Weak-localization effects and conductance fluctuations: Implications of inhomogeneous magnetic fields

Daniel Loss*
IBM Thomas J. Watson Research Center, Yorktown Heights, New York 10598, and Department of Physics, Simon Fraser University, Burnaby, British Columbia, Canada† V5A 1S6

Herbert Schoeller‡
Institut für Theoretische Festkörperphysik, Universität Karlsruhe, Engesserstrasse 7, Postfach 6980, 7500 Karlsruhe, Germany

Paul M. Goldbart§
Department of Physics, University of Illinois at Urbana-Champaign, 1110 West Green Street, Urbana, Illinois 61801-3080
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Low-temperature transport in disordered conductors exhibits a variety of fascinating quantum-mechanical interference effects associated with the phenomenon of weak localization. Such effects are typically isolated and probed by virtue of their sensitivity to applied homogeneous magnetic fields, which introduce Aharonov-Bohm phase factors into quantum-mechanical amplitudes. Analogous interference effects have been proposed in the context of the quantum transport of (possibly electrically neutral) particles with spin in the presence of inhomogeneous magnetic fields, which have the effect of introducing Berry phases. Thus, the possibility is raised of isolating and probing quantum interference effects through their sensitivity to the *inhomogeneity* of applied magnetic fields. In this paper we develop an approach to the study of quantum transport in disordered conductors in the presence of almost arbitrarily inhomogeneous magnetic fields, which is based on diagrammatic and semiclassical path-integral techniques and a subsequent adiabatic approximation. We illustrate these ideas with applications to three examples: anomalous weak-field magnetoconductance, conductance oscillations in mesoscopic multiply connected structures, and sample-dependent mesoscopic conductance fluctuations. Among other things, we find that while in the context of the disorder-averaged conductance it is accurate to regard systems as being composed of two independent subsystems (having spins aligned or antialigned with the local external magnetic field) a more interesting and refined structure emerges in the context of conductance fluctuations.

I. INTRODUCTION AND OVERVIEW

The investigation of quantum-mechanical contributions to the low-temperature conductivity of disordered conductors has revealed a number of fascinating quantum interference phenomena that can be observed in experiments on macroscopic and mesoscopic samples. Notable examples include the anomalous weak-field magnetoconductance of macroscopic metallic films; magnetic-flux-dependent oscillations (with period $hc/2e$) in the conductance of mesoscopic metallic hollow cylinders; and stochastic variations (e.g., with magnetic field) in the conductance of mesoscopic metallic samples, with a range of order e^2/h, commonly referred to as universal conductance fluctuations (UCF's). Typically, these phenomena can be probed and isolated by taking advantage of their sensitivity to *homogeneous* magnetic fields that are weak (on the classical scale) but which nevertheless introduce Aharonov-Bohm phase factors that modify nonclassical (i.e., interference) contributions to the conductance. Several reviews of such phenomena exist; see, e.g., Refs. 1–6.

It was recently anticipated that analogous quantum interference phenomena should be observable in disordered conductors, due to the orbital motion of spinning (although possibly neutral) particles through *orientationally inhomogeneous* magnetic fields, rather than of charged particles through homogeneous magnetic fields.[7,8] Quantum interference, it was argued, should be modified through the acquisition of Berry (i.e., geometric rather than Aharonov-Bohm) phase factors,[9,10] which are sensitive to the field inhomogeneity. Thus implications of the Berry phase should be identifiable in experiments on the conductance of macroscopic and mesoscopic samples of condensed matter, and it should be possible to isolate them by varying the inhomogeneity of the field. Moreover, in multiply connected structures the geometric (rather than topological) character of the Berry phase should allow the coherent relative modification of quantum amplitudes from Feynman paths of, say, differing winding number, so that it should be possible to observe oscillatory manifestations of such quantum interference in condensed matter.

This point of view was also adopted in a recent paper and thesis by Stern,[11] who analyzed a model consisting of a one-dimensional conducting ring in the special case of a cylindrically symmetric inhomogeneous field. By adopting an adiabatic approximation and subsequently regarding the system as comprising two uncoupled electron gases (with spins parallel or antiparallel to the local magnetic field), Stern concurred that implications of the Berry phase should arise in the context of mesoscopic rings. In particular, the conductance was indeed seen to oscillate as the field inhomogeneity is varied, a criterion for the validity of the adiabatic approximation was discussed, and some comments were made regarding sample-specific conductance fluctuations.

In addition to influencing transport phenomena, it has long been appreciated that there are equilibrium implications of Aharonov-Bohm phase factors, most notably the occurrence of persistent charge currents during phase-coherent ballistic or diffusive orbital motion in multiply connected structures threaded by magnetic flux; see, e.g., Refs. 4, and 12–14. Similarly, equilibrium implications of the Berry phase have been investigated, and it has been predicted that persistent currents of charge and spin should occur in multiply connected mesoscopic conductors in the presence of inhomogeneous magnetic fields.[7,15–17]

The aim of the present paper is to give a detailed theory of quantum transport in disordered conductors, incorporating the effects of inhomogeneous magnetic fields on the motion of particles with spin. As we shall see, analogous quantum interference phenomena do indeed emerge as a result of such fields. We illustrate our results by focusing on three examples: the anomalous weak-field magnetoconductance of macroscopic metallic films, oscillations in the conductance of metallic mesoscopic rings and hollow cylinders, and, most importantly, stochastic variations in the conductance of mesoscopic metallic structures. Although we also anticipate implications of quantum interference in the low-temperature transport of mass and spin by *neutral* particles through disordered media (such as the normal Fermi liquid ^{3}He through Vycor) our focus here will be on the low-temperature conductivity of normal disordered metallic conductors.

Our strategy will be to adopt the diagrammatic impurity technique[19] in combination with an extension of the semiclassical Feynman-path-integral approach to weak-localization theory advocated by Chakravarty and Schmid[3] in their substantial elaboration of ideas originally formulated by Larkin and Khmel'nitskii[18] and by Bergmann.[1] Among the virtues of this approach, which we extend to the present situation, are (i) its suitability for the formulation of the adiabatic approximation to the spin dynamics and, in particular, the ability to address essentially *arbitrary* magnetic fields (and not solely field configurations with a high degree of symmetry); and (ii) the opportunity of deferring the adiabatic approximation to the spin dynamics until after the orbital motion has been approximated as diffusive and, thus, the opportunity to assess the range of validity of the adiabatic approximation *within* the weak-localization regime in a self-consistent and controlled manner. What emerges is a criterion for the validity of this adiabatic approximation that is considerably weaker than the condition derived by Stern.[11] Hence, even without invoking an enormous g factor, the opposing requirements that the magnetic field be strong enough to enforce adiabaticity of the spin dynamics but not strong enough to cause the breakdown of the conventional theory for weak localization and UCF's are seen to be quite compatible, which is crucial for the experimental observability of effects due to the inhomogeneous field.

The organization of this paper is as follows. In Sec. II we define the model and introduce appropriate correlators, i.e., disorder-averaged products of single-particle Green functions. We evaluate these correlators explicitly using diagrammatic and semiclassical Feynman-path-integral techniques in Sec. III, in which we also analyze the adiabatic approximation and its range of validity. In Sec. IV we illustrate the results of Secs. II and III with applications to systems embedded in inhomogeneous magnetic fields. First, we consider a geometrical analogue of anomalous weak-field magnetoconductance due to weak-localization effects in simply connected structures. Then we address weak-localization effects in multiply connected structures, i.e., rings and cylinders, and the consequent magnetoconductance oscillations. Finally, and most importantly, we turn to implications of unconjugated quantum Aharonov-Bohm and Berry phases, and analyze issues concerning conductance fluctuations and associated oscillations in multiply connected structures. By and large, technical details concerning diagrammatic computations and the semiclassical method are relegated to a pair of appendixes.

II. MODEL HAMILTONIAN AND CORRELATORS

A. Model

We consider a d-dimensional sample of disordered conductor, which is embedded in an *inhomogeneous* magnetic field $\mathbf{B}(\mathbf{r})$ that couples to the spin of the electrons via the Zeeman interaction and leads to a nontrivial effective coupling between orbital and spin motion. The single-particle Hamiltonian H for this system is given by

$$H = \frac{1}{2m}\left|\mathbf{p} - e\mathbf{A}^{\mathrm{em}}(\mathbf{r})\right|^2 + V(\mathbf{r}) - \frac{1}{2}g\mu_B\,\mathbf{B}(\mathbf{r})\cdot\boldsymbol{\sigma}, \tag{2.1}$$

where m, e, $\mathbf{p}$, $\mathbf{r}$, g, and $\hbar\boldsymbol{\sigma}/2$ are, respectively, the electron mass, charge, canonical momentum, position, g factor, and spin. The operator $V(\mathbf{r})$ represents the (spin-independent) random impurity potential, μ_B is the Bohr magneton, $\mathbf{A}^{\mathrm{em}}$ is the electromagnetic vector potential, with $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}^{\mathrm{em}}$ relating it to the magnetic field, and σ^i (with $i = 1, 2, 3$) are the Pauli spin operators. We shall neglect electron-electron interactions and band-structure effects, as well as alternative spin-scattering mechanisms.

As is well known (see, e.g., Ref. 4), the electromagnetic potential $\mathbf{A}^{\mathrm{em}}$ occurring in the kinetic energy gives rise to a spin-independent Aharonov-Bohm phase, when evaluated in the semiclassical approximation, whereas the Zeeman interaction, when evaluated in an adiabatic approximation defined below, results in a spin-dependent Berry (i.e., geometric) phase. If the magnetic field is homogeneous then the Berry phase vanishes (modulo 2π).

In previous work[15,16] the model defined by Eq. (2.1) has been studied in the context of a one-dimensional isolated metal ring embedded in an inhomogeneous magnetic field. It has been shown that at low temperatures this system carries persistent charge and spin currents in thermodynamic equilibrium, caused by the Berry phase. These persistent currents have been calculated explicitly for the ballistic regime, i.e., in the absence of the impurity potential V.

Here, on the other hand, we are interested in the influence of the spatial inhomogeneity of the static magnetic field (via the Berry phase) on the low-temperature diffusive transport (i.e., nonequilibrium) properties of systems with a variety of geometries, i.e., their linear response to external current-generating forces. In particular, we shall explore the influence of the Berry phase on various phenomena associated with the physics of weak localization, e.g., caused by interference of Feynman paths with their time-reversed partners. As usual, the weak-localization regime is characterized by the requirement that the electronic mean free path l, due to elastic collisions with randomly located impurities, be much larger than the Fermi wavelength λ_F, with the consequence that electron propagation through the disordered medium can be treated semiclassically. Moreover, as we shall consider the diffusive regime, we impose the conditions $l \ll L \ll \xi$, where L is a characteristic sample dimension and (where appropriate) ξ is the localization length. In addition, we shall only consider situations where the (nonquantal) dynamical effect of the magnetic field on the classical trajectory of the particle is negligible. Such situations are readily met, requiring (i) that the cyclotron radius $r_c = mv_F/eB$, in which v_F is the Fermi velocity, be much larger than l (i.e., that the Lorentz force at most barely curves the classical particle trajectories between elastic scattering events); and (ii) that the typical variation of the magnetic field between two such events be much less than the characteristic magnetic field $B_F = \epsilon_F/g\mu_B$ associated with the Fermi energy ϵ_F (so that neither does the Zeeman force cause significant curvature). For typical metals it is only the first condition that is at all relevant; but it is still not restrictive, as it can be fulfilled for magnetic fields as large as 10 T.

B. Correlators and conductivity

As we neglect electron-electron interactions we shall work in the single-particle Hilbert space and concentrate on the evaluation of the following general frequency-dependent current-current correlator:

$$\gamma^{\mu\nu}(\omega) = \frac{\hbar}{2\pi\Omega m^2 d}\langle \mathrm{Tr}\, S^\mu \mathbf{p} G^r(\epsilon_F + \hbar\omega) S^\nu \mathbf{p} G^a(\epsilon_F)\rangle. \tag{2.2}$$

Here, ω is the frequency, Ω is the volume of the system, $G^{r/a}(\epsilon) = (\epsilon - H \pm i0)^{-1}$ is the retarded/advanced single-particle Green function, and Tr denotes a trace taken over all single-particle states. The angle brackets $\langle\ \rangle$ denote an average taken over the distribution of randomly located static impurities. Furthermore, $S^0 = 1$ and $S^i = \hbar\sigma^i/2$ (with $i = 1, 2, 3$). Note that $\gamma^{\mu\nu}(\omega = 0)$ is real. Correspondingly, $e^2\gamma^{00}$ is the charge-current–charge-current correlator, which at low temperature and small frequencies represents the leading contribution to the electrical conductivity σ.[19] Indeed, following Ref. 20 the exact Kubo formula for the conductivity can be written as

$$\sigma(\omega) = \frac{e^2\hbar}{2\pi m^2\Omega d}\int_{-\infty}^{\infty} dE\big[n(E+\hbar\omega) - n(E)\big]\,\langle \mathrm{Tr}\,\mathbf{p}G^r(E+\hbar\omega)\mathbf{p}G^a(E)\rangle + i\frac{e^2 n_e}{m\omega} + \frac{e^2\hbar^2}{2\pi m^2\omega\Omega d}\int_{-\infty}^{\infty} dE\, n(E)\,[\langle \mathrm{Tr}\,\mathbf{p}G^a(E)\mathbf{p}G^a(E-\hbar\omega)\rangle - \langle \mathrm{Tr}\,\mathbf{p}G^r(E+\hbar\omega)\mathbf{p}G^r(E)\rangle], \tag{2.3}$$

where n_e is the number of particles per unit volume and $n(E)$ is the Fermi function. In the limit of zero frequency and temperature this expression simplifies considerably. In fact, using the identities $G^r\mathbf{p}G^r = im[G^r, \mathbf{x}]/\hbar$ and $[x_i, p_j] = i\hbar\delta_{ij}$, and writing $\Delta G = G^r(\epsilon_F) - G^a(\epsilon_F)$, we obtain the well-known Greenwood formula[21] for the zero-temperature dc conductivity:

$$\sigma(0) = -\frac{e^2\hbar}{4\pi m^2\Omega d}\langle \mathrm{Tr}\,\mathbf{p}\Delta G\mathbf{p}\Delta G\rangle = e^2\gamma^{00}(0) - \frac{e^2\hbar}{2\pi m\Omega}\mathrm{Re}\,\langle \mathrm{Tr}\, G^r(\epsilon_F)\rangle, \tag{2.4}$$

where Re denotes the real part. The final representation for the conductivity in Eq. (2.4) provides a most convenient starting point for the following analysis. As its second term (i.e., $\mathrm{Re}\,\langle \mathrm{Tr}\, G^r\rangle$) is of higher order in $1/k_F l$,[22] where k_F is the Fermi wave vector, we need only retain the current-current correlator $e^2\gamma^{00}$, which turns out to contain the Boltzmann value as well as the weak-localization correction (see below). This also holds in the calculation of conductance fluctuations, where only the current-current correlator contributes, to leading order (see Appendix B).

γ^{ij}, on the other hand, is the spin-current–spin-current correlator, which determines the spin diffusion coefficient at zero temperature,[23] i.e., $D_s^{ij} = \gamma^{ij}(\omega = 0)$. Although we are, in the present paper, primarily concerned with electronic systems and their characterization by electrical conductivity σ, we give results for the general correlator $\gamma^{\mu\nu}$.

III. COOPERON/DIFFUSON PROPAGATORS AND ADIABATIC APPROXIMATION

A. Cooperon/diffuson propagators

In this section we analyze the current-current correlator $\gamma^{\mu\nu}$ explicitly, in the weak-localization regime, by isolating the leading quantum correction $\Delta\gamma^{\mu\nu}$ to the classical (i.e., Boltzmann) value. We shall focus our discussion on the quantum correction (and not the Boltzmann contribution) because it is this quantity that is most sensitive to the presence of Aharonov-Bohm and Berry phase factors. As we shall show, this quantum correction is determined by a generalized "cooperon" propagator, which acquires spin dependence via the Zeeman interaction. We also calculate the "diffuson" propagator, which will be used in the following section for the discussion of UCF's in the presence of an inhomogeneous magnetic field.

We perform the calculation using the diagrammatic impurity technique[19,24] in combination with a semiclassical analysis in terms of path integrals.[18,1,3] This approach allows us to treat the Zeeman interaction with the inhomogeneous magnetic field in such a manner that we can postpone the adiabatic approximation until the orbital motion has been approximated as Brownian. Such a treatment has the important virtue that it allows us to assess the validity of the adiabatic approximation self-consistently, within the context of weak-localization theory. Deferring technical details of the diagrammatic and path-integral calculation to Appendix A, we find that the leading quantum correction to $\gamma^{\mu\nu}$ is given by the cooperon:

$$\Delta\gamma_C^{\mu\nu}(\omega) = -\frac{D}{\pi\hbar}\int dt\, e^{i\omega t}\, C^{\mu\nu}(t),$$
$$C^{\mu\nu}(t) = \frac{1}{\Omega}\int d\mathbf{x} \sum_{\alpha_1,\ldots,\alpha_4} S^{\mu}_{\alpha_1\alpha_4} S^{\nu}_{\alpha_3\alpha_2}\, \chi^{C}_{\alpha_1\alpha_2,\alpha_3\alpha_4}(\mathbf{x},\mathbf{x};t,0), \tag{3.1}$$

where $\alpha_k = \pm 1$ are spin indices, and where the time integral carries implicit lower and upper limits of the elastic scattering time τ and the dephasing time τ_φ, respectively. The diffusion constant for particles at the Fermi level is given by $D = v_F^2\tau/d$. As discussed in Appendix A, the propagator $\chi^{C/D}$ obeys the cooperon/diffuson differential equation with the *exact* Zeeman term, the path-integral solution of which [see Eqs. (A16) and (A18)] is explicitly given by

$$\chi^{C/D}_{\alpha_1\alpha_2,\alpha_3\alpha_4}(\mathbf{z}_f,\mathbf{z}_i;t_f,t_i)$$
$$= \theta(t_f - t_i)\int_{\mathbf{R}(t_i)=\mathbf{z}_i}^{\mathbf{R}(t_f)=\mathbf{z}_f} \mathcal{D}\mathbf{R}\exp\left\{-\frac{1}{4D}\int_{t_i}^{t_f} dt|\dot{\mathbf{R}}|^2 + i\frac{e}{\hbar}\int_{t_i}^{t_f} dt\,\dot{\mathbf{R}}\cdot\{\mathbf{A}^{\rm em}(\mathbf{R}(t)) \pm \tilde{\mathbf{A}}^{\rm em}(\mathbf{R}^{\mp}(t))\}\right\}$$
$$\times\left\langle\alpha_4\left|\mathcal{T}\exp\left\{i\frac{g\mu_B}{2\hbar}\int_{t_i}^{t_f} dt\,\mathbf{B}(\mathbf{R}(t))\cdot\boldsymbol{\sigma}\right\}\right|\alpha_3\right\rangle\left\langle\alpha_1\left|\mathcal{T}\exp\left\{i\frac{g\mu_B}{2\hbar}\int_{t_i}^{t_f} dt\,\tilde{\mathbf{B}}(\mathbf{R}^{\mp}(t))\cdot\boldsymbol{\sigma}\right\}\right|\alpha_2\right\rangle^{*} \tag{3.2}$$

where $\mathcal{T}$ is the time-ordering operator and $\mathbf{R}^{\alpha}(t) = \mathbf{R}[\alpha t + (1-\alpha)(t_f + t_i)/2]$, so that $\mathbf{R}^{-}(t)$ is the time-reversed (i.e., conjugate) partner of the Brownian path $\mathbf{R}^{+}(t)$ $[= \mathbf{R}(t)]$. Note that for later convenience we have allowed for the possibility that conjugate paths $\mathbf{R}(t)$ propagating between $\mathbf{z}_i$ and $\mathbf{z}_f$ do not necessarily experience the same fields, i.e., $\tilde{\mathbf{A}}^{\rm em}$ and $\mathbf{A}^{\rm em}$ can differ. This has the consequence that there can be nontrivial interference terms (due to phase factors) for not only the cooperon but also the diffuson propagator. Such a situation with different fields is met, e.g., in the calculation of conductance fluctuations, as we shall see in the next section. For the weak-localization correction to the disorder-averaged conductance itself, however, the fields should be chosen identical. Also note that at this stage propagation through the random medium is described as the Brownian motion of a particle with diffusion constant D.

B. Adiabatic approximation

We are now in a position to address the evaluation of the cooperon/diffuson propagator, Eq. (3.2), in the presence of the Zeeman term that depends on the path $\mathbf{R}$ in a nontrivial way via the inhomogeneous magnetic field $\mathbf{B}(\mathbf{R})$. In general, this path integral cannot be evaluated exactly. However, we shall now concentrate our further considerations on systems that are in the adiabatic regime. Roughly speaking, in this regime the orbital diffusive motion of the particle and the magnetic field must be such that, as the particle diffuses, the field variation occurs over a sufficiently long length scale so as to allow the spin to maintain alignment along the local direction of the magnetic field (a detailed elaboration of this issue follows below). If this is the case then we can make considerable progress by evaluating the Zeeman term using the adiabatic approximation, in a manner similar to that first discussed by Berry.[9] In particular, the path integral in Eq. (3.2) is very similar in form to one studied previously[16] in the context of imaginary-time propagators for one-dimensional rings in thermodynamic equilibrium. The results derived there (up to Sec. V of Ref. 16) can readily be extended to the present situation of arbitrary sample dimension and shape. The technically trivial but physically important alteration is that when importing results from Ref. 16 we must make two formal replacements: (i) $\hbar^2/2m \Rightarrow D$ (the diffusion constant); and (ii) $B \Rightarrow B/\hbar$ [as can be seen, e.g., by comparing the effective Hamiltonian occurring in the cooperon/diffuson differential equation, Eq. (A19) of Appendix A, with the Hamiltonian studied in Ref. 16]. Furthermore, introducing the instantaneous eigenstates of the Zeeman interaction, $\mathbf{B}(\mathbf{R}(t))\cdot\boldsymbol{\sigma}|\mathbf{B}(\mathbf{R}(t)),\alpha\rangle = \alpha|\mathbf{B}(\mathbf{R}(t)),\alpha\rangle$, we find for the spin propagator in the adiabatic approximation[16]

$$\left\langle \alpha_f \middle| \mathcal{T}\exp\left\{ \pm i\frac{g\mu_B}{2\hbar}\int_{t_i}^{t_f} dt\,\mathbf{B}(\mathbf{R}(t))\cdot\boldsymbol{\sigma}\right\}\middle|\alpha_i\right\rangle \approx \sum_{\alpha=\pm1}\langle\alpha_f|\mathbf{B}(\mathbf{R}(t_f)),\alpha\rangle\exp\left\{i\Gamma_\alpha[\mathbf{R}]\right\}\exp\left\{\pm i\alpha\frac{g\mu_B}{2\hbar}(t_f-t_i)B\right\}\langle\mathbf{B}(\mathbf{R}(t_i)),\alpha|\alpha_i\rangle, \tag{3.3}$$

where we have specialized to the case of magnetic fields $\mathbf{B}$ with constant magnitude B, and have parametrized $\mathbf{B}$ in terms of the spherical polar angles χ and η, so that it has Cartesian components $B(\sin\chi\cos\eta, \sin\chi\sin\eta, \cos\chi)$, with the angles χ and η being smooth (but otherwise arbitrary) functions of position. The instantaneous eigenstates can then be expressed in terms of χ, η, and the eigenstates $|\alpha=\pm1\rangle$ of σ_z:

$$|\mathbf{B}(\mathbf{R}),\alpha\rangle = |\alpha\rangle\cos(\chi/2) + \alpha|-\alpha\rangle\, e^{i\alpha\eta}\sin(\chi/2). \tag{3.4}$$

The Berry phase associated with the path $\mathbf{R}$ then becomes

$$\Gamma_\alpha[\mathbf{R}] = -\mathrm{Im}\int_{t_i}^{t_f} dt\left\langle \mathbf{B}(\mathbf{R}(t)),\alpha\middle|\frac{\partial}{\partial t}\middle|\mathbf{B}(\mathbf{R}(t)),\alpha\right\rangle = \frac{\alpha}{2}\int_{[\mathbf{R}(\cdot)]} d\mathbf{R}\cdot\boldsymbol{\nabla}\eta(\mathbf{R})\left\{\cos\chi(\mathbf{R})-1\right\}, \tag{3.5}$$

where Im denotes the imaginary part and $[\mathbf{R}(\cdot)]$ denotes the path of integration. By introducing the spin- and position-dependent gauge potential

$$\mathbf{A}^g_\alpha(\mathbf{R}) = -\mathrm{Im}\langle\mathbf{B}(\mathbf{R}),\alpha|\boldsymbol{\nabla}_\mathbf{R}|\mathbf{B}(\mathbf{R}),\alpha\rangle = \frac{\alpha}{2}\boldsymbol{\nabla}\eta(\mathbf{R})\left\{\cos\chi(\mathbf{R})-1\right\}, \tag{3.6}$$

we can express the Berry phase in the form

$$\Gamma_\alpha[\mathbf{R}] = \int_{t_i}^{t_f} dt\,\dot{\mathbf{R}}\cdot\mathbf{A}^g_\alpha(\mathbf{R}(t)), \tag{3.7}$$

analogous to the form of the phase factor resulting from the electromagnetic gauge potential. Thus we see that the Berry phase changes sign under time reversal, i.e.,

$$\Gamma_\alpha[\mathbf{R}^\pm] = \pm\Gamma_\alpha[\mathbf{R}], \tag{3.8}$$

a reflection of the fact that the exact Zeeman term also changes sign under time reversal. It is this change of sign that leads to interference effects in the cooperon propagator, whereas no such effects occur in the diffuson propagator, provided both paths are affected by the same magnetic field (which is not the case when conductance fluctuations are computed; see the following section).

Next we insert the adiabatic approximation for the spin propagator, Eq. (3.3), into the cooperon/diffuson, Eq. (3.2), and obtain

$$\chi^{C/D}_{\alpha_1\alpha_2,\alpha_3\alpha_4}(\mathbf{z}_f,\mathbf{z}_i;t_f,t_i) = \sum_{\alpha\tilde\alpha}\langle\alpha_4|\mathbf{B}(\mathbf{z}_f),\alpha\rangle\,\langle\mathbf{B}(\mathbf{z}_i),\alpha|\alpha_3\rangle\,\langle\tilde{\mathbf{B}}(\mathbf{z}_{i/f}),\tilde\alpha|\alpha_1\rangle\langle\alpha_2|\tilde{\mathbf{B}}(\mathbf{z}_{f/i}),\tilde\alpha\rangle\,\chi^{C/D}_{\alpha\tilde\alpha}(\mathbf{z}_f,\mathbf{z}_i;t_f,t_i), \tag{3.9}$$

where

$$\chi^{C/D}_{\alpha\tilde\alpha}(\mathbf{z}_f,\mathbf{z}_i;t_f,t_i) = \theta(t_f-t_i)\int_{\mathbf{R}(t_i)=\mathbf{z}_i}^{\mathbf{R}(t_f)=\mathbf{z}_f}\mathcal{D}\mathbf{R}\,\exp\left\{-\frac{1}{4D}\int_{t_i}^{t_f}dt|\dot{\mathbf{R}}|^2 + i\int_{t_i}^{t_f}dt\,\dot{\mathbf{R}}\cdot\{\mathbf{A}_\alpha(\mathbf{R}(t))\pm\tilde{\mathbf{A}}_{\tilde\alpha}(\mathbf{R}(t))\} + i\frac{g\mu_B}{2\hbar}(t_f-t_i)\left(\alpha B-\tilde\alpha\tilde B\right)\right\}. \tag{3.10}$$

Note that the electromagnetic and geometric gauge potentials simply add, and have therefore been combined into a single spin-dependent effective gauge potential $\mathbf{A}_\alpha$, defined by

$$\mathbf{A}_\alpha = \frac{e}{\hbar}\mathbf{A}^{\rm em} + \mathbf{A}^g_\alpha. \tag{3.11}$$

When necessary, explicit expressions for the spin matrix elements occurring in Eq. (3.9) can be obtained from Eq. (3.4). However, we shall see later that, as a result of various spin sums, these spin matrix elements do not appear in the final expressions describing the transport of charge.

Finally, reversing the earlier procedure of solving the cooperon/diffuson differential equation in terms of a path integral, it is not difficult to see that the propagator $\chi^{C/D}_{\alpha\tilde\alpha}$, given in Eq. (3.10), obeys a new cooperon/diffuson differential equation, which is characterized by the presence of this spin-dependent gauge potential $\mathbf{A}_\alpha$:

$$\left\{\frac{\partial}{\partial t'} + D\left[-i\frac{\partial}{\partial \mathbf{x}'} - \left\{\mathbf{A}_\alpha(\mathbf{x}') \pm \tilde{\mathbf{A}}_{\tilde\alpha}(\mathbf{x}')\right\}\right]^2 - i\frac{g\mu_B}{2\hbar}(\alpha B - \tilde\alpha\tilde B)\right\}\chi^{C/D}_{\alpha\tilde\alpha}(\mathbf{x}',\mathbf{x};t',t) = \delta(\mathbf{x}'-\mathbf{x})\,\delta(t'-t). \tag{3.12}$$

It is quite striking that in the adiabatic limit the effect of an essentially *arbitrary* inhomogeneous magnetic field can be taken into account in such a compact way via the geometric gauge potential.

C. Validity of adiabatic approximation and weak-localization framework

Before proceeding to specific applications we pause to discuss the conditions under which the adiabatic approximation is valid, and their relationship with the conditions for the validity of the entire framework of weak-localization theory. This is a particularly important aspect of the present work because it enables us to assess the constraints that would have to be satisfied in experiments in order to observe the effects that we shall discuss. What will emerge is a criterion for the validity of this adiabatic approximation that is considerably weaker than the condition derived by Stern.[11] Most significantly, we shall see that even without invoking an enormous g factor the opposing requirements that the magnetic field be (i) strong enough to enforce adiabaticity of the spin dynamics but (ii) not strong enough to cause the breakdown of conventional weak-localization theory are quite compatible.

The physical picture underlying adiabaticity[25] is as follows. As the orbital motion of a particle carries the spin of the particle to a region in which the magnetic field has a significantly different orientation, enough time must elapse for the spin to have undergone many complete precessions, i.e., $\omega_B\tau_o \gg 2\pi$, where the Bohr frequency ω_B is given by $\omega_B = g\mu_B B/\hbar$ and τ_o is the duration of the orbital motion.

We first give a qualitative discussion of the adiabatic criterion appropriate for generic situations of quantum transport, which involves a physical argument concerning precession times. Later we shall verify our conclusions, at least in restricted contexts, via an analogy with the equilibrium statistical mechanics of quasi-one-dimensional rings in cylindrically symmetric magnetic fields. We begin by introducing the length scale ℓ_B (which we assume to be roughly homogeneous) characterizing the typical distance over which the variation of the orientation of the magnetic field is of order unity. A ring of circumference L on which the field reorients of order once would have $\ell_B \sim L$. Next, consider a typical diffusive path of spatial size Λ, which is completed in a time $t_\Lambda \approx \Lambda^2/D$. Adiabaticity then requires $\omega_B t_\Lambda \gg 2\pi\Lambda/\ell_B$, where the factor Λ/ℓ_B accounts for the number of field reorientations across the distance Λ, i.e.,

$$\omega_B\tau \gg \frac{2\pi}{d}\,\frac{l}{\ell_B}\,\frac{l}{\Lambda}, \tag{3.13}$$

where we have used $D = v_F^2\tau/d$. The factors l/ℓ_B and l/Λ are most significant: typically they make the criterion *far less stringent* than $\omega_B\tau \gg 2\pi/d$ (which would demand many complete precessions between each elastic scattering event).

We now show how an equivalent criterion emerges from a formal relationship between the cooperon of the present transport problem and the partition function of Ref. 16, in which the criterion for adiabaticity was discussed in the context of equilibrium statistical mechanics. There it was found that for ballistic motion on a one-dimensional ring of radius r adiabaticity required

$$\frac{g\mu_B B}{k_B T} \gg \left(\frac{mr^2 k_B T}{\hbar^2}\right)^{-1/2}, \tag{3.14}$$

where k_B is Boltzmann's constant and T is the temperature, provided that the geometric flux was of order unity (i.e., the orientation of the magnetic field makes of order one complete variation around the ring), i.e., $\omega_B\sqrt{mr^2/k_BT} \gg 1$. Physically, what is required is that the magnitude of the magnetic field should be sufficiently large that the spin should precess many times during the time taken by a particle with kinetic energy of order k_BT to orbit the ring. Particularly noteworthy is the fact that the well-known roughness of the (imaginary-time) Feynman paths does not itself immediately lead to the violation of the adiabatic approximation, at least in the context of the calculation of the thermal Green function G (and hence the partition function Z). To import results from Ref. 16 we note the formal relationship between the cooperon, Eq. (3.2)—a transport quantity—and the equilibrium thermal Green function, Eq. (4.15) of Ref. 16: (i) $\chi^C \sim G$; (ii) $t \sim \hbar/k_BT$; (iii) $D \sim \hbar/2m$. This relationship emerges after the Feynman path integral for the orbital motion of the transport problem has been approximated semiclassically, and the disorder average has been performed, so that the orbital motion is represented by Brownian motion and the cooperon is expressed as a Wiener path integral. The crucial point is that the spin motion has thus far been treated exactly.

Suppose that we choose to make the adiabatic approximation at this stage, and ask the question: under what criterion is the adiabatic approximation valid? The answer *within the context of weak-localization theory* may be found by analogy with the equilibrium case, i.e., by comparison of the kinetic and Zeeman terms in the cooperon path integral, Eq. (3.2). Thus, by using Eq. (3.14), we see that the validity of the adiabatic approximation for the computation of the cooperon χ^C for diffusive motion of duration t on a ring of radius r requires $g\mu_B Bt/\hbar \gg (r^2/Dt)^{-1/2}$, i.e.,

$$\omega_B\tau \gg \sqrt{\frac{\tau}{t}}\,\frac{l}{r}, \tag{3.15}$$

which is equivalent to the criterion obtained above, Eq. (3.13), for the appropriate choice $\ell_B \approx 2\pi r$. Just as the roughness of the (imaginary-time) Feynman paths does not violate adiabaticity, neither does the roughness of the Wiener paths that simulate the Brownian motion. And just as for the equilibrium case, this criterion can (in principle) be confirmed by an exact calculation of the cooperon with the Zeeman term for the case of a cylindrically symmetric field configuration, the only modification being that the doubling of the number of spin degrees of freedom requires the diagonalization of a 4×4 matrix. Since no new insight is gained from such a calculation we shall not pursue this direction here.

Thus far we have been considering adiabaticity with regard to the cooperon. We now address the fact that to compute the leading quantum correction to the conductivity the cooperon must be integrated over times from τ to τ_φ, i.e., contributions must be collected from paths of all lengths Λ for which our description is valid, i.e., $l \lesssim \Lambda \lesssim L_\varphi$, where L_φ $\left[\equiv \sqrt{D\tau_\varphi}\right]$ is the diffusive dephasing length. In one and two dimensions the dominant contribution arises from times of order τ_φ. Observe that the criterion $\omega_B\tau \gg (2\pi/d)\,(l/\ell_B)\,(l/\Lambda)$ is softer for longer diffusive paths: in diffusive processes extra time produces disproportionately little extra distance. For the shortest paths $\Lambda \sim l$ so that adiabaticity requires $\omega_B\tau \gg (2\pi/d)\,(l/\ell_B)$; for the longest paths $\Lambda \sim L_\varphi$ so that adiabaticity requires $\omega_B\tau \gg (2\pi/d)\,(l/\ell_B)(l/L_\varphi)$.

Now envisage increasing the degree of field inhomogeneity (i.e., decreasing ℓ_B) until $\ell_B \sim L_\varphi$, so that the longest of paths contributing to the conductivity correction begin to be affected by the field inhomogeneity. Provided the adiabatic criterion is satisfied for such paths we will begin to see effects of the field inhomogeneity as described in this paper, e.g., in the magnetoconductance of a film. In other words, at least when $\ell_B \sim L_\varphi$ only the very mild criterion $\omega_B\tau \gg (2\pi/d)\,(l/L_\varphi)^2$ must be satisfied. As the field inhomogeneity is increased (i.e., ℓ_B is decreased) the criterion does harden, but it only reaches the extreme case $\omega_B\tau \gg 2\pi/d$ under the most extreme of conditions, i.e., when $\ell_B \sim l$ and the shortest paths $\Lambda \sim l$ are considered.

Multiply connected geometries afford a useful simplification of this issue of the validity of the adiabatic approximation because the Brownian paths then fall into discrete sectors according to winding number. The essential consequence is that only topologically nontrivial paths contribute to the flux sensitivity, e.g., of the magnetoconductance of a structure with a hole of circumference L, and the shortest of these paths has an associated time scale of order L^2/D, i.e., the time taken to diffuse around the hole, rather than τ. All contributions to the flux sensitivity then obey the adiabaticity criterion provided that $\omega_B L^2/D \gg 2\pi\,(L/\ell_B)$, i.e., $\omega_B\tau \gg (2\pi/d)\,(l/\ell_B)(l/L)$, if the field reorients roughly L/ℓ_B times as the ring is circumnavigated. Equivalently, this criterion reads $g\mu_B B \gg (L/\ell_B)E_{\rm Th}$, where $E_{\rm Th} \equiv hD/L^2$ is the so-called Thouless energy. This criterion agrees with the one identified by analogy with the equilibrium case.

Now, if the criterion for the validity of the adiabatic approximation had turned out to be[11] $\omega_B\tau \gg 1$, rather than the much softer criterion that we have found, then the whole theory considered here would break down for the following reasons. Recall that for electrons the Bohr frequency ω_B and the Larmor frequency ω_L differ by the g factor. Thus, unless g were vastly larger than unity, the adiabatic criterion $\omega_B\tau \gg 1$ would enforce $\omega_L\tau \gg 1$. Then the high magnetic field, while good for producing rapid spin precession, would cause strong curvature of the semiclassical paths between elastic scattering events, and the entire weak-localization picture would be in need of refinement. In particular, the *orbital* dependence of the disorder-averaged single Green function could no longer be accurately approximated by its zero-field value, and in fact a Landau level picture would be a more appropriate starting point[11] (for a discussion of UCF's in strong homogeneous magnetic fields see Refs. 36 and 37). Thus, save the demand of an enormous g factor, a criterion $\omega_B\tau \gg 1$ would self-consistently predict the breakdown of the entire theory.

Indeed, the criterion $\omega_B\tau \gg 1$ would be fatal to the theory for an even less avoidable reason. Even if the condition $\omega_L\tau \gg 1$ (and the consequent strong perturbation of the orbital motion) could be evaded by virtue of a large g factor, to have $\omega_B\tau \gg 1$ would invalidate the standard assumption that the *spin* dependence of the disorder-averaged single Green function is negligibly affected by the magnetic field. More precisely, although this Green function has a range of order l, such a strong Zeeman coupling would perturb the *spin* motion at this length scale, the resulting spin precession causing significant alteration of the spin dependence of this Green function. Consequently, in order to avoid such nontrivial alterations of the whole formalism, we are actually forced to assume that $\omega_B\tau \ll 1$, which allows us then to approximate $\langle G\rangle$ by its zero-magnetic-field form (except in the integral equation for the cooperon/diffuson propagator), as we have done throughout our analysis in Appendixes A and B. However, by first approximating the motion as diffusive and then making the adiabatic approximation we have found that the self-consistent criterion for the validity of the adiabatic approximation is very considerably weaker than $\omega_B\tau \gg 1$.

We emphasize again that everything said so far is valid for all spatial dimensions d and for *arbitrary* magnetic textures satisfying the adiabaticity criterion stated above. We have been able to derive this criterion because we have made the adiabatic approximation *after* the semiclassical approximation in the corresponding path-integral representation of the retarded and advanced Green functions. This order of approximation steps seems to be more difficult to perform within the conventional Green function formalism because the Zeeman term cannot be treated in the same way as the electromagnetic gauge potential.[26]

Finally, for the sake of illustration, we consider the implications of the adiabatic criterion for parameters typical of metallic cylinders studied experimentally.[14] Supposing that $L \approx \ell_B$ we find the criterion $\omega_B \tau_L \gg 2\pi$, where τ_L is the time taken to diffuse once around the ring. If we further assume that $L \approx 7\,\mu$m, and that $v_F \approx 1.4 \times 10^6$ ms^{-1} and $l \approx 70$ nm so that $\tau \approx 50$ fs, then adiabaticity requires magnetic fields of magnitude greater than 240 G.[27]

For the sake of comparison we remark that the criterion $\omega_B \tau \gg 2\pi/d$ would require unattainably strong magnetic fields, exceeding 240 T, i.e., a factor of 10^4 larger, rendering the Berry phase effects described here completely unobservable. The physical picture corresponding to this important conclusion is that the diffusive length L^2/l is two orders of magnitude larger than the circumference L, so that the strength of the magnetic field required for adiabaticity is two orders of magnitude *smaller* for diffusive systems than it is for ballistic systems. Thus diffusive motion requires more time for the completion of orbits, and hence provides more time for spin precession around the local field direction, which preserves adiabatic alignment.

IV. APPLICATIONS

In this section we shall apply the results of the previous sections, together with certain standard results, to three illustrative examples of experimental interest concerning transport in disordered conductors in inhomogeneous magnetic fields. First we make some qualitative remarks regarding a geometrical analogue of anomalous weak-field magnetoconductance due to weak-localization effects in simply connected structures. Then we address weak-localization effects in multiply connected structures, i.e., rings and cylinders, and the consequent magnetoconductance oscillations. Finally we turn to implications of unconjugated quantum Aharonov-Bohm and Berry phases, analyzing issues concerning conductance fluctuations and associated oscillations in multiply connected structures.

A. Analogue of anomalous weak-field magnetoconductance

Perhaps the simplest application of conventional (i.e., Aharonov-Bohm phase controlled) weak-localization ideas concerns the anomalous sensitivity of the low-temperature conductance of a macroscopic disordered metallic film to a homogeneous perpendicular magnetic field. As discussed, e.g., in Ref. 24, in the absence of spin-orbit scattering the leading quantum correction to the Boltzmann value *decreases* the conductance (in other words, the interference of time-reversed paths enhances backscattering, i.e., suppresses diffusion). Via the Aharonov-Bohm phase the application of a perpendicular magnetic flux can moderate this quantum correction, as it destroys the constructive interference between terms each associated with a pair of conjugate paths. As the strength of the magnetic field is increased, the quantum correction begins to be eliminated at the scale of fields B_0 for which the largest phase-coherent paths enclose roughly one quantum of electromagnetic flux, i.e., $B_0 D\tau_\varphi \sim \Phi_0$, where Φ_0 $\left[\, \equiv hc/e\right]$ is the elementary flux quantum. As the strength of the field is increased beyond B_0 (while remaining weak on the classical scale) the conductance continuously increases towards its Boltzmann value.

What is the analogue of this suppression of the quantum correction to conductance in the presence of an inhomogeneous magnetic field, i.e., due instead to the geometric phase? To answer this question it is useful to note from Eq. (3.6) that $A^g \sim |\boldsymbol{\nabla}(\mathbf{B}/B)| \equiv \ell_B^{-1}$, i.e., the inverse field-reorientation length (up to a pure geometric gauge term that does not contribute to the geometric flux through the path). The essence is then identical: the quantum correction begins to be eliminated when A^g rises to such a scale that the largest phase-coherent paths enclose roughly one quantum of Berry flux, i.e., $A^g\sqrt{D\tau_\varphi} \sim 1$ or, equivalently, $\ell_B \sim L_\varphi$. In other words, just as one would anticipate, as ℓ_B is decreased (i.e., the inhomogeneity of the field is increased) the quantum contribution begins to be eliminated when ℓ_B is reduced below L_φ. Further increase of the inhomogeneity causes the conductance to increase continuously towards its Boltzmann value. (For an estimate of the scale of magnetic fields typically required to observe this effect see the discussion at the end of the following section.)

B. Magnetoconductance oscillations

We now turn to implications of weak localization in contexts where the paths can be classified into sectors so that, under appropriate circumstances, all (or at least most) paths in a given sector acquire more or less the same modulation of quantum phase under the application of an inhomogeneous field, and the phases acquired by paths in distinct sectors are simply related. One then has the possibility of observing oscillatory quantum effects, because increasing the (electromagnetic or geometric) flux will sequentially smear and then resharpen the interference contributions from conjugate pairs of paths accumulated in different winding number sectors.

Perhaps the simplest condensed matter contexts in which such effects may arise are multiply connected structures, e.g., as rings and cylinders. Optimally, the circumference of the holes should not greatly exceed the dephasing length L_φ (to avoid strong attenuation of the amplitude of oscillations), and the radial (i.e., wall) thickness should be sufficiently small that the geometric flux penetrating the sample itself (e.g., due to radial variations in the field inhomogeneity) does not strongly smear the contributions from a given winding number sector.

We introduce the conductance $g = a_d\sigma$ of a d-dimensional system, where the appropriately chosen geometrical factor a_d relates the conductance to the conductivity σ. We denote the weak-localization correction to this conductance by $\Delta g(\omega) = a_d e^2 \Delta\gamma_C^{00}(\omega)$, where $\Delta\gamma_C^{00}$ is given by Eq. (3.1), with the cooperon propagator χ^C of Eq. (3.12) carrying identical fields, i.e., $\mathbf{A}^{\rm em} = \tilde{\mathbf{A}}^{\rm em}$ and $\mathbf{B} = \tilde{\mathbf{B}}$. Under these particular circumstances (viz., identical fields and closed paths), the sum over spin matrix elements is elementary, and leads to

$$\Delta g = -a_d \frac{e^2 D}{\pi\hbar}\int_\tau^{\tau_\varphi} dt\, C^{00}(t),$$

$$C^{00}(t) = \sum_\alpha \chi^C_{\alpha\alpha}(\mathbf{x},\mathbf{x};t,0) = \sum_\alpha \int_{\mathbf{R}(0)=\mathbf{x}}^{\mathbf{R}(t)=\mathbf{x}} \mathcal{D}\mathbf{R}\exp\left\{-\frac{1}{4D}\int_0^t d\tau|\dot{\mathbf{R}}|^2 + 2i\int_0^t d\tau\,\dot{\mathbf{R}}\cdot\mathbf{A}_\alpha(\mathbf{R}(\tau))\right\}, \tag{4.1}$$

where we have invoked the translational invariance of χ^C. Note that the dynamical Zeeman terms have canceled, whereas the gauge-field terms have reinforced, thus producing the factor of 2 in the gauge-field term in the exponent. From this form of the cooperon it is evident that the "up-spin" (i.e., the spin component parallel to the local field direction) and "down-spin" (i.e., the antiparallel spin component) contributions *are entirely uncorrelated* and thus can be treated as independent species,[7] as observed by Stern.[11] We emphasize, however, that such a decoupling of the two spin channels occurs only under the particular circumstances mentioned prior to Eq. (4.1); as we shall see shortly, this property is not shared, for instance, by the expression describing the conductance fluctuations.

We now evaluate the spin-averaged cooperon propagator C^{00} for the illustrative cases of hollow metallic rings and cylinders, having circumference $L = 2\pi r$, height b, and thin walls of thickness $a \ll L_\varphi$. The magnetic field is assumed to have constant magnitude, and not to vary appreciably either radially across the wall or parallel to the (ring or cylinder) axis. In this special case, the effective gauge potential $\mathbf{A}_\alpha$ can be replaced by its tangential component A^ϕ_α (where ϕ is the azimuthal angle), which can be expressed in terms of the associated spin-dependent fluxes (see also Ref. 16):

$$2\pi r A^\phi_\alpha = \Phi_\alpha = \Phi^{\rm em}/\Phi_0 + \alpha\Phi^g, \tag{4.2}$$

where r is the radius of the structure, $\Phi^{\rm em}$ is the electromagnetic flux through the area πr^2, which contributes in units of Φ_0, and the geometric flux (i.e., Berry phase) is given by

$$\Phi^g = -\frac{1}{4\pi}\int_\Sigma d\Omega = \frac{1}{4\pi}\int_0^{2\pi} d\phi\,[\cos\chi(\phi) - 1]\left(\frac{d\eta}{d\phi}\right). \tag{4.3}$$

This formula shows that (up to a factor of minus 4π) the Berry phase is equal to the area of the surface Σ on the unit sphere whose boundary is described by the *direction* of the magnetic field $\mathbf{B}/B$, as this field varies around the circumference of the ring or cylinder. The Berry phase takes a particularly simple form in the case of a cylindrically symmetric texture (i.e., $\eta = \phi$), namely, $\Phi^g = (\cos\chi - 1)/2$, where χ is the (constant) tilt angle away from the cylinder axis.

The subsequent evaluation of the cooperon propagator C^{00} is standard.[28,3,4] As is evident from Eq. (4.1), one simply has to replace the Aharonov-Bohm flux $\Phi^{\rm em}/\Phi_0$ of previous (spinless) calculations by the spin-dependent flux Φ_α and then sum over $\alpha = \pm 1$. We remark that a similar property emerges in the calculation of equilibrium quantities such as the free energy and persistent charge current;[16] this simple replacement rule has therefore been correctly anticipated in Ref. 7 for the case of arbitrary textures and also in Ref. 11 for the special case considered in this subsection. As we shall see, however, this rule fails for conductance fluctuations.

For the sake of completeness we state without elaboration the resulting expressions for the magnetoconductance, following Ref. 4. For this purpose, and also for later use, we note that the cooperon/diffuson propagators are explicitly given by

$$\chi^{C/D}_{\alpha\tilde{\alpha}}(\mathbf{x},\mathbf{x}';t,t') = \frac{1}{\Omega}\sum_{\mathbf{k}}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,\chi^{C/D}_{\alpha\tilde{\alpha}}(\mathbf{k},\omega)\,e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')-i\omega(t-t')}, \tag{4.4a}$$

$$\chi^{C/D}_{\alpha\tilde{\alpha}}(\mathbf{k},\omega) = \frac{1}{-i(\omega+\varpi_{\alpha\tilde{\alpha}}) + \tau_\varphi^{-1} + D\mathbf{k}_\perp^2 + Dr^{-2}[n - (\Phi_\alpha \pm \tilde{\Phi}_{\tilde{\alpha}})]^2}. \tag{4.4b}$$

Here, we have used the definitions

$$\varpi_{\alpha\tilde{\alpha}} = \frac{g\mu_B}{2\hbar}(\alpha B - \tilde{\alpha}\tilde{B}), \tag{4.4c}$$

$\mathbf{k} = (2\pi n/L, \mathbf{k}_\perp)$ and $\mathbf{k}_\perp = (2\pi m/a, 2\pi p/b)$ with n, m, p integral, and the volume Ω is given by $\Omega = abL$. For convenience we have incorporated the upper cutoff of the time integration, τ_φ, directly into the propagator. Note that for the present case of magnetoconductance oscillations the dynamical Zeeman terms cancel because $\varpi_{\alpha\alpha} = 0$ for $B = \tilde{B}$. (This, however, is no longer the case in the expression describing the conductance fluctuations; see below.)

To study the case of a ring of small height $b \ll L_\varphi$, in which diffusion is effectively one dimensional, we insert Eq. (4.4b) into Eq. (4.1) and retain only the $\mathbf{k}_\perp = \mathbf{0}$ term in the summation over wave vectors. Thus, for the weak-localization correction to the conductance (per unit length) along the circumference of a quasi-one-dimensional ring, we find

$$L\,\Delta g(\omega = 0) = -\frac{e^2}{\pi\hbar} L_\varphi^B \frac{1}{2} \sum_\alpha \frac{\sinh(L/L_\varphi^B)}{\cosh(L/L_\varphi^B) - \cos(4\pi\Phi_\alpha)}, \tag{4.5}$$

where we have used $a_d = ab/L$, the magnetic dephasing length L_φ^B is given by

$$\frac{1}{(L_\varphi^B)^2} = \frac{1}{(L_\varphi)^2} + \frac{1}{(L_B)^2}, \tag{4.6}$$

and $L_B = \sqrt{3}\Phi_0/2\pi a B$ accounts for the dephasing effect of magnetic flux penetrating into the sample. The geometric phase gives no contribution to the dephasing length because we have assumed that the field configuration excludes the geometric flux from the sample. Note that the conductance correction is periodic in Φ_α with period 1/2; this includes the well-known periodicity in $\Phi^{\rm em}$ with period $\Phi_0/2$.[28,4] (For different periodicities in $\Phi^{\rm em}$ in *ballistic* rings see Ref. 29.) Equation (4.5) was given by Stern[11] for the case of a certain cylindrically symmetric field configuration.

We remark that L_B should be regarded as a *lower* bound to the true magnetic dephasing length, which ought to incorporate the fact that the penetrating field is generally not parallel to the cylinder axis. For instance, for a cylindrically symmetric texture one would expect that the magnitude B occurring in the above definition of L_B should be replaced by $|B_z|$, as only the z component contributes to the dephasing effect caused by the width of the distribution of magnetic fluxes enclosed by conjugate pairs of paths in the cooperon/diffuson description. The resulting dephasing length L_{B_z} is then in general *larger* than L_B—a situation more favorable for possible experiments. However, we shall ignore such refinements and use the more conservative length L_B for the quantitative estimates given below. For further refinements due to various other possible dephasing mechanisms see Ref. 4.

To illustrate the result (4.5), consider a planar configuration, i.e., $\chi = \pi/2$, say for the ring, with $\eta(\phi)$ equivalent in winding number to ϕ. An electron that orbits the ring once is rotated through 2π and accordingly acquires a Berry phase factor of minus unity. As, however, the conductance correction comes from *conjugate pairs* of paths, each mate of the pair brings an equal factor of ± 1, for any winding number. Thus such a field configuration has no net effect on the conductance correction, as indicated by Eq. (4.5). As it is unconjugated paths that are responsible for them, such a field configuration would have a net effect on conductance fluctuations.

We now turn to the case of a hollow cylinder of large height $b \gg L_\varphi$, and thin walls of thickness $a \ll L_\varphi$. The longitudinal (i.e., axial) magnetoconductance follows from Eqs. (4.1) and (4.4b) with $a_d = aL/b$, and is given by

$$\Delta g = -\frac{e^2}{\pi^2\hbar}\frac{L}{b}\left\{\ln\left(\frac{L_\varphi^B}{l}\right) + 2\sum_{n=1}^\infty K_0\left(\frac{nL}{L_\varphi^B}\right)\cos\left(\frac{4\pi n\Phi^{\rm em}}{\Phi_0}\right)\cos(4\pi n\Phi^g)\right\}, \tag{4.7}$$

where K_0 denotes a modified Bessel function. Note that the spin sum has been performed explicitly, leading to a Berry phase factor that modulates the magnetoconductance oscillations due to the Aharonov-Bohm phase. In particular, this formula predicts that, with regard to the conductance, the addition of a Berry flux of $-1/4$ is equivalent to the addition of $\Phi_0/4$ to the electromagnetic flux. For a cylindrically symmetric texture, for instance, such a Berry flux requires a tilt angle χ away from the z axis of $\pi/3$. Thus in a possible experiment one could search for this effective electromagnetic flux shift by comparing the magnetoconductance at, say, $(\Phi^{\rm em}/\Phi_0, \Phi^g) = (0,0)$ with that at $(\Phi^{\rm em}/\Phi_0, \Phi^g) = (1/4, -1/4)$. The situation is somewhat simpler if the terms with $n \geq 2$ are negligible, i.e., for large ratios L/L_φ^B. To be able to attribute unambiguously the measured phase shift to the presence of the Berry phase one has, of course, to take into account a possible change of the Aharonov-Bohm phase (i.e., the flux through the cylinder) which might simultaneously occur when changing the configuration of the magnetic field.

In Sec. III C, we illustrated the adiabatic criterion using parameters roughly pertaining to recent experiments on Au loops, Ref. 14, and found a field scale of order 240 G. If we additionally suppose that the linewidth $a \approx 100$ nm then we find that one electromagnetic flux quantum penetrates the sample at field scales of order 60 G. Thus for the stated geometry we anticipate that the scale for adiabaticity is in the crossover regime, where the weak-localization effects start to become suppressed. More precisely, electromagnetic dephasing becomes relevant when the magnetic field becomes larger than roughly 60 G, as then the magnetic dephasing length L^B no longer exceeds the observed dephasing length L_φ, which is reported to exceed the ring circumference L at $T = 40$ mK.[14] It is reasonable, however, to expect that the magnetoconductance oscillations induced by an inhomogeneous magnetic field via the Berry phase would remain experimentally observable, albeit with a somewhat reduced amplitude.

Arguments similar to those given in the preceding paragraph also apply to the observability of the analogue of the anomalous magnetoconductance caused by the Berry phase mechanism (see Sec. IV A). In particular, if we assume that typically $\ell_B \sim L_\varphi \approx 7$ μm, then adiabaticity for the longest of phase-coherent paths (i.e., $\Lambda \sim L_\varphi$) requires magnetic fields stronger than about 200 G. On the other hand, in a two-dimensional metallic film, weak-localization effects begin to be suppressed when roughly one quantum of (electromagnetic) flux penetrates each phase-coherent area (roughly L_φ^2). In other words, the *onset* of suppression occurs at fields of the order $\Phi_0/L_\varphi^2 \sim 1$ G.[30] However, we emphasize that this suppression of weak localization is far from complete at such field strengths. The suppression follows a power law, behaving asymptotically as $1/B$ [see, e.g., Eq. (7.11) of Ref. 3], in marked contrast to the exponential suppression found in multiply connected geometries. Consequently, the suppression of weak-localization effects in two-dimensional films becomes complete only at much higher magnetic fields, typically of the order of a few thousand G (see, e.g., the measurements by Bergmann[1] on the anomalous magnetoconductance in thin Mg films in homogeneous magnetic fields). Thus one should comfortably be able to satisfy the adiabatic criterion *within* the weak-localization regime and, consequently, observe the analogue of the anomalous magnetoconductance described in Sec. IV A.

C. Universal conductance fluctuations

We now turn to our third example, namely, conductance fluctuations in mesoscopic metallic rings embedded in spatially inhomogeneous magnetic fields. In the context of the Berry phase there is, as we now explain, an essential advantage to studying fluctuations of the conductance about the Boltzmann value at higher magnetic fields, rather than the oscillatory weak-localization corrections to the Boltzmann value (discussed in the previous example) at lower fields. The latter oscillations (with period $\Phi_0/2$) are associated with the coherent contributions of *conjugate* paths and are described by the cooperon; for charged particles they are completely suppressed at high fields, due to smearing by magnetic flux passing through the sample itself. However, the former fluctuations are due to the incoherent contributions of *unconjugated* paths. They include *oscillatory* Aharonov-Bohm variations at the *fundamental* period Φ_0. They also include *stochastic* variations with flux, on the one-quantum-per-sample flux scale (at which the $\Phi_0/2$ oscillations become suppressed), but these stochastic variations *are not suppressed by such fields*. In particular, if the sample area is significantly smaller than the hole then the conductance will show Φ_0 oscillations at a correspondingly higher frequency, superposed on the slower stochastic variations, and they will continue to be observable even up to very high magnetic fields (e.g., more than 8 T, with r_c remaining much larger than ℓ).[31,32]

Now, from the adiabatic criterion given above it is apparent that high magnetic fields are precisely what are required to guarantee the accuracy of the adiabatic approximation invoked in our formalism. This means that even though it may be difficult to avoid the suppression (by electromagnetic flux penetration) of magnetoconductance oscillations resulting from *conjugate* paths, implications of the Berry phase should remain accessible through the modulation that it causes in the contributions to the conductance of mesoscopic samples arising from unconjugated paths. In particular, the conductance of simply connected mesoscopic samples should vary stochastically, with a scale of order e^2/h, as alterations of the field inhomogeneity (at fixed uniform electromagnetic flux) change the geometric flux penetrating the sample. These stochastic variations should have the statistical property that the value of the conductance remains correlated, as the field inhomogeneity is varied, until of order one additional quantum of geometric flux penetrates the sample. The conductance of multiply connected mesoscopic samples should show oscillations (with unit period in the geometric flux) superposed on the stochastic variations, as the field inhomogeneity is altered.

The subject of conductance fluctuations in mesoscopic systems originating from the absence of self-averaging has been addressed extensively over the last few years, both theoretically[4,5,33–35,38] and experimentally.[31,32] We therefore confine our discussion to the novel theoretical aspects that result from the inhomogeneity of the magnetic field. Most notably, the Zeeman interaction causes nontrivial correlations between the "up" and "down" spin channels, and thus a careful analysis of the conductivity-conductivity correlator is required in order to determine the complete spin dependence of the relevant propagators. The purpose of this subsection is to provide such an analysis, which we perform along lines similar to those introduced for the above computation of the disorder-averaged conductivity. In passing, we note that in the absence of the Zeeman interaction conductance fluctuations have been calculated by several authors. However, results have been obtained that differ in the numerical prefactors, partly as a consequence of the retention of different sets of diagrams.

To be specific, we focus on the evaluation of the conductivity-conductivity correlator defined by

$$\delta\sigma^{(2)}(\mathbf{B}, \tilde{\mathbf{B}}) = \langle \sigma_B \sigma_{\tilde{B}} \rangle - \langle \sigma_B \rangle \langle \sigma_{\tilde{B}} \rangle, \tag{4.8}$$

where $\sigma_{B/\tilde{B}}$ are now conductivities in the presence of the (possibly different) magnetic fields $\mathbf{B}$ and $\tilde{\mathbf{B}}$, as given in Eq. (2.2) but now without the disorder average. The dominant contribution once again comes from the current-current correlator $e^2\gamma^{00}$. We evaluate this quantity explicitly in Appendix B by making use of diagrammatic and path-integral techniques with subsequent adiabatic approximation. The result of this calculation is [see Eqs. (B8)–(B10)]

$$\delta\sigma^{(2)} = \left(\frac{e^2}{h}\right)^2 (2D)^2 \int d\epsilon\, d\epsilon' n'(\epsilon) n'(\epsilon') \int \frac{d\mathbf{x}}{\Omega} \sum_{\alpha\beta} \left\{ \frac{1}{d} [|\chi^C_{\alpha\beta}(\mathbf{0},\mathbf{x};\omega)|^2 + |\chi^D_{\alpha\beta}(\mathbf{0},\mathbf{x};\omega)|^2] + 2\mathrm{Re}[\chi^C_{\alpha\beta}(\mathbf{0},\mathbf{x};\omega)\chi^C_{\alpha\beta}(\mathbf{x},\mathbf{0};\omega) + \chi^D_{\alpha\beta}(\mathbf{0},\mathbf{x};\omega)\chi^D_{\alpha\beta}(\mathbf{x},\mathbf{0};\omega)] \right\}, \quad (4.9)$$

where $\chi^{C/D}_{\alpha\beta}$ is given by Eq. (3.10) or, equivalently, by the cooperon/diffuson differential equation, Eq. (3.12), and $\hbar\omega = \epsilon - \epsilon'$. We note that the term whose real part is taken can be interpreted as arising from fluctuations in the density of states, whereas the other terms arise from fluctuations of the diffusion coefficient. The fluctuation $\delta\sigma^{(2)}_{kk}$ of a single diagonal component σ_{kk} (with $k = x, y, z$) is obtained from Eq. (4.9) by setting $d = 1$. In the limit of vanishing Zeeman interaction our result comes closest to the one reported by Aronov and Sharvin;[4] the structure of the expressions is the same, only the numerical prefactors differ slightly.

It is apparent from Eq. (4.9) that in the present case of conductance fluctuations the two spin channels "up" and "down" are in general no longer uncorrelated and instead are mixed, in contrast with the case of magnetoconductance oscillations. As a consequence, the cooperon part now includes a contribution involving the *sum* of the two Berry phases as well as a contribution involving their *difference*; and similarly for the diffuson part.

Evidently, this spin-channel mixing invalidates the simple replacement rule, mentioned above, for incorporating the effect of an inhomogeneous field into the magnetoconductance fluctuations via the Berry phase. However, it is clear from our result, Eq. (4.9), how this rule is to be generalized to the case of the universal conductance fluctuations. Indeed, in the adiabatic regime the effect of the inhomogeneous magnetic field can simply be accounted for by replacing $F[\chi^{C/D}]$ by $\sum_{\alpha\beta} F[\chi^{C/D}_{\alpha\beta}]$, where F is a certain functional that is quadratic in $\chi^{C/D}$, and where $\chi^{C/D}$ obeys the generalized cooperon/diffuson differential equation, Eq. (3.12), which includes the Berry phases as well as the dynamical Zeeman terms. The dynamical Zeeman terms lead to a suppression of the spin-channel mixing for sufficiently high magnetic fields and/or low temperatures, as we shall discuss below.

To exhibit the physical consequences of the channel-mixing result in more detail, we now evaluate the conductance correlator, Eq. (4.9), for the illustrative example of a quasi-one-dimensional mesoscopic ring of circumference L, height $b \ll L_\varphi$, and wall thickness $a \ll L_\varphi$, embedded in an inhomogeneous magnetic field. Moreover, we shall focus on the limiting regime typically encountered in metals: $L_T \ll L_\varphi < L$, where $L_T = \sqrt{D\hbar\beta}$ is the thermal diffusion length, which provides a measure of the smearing of the conductance fluctuations due to nonzero temperature.[4,5,38] This example will then allow us to make quantitative statements about the regime in which correlations between the spin channels may be found. We begin by noting that in the specified regime the second term in Eq. (4.9), i.e., the term whose real part is taken, turns out to be of higher order in L_T/L_φ, and thus is omitted from the following analysis (see also Ref. 4). Focusing, then, on the first two terms, $|\chi^{C/D}|^2$, we insert the explicit solution for the cooperon/diffuson propagator, Eq. (4.4b), into Eq. (4.9). The remaining integrations over ϵ and ϵ' can then be evaluated by standard Matsubara techniques, and for the fluctuations of the tangential conductance per unit length $L^2\delta g^{(2)} \equiv (ab)^2\delta\sigma^{(2)}$ we obtain the result

$$L^2\delta g^{(2)} = \frac{1}{\pi^3}\left(\frac{e^2}{h}\right)^2 L_T^2 \mathrm{Re}\sum_{\alpha,\tilde\alpha}\sum_{k,m,n}{}' \left\{ \frac{1}{\delta^C_{\alpha\tilde\alpha}(k)\left[\frac{1}{2}(m+n) + 2\pi\delta^C_{\alpha\tilde\alpha}(k)\,(L_T/L)^2 + i\,(\beta/2\pi)\,\varpi_{\alpha\tilde\alpha}\right]^3} + [\delta^C_{\alpha\tilde\alpha} \to \delta^D_{\alpha\tilde\alpha}] \right\},$$

$$\delta^{C/D}_{\alpha\tilde\alpha}(k) = (r/L_\varphi)^2 + [k - (\Phi_\alpha \pm \tilde\Phi_{\tilde\alpha})]^2. \quad (4.10)$$

Here, the prime on the summation indicates the constraint that m and n are to be positive, odd integers. Next, we expand the foregoing result in powers of L_T/L and L_T/L_φ and, passing to the Fourier representation, for the tangential conductance fluctuations (to in leading order) we finally obtain

$$\delta g^{(2)} = \delta g^{(2)}_C + \delta g^{(2)}_D, \quad (4.11)$$

$$\delta g^{(2)}_{C/D} = \frac{2}{\pi}\left(\frac{e^2}{h}\right)^2 \frac{L_T^2 L_{C/D}}{L^3} \sum_{\alpha,\tilde\alpha} I_{\alpha\tilde\alpha} \left\{ 1 + 2\sum_{n=1}^{\infty} e^{-nL/L_{C/D}} \cos 2\pi n(\Phi_\alpha \pm \tilde\Phi_{\tilde\alpha}) \right\}, \quad (4.12)$$

where we have introduced the magnetic dephasing lengths associated with the cooperon and diffuson propagators (see, e.g., Ref. 4)

$$\frac{1}{L_{C/D}^2} = \frac{1}{L_\varphi^2} + \frac{a^2}{12}\left|\frac{\mathbf{B} \pm \tilde{\mathbf{B}}}{2\pi\Phi_0}\right|^2 . \quad (4.13)$$

Furthermore, the spin-dependent prefactor resulting from the dynamical Zeeman term measures the degree of the spin-channel mixing, and is explicitly given by

$$I_{\alpha\tilde{\alpha}} = \mathrm{Re}\,\frac{1}{[1 + i\beta g\mu_B(\alpha B - \tilde{\alpha}\tilde{B})/4\pi]^3}. \quad (4.14)$$

This expression now shows that the spin channels, "up" and "down," are mixed provided that $I_{\alpha\tilde{\alpha}}$ is of order unity for all $\alpha \neq \tilde{\alpha}$. More explicitly, we see that $|I_{\alpha\tilde{\alpha}}| \leq \frac{1}{4}$, if $\beta g\mu_B|\alpha B - \tilde{\alpha}\tilde{B}| \geq 4\pi$. For instance, if $B \approx \tilde{B} \approx 5$ T and if the temperature $T \approx 40$ mK then the spin weighting factor $I_{\alpha,-\alpha}$ becomes exceedingly small and the spin-channel mixing is completely suppressed, which reduces the magnitude of the conductance fluctuations by a factor of 2. An additional reduction by a factor of 2 is provided by the suppression of the weak-localization contribution (i.e., the cooperon part $\delta g_C^{(2)}$), as typically $L_C \ll L_D < L$ for fields higher than a few hundred G, as follows from Eq. (4.13). This reduction of the universal conductance fluctuation is well known for homogeneous magnetic fields.[38]

On the other hand, if the magnetic fields are smaller and/or the temperature higher, then the "up" and "down" spin channels *do* mix. This is the case, for instance, if $B \approx \tilde{B} \approx 0.5$ T, and $T \approx 0.4$ K, at which fields the weak-localization contribution $\delta g_C^{(2)}$ is still suppressed. This magnetic field strength is by far sufficient to satisfy the adiabaticity criterion, which requires the field to be greater than roughly 100 G.

As with the case of magnetoconductance oscillations, the Berry phase leads to a phase shift of the conventional Aharonov-Bohm oscillations. We emphasize that (in comparison with the Berry phase sensitivity of weak-localization corrections to the conductance) the primary advantage of conductance fluctuations is that their sensitivity to the Berry phase persists to much higher regimes of magnetic field, in which the the adiabaticity criterion can be readily satisfied.

V. CONCLUSIONS

In this paper we have aimed at presenting a detailed theory of quantum interference effects in the transport of particles with spin through disordered conductors in the presence of spatially inhomogeneous but static magnetic fields. By using diagrammatic techniques, along with the semiclassical Feynman-path-integral formulation of weak-localization theory and an adiabatic approximation, the present approach is capable of describing the effect of essentially arbitrary magnetic fields, by encoding their impact into an effective spin-dependent geometric gauge potential.

We have seen that there arise analogues of quantum interference phenomena that are familiar from the transport of charged particles through disordered conductors in the presence of homogeneous magnetic fields, with the role of the Aharonov-Bohm phase essentially being played by the Berry phase. In particular, analogues have been examined of the anomalous weak-field magnetoconductance of macroscopic metallic films, oscillations in the conductance of metallic mesoscopic rings and hollow cylinders, and stochastic variations in the conductance of mesoscopic metallic structures. Although our focus has been on the low-temperature conductivity of normal disordered metallic conductors, we stress that implications of quantum interference should be anticipated in the low-temperature transport of mass and spin by *neutral* particles through disordered media (such as the normal Fermi liquid ^{3}He through Vycor).

For certain properties, e.g., conductance oscillations in cylinders, it has been found that the system can be regarded as comprising two independent subsystems that experience a common electromagnetic flux but an opposite geometric flux. Such properties can be described by importing results derived for charged but spinless particles and simply superposing the results for a pair of shifted electromagnetic fluxes: $\sum_\alpha Q(\Phi^{\mathrm{em}} + \Phi^g_\alpha)$. However, for other properties, e.g., conductance fluctuations, such a simple structure does not emerge and, instead, off-diagonal aspects of the spin subsystems are needed to complete the description.

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APPENDIX A: WEAK-LOCALIZATION CORRECTION TO TRANSPORT COEFFICIENTS

In this appendix we derive the formula for the weak-localization correction to the spin and charge current-current correlators, using the diagrammatic impurity technique[19,24] in combination with a semiclassical analysis in terms of path integrals.[3,18] We emphasize that the Zeeman interaction with the inhomogeneous magnetic field is treated exactly in this semiclassical approximation. (The subsequent adiabatic approximation is introduced in Sec. III.)

We start with Eq. (2.1), which, evaluated in the position and spin representation, becomes

$$\gamma^{\mu\nu}(\omega) = \frac{\hbar}{2\pi\Omega m^2 d} \sum_{\alpha_1,\ldots,\alpha_4} S^{\mu}_{\alpha_1\alpha_4} S^{\nu}_{\alpha_3\alpha_2} \int d\mathbf{x}_1\, d\mathbf{x}_3 \lim_{\substack{\mathbf{x}_4\to\mathbf{x}_1\\ \mathbf{x}_2\to\mathbf{x}_3}} \frac{\hbar}{i}\frac{\partial}{\partial \mathbf{x}_2}\frac{\hbar}{i}\frac{\partial}{\partial \mathbf{x}_4} \langle G^r_{\alpha_4\alpha_3}(\mathbf{x}_4,\mathbf{x}_3;\epsilon_F+\hbar\omega)\, G^a_{\alpha_2\alpha_1}(\mathbf{x}_2,\mathbf{x}_1;\epsilon_F)\rangle. \tag{A1}$$

Due to the presence of the Zeeman term the disorder-averaged Green functions are no longer translationally invariant, and it is thus more convenient to work in position space than momentum space. After performing a perturbation expansion in the impurity potential, we approximate the impurity-averaged product of Green functions [represented by Feynman diagram (a)], as shown graphically in Fig. 1. In this diagrammatic iteration procedure the impurity-averaged product of Green functions also appears on the right-hand side of the graphical equation shown in Fig. 1.

This graphical scheme includes the Boltzmann contributions [Figs. 1(b)–1(d)] and a contribution containing all maximally crossed diagrams [Fig. 1(e)]. As we shall show, the latter generate the leading weak-localization correction to the transport coefficient. [Although it is not needed for the calculation of the leading quantum correction to $\gamma^{\mu\nu}$ we also evaluate Fig. 1(d), as it occurs in the calculation of conductance fluctuations.] For use in the calculation of the conductance fluctuations we have also included higher-order diagrams [Figs. 1(f) and 1(g)]. The dashed lines resulting from the averaging of impurity scattering correspond to a factor $n_i V_0^2 \delta(\mathbf{x}-\mathbf{y})$, where n_i is the impurity concentration and V_0 is the interaction strength. (For simplicity, we model the impurity potential by an uncorrelated distribution of randomly located isotropic δ-function potentials of strength V_0.) The full lines in the Feynman diagrams correspond to exact disorder-averaged retarded/advanced Green functions, with the convention that retarded propagators (upper lines) carry energy $\epsilon = \epsilon_F + \hbar\omega$ (with $\hbar\omega \ll \epsilon_F$), while advanced propagators (lower lines) carry energy $\epsilon' = \epsilon_F$. As usual, we are interested only in the field dependence resulting from interfering paths (see below). Therefore we ignore the less important field dependence of the disorder-averaged single Green functions themselves, as explained in Sec. III C. Then, in the standard Born approximation to the self-energy[19] the disorder-averaged single Green functions read

$$\langle G^{r/a}_{\alpha\beta}(\mathbf{x},\mathbf{y})\rangle \approx \langle G^{r/a}_{\alpha\beta}(\mathbf{x},\mathbf{y})\rangle_{\mathbf{A}^{\rm em}=\mathbf{0}} \equiv \delta_{\alpha\beta}\, g^{r/a}(\mathbf{x}-\mathbf{y}),$$
$$g^{r/a}(\mathbf{k}) \approx \frac{1}{E-\epsilon_{\mathbf{k}} \pm i\hbar/2\tau}. \tag{A2}$$

Here, $\epsilon_{\mathbf{k}}$ is the free one-particle energy spectrum, and the elastic scattering time τ is given by $1/\tau = n_i V_0^2 2\pi N_F/\hbar = v_F/\ell$, where N_F is the density of states at the Fermi level (per spin and d-dimensional unit volume; i.e., $mk_F/2\pi^2\hbar^2$ in $d=3$). Figure 1(d) then translates into the following expression:

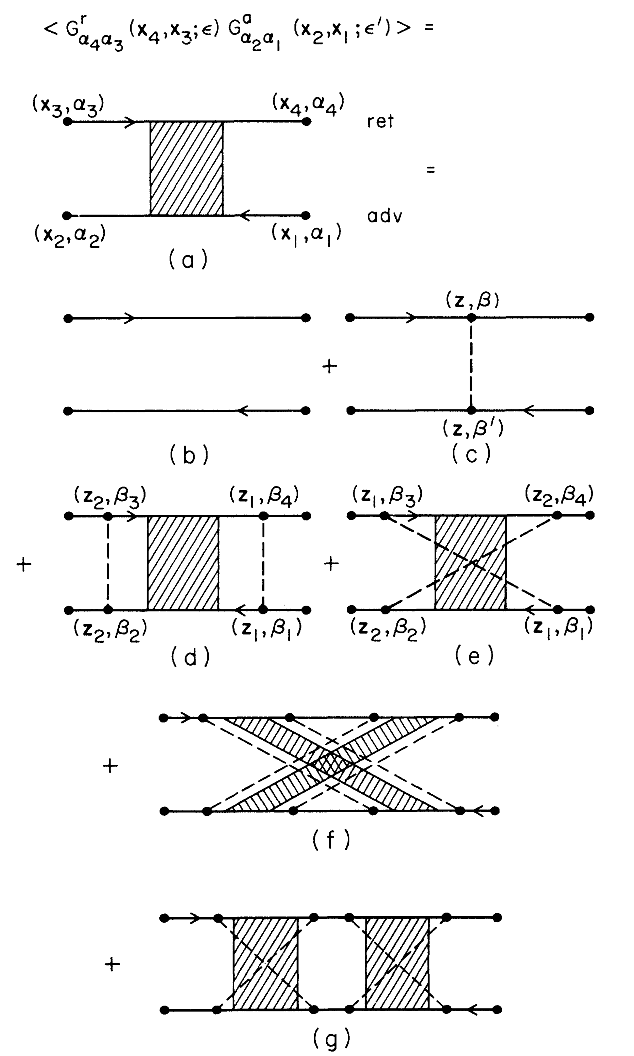

FIG. 1. Diagrammatic representation of the leading contributions in the perturbation expansion of the current correlator. For explanations see text.

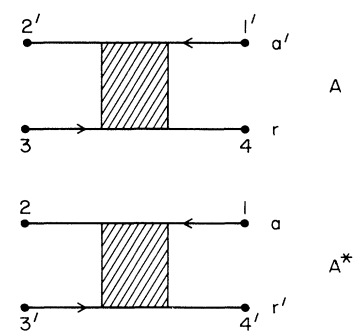

FIG. 2. Diagrammatic representation of the leading contribution in the perturbation expansion of the mean square value of the conductivity fluctuation. We use the abbreviations $1 = (\mathbf{x}_1, \alpha_1)$, etc., and $a = (adv, \mathbf{B}, \epsilon)$, $a' = (adv, \tilde{\mathbf{B}}, \epsilon')$, etc. Note that $\alpha_1 = \alpha_4$, $\alpha_2 = \alpha_3$, and $\alpha_1{}' = \alpha_4{}'$, $\alpha_2{}' = \alpha_3{}'$.

$$\Delta\gamma_D^{\mu\nu}(\omega) = \frac{\hbar n_i^2 V^4}{2\pi\Omega m^2 d}\int d\mathbf{z}_1 d\mathbf{z}_2 \langle \mathbf{z}_1|g^r \mathbf{p} g^a|\mathbf{z}_1\rangle\langle \mathbf{z}_2|g^a \mathbf{p} g^r|\mathbf{z}_2\rangle \sum_{\alpha_1,\ldots,\alpha_4} S^\mu_{\alpha_1\alpha_4} S^\nu_{\alpha_3\alpha_2}\bar{\chi}^D_{\alpha_1\alpha_2,\alpha_3\alpha_4}(\mathbf{z}_1,\mathbf{z}_2;\omega), \tag{A3}$$

and Fig. 1(e) corresponds to

$$\Delta\gamma_C^{\mu\nu}(\omega) = \frac{\hbar n_i^2 V^4}{2\pi\Omega m^2 d}\int d\mathbf{z}_1 d\mathbf{z}_2 \langle \mathbf{z}_1|g^r \mathbf{p} g^a|\mathbf{z}_2\rangle\langle \mathbf{z}_1|g^a \mathbf{p} g^r|\mathbf{z}_2\rangle \sum_{\alpha_1,\ldots,\alpha_4} S^\mu_{\alpha_1\alpha_4} S^\nu_{\alpha_3\alpha_2}\bar{\chi}^C_{\alpha_1\alpha_2,\alpha_3\alpha_4}(\mathbf{z}_1,\mathbf{z}_2;\omega), \tag{A4}$$

which represents the leading weak-localization correction. We have used the definition

$$\bar{\chi}^{C/D}_{\alpha_1\alpha_2,\alpha_3\alpha_4}(\mathbf{z}_1,\mathbf{z}_2;\omega) = \left\langle G^r_{\alpha_4\alpha_3}(\mathbf{z}_2,\mathbf{z}_1;\epsilon_F+\hbar\omega)\, G^a_{\alpha_2\alpha_1}(\mathbf{z}_{2/1},\mathbf{z}_{1/2};\epsilon_F)\right\rangle. \tag{A5}$$

Note the difference in the position arguments in the above expressions.

Next, following Ref. 3, we evaluate the propagators $\bar{\chi}^{C/D}$ in the semiclassical limit, and show that they obey a cooperon/diffuson type differential equation that is modified by the presence of the exact Zeeman term. For this purpose we first write

$$G^r(\epsilon_F+\hbar\omega)\, G^a(\epsilon_F) = \frac{1}{\hbar^2}\int_0^\infty dt\, e^{i\omega t}\int_{-\infty}^t dt'\, e^{i\epsilon_F t'/\hbar}\, e^{-iHt/\hbar}\, e^{iH(t-t')/\hbar}, \tag{A6}$$

and then express the orbital part of the transition amplitude as a Feynman path integral by formally decoupling orbital and spin motion (for details see Ref. 3 or 16):

$$\begin{aligned}\langle\alpha\mathbf{x}|e^{-iHt/\hbar}|\alpha'\mathbf{x}'\rangle &= \int_{\mathbf{x}(0)=\mathbf{x}'}^{\mathbf{x}(t)=\mathbf{x}}\mathcal{D}\mathbf{x}\, e^{iS_0[\mathbf{x}]/\hbar}\langle\alpha|\mathcal{T}e^{iS_1[\mathbf{x}]/\hbar}|\alpha'\rangle,\\ S_0[\mathbf{x}] &= \int_0^t dt'\{\tfrac{1}{2}m|\dot{\mathbf{x}}|^2 - V(\mathbf{x})\},\\ S_1[\mathbf{x}] &= \int_0^t dt'\{e\dot{\mathbf{x}}\cdot\mathbf{A}^{\rm em}(\mathbf{x}) + \tfrac{1}{2}g\mu_B\,\mathbf{B}\cdot\boldsymbol{\sigma}\}.\end{aligned} \tag{A7}$$

Here, $\mathcal{T}$ denotes time ordering with respect to the spin operator. Next, we invoke the semiclassical approximation for this path integral,

$$\int_{\mathbf{x}(0)=\mathbf{x}'}^{\mathbf{x}(t)=\mathbf{x}}\mathcal{D}\mathbf{x}\, e^{iS[\mathbf{x}]/\hbar} \approx \sum_{\mathbf{x}_{\rm cl}}\mathcal{A}[\mathbf{x}_{\rm cl}]\, e^{iS[\mathbf{x}_{\rm cl}]/\hbar}, \tag{A8}$$

where the sum runs over all classical paths $\mathbf{x}_{\rm cl}$ that start at $\mathbf{x}'$, end at $\mathbf{x}$, and make stationary the *orbital* part of the action. [As mentioned above, in the semiclassical limit we can safely ignore the (nonquantal) dynamical effects of the field-dependent terms on the orbital motion in the semiclassical limit.] The prefactor $\mathcal{A}$ accounts for the inclusion of (Gaussian) quantum fluctuations around the classical paths. Collecting the above results and inserting them into Eq. (A5) we obtain

$$\begin{aligned}\bar{\chi}^{C/D}_{\alpha_1\alpha_2,\alpha_3\alpha_4}(\mathbf{z}_1,\mathbf{z}_2;\omega) &= \frac{1}{\hbar^2}\int_0^\infty dt\, e^{i\omega t}\int_{-\infty}^t dt'\, e^{i\epsilon_F t'/\hbar}\\ &\times\left\langle\sum_{\mathbf{x}_{\rm cl},\,\mathbf{y}_{\rm cl}}\mathcal{A}[\mathbf{x}_{\rm cl}]\mathcal{A}^*[\mathbf{y}_{\rm cl}]\, e^{i\{S_0[\mathbf{x}_{\rm cl}]-S_0[\mathbf{y}_{\rm cl}]\}/\hbar}\langle\alpha_4|\mathcal{T}e^{iS_1[\mathbf{x}_{\rm cl}]/\hbar}|\alpha_3\rangle\langle\alpha_1|\mathcal{T}e^{iS_1[\mathbf{y}_{\rm cl}]/\hbar}|\alpha_2\rangle^*\right\rangle,\end{aligned} \tag{A9}$$

where the paths $\mathbf{x}_{\rm cl}$ and $\mathbf{y}_{\rm cl}$ satisfy the boundary conditions $\mathbf{x}_{\rm cl}(0)=\mathbf{z}_1$, $\mathbf{x}_{\rm cl}(t)=\mathbf{z}_2$, $\mathbf{y}_{\rm cl}(0)=\mathbf{z}_{2/1}$, and $\mathbf{y}_{\rm cl}(t-t')=\mathbf{z}_{1/2}$.

Next, observe that $t'\sim\hbar/\epsilon_F\ll 1/\omega\sim t$, i.e., $t'\ll t$. Thus we can expand the action around t:

$$S_0(t-t') = S_0(t) + t'\epsilon_t[\mathbf{x}_{\rm cl}] + \mathcal{O}(t'^{\,2}), \tag{A10}$$

where $\epsilon_t[\mathbf{x}_{\rm cl}] = \frac{1}{2}m|\dot{\mathbf{x}}_{\rm cl}(t)|^2 + V(\mathbf{x}_{\rm cl}(t))$ (which is a constant of motion), and where we have used the Hamilton-Jacobi equation $\partial S_0/\partial t = -\epsilon_t$. By neglecting the t' dependence of S_1 we arrive at

$$\begin{aligned}\bar{\chi}^{C/D}_{\alpha_1\alpha_2,\alpha_3\alpha_4}(\mathbf{z}_1,\mathbf{z}_2;\omega) &= \frac{2\pi}{\hbar^2}\int_0^\infty dt\, e^{i\omega t}\Bigg\langle\sum_{\mathbf{x}_{\rm cl},\,\mathbf{y}_{\rm cl}}\mathcal{A}[\mathbf{x}_{\rm cl}]\mathcal{A}^*[\mathbf{y}_{\rm cl}]\,\delta(\epsilon_t[\mathbf{y}_{\rm cl}]-\epsilon_F)\\ &\times e^{i\{S_0[\mathbf{x}_{\rm cl}]-S_0[\mathbf{y}_{\rm cl}]\}/\hbar}\,\langle\alpha_4|\mathcal{T}e^{iS_1[\mathbf{x}_{\rm cl}]/\hbar}|\alpha_3\rangle\langle\alpha_1|\mathcal{T}e^{iS_1[\mathbf{y}_{\rm cl}]/\hbar}|\alpha_2\rangle^*\Bigg\rangle.\end{aligned} \tag{A11}$$

Note that the boundary conditions in the sums over classical paths have become $\mathbf{x}_{\rm cl}(0) = \mathbf{y}_{\rm cl}(t) = \mathbf{z}_1$ and $\mathbf{x}_{\rm cl}(t) = \mathbf{y}_{\rm cl}(0) = \mathbf{z}_2$ (for $\bar{\chi}^C$), and $\mathbf{x}_{\rm cl}(0) = \mathbf{y}_{\rm cl}(0) = \mathbf{z}_1$ and $\mathbf{x}_{\rm cl}(t) = \mathbf{y}_{\rm cl}(t) = \mathbf{z}_2$ (for $\bar{\chi}^D$).

Now, in Eq. (A11) we retain only those paths $\mathbf{x}_{\rm cl}$ and $\mathbf{y}_{\rm cl}$ with $S_0[\mathbf{x}_{\rm cl}] = S_0[\mathbf{y}_{\rm cl}]$, because they give the dominant contribution to the sums over classical paths. In $\bar{\chi}^C$ one must take pairs of paths consisting of a path and its time-reversed partner; in $\bar{\chi}^D$ one must take pairs of paths that are identical. The remaining sum over a single Boltzmannian path is described as Brownian motion in terms of a Wiener path integral [see, in particular, Eqs. (4.9), (4.12), and (6.1) of Ref. 3]. The result is

$$\left\langle \sum_{\mathbf{x}_{\rm cl}} |\mathcal{A}[\mathbf{x}_{\rm cl}]|^2 \delta(\epsilon_t[\mathbf{x}_{\rm cl}] - \epsilon_F) \cdots \right\rangle \approx N_F \int_{\mathbf{R}(0)=\mathbf{z}_1}^{\mathbf{R}(t)=\mathbf{z}_2} \mathcal{D}\mathbf{R}\, e^{-(1/4D)\int_0^t d\tau |\dot{\mathbf{R}}|^2} \cdots , \tag{A12}$$

where the dots stand for appropriate phase factors. Thus we obtain

$$\bar{\chi}^{C/D}_{\alpha_1\alpha_2,\alpha_3\alpha_4}(\mathbf{z}_1,\mathbf{z}_2;\omega) = \frac{2\pi}{\hbar^2} N_F \int_0^\infty dt\, e^{i\omega t} \int_{\mathbf{R}(0)=\mathbf{z}_1}^{\mathbf{R}(t)=\mathbf{z}_2} \mathcal{D}\mathbf{R}\, e^{-(1/4D)\int_0^t d\tau |\dot{\mathbf{R}}|^2} \langle \alpha_4 | \mathcal{T} e^{iS_1[\mathbf{R}]/\hbar} | \alpha_3 \rangle \langle \alpha_1 | \mathcal{T} e^{iS_1[\mathbf{R}^{\mp}]/\hbar} | \alpha_2 \rangle^* , \tag{A13}$$

where $\mathbf{R}^\alpha(\tau) \equiv \mathbf{R}\left[\alpha\tau + \frac{1}{2}(1-\alpha)(t'+t)\right]$ with $\alpha = -1$ giving the time-reversed path associated with $\mathbf{R}^+$ [$= \mathbf{R}$].

Next, by using Eq. (A2) for $g^{r/a}$, setting $E = \epsilon_F$ and passing to Fourier space, we find that

$$\langle \mathbf{x} | g^r p_k g^a | \mathbf{x}' \rangle \langle \mathbf{x} | g^a p_m g^r | \mathbf{x}' \rangle \approx -\delta_{km}\, \delta(\mathbf{x}-\mathbf{x}')\, m^2 v_F^2\, 4\pi N_F \tau^3 / d\hbar^3 , \tag{A14}$$

where we have anticipated the fact that $\bar{\chi}^{C/D}$ only gives an essential contribution for $|\mathbf{q}| \ll k_F$. Observe the minus sign here, which reflects the fact that coherent back-scattering *reduces* the conductivity. (A plus sign would be obtained if one of the factors on the left-hand side were replaced by its complex conjugate; such a situation arises in the calculation of the conductance fluctuations.) Using this result, and inserting $\bar{\chi}^C$ into $\Delta\gamma_C^{\mu\nu}$ we obtain

$$\Delta\gamma_C^{\mu\nu}(\omega) = -\frac{D}{\pi\hbar\Omega} \int_0^\infty dt\, e^{i\omega t} \int d\mathbf{z} \sum_{\alpha_1,\ldots,\alpha_4} S^\mu_{\alpha_1\alpha_4} S^\nu_{\alpha_3\alpha_2}\, \chi^C_{\alpha_1\alpha_2,\alpha_3\alpha_4}(\mathbf{z},\mathbf{z};t,0), \tag{A15}$$

where we have introduced the inverse Fourier transform of $\chi^{C/D}(\mathbf{z}_1,\mathbf{z}_2;\omega)$ [$\equiv \hbar^2 \bar{\chi}^{C/D}(\mathbf{z}_2,\mathbf{z}_1;\omega)/2\pi N_F$, in which we have for convenience reversed the order of the arguments], which is explicitly given by

$$\begin{aligned} \chi^{C/D}_{\alpha_1\alpha_2,\alpha_3\alpha_4}(\mathbf{z}',\mathbf{z};t',t) = \theta(t'-t) \int_{\mathbf{R}(t)=\mathbf{z}}^{\mathbf{R}(t')=\mathbf{z}'} \mathcal{D}\mathbf{R}\, \exp\left\{ -\frac{1}{4D} \int_t^{t'} d\tau |\dot{\mathbf{R}}|^2 \right\} \\ \times \exp\left\{ i\frac{e}{\hbar} \int_t^{t'} d\tau\, \dot{\mathbf{R}} \cdot \left\{ \mathbf{A}^{\rm em}(\mathbf{R}(\tau)) \pm \tilde{\mathbf{A}}^{\rm em}(\mathbf{R}^{\pm}(\tau)) \right\} \right\} \\ \times \left\langle \alpha_4\alpha_2 \middle| \mathcal{T} \exp\left\{ i\frac{g\mu_B}{2\hbar} \int_t^{t'} d\tau \{ \mathbf{B}(\mathbf{R}(\tau)) \cdot \boldsymbol{\sigma}_1 - \tilde{\mathbf{B}}(\mathbf{R}^{\pm}(\tau)) \cdot \boldsymbol{\sigma}_2 \} \right\} \middle| \alpha_3\alpha_1 \right\rangle . \end{aligned} \tag{A16}$$

We have included the possibility of propagation through different fields (i.e., $\tilde{\mathbf{A}}^{\rm em}$ and $\mathbf{A}^{\rm em}$ can differ) to encompass conductance fluctuations, for which the path and its conjugate partner generally experience different fields[33,34,4] (see Appendix B). Furthermore, we have used the fact that

$$\begin{aligned} \left\langle \alpha_1 \middle| \mathcal{T} \exp\left\{ i\frac{g\mu_B}{2\hbar} \int_t^{t'} d\tau\, \mathbf{B}(\mathbf{R}^{\mp}(\tau)) \cdot \boldsymbol{\sigma} \right\} \middle| \alpha_2 \right\rangle^* \\ = \left\langle \alpha_2 \middle| \mathcal{T} \exp\left\{ -i\frac{g\mu_B}{2\hbar} \int_t^{t'} d\tau \mathbf{B}(\mathbf{R}^{\pm}(\tau)) \cdot \boldsymbol{\sigma} \right\} \middle| \alpha_1 \right\rangle , \end{aligned} \tag{A17}$$

and thus that

$$\begin{aligned} \left\langle \alpha_4 \middle| \mathcal{T} \exp\left\{ i\frac{g\mu_B}{2\hbar} \int_t^{t'} d\tau\, \mathbf{B}(\mathbf{R}(\tau)) \cdot \boldsymbol{\sigma}_1 \right\} \middle| \alpha_3 \right\rangle \left\langle \alpha_1 \middle| \mathcal{T} \exp\left\{ i\frac{g\mu_B}{2\hbar} \int_t^{t'} d\tau \tilde{\mathbf{B}}(\mathbf{R}^{\mp}(\tau)) \cdot \boldsymbol{\sigma}_2 \right\} \middle| \alpha_2 \right\rangle^* \\ = \left\langle \alpha_4\alpha_2 \middle| \mathcal{T} \exp\left\{ i\frac{g\mu_B}{2\hbar} \int_t^{t'} d\tau [\mathbf{B}(\mathbf{R}(\tau)) \cdot \boldsymbol{\sigma}_1 - \tilde{\mathbf{B}}(\mathbf{R}^{\pm}(\tau)) \cdot \boldsymbol{\sigma}_2] \right\} \middle| \alpha_3\alpha_1 \right\rangle , \end{aligned} \tag{A18}$$

with the notation $|\alpha_3\alpha_1\rangle = |\alpha_3\rangle \otimes |\alpha_1\rangle$. Finally, it is not difficult to see that $\Delta\gamma_D^{\mu\nu}$ vanishes upon insertion of χ^D, which simply reflects the well-known fact that the vertex corrections [diagrams (c) and (d) in Fig. 1] to the Boltzmann value [Fig. 1(b)] give a vanishing contribution to the transport coefficient for the case of isotropic scattering considered here. (Consequently, the elastic scattering time and the mean free time become identical.)

The path-integral representation for χ^C can be transformed into a differential equation, which eventually brings us to the cooperon differential equation with the exact Zeeman term

$$\left\{\frac{\partial}{\partial t'} + D\left[-i\frac{\partial}{\partial \mathbf{x}'} - \frac{e}{\hbar c}[\mathbf{A}^{\rm em}(\mathbf{x}') + \tilde{\mathbf{A}}^{\rm em}(\mathbf{x}')]\right]^2 - i\frac{g\mu_B}{2\hbar}[\mathbf{B}(\mathbf{x}')\cdot\boldsymbol{\sigma}_1 - \tilde{\mathbf{B}}(\mathbf{x}')\cdot\boldsymbol{\sigma}_2]\right\}\hat{\chi}^C(\mathbf{x}',\mathbf{x};t',t) = \delta(\mathbf{x}'-\mathbf{x})\,\delta(t'-t)\,\hat{\mathbb{1}}, \quad \text{(A19)}$$

where $\hat{\chi}^C(\mathbf{x}',\mathbf{x};t',t)$ is a 4-spinor in a four-dimensional spin space. The diffuson satisfies a similar equation. Note that this partial differential equation can be interpreted as a Schrödinger equation in imaginary time for a particle with effective mass $m = \hbar/2D$. The corresponding effective Hamiltonian, however, is no longer Hermitian, due to the presence of the Zeeman terms.

APPENDIX B: CONDUCTIVITY FLUCTUATIONS

In this appendix we shall outline the calculation of the conductivity fluctuations using our scheme of combining diagrammatic and path-integral techniques. The quantity of interest is the conductance correlator given in Eq. (4.8). As with the case of the average conductance, the leading contribution comes from the correlator $e^2\gamma^{00}$; other terms involve smaller numbers of current operators and lead to smaller powers of Fermi wave vectors; thus they can be seen to be of higher order in $1/k_F l$ (as we have checked explicitly). Restoring the Fermi function (to account for thermal dephasing effects) and using Eq. (A1) prior to disorder averaging, we then have

$$\langle\sigma_B\sigma_{\tilde{B}}\rangle = \left(\frac{e^2\hbar}{2\pi\Omega m^2 d}\right)^2 \sum_{\alpha_1,\ldots,\alpha_4}\int d\mathbf{x}_1\,d\mathbf{x}_3\,d\mathbf{x}_1'\,d\mathbf{x}_3' \lim_{\substack{\mathbf{x}_4\to\mathbf{x}_1\\ \mathbf{x}_2\to\mathbf{x}_3}}\lim_{\substack{\mathbf{x}_4'\to\mathbf{x}_1'\\ \mathbf{x}_2'\to\mathbf{x}_3'}}$$
$$\times(\hbar/i)^4(\partial_2\cdot\partial_4)\,(\partial_{2'}\cdot\partial_{4'})\int d\epsilon\,d\epsilon'\,n'(\epsilon)\,n'(\epsilon')\big\langle G^{r,B}_{\alpha_1\alpha_2}(\mathbf{x}_4,\mathbf{x}_3;\epsilon/\hbar)\,G^{a,B}_{\alpha_2\alpha_1}(\mathbf{x}_2,\mathbf{x}_1;\epsilon/\hbar)$$
$$\times G^{r,\tilde{B}}_{\alpha_3\alpha_4}(\mathbf{x}_4',\mathbf{x}_3';\epsilon'/\hbar)\,G^{a,\tilde{B}}_{\alpha_4\alpha_3}(\mathbf{x}_2',\mathbf{x}_1';\epsilon'/\hbar)\big\rangle, \quad \text{(B1)}$$

where $\partial_i = \partial/\partial\mathbf{x}_i$ and $n'(\epsilon) = \partial n/\partial\epsilon$, and where we have omitted higher-order terms. Note that the Green functions depend on different magnetic fields. The four-point correlation function occurring in the above expression can now be factorized into a number of terms, of which we retain only the contributions that pair: (i) $G^{r,B}$ with $G^{a,\tilde{B}}$; and (ii) $G^{a,B}$ with $G^{r,\tilde{B}}$. Other pairings are either cancelled by $\langle\sigma_B\rangle\langle\sigma_{\tilde{B}}\rangle$ or involve averages of G^aG^a and/or of G^rG^r, which lead to smaller contributions, as they only involve terms with common analytic structure, and do not contribute at leading order in $1/k_F l$. In terms of diagrams this means that we retain only the contribution shown in Fig. 2, where the blocks A and A^* can now be evaluated by the technique introduced in Appendix A. In fact, with the relation

$$G^{r,\tilde{B}}_{\alpha_3\alpha_4}(\mathbf{x}_4',\mathbf{x}_3';\epsilon'/\hbar) = G^{a,\tilde{B}}_{\alpha_4\alpha_3}(\mathbf{x}_3',\mathbf{x}_4';\epsilon'/\hbar)^* \quad \text{(B2)}$$

the diagram in Fig. 2 translates into the following expression:

$$\delta\sigma^{(2)} = \left(\frac{e^2\hbar}{2\pi\Omega m^2 d}\right)^2 \sum_{\alpha_1,\ldots,\alpha_4}\sum_{k,m=1}^{d}\int d\epsilon\,d\epsilon'\,n'(\epsilon)\,n'(\epsilon')\,I \quad \text{(B3)}$$

where

$$I = \int d\mathbf{x}_1\cdots d\mathbf{x}_4\,(\hbar/i)^4(\partial_{4,k}\,\partial_{2,m}\,A)(\partial_{1,m}\,\partial_{3,k}\,A^*),$$
$$A = \big\langle G^{r,B}_{\alpha_4\alpha_3}(\mathbf{x}_4,\mathbf{x}_3;\epsilon/\hbar)\,G^{a,\tilde{B}}_{\alpha_2\alpha_1}(\mathbf{x}_2,\mathbf{x}_1;\epsilon'/\hbar)\big\rangle, \quad \text{(B4)}$$

where A is precisely the propagator occurring in Eq. (A1), but now with different magnetic fields. We first concentrate on the contribution coming from the retention in A *and* A^* of diagrams (d) and (e) of Fig. 1. Denoting this contribution by I_1 we find

$$I_1 = (n_i V_0^2)^4 \int d\mathbf{z}_1 \cdots d\mathbf{z}_4 \, \langle \mathbf{z}_1 | g^r p_k g^a | \mathbf{z}_2 \rangle \langle \mathbf{z}_3 | g^a p_k g^r | \mathbf{z}_4 \rangle$$
$$\times \Big\{ \langle \mathbf{z}_3 | g^r p_m g^a | \mathbf{z}_4 \rangle \langle \mathbf{z}_1 | g^a p_m g^r | \mathbf{z}_2 \rangle \, \bar{\chi}^C_{\alpha_1\alpha_2,\alpha_3\alpha_4}(\mathbf{z}_1, \mathbf{z}_4; \omega) \, \bar{\chi}^C_{\alpha_1\alpha_2,\alpha_3\alpha_4}(\mathbf{z}_2, \mathbf{z}_3; \omega)^*$$
$$+ \langle \mathbf{z}_2 | g^r p_m g^a | \mathbf{z}_1 \rangle \langle \mathbf{z}_4 | g^a p_m g^r | \mathbf{z}_3 \rangle \, \bar{\chi}^D_{\alpha_1\alpha_2,\alpha_3\alpha_4}(\mathbf{z}_1, \mathbf{z}_4; \omega) \, \bar{\chi}^D_{\alpha_1\alpha_2,\alpha_3\alpha_4}(\mathbf{z}_2, \mathbf{z}_3; \omega)^* \Big\}, \quad \text{(B5)}$$

where $\hbar\omega = \epsilon - \epsilon'$, the Cartesian index m is not summed, and where we have dropped the cross terms involving $\bar{\chi}^C \bar{\chi}^D$ as they are of smaller order in $1/k_F l$ (which can be seen by going to Fourier space and estimating the available phase space). $\bar{\chi}^{C/D}$ is given in Eq. (A13).

A second contribution of the same order in $1/k_F l$ is obtained by retaining Figs. 1(f) and 1(g) in A, while retaining only Fig. 1(a) in A^*, and vice versa. Denoting this contribution by $I_2 = I_2^C + I_2^D$ we find

$$I_2^C = (n_i V_0^2)^4 \, 2\,\mathrm{Re} \int d\mathbf{z}_1 \cdots d\mathbf{z}_4 \langle \mathbf{z}_3 | g^a | \mathbf{z}_2 \rangle \langle \mathbf{z}_1 | g^r | \mathbf{z}_4 \rangle \langle \mathbf{z}_1 | g^a p_k g^r p_k g^a | \mathbf{z}_4 \rangle \langle \mathbf{z}_3 | g^r p_m g^a p_m g^r | \mathbf{z}_2 \rangle$$
$$\times \bar{\chi}^C_{\alpha_1\alpha_2,\alpha_3\alpha_4}(\mathbf{z}_1, \mathbf{z}_2; \omega) \, \bar{\chi}^C_{\alpha_2\alpha_1,\alpha_4\alpha_3}(\mathbf{z}_3, \mathbf{z}_4; \omega) \quad \text{(B6)}$$

and

$$I_2^D = (n_i V_0^2)^4 \, 2\,\mathrm{Re} \int d\mathbf{z}_1 \cdots d\mathbf{z}_4 \langle \mathbf{z}_2 | g^a | \mathbf{z}_3 \rangle \langle \mathbf{z}_1 | g^r | \mathbf{z}_4 \rangle \langle \mathbf{z}_4 | g^a p_k g^r p_k g^a | \mathbf{z}_1 \rangle \langle \mathbf{z}_3 | g^r p_m g^a p_m g^r | \mathbf{z}_2 \rangle$$
$$\times \bar{\chi}^D_{\alpha_1\alpha_2,\alpha_3\alpha_4}(\mathbf{z}_1, \mathbf{z}_2; \omega) \, \bar{\chi}^D_{\alpha_2\alpha_1,\alpha_4\alpha_3}(\mathbf{z}_3, \mathbf{z}_4; \omega). \quad \text{(B7)}$$

Inserting $I = I_1 + I_2$ back into Eq. (B3) and making use of Eq. (A14), and similar relations for the other matrix elements, we obtain

$$\delta\sigma^{(2)} = \left(\frac{e^2 D}{\Omega \pi \hbar} \right)^2 \int d\epsilon \, d\epsilon' n'(\epsilon) n'(\epsilon') \int d\mathbf{x} \, d\mathbf{x}'$$
$$\times \sum_{\alpha_1, \ldots, \alpha_4} \Big\{ \frac{1}{d} \left[|\chi^C_{\alpha_1\alpha_2,\alpha_3\alpha_4}(\mathbf{x}, \mathbf{x}'; \omega)|^2 + |\chi^D_{\alpha_1\alpha_2,\alpha_3\alpha_4}(\mathbf{x}, \mathbf{x}'; \omega)|^2 \right]$$
$$+ 2\,\mathrm{Re}[\bar{\chi}^C_{\alpha_1\alpha_2,\alpha_3\alpha_4}(\mathbf{x}, \mathbf{x}'; \omega) \, \bar{\chi}^C_{\alpha_2\alpha_1,\alpha_4\alpha_3}(\mathbf{x}', \mathbf{x}; \omega) + \bar{\chi}^D_{\alpha_1\alpha_2,\alpha_3\alpha_4}(\mathbf{x}, \mathbf{x}'; \omega) \, \bar{\chi}^D_{\alpha_2\alpha_1,\alpha_4\alpha_3}(\mathbf{x}', \mathbf{x}; \omega)] \Big\}, \quad \text{(B8)}$$

where we have used the relations $D = v_F^2 \tau / d$, and $2\pi N_F n_i V_0^2 = \hbar/\tau$. Up to this point the Zeeman interaction in the cooperon/diffuson (as opposed, of course, to the averaged single Green function) has been treated exactly. We now introduce the adiabatic approximation for the propagators occurring in the foregoing equation and thus, with Eq. (3.9), we find

$$\sum_{\alpha_1, \ldots, \alpha_4} |\chi^{C/D}_{\alpha_1\alpha_2,\alpha_3\alpha_4}(\mathbf{x}, \mathbf{x}'; \omega)|^2 = \sum_{\alpha\beta} |\chi^{C/D}_{\alpha\beta}(\mathbf{x}, \mathbf{x}'; \omega)|^2, \quad \text{(B9)}$$

and similarly,

$$\sum_{\alpha_1, \ldots, \alpha_4} \chi^{C/D}_{\alpha_1\alpha_2,\alpha_3\alpha_4}(\mathbf{x}, \mathbf{x}'; \omega) \, \chi^{C/D}_{\alpha_2\alpha_1,\alpha_4\alpha_3}(\mathbf{x}', \mathbf{x}; \omega) = \sum_{\alpha\beta} \chi^{C/D}_{\alpha\beta}(\mathbf{x}, \mathbf{x}'; \omega) \, \chi^{C/D}_{\alpha\beta}(\mathbf{x}', \mathbf{x}; \omega). \quad \text{(B10)}$$

It is quite remarkable that the overlap matrix elements of the instantaneous spin states disappear upon performing the spin sums. Finally, inserting the last result back into Eq. (B8) and assuming translational invariance [which holds under the assumptions stated in the text before Eq. (4.2)], we arrive at Eq. (4.9).

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*Electronic address: dloss@sfu.ca
†Permanent address.
‡Electronic address: schoelle@tfp.physik.uni-karlsruhe.de
§Electronic address: p-goldbart@uiuc.edu; also at Materials Research Laboratory, 104 South Goodwin Avenue, Urbana, IL 61801; and Beckman Institute for Advanced Science and Technology, 405 North Mathews Avenue, Urbana, IL 61801.

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