscillations as the magnetic field or the induced charge is varied. The corresponding current and the voltage carried by the system are found to display characteristic behaviors: The persistent current shows the periodic oscillation in the superconducting phase, while it is vanishingly small in the insulating phase. Conversely, the persistent voltage is present in the insulating phase but suppressed in the superconducting phase.

The current and the voltage in the system have simple limiting behaviors in the strong-coupling limit ($\alpha\rightarrow 0$) and in the weak-coupling limit ($\alpha\rightarrow\infty$). In the weak-coupling limit where the charging energy is dominant over the Josephson coupling energy, we have the persistent voltage $V=(E_C/2e)q$ without a persistent current, manifesting the insulating phase. This linear behavior with respect to q reflects that the system reduces to a free particle on a circular loop. In the opposite strong-coupling limit, the voltage decays to zero, while the persistent current is given by $I=(2e/\hbar)E_J\sin(2\pi f)$. Recall that f has been defined to be the number of flux quanta per junction, and scales like N^{-1}. For large N, therefore, the current takes the form $I=(2eE_J/\hbar)2\pi f$, which again reflects that the system behaves like a free particle. Thus, the two limits can be mapped into each other, since in both limits the necklace of sufficiently many junctions reduces to a free particle on a circular loop. This behavior, which stems from the two nontrivial topologies of the necklace, suggests the duality present in the system.[6] In particular, the insulating phase carrying persistent voltage apparently maps into the superconducting phase carrying a persistent current, with the correspondence $q\leftrightarrow 2\pi f$.

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[15]It is of interest to note that the action in Eq. (7) can be written in the form

$$\sum_{\langle \mathbf{RR}' \rangle} [V(S_{\mathbf{R}}-S_{\mathbf{R}'})-i(2\pi f_{\mathbf{RR}'}-iE_C q_{\mathbf{RR}'}/\Delta)(S_{\mathbf{R}}-S_{\mathbf{R}'})] ,$$

which shows that the induced charge q corresponds to the *imaginary* bond angle $iE_C q/\Delta$ in the τ direction.
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