Classification of N-electron states in a quantum dot

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The physical symmetries of an N-electron gas in a d-dimensional quantum dot are studied. The symplectic group framework is used to discuss the dynamical modes of the system and to classify the N-electron states including spin. We focus on the results for a few-electron ($N \leq 12$) dot in $d=2$ dimensions. For $N<4$ electrons the types of configurations appearing are strongly N dependent. The allowed configurations for $N<4$ are also highly restricted compared to those for $N \geq 4$. These findings suggest that results of few-electron ($N<4$) cluster calculations may have limited applicability to large-N systems.

I. INTRODUCTION

In recent years there has been much progress in the fabrication of so-called quantum devices[1] with physical characteristics that derive directly from the quantum behavior of an N-electron system. Such systems may involve small or large numbers of electrons as well as being confined in one or more dimensions, and they possess properties that have the potential for application in future electronic technology. There is now a great deal of data emerging from transport and optical measurements on quantum dots containing anywhere between 1 and 100 electrons.[2] Attributing the various experimental peaks to a specific number of electrons requires full knowledge of the N-electron energies of the quantum dot. At the present time this is an unsolved problem for arbitrary N, although there have been numerical studies of the few-electron ($N \leq 4$) quantum dot,[3,4] an analytical solution of an N-electron quantum dot in a magnetic field with a harmonic potential interaction,[5] and a discussion of the $N=2$ "helium" dot.[6] Given the wide possible variation of N values in a given quantum dot experiment, it would be a significant advance if some "signatures" could be obtained whereby one would know that the occurrence of a given multiplet of conduction peaks in the experimental data signaled that the quantum dot contained $6 \leq N \leq 8$ electrons, as opposed to $2 \leq N \leq 4$.

This paper takes a first step towards this goal by providing a classification scheme for the wide variety of possible N-particle states in a quantum dot with parabolic confinement. Unlike previous classification schemes[7] used early on in nuclear and atomic physics, the states can be labeled using Schur function techniques which are valid for any N. In this way, the appearance (or disappearance) of various configurations can be charted as a function of N. We find that the types of configurations appearing varies nontrivially as N is increased. Indeed, in $d=2$ dimensions it is not until $N \geq 4$ that the types of states occurring become fairly N independent. This finding therefore injects an element of caution into interpolation between the results obtained by diagonalizing few-electron systems and the true physics of the large-N system.

In the description of the N-electron quantum system there is a strong resemblance to the N-body nuclear system of protons and neutrons which we want here to exploit. In nuclear physics, there exists a large body of literature describing the collective modes of the N-nucleon system but also establishing a microscopic description based on the Heisenberg algebra (see, for example, Moshinsky,[8,9] Kramer and Moshinsky,[10] Rowe,[11] Vanagas,[12] Filippov, Chovposky, and Vasilevsky,[13] Arima and Iachello[14] and their respective collaborators). Such a description in terms of the Heisenberg algebra is called a *microscopic* model. The advantage of such a microscopic model is that it contains various physically appealing submodels, such as the shell model, collective model, and cluster model. The basic idea of the shell model[8,15] is that as a first approximation the N-particle states can be built from independent single-particle states in some common potential, usually taken in practical applications as a harmonic-oscillator potential. In this sense, the shell model resembles a mean-field theory. In confined systems of electrons, the two competing energy scales are the kinetic energy of the particles (which scales as the inverse square of the confinement length) and the electron-electron interaction energy (which scales as the inverse of the electron separation for a Coulombic interaction). In systems where the confinement length scale is small (e.g., atoms), the kinetic energy tends to dominate. Perturbative treatments of the electron-electron energy, such as mean-field approximations or central-field potentials which underlie the shell model, are therefore reasonable and have had much success. However in the mesoscopic quantum dot systems, the kinetic energy and potential energy are comparable[16] and the shell-model approach is therefore too crude. The collective model, on the other hand, views the collection of particles as a liquid droplet described by various phenomenological, hydrodynamic parameters.[17] The cluster model approach correlates subsets of the N particles, as in interacting boson approximation models.[14] Typical exam-

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ples of the latter in condensed-matter physics are Cooper pairing in superconductivity, and electron-hole pairing in excitonic systems. A common framework for all these models is provided by studying the symmetry groups of the N-particle system. The present work transports these ideas of nuclear physics to the realm of N-electron quantum dots.

Based on the generalized Kohn theorem introduced in Refs. 18 and 3, recent far-infrared optical measurements[19] have shown the confinement potential in electrostatically defined quantum dots to be nearly quadratic.[20] Additional terms in the Hamiltonian arising from a uniform external magnetic field will also be of bilinear form. Physically meaningful Hamiltonians can therefore be constructed from bilinear operators (e.g., XX where X is the position coordinate operator). A Lie algebra based on such bilinear operators generates the symplectic group. This fact, coupled with the success of the symplectic group in describing collective nuclear behavior,[21,22] implies that the symplectic group $\mathrm{Sp}(2Nd,R)$ is an appropriate dynamical group for a many-body theory of N electrons in a d-dimensional quantum dot. It should be noted that there is an important difference between a *dynamical* group and a *symmetry* group.[23] A symmetry group of transformations leaves the Hamiltonian invariant and gives rise to degenerate multiplets of states which carry representations of the group. Examples of this are the rotation group SO(3) and the spin group SU(2). A dynamical group, on the other hand, requires only that energy eigenstates belong to a single irreducible representation of the group but does not require that all states of an irreducible representation be degenerate. A familiar example is the simple three-dimensional harmonic oscillator for which the dynamical group is $\mathrm{Sp}(6,R)$ while the symmetry group is its SU(3) subgroup.[24]

The advantage of dealing directly with the dynamical group, as we will in this paper, is that its algebra can be chosen to include not only the operators appearing in the Hamiltonian, but also various physically meaningful operators (including certain forms of anharmonic potential terms and particle-particle interactions[25]) which would usually have to be accounted for in perturbation theory. A dynamical group will contain the symmetry group of the Hamiltonian as a subgroup, and will lie above it in the group chain. The basic philosophy is therefore to classify states of the many-body system using labels obtained by looking down the various group-subgroup chains, starting with the dynamical group at the top. Figure 1 shows the various group-subgroup chains of interest in the present problem.

We begin by formulating in Sec. II a microscopic model of an N-particle system in algebraic terms, and then discussing the algebra of the dynamical group which is the noncompact Lie algebra of $\mathrm{Sp}(2Nd,R)$.[8] As mentioned above, physical quantities associated with the system, e.g., total kinetic energy, total angular momentum, simple types of particle-particle interaction, external confining potential and magnetic field, can be expressed in terms of this algebra. Physically relevant subalgebras can then be identified and the N-particle states classified

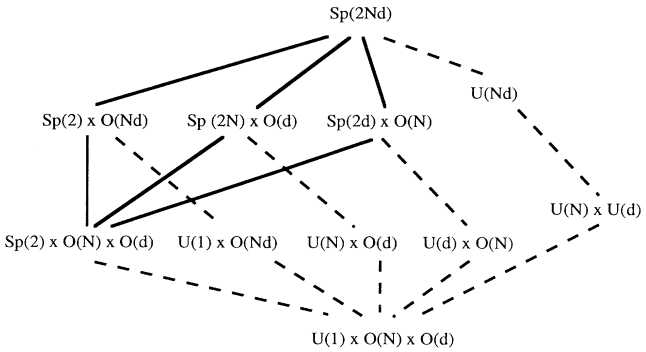

FIG. 1. The group-subgroup chains of an N-particle system in d dimensions. For simplicity we denote $\mathrm{Sp}(2n,R)$ as $\mathrm{Sp}(2n)$.

in terms of the group representations. It can be seen from the subgroup chains in Fig. 1 that one appropriate subalgebra is that of $\mathrm{Sp}(2,R)\times\mathrm{O}(N)\times\mathrm{O}(d)$. The $\mathrm{O}(d)$ group describes the angular momentum of the system, and $\mathrm{O}(N)$ contains the permutation symmetry of the states via the subgroup $\mathrm{S}(N)\subset\mathrm{O}(N)$. Various intermediate subalgebras between the $\mathrm{Sp}(2Nd,R)$ and $\mathrm{Sp}(2,R)\times\mathrm{O}(N)\times\mathrm{O}(d)$ are possible (see Fig. 1). The relative values of the system parameters, e.g., strength of the magnetic field, will determine the appropriate chain of groups and, hence, basis states for describing the system. The microscopic model, and, hence, the N-particle system itself, is then characterized as unitary *irreducible representations* (irreps) of its Lie algebra and subalgebras. However, apart from the trivial one-dimensional representations, all unitary irreps of noncompact Lie groups are necessarily infinite dimensional. A complete classification scheme of such irreps is not yet available although there exists an extensive mathematical literature on certain types of unitary irreps.[26–28] The symplectic group representations we require here are the unitary discrete series irreps. In the Appendixes we briefly discuss the representation theory of such irreps employing the powerful Schur function techniques[29,30] which have the advantage of being valid for any N or d.

In Secs. III and IV, we apply the various group-theoretic results to the physical system of N-interacting particles in d dimensions, specifically classifying the states of the few-electron dot in $d=2$ dimensions. In order to implement the Pauli principle, we need spatial states of well-defined permutational symmetry. This information is contained in the $\mathrm{O}(N)$ group and its reduction to $\mathrm{S}(N)$. We analyze this group reduction in Appendix A. We note that in the process of presenting a classification of states of the N-electron quantum dot, this paper also quotes group-theoretic results. The derivation of these results is presented elsewhere.[31]

II. DYNAMICAL GROUP OF THE N-PARTICLE SYSTEM

In this section we formally develop the algebras that generate the dynamical group of the system of N-interacting particles. The spatial part of the Hamiltonian

of the N-particle system is a function of coordinate and momentum operators of these individual particles. We denote the coordinate and momentum components of the rth particle ($r=1,\ldots,N$) by x_{ri} and p_{ri} ($i=1,\ldots,d$), respectively. The space dimension is d where $d=1$, 2, or 3. The associated operators X_{ri} and P_{ri} obey the Heisenberg algebra,

$$[X_{ri},X_{sj}]=0,\quad [X_{ri},P_{sj}]=i\hbar I\delta_{rs}\delta_{ij}\ ,\quad [P_{ri},P_{sj}]=0 \tag{1}$$

with N distinct, commuting copies of the one-particle Heisenberg algebra. So far the particles are distinguishable and their particle and spatial indices are treated in the same manner. This reflects the fact that the spatial part of an N-particle system in d dimensions behaves as Nd quanta in one dimension. The $(2Nd)^2$-dependent set of bilinear operators, $\{X_{ri},X_{sj},X_{ri}P_{sj},P_{ri},X_{sj},P_{ri}P_{sj}\}$, can be shown to close under commutation. Since $P_{ri}X_{sj}=X_{sj}P_{ri}-i\hbar I\delta_{rs}\delta_{ij}$, there are $2(Nd+1)Nd$-independent generators. The sympletic group $\mathrm{Sp}(2Nd,R)$ has $(2Nd+1)Nd$-independent generators which can be taken as

$$\begin{aligned} &Q_{risj}\equiv\tfrac{1}{2}\{X_{ri},X_{sj}\},\quad V_{risj}\equiv\tfrac{1}{2}\{X_{ri},P_{sj}\}\ ,\\ &K_{risj}\equiv\tfrac{1}{2}\{P_{ri},P_{sj}\}\ , \end{aligned} \tag{2}$$

while each of the Nd nonzero generators $i\hbar I\delta_{rs}\delta_{ij}$ provides a phase, one for each of the Nd quanta. As mentioned above, there are various subalgebras of possible interest. Here we focus on the algebras described explicitly in terms of the Cartesian operators X_{ri},P_{ri} leading to subalgebras of symplectic groups and orthogonal groups (see Fig. 1). Since the X's and P's separately commute among themselves, so do the Q's and K's which in turn generate two Abelian subalgebras $R^{(Nd+1)Nd/2}$. Moreover, the V's close under commutation and generate a $GL(Nd,R)$ subalgebra. Other subalgebras can be formed by contracting on the particle or spatial indices. The two sets of operators (summing on twice repeated indices)

$$\begin{aligned} &Q_{ij}\equiv X_{ri}X_{rj},\quad L_{ij}\equiv X_{ri}P_{rj}-X_{rj}P_{ri}\ ,\quad K_{ij}\equiv P_{ri}P_{rj}\ ,\\ &T_{ij}\equiv\tfrac{1}{2}(X_{ri}P_{rj}+X_{rj}P_{ri}+P_{ri}X_{rj}+P_{rj}X_{ri}) \end{aligned} \tag{3}$$

and

$$\begin{aligned} &Q_{rs}\equiv X_{ri}X_{si},\quad L_{rs}\equiv X_{ri}P_{si}-X_{si}P_{ri}\ ,\quad K_{rs}\equiv P_{ri}P_{si}\\ &T_{rs}\equiv\tfrac{1}{2}(X_{ri}P_{si}+X_{si}P_{ri}+P_{ri}X_{si}+P_{si}X_{ri}) \end{aligned} \tag{4}$$

close under commutation and form the algebras of $\mathrm{Sp}(2d,R)$ and $\mathrm{Sp}(2N,R)$, respectively. Note that the operators forming the T's and L's are the symmetric and antisymmetric parts of the contracted V's. Furthermore, the $\mathrm{Sp}(2d,R)$ algebra acts only on space indices ($i=1,\ldots,d$) and due to the summation over particle index is indifferent to particle indices ($r=1,\ldots,N$). Similarly, the $\mathrm{Sp}(2N,R)$ algebra acts only on particle indices and is indifferent to space indices. The two algebras do not commute with one another. This can be easily seen by calculating the commutator of Q_{ij} with T_{rs}, for example. However, the subsets $\{L_{ij}\}$ and $\{L_{rs}\}$ separately close under commutation

$$\begin{aligned} &[L_{ij},L_{kl}]=i\hbar(L_{ik}\delta_{jl}-L_{il}\delta_{jk}+L_{jk}\delta_{il}-L_{jl}\delta_{ik})\ ,\\ &[L_{rs},L_{tu}]=i\hbar(L_{rt}\delta_{su}-L_{ru}\delta_{st}+L_{st}\delta_{ru}-L_{su}\delta_{rt})\ , \end{aligned} \tag{5}$$

and form, respectively, the subalgebras of $\mathrm{O}(d)$ and $\mathrm{O}(N)$. Now, the operators of $\mathrm{Sp}(2d,R)$ and $\mathrm{O}(N)$, and those of $\mathrm{Sp}(2N,R)$ and $\mathrm{O}(d)$, do commute. Hence, we have identified as subgroups of $\mathrm{Sp}(2Nd,R)$ the direct product groups $\mathrm{Sp}(2d,R)\times\mathrm{O}(N)$ and $\mathrm{Sp}(2N,R)\times\mathrm{O}(d)$ (see Fig. 1). These subgroups are physically significant as they separate effects arising from the dimensionality d from those depending solely on particle number N. The $\mathrm{O}(d)$ group determines the total angular momentum of the system in the d-dimensional space. $\mathrm{O}(N)$ also contains various subgroups, but of greatest physical importance is the group-subgroup chain $\mathrm{O}(N)\supset\mathrm{O}(N-1)\supset\mathrm{S}(N)$ where $\mathrm{S}(N)$ is the symmetric group which gives the permutational symmetry of the many-particle system. $\mathrm{O}(N-1)$ extracts the center-of-mass behavior and necessitates a Jacobi-type transformation.

A further contraction over the remaining free indices of the $\mathrm{Sp}(2N,R)$ and $\mathrm{Sp}(2d,R)$ operators can be made. In so doing we obtain a subalgebra of both $\mathrm{Sp}(2d,R)$ and $\mathrm{Sp}(2N,R)$, which is that of $\mathrm{Sp}(2,R)$ defined by

$$Q\equiv X_{ri}X_{ri},\quad T\equiv X_{ri}P_{ri}+P_{ri}X_{ri},\quad K\equiv P_{ri}P_{ri} \tag{6}$$

with nonzero commutation relations[24]

$$[Q,K]=2i\hbar T,\quad [Q,T]=4i\hbar Q,\quad [K,T]=-4i\hbar K\ . \tag{7}$$

The $\mathrm{Sp}(2,R)$ generators are found to commute with the orthogonal groups $\mathrm{O}(N)$ and $\mathrm{O}(d)$, and, hence, the symplectic groups $\mathrm{Sp}(2N,R)$ and $\mathrm{Sp}(2d,R)$ contain the subgroups $\mathrm{Sp}(2,R)\times\mathrm{O}(N)$ and $\mathrm{Sp}(2,R)\times\mathrm{O}(d)$, respectively. The $\mathrm{Sp}(2,R)$ group is capable of describing collective vibrational excitations of the system, as in the nuclear problem. The generators of $\mathrm{Sp}(2,R)$ and $\mathrm{O}(Nd)$ also commute and, hence, generate the subalgebra of $\mathrm{Sp}(2,R)\times\mathrm{O}(Nd)$. We can then recover the physically interesting groups by reducing to $\mathrm{O}(Nd)\supset\mathrm{O}(N)\times\mathrm{O}(d)$ (see Fig. 1). One can immediately see that there exists in all three different reduction chains a common subgroup $\mathrm{Sp}(2,R)\times\mathrm{O}(N)\times\mathrm{O}(d)$ which would appear fundamental in describing the collective behavior of the Nd quantal system. Other subgroups can, of course, occur, but this paper will only focus on the physically relevant subgroups discussed above.

The main point that we wish to emphasize is that the symplectic group chains have many interesting model-based possibilities. The $\mathrm{Sp}(2,R)$ algebra already includes the total kinetic-energy term $(1/2m)(P_{ri}P_{ri})=(1/2m)K$, and the potential term $(m\omega^2/2)(X_{ri}X_{ri})=(m\omega^2/2)Q$ of the isotropic harmonic-oscillator Hamiltonian $H_0=(1/2m)K+(m\omega^2/2)Q$. The spatial part of the N-particle Landau Hamiltonian for N electrons (charge q, mass m) in two dimensions in a homogeneous magnetic field,

$$H_L=\frac{1}{2m}K+\frac{B^2q^2}{8mc^2}Q-\frac{qB}{2mc}L_{12}\ , \tag{8}$$

consists of operators from $\mathrm{Sp}(2,R)$ and $\mathrm{O}(d)$. The Hamil-

tonian of Johnson and Payne[5] describing an electron gas in a parabolic quantum dot ($\hbar\omega$) in an external magnetic field with an electron-electron interaction term

$$N(N-1)V_0-\frac{m\Omega^2}{4}\sum_{rsi}(X_{ri}-X_{si})(X_{ri}-X_{si}) \tag{9}$$

can also be expressed as

$$H_{\mathrm{JP}}=\frac{1}{2m}K+\frac{m\Omega_0^2}{2}Q-\frac{qB}{4mc}L_{12}+N(N-1)V_0+\frac{m\Omega^2}{2}\sum_{rs}Q_{rs} \tag{10}$$

with $\Omega_0^2\equiv\omega^2+(qB/2mc)^2-N\Omega^2$. The first three terms of H_{JP} contain operators in the Sp(2,R) and O(d) algebras, while the last term is expressed as generators of Sp($2N,R$). We note that Q_{rs} generates the Abelian algebra of $R^{(N+1)N/2}$ making possible an exact solution of the Hamiltonian H_{JP}.[5]

So far we have identified the symplectic group Sp($2Nd,R$) as the dynamical group of the N-particle system, and we have discussed the subgroups of physical interest. We now need the representation theory of these groups in order to classify and label the N-particle states of the system, and, hence, to identify a suitable set of basis sets for quantum dot Hamiltonians and their commuting operators. We exploit the power of Schur function techniques to generate results that are independent of the rank of the groups involved. Physically this means that we classify the states of the N-electron quantum system in a way that gives an explicit algebraic dependence on the number N of particles present. The salient mathematical details related to the representation theory of the noncompact group Sp($2n,R$) have been presented in Ref. 31.

Before proceeding we give a brief discussion of terminology and notation. For compact groups, irreps are denoted by partitions, and their properties, such as branching rules and Kronecker products, are given by certain operations of Schur functions which describe the characters of the irreps. A brief discussion is given in Appendix A. For full details, the reader is referred to Refs. 24, 29, and 30. From the representation theory of finite and compact groups, every finite-dimensional nonequivalent irrep can be denoted by a partition (μ)—an ordered set of positive integers $\mu_1\geq\cdots\geq\mu_k>0$. The integers μ_i are called the parts of the partition while the weight of the partition is the sum of its parts $w(\mu)=\mu_1+\cdots+\mu_k$. As shown in Fig. 2 the partition ($\tilde{\mu}$) is called the conjugate partition of (μ) formed by interchanging columns and rows of the associated Young diagram.[32] We use the convention of distinguishing characters and irreps of the orthogonal groups O(N) and symmetric groups S(N) by square brackets [ ], those of the sympletic groups Sp($2n$) by angular brackets ⟨ ⟩, and those of the unitary group U(n) by curly brackets { }. For clarify, we shall often subscript the rank of the group after the brackets. We shall also employ a reduced notation for the symmetric groups which is denoted by angular brackets ⟨ ⟩. The relation to the standard notation is the removal of the first part μ_1 of $[\mu_1,\mu_2,\ldots,\mu_k]=\langle\mu_2,\ldots,\mu_k\rangle$, and has the advantage of presenting many results of S(N) N independently. Given N one recovers the standard notation by the inverse process $\langle\mu\rangle=[N-w,\mu]$ where w is the weight of (μ). There should be no confusion with Sp($2n$) group irreps as the context will clearly indicate which group is to be used. For the sake of brevity, we shall denote the noncompact symplectic group Sp($2n,R$) by Sp($2n$); the latter is often used for the compact symplectic group Sp($2n,C$)∩U(n) which is not considered here at all.

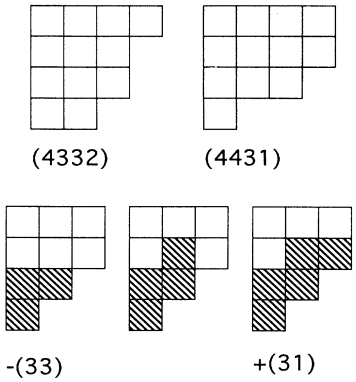

FIG. 2. Conjugate partitions (top) and O(n) hook-length modification rules (bottom). See text for a full explanation.

III. SPIN AND SPATIAL PERMUTATION SYMMETRIES

In classifying the N-particle states under the spatial symmetry groups, one must consider the spatial permutation symmetry. This information is contained in the O(N) group and its reduction to S(N). The O(N)⊃S(N) reduction is given in Appendix A. It is important because the requirement of overall antisymmetry (symmetry) on the total wave function for an N-fermion (boson) system implies spatial states of a well-defined permutation symmetry. For an N-electron system with each electron taking two spin values, we need to couple N times the defining spin irrep $[s;0]_3=[\frac{1}{2}]_3$ of SO(3), and decompose this into symmetrized tensor irreps of SO(3). This decomposition provides the total spin S of the N-particle system and its spin permutation symmetry. With the downward-pointing arrow denoting irrep character reduction, we have

$$[s;0]_3^{\times N}\downarrow\sum_S\left[\frac{N}{2}+S,\frac{N}{2}-S\right]_N\times[S]_3 \tag{11}$$

where the summation is $S=(N/2),(N/2)-1,\ldots,0$ or $\frac{1}{2}$ depending whether N is even or odd. The decomposition incorporates particle number N and total spin S directly. The corresponding S(N) irrep label is, in general, a two-part partition, $[(N/2)+S,(N/2)-S]=\langle(N/2)-S\rangle$, the latter being the reduced notation for this irrep. The Pauli principle clearly requires that the spatial permutation

symmetry of an N-electron system must now be

$$\left\langle \widetilde{\frac{N}{2}-S} \right\rangle = \left[\widetilde{\frac{N}{2}+S, \frac{N}{2}-S} \right] = [2^{(N/2)-S}, 1^{2S}] = \langle 2^{(N/2)-S-1}, 1^{2S} \rangle \ , \tag{12}$$

and it is the states labeled by these partitions that couple to the spin states to give totally antisymmetric states. For an N-electron system in a strong magnetic field, the spins are aligned with the field, and hence we have the spin-polarized case where $S=(N/2)$. The spin permutation label (using the reduced notation discussed in Sec. II) becomes $[N]=\langle 0 \rangle$ and the spatial permutation label becomes $[1^N]=\langle 1^{N-1} \rangle$.

IV. FEW-ELECTRON STATES IN A TWO-DIMENSIONAL DOT

This section uses the group-theory results of Appendices A and B to classify few-particle states in a two-dimensional quantum dot. In addition to a description of a quantum dot containing just N electrons, the discussion of N-particle states for small N has a further important application. These few-particle states can be used as basis states for describing correlated subsets of electrons in a cluster-type calculation of a many-electron dot. Such cluster models, particularly involving two particles, have been of great use in condensed-matter physics for describing phenomena such as superconductivity and superfluidity in extended many-body systems. Investigation of the analogs of such effects in confined systems such as quantum dots may provide interesting future developments.

We begin by giving a classification of the two-dimensional single-particle states. These states could form the starting point for a noninteracting particle description of a two-dimensional parabolic dot in the presence of any additional perturbations expressible in the Sp(4) algebra.[25] The states are taken to transform as the spin representation $\langle s;0 \rangle_2$ of Sp(4). The symplectic group character theory [Eq. (B2) of Appendix B with $n=2$ and $k=1$] gives

$$\langle s;0 \rangle_2 \downarrow \sum_{\nu} \langle \tfrac{1}{2}(\nu) \rangle_2 \times [\nu]_1 \ . \tag{13}$$

The only possibilities for the partition (ν) are (0) and (1), and hence Eq. (13) becomes

$$\langle s;0 \rangle_2 \downarrow \langle \tfrac{1}{2}(0) \rangle_2 \times [0]_1 + \langle \tfrac{1}{2}(1) \rangle_2 \times [0^*]_1 \ , \tag{14}$$

where the standard labeling of [1] in O(1) is $[0^*]$ (see Appendix A). The O(1)⊃S(1) reduction yields $[0]\downarrow[1]$ and $[0^*]\downarrow[1]$. Obviously, there is no permutation symmetry but there is a nontrivial O(1) symmetry. The total angular momentum is extracted by reduction of $\langle \frac{1}{2}(0) \rangle_2$ and $\langle \frac{1}{2}(1) \rangle_2$ to the subgroup Sp(2)×O(2) [Eq. (B6) with $n=1$ and $k=2$],

$$\langle \tfrac{1}{2}(0) \rangle_2 \downarrow \sum_{l \text{ even}} \langle 1(l) \rangle_1 \times [l]_2 \ , \tag{15}$$

$$\langle \tfrac{1}{2}(1) \rangle_2 \downarrow \sum_{l \text{ odd}} \langle 1(l) \rangle_1 \times [l]_2 \ . \tag{16}$$

It is quite clear that all even angular momentum states are contained in one basic spin irrep of Sp(4) while all odd angular momentum states are contained in a second. It is also evident that there is no multiplicity; for each angular momentum quantum number there exists one Sp(2) quantum number and vice versa. This is the so-called complementarity in nuclear physics.[33]

The one-particle states are hence labeled by the following ket:

$$\left| \begin{array}{c} \langle s;0 \rangle_2 \\ 0 \\ \langle \frac{1}{2}(0) \rangle_2 \times [0]_1 \\ 0 \quad\quad 0 \\ \langle 1(l) \rangle_1 \times [l]_2 \times [1]_1 \\ 0 \quad\quad 0 \quad\quad 0 \\ \{l+2j+1\}_1 \times \{\pm l\}_1 \times [0] \end{array} \right\rangle \tag{17}$$

with l even, and

$$\left| \begin{array}{c} \langle s;0 \rangle_2 \\ 0 \\ \langle \frac{1}{2}(1) \rangle_2 \times [0^*]_1 \\ 0 \quad\quad 0 \\ \langle 1(l) \rangle_1 \times [l]_2 \times [1]_1 \\ 0 \quad\quad 0 \quad\quad 0 \\ \{l+2j+1\}_1 \times \{\pm l\}_1 \times [0] \end{array} \right\rangle \tag{18}$$

with l odd. In both Eqs. (17) and (18), l and j are nonnegative integers. There is obviously some redundancy; the states are normally labeled by j and $\pm l$, but we have maintained all labels to show the full group-theoretic content. The labels on the bottom lines of Eqs. (17) and (18) correspond to the well-known Fock-Darwin labels for a particle in a harmonic-oscillator plus magnetic field [i.e., setting $l \rightarrow |l|$, and $j \rightarrow n$ in Eqs. (17) and (18) yields the labels of Ref. 3].

We now move on to a description of two-electron states in the quantum dot. These two-electron states will be labeled by the chain (see Fig. 1 with $N=2$ and $d=2$),

$$\mathrm{Sp}(8) \supset \mathrm{Sp}(4) \times \mathrm{O}(2) \supset \mathrm{Sp}(2) \times \mathrm{O}(2) \times \mathrm{O}(2) \ .$$

Using Eq. (B2) with $n=2$ and $k=2$, the first reduction yields

$$\langle s;0 \rangle_4 \downarrow \sum_{\nu} \langle 1(\nu) \rangle_2 \times [\nu]_2 \ . \tag{19}$$

The possible partitions (ν) are then of the form (0), (1), (1^2), and (n) with $n \geq 2$. Note the occurrence of the one- and two-part partitions which do not appear in the one-particle classification. At this stage we can extract the S(2) permutation symmetry of the states by the character reduction of the O(2) irreps. These are given as follows:

$$\begin{aligned} &[0]\downarrow[2], \quad [1]\downarrow[2]+[1^2] \ , \\ &[0^*]\downarrow[1^2], \quad [n]\downarrow[2]+[1^2] \ , \end{aligned} \tag{20}$$

where the standard labeling of $[1^2]$ in O(2) is $[0^*]$. It can

be seen that the O(2) scalar (i.e., [0]) and pseudoscalar (i.e., [0*]) give, respectively, only a symmetric (i.e., [2]) and an antisymmetric (i.e., $[1^2]$) spatial permutational symmetry type while the rest give states of each. Since both [2] and $[1^2]$ spin permutational symmetries can occur, all spatial states are allowed and none are excluded.

The Sp(4) irreps labeled by (0), (1), (1^2) are highly standard,[28] and, hence, are more easily reduced to Sp(2)×O(2) yielding the total angular momentum quantum number which can be obtained from the analysis of Eq. (B6) with $k=2$, $n=2$, and Eq. (B7). Expanding Eq. (B6) in leading lowest weight terms yields

$$\begin{aligned}
&\langle 1(0)\rangle_2\downarrow\langle 2(0)\rangle_1\times[0]_2+\langle 2(2)\rangle_1\times[2]_2\\
&\quad+\langle 2(4)\rangle_1\times[4+0]_2+\langle 2(6)\rangle_1\times[6+2]_2\\
&\quad+\langle 2(8)\rangle_1\times[8+4+0]_2+\cdots,\\
&\langle 1(1)\rangle_2\downarrow\langle 2(1)\rangle_1\times[1]_2+\langle 2(3)\rangle_1\times[3+1]_2\\
&\quad+\langle 2(5)\rangle_1\times[5+3+1]_2\\
&\quad+\langle 2(7)\rangle_1\times[7+5+3+1]_2+\cdots,\\
&\langle 1(1^2)\rangle_2\downarrow\langle 2(2)\rangle_1\times[0^*]_2+\langle 2(4)\rangle_1\times[2]_2\\
&\quad+\langle 2(6)\rangle_1\times[4+0^*]_2\\
&\quad+\langle 2(8)\rangle_1\times[6+2]_2+\cdots.
\end{aligned}\tag{21}$$

We have for the case (n) with $n\geq 2$ the signed sequence ${}_s(n)^2=(n)-(n2)$. To leading lowest weight terms the expansion is given by

$$\begin{aligned}
&\langle 1(n)\rangle_2\downarrow\langle 2(n)\rangle_1\times[(n)+(n-2)+\cdots+(\xi)]_2\\
&\quad+\langle 2(n+2)\rangle_1\times[(n+2)+(n)+\cdots\\
&\qquad+(\xi)]_2+\cdots,
\end{aligned}\tag{22}$$

where $(\xi)=(1)$ if n is odd. For n even, (ξ) alternates between (0) and (0*) starting with (0) in the zeroth leading term. Note that the reductions remain multiplicity free but are not complementary as there are several angular momenta O(2) irrep labels for each Sp(2) irrep label. We also observe that, in general, given an Sp(2) irrep label $\langle 2(n')\rangle_1$, the angular momentum quantum numbers are limited to $0\leq l\leq n'$, stepping in units of 2 except for the cases (0) and (1^2) which step in units of 4. A consequence of the stepping by 2 and 4 is that there is no mixing of even and odd angular momenta within a given Sp(4) irrep. In fact, this even or odd nature is determined, as we shall clearly see below, by the weight of the Sp(4) partition. The difference between these two-particle states and the one-particle states is that there are now more angular momentum quantum numbers possible. The general states, which we have separated into symmetric $[2]_2$ and antisymmetric $[1^2]_2$ states, are labeled as

$$\left|\begin{array}{c}
\langle s;0\rangle_4\\
0\\
\langle 1(\nu)\rangle_2\times[\nu]_2\\
0\quad\ 0\\
\langle 2(n+2i)\rangle_1\times[l]_2\times[2]_2\\
0\quad\ 0\quad\ 0\\
\{n+2+2i+2j\}_1\times\{\pm l\}_1\times[0]
\end{array}\right\rangle \tag{23}$$

$$\left|\begin{array}{c}
\langle s;0\rangle_4\\
0\\
\langle 1(\nu)\rangle_2\times[\nu]_2\\
0\quad\ 0\\
\langle 2(n+2i)\rangle_1\times[l]_2\times[1^2]_2\\
0\quad\ 0\quad\ 0\\
\{n+2+2i+2j\}_1\times\{\pm l\}_1\times[0]
\end{array}\right\rangle . \tag{24}$$

In both cases we have set $n=w(\nu)$. If $(\nu)=(0)$ then only symmetric states occur while for $(\nu)=(1^2)$ only antisymmetric states occur.

We now turn to three-electron states, labeling them once more according to the chain (see Fig. 1 with $N=3$ and $d=2$)

$$\mathrm{Sp}(12)\supset\mathrm{Sp}(4)\times\mathrm{O}(3)\supset\mathrm{Sp}(2)\times\mathrm{O}(2)\times\mathrm{O}(3) .$$

The first reduction [Eq. (B2) with $n=2$ and $k=3$] is written as

$$\langle s;0\rangle_6\downarrow\sum_{\nu}\langle\tfrac{3}{2}(\nu)\rangle_2\times[\nu]_3 . \tag{25}$$

The possible partitions (ν) are now of the form (0), (1), (1^2), (n), and $(n,1)$ with $n\geq 2$. Note the occurrence of the new two-part partitions which do not appear in the two-particle classification. Extraction of the S(3) permutation symmetry of the states by the character reduction of the O(3) irreps leads to the following analysis:

$$\begin{aligned}
&[0]\downarrow[3]\\
&[1]\downarrow[3]+[21]\\
&[1^*]\downarrow[21]+[1^3]\\
&[2]\downarrow[3]+(2)[21]\\
&[2^*]\downarrow(2)[21]+[1^3]\\
&[n]\downarrow(a+1)[3]+(n-a)[21]+(a)[1^3]\\
&[n^*]\downarrow(a)[3]+(n-a)[21]+(a+1)[1^3],
\end{aligned}\tag{26}$$

where the standard labelings of $[1^2]$ and $[n,1]$ in O(3) are $[1^*]$ and $[n^*]$, respectively, the multiplicities of S(3) irreps are placed in parentheses, and $a=\lfloor n/3\rfloor$ which denotes the integer part of $n/3$. Note that, excluding $[1^*]$ and $[2^*]$, all the O(3) irreps yield states of totally symmetric permutational symmetry. However, to couple with the appropriate spin permutation symmetry, only states with spatial permutation symmetry [21] and $[1^3]$ can exist. This is because three electrons cannot form a

totally antisymmetric spin state.[34] The totally symmetric spatial permutation [3] will therefore not arise and leads to the complete exclusion of the states transforming as $\langle \frac{3}{2}(0) \rangle$ of Sp(4), and states transforming as $[3]_3$ of other Sp(4) irreps. This represents a massive restriction to the number of possible three-electron states.

From Eq. (B6) the reduction of the Sp(4) irreps in leading lowest weight terms expands to

$$\begin{aligned}
\langle \tfrac{3}{2}(0) \rangle_2 \downarrow & \langle 3(0) \rangle_1 \times [0]_2 + \langle 3(2) \rangle_1 \times [2]_2 \\
& + \langle 3(4) \rangle_1 \times [4+0]_2 \\
& + \langle 3(6) \rangle_1 \times [6+2]_2 \\
& + \langle 3(8) \rangle_1 \times [8+4+0]_2 + \cdots \\
\langle \tfrac{3}{2}(1) \rangle_2 \downarrow & \langle 3(1) \rangle_1 \times [1]_2 + \langle 3(3) \rangle_1 \times [3+1]_2 \\
& + \langle 3(5) \rangle_1 \times [5+3+1]_2 \\
& + \langle 3(7) \rangle_1 \times [7+5+3+1]_2 + \cdots \\
\langle \tfrac{3}{2}(1^2) \rangle_2 \downarrow & \langle 3(2) \rangle_1 \times [0^*]_2 + \langle 3(4) \rangle_1 \times [2]_2 \\
& + \langle 3(6) \rangle_1 \times [4+0^*]_2 \\
& + \langle 3(8) \rangle_1 \times [6+2]_2 + \cdots \qquad (27) \\
\langle \tfrac{3}{2}(n) \rangle_2 \downarrow & \langle 3(n) \rangle_1 \times [(n)+(n-2)+\cdots+(\xi)]_2 \\
& + \langle 3(n+2) \rangle_1 \times [(n+2)+(n)+2(n-2) \\
& \qquad + \cdots + (\xi')]_2 + \cdots \\
\langle \tfrac{3}{2}(n,1) \rangle_2 \downarrow & \langle 3(n+1) \rangle_1 \\
& \times [(n-1)+(n-3)+\cdots+(\xi^*)]_2 \\
& + \langle 3(n+3) \rangle_1 \times [(n+1)+(n-1)+2(n-3) \\
& \qquad + \cdots + (\xi')]_2 + \cdots ,
\end{aligned}$$

where $(\xi)=(0)$ or (1) and $(\xi')=(0^*)+(0)$ or 2(1) depending on the even or odd nature of the O(2) sequence of labels. We note the appearance of multiplicities of angular momentum quantum numbers. The angular momentum content for the partitions (0), (1), (1^2) does not change from that of the two-particle system. Also, depending whether the weight of the Sp(4) partition is even or odd, the angular momentum quantum numbers are all even or all odd, hence no mixing occurs. With i,j being nonnegative integers and $n=w(\nu)$, the three-particle states can therefore be labeled as follows:

$$\left|\begin{array}{c}
\langle s;0 \rangle_6 \\
0 \\
\langle \frac{3}{2}(\nu) \rangle_2 \times [\nu]_3 \\
c \qquad d \\
\langle 3(n+2i) \rangle_1 \times [l]_2 \times [21]_3 \\
0 \qquad 0 \qquad k \\
\{n+3+2i+2j\}_1 \times \{\pm l\}_1 \times [0]
\end{array}\right\rangle \qquad (28)$$

with $0 \le l \le n$ and $k=1,2$, and

$$\left|\begin{array}{c}
\langle s;0 \rangle_6 \\
0 \\
\langle \frac{3}{2}(\nu) \rangle_2 \times [\nu]_3 \\
c \qquad d \\
\langle 3(n) \rangle_1 \times [l]_2 \times [1^3]_3 \\
0 \qquad 0 \qquad 0 \\
\{n+3+2i+2j\}_1 \times \{\pm l\}_1 \times [0]
\end{array}\right\rangle . \qquad (29)$$

We pause here to connect the present work to the three-electron calculations performed by Laughlin[35] in connection with the fractional quantum Hall effect, and the N-electron calculations of Stone, Wyld, and Schult.[36] Both groups considered the high magnetic-field limit where the system is spin polarized. They identified the number m of N (noninteracting) electron states of a given angular momentum within the lowest Landau level and then diagonalized the Coulomb interaction for each angular momentum. The present formalism provides a rigorous counting procedure for evaluating the number m of possible states of a given angular momentum for a given N, and reproduces[37] the m values of both Stone, Wyld, and Schult[36] and Laughlin.[35] More importantly, the present counting scheme is both systematic and manageable for *any N*.

For the four-electron states in a quantum dot the complexity of the analysis increases significantly. In fact, the four-electron system can be considered as the generic case since all the possible types of Sp(4) partitions that can occur under the dimensional constraint $\tilde{\nu}_1 \le 2$ have appeared. For *five or more electrons no other types of partitions will arise.* We will once more label the four-electron states according to the chain (see Fig. 1 with $N=4$ and $d=2$)

$$\mathrm{Sp}(16) \supset \mathrm{Sp}(4) \times \mathrm{O}(4) \supset \mathrm{Sp}(2) \times \mathrm{O}(2) \times \mathrm{O}(4) .$$

The first reduction is written [Eq. (B2) with $n=2$ and $k=4$] as

$$\langle s;0 \rangle_8 \downarrow \sum_{\nu} \langle 2(\nu) \rangle_2 \times [\nu]_4 . \qquad (30)$$

The possible partitions (ν) are then of the form (0), (1), (1^2), (n), and $(n,1)$ with $n \ge 2$, and (n_1,n_2) with $n_1 \ge n_2 \ge 2$. Note the occurrence of the general two-part partitions which do not appear in the three-particle classification. Extraction of the S(4) permutation symmetry of the states by the character reduction of the O(4) irreps leads to the following analysis:

$$\begin{aligned}
& [0] \downarrow [4] \\
& [1] \downarrow [4]+[31] \\
& [1^2] \downarrow [31]+[21^2] \\
& [2] \downarrow [4]+2[31]+[2^2] \\
& [21] \downarrow 2[31]+2[2^2]+2[21^2] \\
& [3] \downarrow 2[4]+3[31]+[2^2]+[21^2] \\
& [31] \downarrow [4]+4[31]+2[2^2]+4[21^2]+[1^4]
\end{aligned} \qquad (31)$$

$$[2^2]\downarrow[31]+2[2^2]+[21^2]$$

$$[n]\downarrow a[4]+b[31]+c[2^2]+d[21^2]+e[1^4]$$

$$[n_1,n_2]\downarrow a[4]+b[31]+c[2^2]+b[21^2]+a[1^4]\ ,$$

where the multiplicities of the S(4) irreps are placed before the irrep label. To couple with the appropriate spin permutation symmetry, only states with spatial permutation symmetry $[2^2]$, $[21^2]$, and $[1^4]$ can be retained. The totally symmetric partition [4] and in addition [31] will not suffice and lead to the total exclusion of the states transforming as $\langle 2(0)\rangle$ and $\langle 2(1)\rangle$ of Sp(4). From Eq. (B6) the reduction of the Sp(4) irreps in leading lowest weight terms is given by

$$\begin{aligned}
&\langle 2(1^2)\rangle_2\downarrow\langle 4(2)\rangle_1\times[0^*]_2+\langle 4(4)\rangle_1\times[2]_2+\langle 4(6)\rangle_1\times[4+0^*]_2+\langle 4(8)\rangle_1\times[6+2]_2+\cdots\ ,\\
&\langle 2(n)\rangle_2\downarrow\langle 4(n)\rangle_1\times[(n)+(n-1)+\cdots+(\xi)]_2+\langle 4(n+2)\rangle_1\times[(n+2)+(n)+\cdots+(\xi')]_2+\cdots\ ,\\
&\langle 2(n,1)\rangle_2\downarrow\langle 4(n+1)\rangle_1\times[(n-1)+(n-3)+\cdots+(\xi)]_2\\
&\qquad+\langle 4(n+3)\rangle_1\times[(n+1)+(n-1)+2(n-3)+\cdots+(\xi')]_2+\cdots\ ,\\
&\langle 2(n_1,n_2)\rangle_2\downarrow\langle 4(n_1+n_2)\rangle_1\times[(n_1,n_2)/D]_2+\langle 4(n_1+n_2+2)\rangle_1\\
&\qquad\times[((n_1+2,n_2)+(n_1+1,n_2+1)+(n_1,n_2+2)-(n_1,n_2))/D]_2+\cdots\ .
\end{aligned}\tag{32}$$

Separating the states according to their permutational symmetry, the four-electron states can be written as

$$\left|\begin{array}{c}\langle s;0\rangle_8\\ 0\\ \langle 2(\nu)\rangle_2\times[\nu]_4\\ c\qquad d\\ \langle 4(n+2i)\rangle_1\times[l]_2\times[2^2]_4\\ 0\qquad 0\qquad k\\ \{n+4+2i+2j\}_1\times\{\pm l\}_1\times[0]\end{array}\right\rangle\ ,\tag{33}$$

where $n=w(\nu)$, $(\nu)=(2),(3),\ldots,(21),(31)\ldots$ and $k=1,2$;

$$\left|\begin{array}{c}\langle s;0\rangle_8\\ 0\\ \langle 2(\nu)\rangle_2\times[\nu]_4\\ c\qquad d\\ \langle 4(n+2i)\rangle_1\times[l]_2\times[21^2]_4\\ 0\qquad 0\qquad k\\ \{n+4+2i+2j\}_1\times\{\pm l\}_1\times[0]\end{array}\right\rangle\tag{34}$$

where $n=w(\nu)$, $(\nu)=(3),(4),\ldots,(1^2),(21)\ldots$ and $k=1,2,3$;

$$\left|\begin{array}{c}\langle s;0\rangle_8\\ 0\\ \langle 2(\nu)\rangle_2\times[\nu]_4\\ c\qquad d\\ \langle 4(n+2i)\rangle_1\times[l]_2\times[1^4]_4\\ 0\quad 0\quad 0\\ \{n+4+2i+2j\}_1\times\{\pm l\}_1\times[0]\end{array}\right\rangle\tag{35}$$

where $n=w(\nu)$ and $(\nu)=(4),(5),\ldots,(31),(41)\ldots$.

We have noted from the above analysis that as the number of electrons N increases the admissible Sp(4) partitions increase from only (0) and (1) for $N=1$, to the general case $(\nu)=(\nu_1,\nu_2)$ for $N\geq 4$. It can also been seen that for a given Sp(4) partition (ν) the Sp(2)×O(2) reduction does not change in any essential way with respect to particle number for $N\geq 2$. In fact, for $N\geq 3$ the reduction is independent of N. What does change significantly with N is the $O(N)\supset S(N)$ reduction, and is responsible for the inadmissability or "disappearance" of certain Sp(4)×O(N) irreps. We now briefly discuss the reasons behind this effect of disappearing partitions. This explanation delves into details of the $O(N)\supset S(N)$ Schur function expression for which we refer the reader to Ref. 38. We start with the general $O(N)\supset S(N)$ reduction discussed in Appendix A. The reduction has the general form

$$[\nu]\downarrow\sum_{\rho} m(\nu,\rho)[N-w(\rho),\rho]\ .\tag{36}$$

As noted in Appendix A, the reduction can be carried out N independently. The partition (ρ) has weight $w(\rho)\leq w(\nu)$. For the appropriate permutational symmetry to arise, we need

$$[N-w(\rho),\rho]=[N-2r-s,2^r,1^s]\tag{37}$$

which yields the necessary condition

$$N-2r-s=\begin{cases}1\text{ or }2\text{ if }r=0\ ,\\ 2\text{ if }r\geq 1\ .\end{cases}\tag{38}$$

Thus, given the admissible partitions $(\rho)=(2^r,1^s)$, this condition determines and fixes the allowed values of N. For higher values of N this condition will not be satisfied, and hence to absorb more of the value N other higher weight partitions (ρ) must be sought, or (ν) replaced by another larger partition. The reasons for (ν) not being admissable are therefore twofold: (i) there are no (ρ) of the form $[2^r,1^s]$, and (ii) there exists a (ρ), but it does not

TABLE I. The smallest partitions (π) giving spatial S(N)-type $[2^{r+1}]$ or $[2^{r+1}1]$.

N	1	2	3	4	5	6	7	8	9	10	11	12
$w(\pi)$		0	1	2	3	4	6	8	10	12	14	16
(π)		(0)	(1)	(2)	(21)	(2^2)	(42)	(62)	(73)	(84)	(86)	(8^8)
								(4^4)	(64)	(6^6)		

satisfy condition (38).

Finally we will discuss, for a given number of particles N, the smallest O(N) partition and, hence, by complementarity the Sp(4) partition denoted here by (π) that can give rise to an admissible S(N) irrep label. This can be readily found for the following few-electron cases: $(\pi)=(0)$ for $N=2$, $(\pi)=(1)$ for $N=3$, $(\pi)=(2)$, and (1^2) for $N=4$. The resolution of this problem for general N is complicated because of the complexity of the Schur function plethysm operations involved in the reduction O(N)$\supset$S(N) (see Appendix A). We will now give a procedure by which one can obtain these smallest partitions. It yields a necessary, but not sufficient, condition for the smallest partition. Given an S(N) irrep label $[N-w(\rho),\rho]=[N-2r-s,2^r,1^s]$, with $N-2r-s=1,2$ as above, we must partition $(\rho)=(2^r,1^s)$ in k smaller partitions $(\rho_i)=(2^{r_i},1^{s_i})$ with $i=1,\ldots,k$. Note that each (ρ_i) is of a similar shape to (ρ). Clearly,

$$w(\rho)=\sum_i w(\rho_i) \text{ or } 2r-s=\sum_i 2r_i-s_i \ . \tag{39}$$

Denoting the smallest partition of Sp(4) associated with the S(N) irrep label $[N-w(\rho),\rho]$ by (π), we have the necessary weight condition

$$w(\pi)=\sum_i i w(\rho_i) \ . \tag{40}$$

The Sp(4) restricts (π) to at most two-part partitions. However, we still must sort through the possible (π) by making sure that each plethysm $(i\otimes\rho_i)$ contains a two-part partition (π_i) that is smaller or equal to (π). For example, since $(1\otimes\rho_1)=(\rho_1)$ we have $(\rho_1)=(\pi_1)=(0)$, (1), (2), (1^2), (21), (2^2) while $(2\otimes\rho_2)$ has only two-part partitions for $(\rho_2)=(0)$, (1), (2), (1^2), (21), (1^3), (2^2), (21^2), (2^21), (2^3). For five electrons with spatial permutational symmetry types $[2^2,1]$, $[2,1^3]$, and $[1^5]$, we can partition $[2^2,1]$ into one term, $(\rho_1)=(21)$ itself, so that $w(\pi)=3$ and only $(\pi)=(21)$ is possible from the plethysm constraint. Note that for $[2,1^3]$ the partitioning process gives $(\rho_1)=(1^2)$ and $(\rho_2)=(1)$ so that $w(\pi)=1.2+2.1=4$, while $[1^5]$ yields $w(\pi)=1.2+2.3=8$. For the six-electron system it is spatial permutational symmetry type $[2^3]$ that gives rise to $(\pi)=(2^2)$. Indeed, for the general N-electron system ($6\le N\le 12$), it is spatial permutational symmetry type $[2^{r+1}]$ (for even $N=2r+2$) or $[2^{r+1}1]$ (for odd $N=2r+3$) that gives rise to the weight condition $w(\pi)=2(N-4)$. The smallest partitions (π) giving spatial S(N)-type $[2^{r+1}]$ or $[2^{r+1}1]$ are given in Table I. The smallest partitions (π) giving spatial S(N)-type $[2^{r+1}1^2]$ or $[2^{r+1}1^3]$ are given in Table II. These considerations are important when one wants to evaluate the lowest energy eigenstates for the N-electron system. For a central harmonic potential in zero magnetic field, the energy depend on $n=w(\nu)$ as discussed elsewhere,[25] i.e., the lower the weight of the partition the lower the energy. These partitions are mainly given by the spatial S(N)-type $[2^{r+1}]$ or $[2^{r+1}1]$ which are associated with lowest total spin $S=0$ (N even) and $S=\frac{1}{2}$ (N odd), respectively.

TABLE II. The smallest partitions (π) giving spatial S(N)-type $[2^{r+1}1^2]$ or $[2^{r+1}1^3]$.

N	8	9	10
$w(\pi)$	8	10	12
(π)	(53)	(5^5)	(75)

V. CONCLUSIONS

We have given a general procedure for classifying states of N-particles confined in two dimensions with a view to describing the complex behavior of a quantum dot system. In fact, the general analysis of this paper is relevant to both fermion *and* boson systems, and can possibly even be applied to anyon systems as well. Since we have focused on a confined system of particles, we have only considered the discrete series representations. The basis states corresponding to these representations describe localized states (e.g., vibrational and rotational modes) and *not* extended states (translational modes). We have found that for $N<4$ the pattern of occurrence of various types of configurations is highly nontrivial as N is increased. It is not until $N\ge 4$ that the types of states occurring become fairly N independent. This finding injects an element of caution into the practice of interpolating between results obtained from diagonalization of few-electron systems, and the physics of the large-N system.

Finally we note that, using the concise labeling and counting scheme provided in this paper, we have recently been able to evaluate matrix elements and hence energies for various N-electron Hamiltonians. These results will be presented elsewhere.[25]

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APPENDIX A: SCHUR FUNCTION ANALYSIS OF O(k) AND S(N)

In order to introduce some of the Schur function techniques we require, we give the relevant compact group reductions that appear in the symmetry classification of

the N-electron system in two dimensions (see Fig. 1). For details, see King,[29] Black, King, and Wybourne,[30] Butler and King,[38] Butler and Wybourne.[39] Many tables of Schur function operations, such as the outer and inner multiplications and division, can be found in Ref. 40.

In general, the nonequivalent irreps of the compact groups, both finite and infinite, can be labeled by partitions. For the orthogonal group $O(k)$ each nonequivalent true irrep can be labeled by the so-called standard partitions, $[\mu]$ and $[\mu^*]$, with $\tilde{\mu}_1 \leq \lfloor k/2 \rfloor$ parts. Here, $[\mu^*]$ represents the associated irrep character of $[\mu]$ defined by $[\mu^*] \equiv [0^*] \times [\mu]$, and $[0^*]$ denotes the one-dimensional nontrivial determinantal irrep of $O(k)$. If $[\mu]$ or $[\mu^*]$ is nonstandard with $\tilde{\mu}_1 > \lfloor N/2 \rfloor$, one must apply the $O(k)$ modification rule. This rule removes a hook length of length $h \equiv 2\tilde{\mu}_1 - k$ starting at the bottom of the first column of the associated Young diagram, and removing h contiguous boxes along the boundary of the diagram. If the resulting diagram, denoted symbollically as $[\mu - h]$, is a regular partition then $[\mu] = (-)^{c-1}[\mu - h]^*$ where c is the column in which the removal procedure ends, otherwise if the resulting partition is irregular, it is the null character $\varnothing$. If one begins with the associated character $[\mu]^*$ then $[\mu]^* = (-)^{c-1}[\mu - h]$. The hook length removal procedure must be repeated until a standard irrep label or the null label is obtained. By way of illustration, if $[\mu] = [3321]$ and $h = 3,4,5$ then the Young diagrams are as shown at the bottom of Fig. 2, and $[\mu - h] = -[33]^*$, $\varnothing$ and $[31]^*$, respectively.

In classifying the states of the N-particle system in two dimensions one requires various orthogonal group reductions. The simplest is $O(N) \supset O(N-1)$ and this is symbolically given by the compact Schur function expression,

$$[\mu]_N \downarrow [\mu/M]_{N-1} \equiv \sum_m [\mu/m]_{N-1} , \tag{A1}$$

where M represents the Schur function series containing only those partitions of one part, i.e., m integer, and the symbol "/" denotes the Schur function operation of division.[40] Although the series is, in principle, infinite, the division by definition restricts the series to a finite number of terms. An example of the application of this formula is

$$[311]_N \downarrow [311/(0+1+2+3)]_{N-1} = [311+31+211+21+111+11]_{N-1} . \tag{A2}$$

The advantage of the Schur function techniques is that a general result such as Eq. (A2) is valid for all N. However, to apply this for any particular value, the $O(N)$ partitions on the right-hand side may not be standard irrep labels of $O(N-1)$ and need to be modified accordingly in the manner introduced above.

Implementing the Pauli principle requires the reduction of $O(N)$ irreps to $S(N)$. Using the $O(N) \supset O(N-1)$ reduction from above, we now only need the $O(N-1) \supset S(N)$ irrep reductions. This is actually one of the more difficult reductions to perform as it involves many operations at once, including the difficult plethysm Schur function operation denoted by the symbol $\otimes$. These plethysms are related to the symmetrization process of $S(N)$ permutations, and a discussion can be found in Wybourne[40] with tables given by Butler and Wybourne.[39] The reduction is divided into two parts. The first part introduces a convenient designation $Q(N-1)$, which, it must be emphasized, is not a group, and the Schur function expansion of $O(N-1)$ irrep characters to $Q(N-1)$ given by $[\mu]_{N-1} \downarrow (\mu / \prod_i M_{(i)})_{N-1}$ where $i = 3,4, \ldots \infty$. The plethysm sum is defined as $M_{(i)} \equiv \sum_j (i \otimes j)$ with $j = 0,1, \ldots \infty$. Again, the Schur function division renders finite and manageable the infinite product of infinite series.

The second part describes the reduction of the characters of $Q(N-1)$ to $S(N)$ in terms of the reduced notation of the symmetric group $(\beta)_{N-1} \downarrow \sum_Z \langle (\beta/sp(Z)) \cdot Z \rangle_N$ where the symbol $\cdot$ denotes the Schur function operation of outer multiplication evaluated with the help of the Littlewood-Richardson rules.[40] The summation term $Z \equiv (\zeta_2) \cdot (\zeta_3) \cdot (\zeta_4) \ldots$ is an infinite outer multiplication of partitions (ζ_i), and $sp(Z) \equiv (2 \otimes \zeta_2) \cdot (3 \otimes \zeta_3) \cdot (4 \otimes \zeta_4) \ldots$ is an infinite product of special plethysms of the form $(i \otimes \zeta_i)$. A summation over all possible partitions (ζ_i) is implied. Although the division renders the expansion finite, the various plethysms make the reduction cumbersome to perform. A small table of $O(N-1) \supset S(N)$ reductions is given by Butler and King[38] and the reader is referred to that paper for examples.

Although difficult, the $O(N-1) \supset S(N)$ reduction has been specified without reference to the integer N and, hence, the result is valid for all N. However, in order to use the results for a given N one has to perform $S(N)$ modification to each partition appearing in the expansion. This consists of converting each reduced-notation term $\langle \mu \rangle_N$ to the standard notation $[N-w, \mu]_N$ where w is the weight of (μ), and if the leading part $N-w$ is less than the second part μ_1, the modification rule $[\ldots \mu_i, \mu_{i+1} \ldots] = -[\ldots \mu_{i+1} - 1, \mu_i + 1 \ldots]$ is employed repeatedly until a standard partition is obtained or it is shown to be null. For example, $[13] = -[22]$ in S(4), $[0421] = -[3121] = +[3121] = \varnothing$ in S(7), and $[255] = -[435] = +[444]$ in S(12).

In classifying the N-particle states in Sec. IV, we have used the reduction $O(2) \supset U(1)$. The O(2) irrep label provides the total angular momentum and hence the U(1) irrep label yields the component of this angular momentum. Since all irreps of O(2) are labeled as $[m]_2$ with m a positive integer and are two dimensional, the U(1) irrep content is readily obtained as

$$[m]_2 \downarrow \{+m\}_1 + \{-m\}_1 . \tag{A3}$$

Finally, we remark here that $M_{(1)} \equiv M$, $M_{(2)} \equiv D$, and $M_{-(2)} \equiv C$ where the latter designations are those given by King.[29] These series appear prominently in the symplectic group representation analysis and are briefly discussed here. The M series is already defined above in Eq. (A1). The D series represents an infinite sum of partitions (δ) whose parts δ_i are all even. The lowest weight terms are

$$D = (0) + (2) + (4) + (22) + (6) + (42) + (222) + \cdots . \tag{A4}$$

The C series is formally the inverse series of D, $CD=(0)$, with leading terms

$$C=(0)-(2)+(31)-(41^2)$$
$$-(3^2)+(431)+(51^3)+\cdots . \tag{A5}$$

APPENDIX B: REPRESENTATIONS OF THE NONCOMPACT Sp(2n,R) GROUP

To develop results for Sp($2n$) one uses the fact that it contains the maximal compact subgroup U(n) and hence the character of the infinite-dimensional unitary irreps of Sp($2n$) can be expressed as an infinite sum of characters of the finite-dimensional unitary irreps of U(n). We refer the reader to King and Wybourne[28] for a fuller explanation of the representation theory of Sp($2n$). There exist two basic discrete series representations, $\langle s;0_+\rangle$ and $\langle s;0_-\rangle$, of Sp($2n$). Both are infinite dimensional and unitary, and, hence, nonfaithful. In addition there exists the so-called "spin" representation $\langle s;0\rangle$ which is a faithful unitary irrep of the double-covering group, the so-called metaplectic group Mp($2n$) of Sp($2n$), and is reducible into a direct sum of the two basic spin irreps of Sp($2n$), i.e., $\langle s;0\rangle_n \downarrow \langle s;0_+\rangle_n + \langle s;0_-\rangle_n$. There are two fundamental results associated with the basic spin irrep $\langle s;0\rangle$. The first refers to the fact that the metaplectic group is the usual dynamical group of the isotropic harmonic oscillator. All the harmonic-oscillator states, both even and odd quanta, therefore span the space of the basic spin irrep $\langle s;0\rangle$. Specifically, the even (odd) states span the spin irreps $\langle s;0_+\rangle$ ($\langle s;0_-\rangle$). In Schur function terminology this means the irrep character $\langle s;0\rangle$ can be expressed as a sum of irrep characters of U(n), thus providing the description of the degeneracies of the states of the isotropic harmonic oscillator

$$\langle s;0\rangle_n \downarrow \epsilon_n^{1/2}\cdot M \equiv \epsilon_n^{1/2}\cdot \sum_m \{m\}_n , \tag{B1}$$

where $\{m\}$ denotes a covariant tensor irrep of U(n) and $\epsilon_n \equiv \{1^n\}_n$ denotes the determinantal representation of U(n). The half-power of ϵ_n implies the presence of a projective representation of U(n). The summation is over all non-negative integers, and hence composes the Schur function series M introduced in Appendix A.

The second result is that the tensor powers $\langle s;0\rangle^{\times k}$ decomposes into a direct sum of unitary irreps $\langle (k/2)(\nu)\rangle$ of Sp($2n$) where (ν) is a partition into at most n parts, $\tilde{\nu}_1 \le n$, with the additional constraint $\tilde{\nu}_1+\tilde{\nu}_2 \le k$. These irreps are referred to as unitary discrete series representations, or *harmonic* series representations by King and Wybourne,[28] and are those that are considered here since we are considering a bound system. Furthermore, the reduction of the basic spin irrep $\langle s;0\rangle$ to Sp($2n$)$\times$O(k) is in one-to-one correspondence to the reduction of the k-fold tensor product, i.e.,

$$\langle s;0\rangle_{nk} \downarrow \sum_\nu \left\langle \frac{k}{2}(\nu)\right\rangle_n \times [\nu]_k , \tag{B2}$$

where the summation is over all the partitions (ν) satisfying the above constraints.[41] Note that the branching is multiplicity free and that there is for each irrep label of Sp($2n$) just one irrep label for O(k) and vice versa—a fact known as complementarity in nuclear physics.[33] The constraints imply that the summation is over those standard partitions that label covariant tensor irreps of U(n), and those irrep labels of O(k) that are "near standard"[28] (i.e., $[\nu]$ such that $\llcorner (k/2) \lrcorner < \tilde{\nu}_1 < k$, hence requiring one O($k$) modification step).

It is just these two results given by Eqs. (B1) and (B2) that are needed to determine properties of the unitary discrete series irreps $\langle (k/2)(\nu)\rangle$ of Sp($2n$). For example, the branching rule appropriate for the reduction Sp($2n$)$\supset$U(n) has been shown to take the form

$$\left\langle \frac{k}{2}(\nu)\right\rangle_n \downarrow \epsilon_n^{k/2}\cdot \{({}_s\nu^k\cdot D)_K\}_n \tag{B3}$$

with $K \equiv \min(k,n)$. We have used this result in Sec. IV in performing the Sp(2)$\supset$U(1) reduction. We note first that $\langle (k/2)(\nu)\rangle$ is a unitary irrep and therefore (ν) is a near-standard label of O(k). Second, D is a Schur function series (see Appendix A) and since this is formally an infinite sum of partitions, the U(n) irrep content is infinite. Third, ${}_s(\nu)^k$ is called the "signed sequence"[28] of (ν). The signed sequence involves an infinite sum of partitions $\pm(\mu)$ such that $\pm[\mu]$ is equivalent to $[\nu]$ under the modification rule of O(k).[28] For example,

$${}_s(54)^4=(54)-(542)+(5431)-(543^2)-(54^21^2)+\cdots . \tag{B4}$$

More details are given in King and Wybourne.[28] The term $({}_s\nu^k\cdot D)_K$ involves an outer multiplication of the signed sequence with the D series and is to be carried out with the restriction to at most K parts, as implied by the subscript K. This fact imposes certain limits on the partitions appearing in each of the infinite sequences ${}_s(\nu)^k$ and D. The above example now gives ${}_s(54)^4=(54)-(542)$ with $K=3$. The justification in labeling Sp($2n$) irreps by $\langle (k/2)(\nu)\rangle$ derives from this Sp($2n$)$\supset$U(n) reduction and the fact that ν is the smallest partition in the restriction to U(n), i.e.,

$$\left\langle \frac{k}{2}(\nu)\right\rangle_n \downarrow \epsilon_n^{k/2}\cdot \{\nu\}_n + \cdots , \tag{B5}$$

where the continuation involves larger partitions than ν. Note that from this remark the two basic spin representations $\langle s;0_+\rangle$ and $\langle s;0_-\rangle$ could be denoted by $\langle \frac{1}{2}(0)\rangle$ and $\langle \frac{1}{2}(1)\rangle$, respectively.

Another result which makes all the analysis in Sec. IV possible and which follows from the properties of the basic spin irreps is the general branching rule for Sp($2n$)$\supset$Sp(2)$\times$O(n) which is summarized in the following:

$$\left\langle \frac{k}{2}(\nu)\right\rangle_n \downarrow \sum_m \left\langle \frac{nk}{2}(m)\right\rangle_1 [((m/C)\circ({}_s\nu^k\cdot D)_k)_n/D]_n , \tag{B6}$$

where the symbol $\circ$ signifies the Schur function operation of inner multiplication, known more in association with Kronecker products of symmetric group S(N) irreps. The C series was introduced in Appendix A. Fortunate-

ly, there is some simplification in the evaluation of $[(m/C)\circ({}_s\nu^k\cdot D)_k]_n$. First, since (m) is only a one-part partition, $(m/C)=(m)-(m-2)$ for $m\geq 2$ with the special cases $(0/C)=(0)$, $(1/C)=(1)$. Second, ${}_s(\nu)^k$ with partitions restricted to at most k parts becomes a finite sum. Third, the inner products are particularly easy since for m integer $(m)\circ(\nu)=(\nu)$, where (ν) is any partition of weight m. Hence, for given m the purpose of the inner product is to extract from $({}_s\nu^k\cdot D)_k$ those terms of weight m or $m-2$. The terms arising from the inner products must also be standard labels of U(n) hence the subscript n in $[(m/C)\circ({}_s\nu^k\cdot D)_k]_n$. As a consequence we could write

$$[(m/C)\circ({}_s\nu^k\cdot D)_k]_n=({}_s\nu^k\cdot D)_K^m-({}_s\nu^k\cdot D)_K^{m-2} \tag{B7}$$

with $K=\min(k,n)$, where the superscript after the parentheses determines the weight restriction and the subscript determines the combined part restriction. If (ν) is a partition of even (odd) weight, the weights of the partitions appearing in $({}_s\nu^k\cdot D)$ are also even (odd) and therefore m must be accordingly an even (odd) integer. Finally, the division operation by D renders this series finite by restricting the terms to partitions which are no greater than the partitions appearing in the expression $[(m/C)\circ({}_s\nu^k\cdot D)_k]_n$.

Letting $w(\nu)=w$, the leading term in the reduction as determined by the smallest Sp(2) irrep can be seen to be

$$\left\langle \frac{k}{2}(\nu)\right\rangle_n \downarrow \left\langle \frac{nk}{2}(w)\right\rangle_1 \times[\nu/D]_n+\cdots \tag{B8}$$

giving an alternative justification for the labeling of the Sp($2n$) irreps from an O(k) viewpoint.

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