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Lowest-Landau-level constraint, Goldstone mode, and Josephson effect in a double-layer quantum Hall system

Z. F. Ezawa
Department of Physics, Tohoku University, Sendai 980, Japan

A. Iwazaki
Department of Physics, Nishogakusha University, Ohi 2590, Shonan-machi 277, Japan
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Based on the Chern-Simons gauge theory of the planar electrons together with the lowest-Landau-level projection, we analyze the dynamical mechanism of a Josephson effect predicted recently in a double-layer quantum Hall (QH) system. It is shown that a Goldstone mode exists in a certain double-layer QH system and that the mode induces the Josephson effect through a weak interlayer tunneling.

I. INTRODUCTION

Statistical transmutation is an intrinsic property to the planar system,[1,2] which leads to a remarkable possibility that in the presence of an external magnetic field an electron can be turned into a boson by making an electron-flux composite (bosonized electron) and condensed. It is widely believed that the quantum Hall (QH) state is indeed such a condensed state of bosonized electrons which forms an incompressible liquid.[3] The peculiarity is that the condensate must carry the charge $-e$ of the electron, in clear contrast with the charge $-2e$ of the Cooper pair in a superconductor. An intriguing problem is how to verify such a condensation of unpaired electrons experimentally. Recently, for its verification we have proposed[4] one definite experimental test with some numerical conditions to realize it using a Josephson effect in a double-layer QH system.

It has been argued[5] that in a certain double-layer QH system there is a Goldstone mode associated with the conservation of the electron number difference between the two layers. Because of this fact one naturally anticipates the Josephson effect in this system. However, it is very difficult to make a precise formulation (or prediction) of the Josephson effect without using a bosonic field describing the coherent phase of the condensate. In the case of a superconductor the essence of the Josephson effect is that the coherent current is induced by the phase difference of the two condensates (Cooper pairs). Hence, in proposing a mechanism of the Josephson effect in the QH system, it is essential to identify what is the condensate and to show that the current is actually induced by the coherent tunneling of the condensate.

So far there are two different approaches to this problem. In the scenario of Wen and Zee[6] the tunneling is treated in a dilute-gas approximation of instantons (magnetic monopoles). Their effective Lagrangian, appropriate only for long-range properties of the system, does not contain any variable representing the condensate in the QH state. Thus, a bosonic field is introduced so as to reproduce the Coulomb interaction between instantons, and it is claimed that this field induces the Josephson effect. However, the connection of this field to the phase of the condensate is quite unclear.

On the contrary, the Chern-Simons (CS) formulation of the planar electrons is adequate to analyze the coherent tunneling of unpaired electrons,[7,8] where an electron is regarded as a composite of a boson (bosonized electron) and a CS flux of the statistical field. When this CS flux precisely cancels the external magnetic flux, bosonized electrons see no net magnetic field and can be condensed. Then, the Josephson current follows due to the phase difference between the condensates on the two layers. This formalism presents a microscopic picture of the Josephson effect with some numerical predictions.[4]

The aim of this paper is to clarify the mechanism of the Josephson effect by making the lowest-Landau-level (LLL) projection, which was not imposed explicitly in the previous treatment.[7] There is a merit of the LLL projection since it makes the diagonalization of the Hamiltonian considerably simpler in the mean-field approximation. First, we take the case of the single-layer Hall system,[9] and formulate the LLL projection as a constraint condition on the states in the CS gauge theory. Then, generalizing this to the double-layer case, we make a careful analysis of the Goldstone mode,[10] which is shown to arise at a specific filling factor $\nu = 1/m$ with m an odd integer. It is clarified that this mode is associated with the electron number difference between the two layers, whose canonical conjugate is the phase difference of the condensates on the two layers. It is precisely this mode that carries the Josephson current in the presence of weak interlayer tunneling.

This paper is composed as follows. In Sec. II, from a physical point of view, we explain why unpaired planar electrons may be turned into bosons in the presence of an appropriate external magnetic field in addition to the Coulomb repulsion. In Sec. III the LLL projection is formulated in the CS gauge theory of planar electrons. In Sec. IV we analyze the double-layer QH system together with the LLL projection. In Sec. V we focus our attention to the Goldstone mode and the associated Josephson effect. We use the unit such that $c = 1$ and $\hbar = 1$.

II. BOSONIZATION OF PLANAR ELECTRONS

Let us review a physical picture of how unpaired electrons can be condensed on a plane in the presence of an external magnetic field. It is essential that electrons, making cyclotron motions, are uniformly distributed on the plane due to the Coulomb repulsion. It is reasonable to simulate the system by placing electrons on lattice points with equal spacing and with all lattice points occupied. Due to the Coulomb interaction electrons exchange their positions. Thus, the electrons move around all the lattice points. In this lattice approximation the external magnetic field is also distributed among the lattice points: to each lattice point the magnetic flux B/ρ is attached, with ρ being the electron density. Then, when two electrons exchange their positions, the wave function reads as $\Psi(y,x) = -e^{i\alpha}\Psi(x,y)$ with $\alpha = eB/2\rho$ by acquiring the Aharanov-Bohm phase $e^{i\alpha}$.[11] When the magnetic field B is such that $\alpha = \pi m$ with m being an odd integer, $\Psi(x,y)$ becomes a wave function of bosons, which are nothing but bosonized electrons. Then, these bosonized electrons would make a condensed phase, the QH fluid, at the magic filling factor $\nu \equiv 2\pi\rho/eB = 1/m$.

A field-theoretical realization of this picture of the QH state is given by the CS gauge theory, as we now explain, where the electron is described by the bosonized electron together with the CS gauge field representing the statistics of the electron. Namely, the electron is regarded as a composite of a boson and a CS gauge flux. In the presence of a repulsive Coulomb interaction, the mean-field ground state at the magic filling factor is such that the CS flux is canceled out by the external magnetic flux, and hence the bosonized electrons, thus identified as composites of electrons and magnetic flux, make a condensation.

As pointed out by Girvin and McDonald,[12] bosonized electrons are defined trivially in terms of the wave functions. Let $\widehat{\Psi}(x_1,\ldots,x_N)$ be the wave function of N electrons on a plane. Then, the wave function of the bosonized electrons $\Psi(x_1,\ldots,x_N)$ is defined by

$$\Psi(x_1,\ldots,x_N) \equiv e^{im\sum_{r<s}\theta(x_r-x_s)}\widehat{\Psi}(x_1,\ldots,x_N), \quad (2.1)$$

where $\theta(x_r - x_s)$ is the azimuthal angle between two electrons. We call an odd integer m the statistics parameter. By exchanging any two electrons, Ψ transforms obviously as a bosonic wave function. (The bosonic wave function satisfies the hard-core condition, $\Psi = 0$ for $x_r = x_s$.) The momentum operator of the bosonized electron is given by $i\partial_k^r + a_k(x_r)$, with $a_k(x_r) = m\sum_{s\neq r}\partial_k^r\theta(x_r - x_s)$ or

$$\varepsilon_{ij}\partial_i a_j(x) = 2\pi m\sum_s \delta(x - x_s). \quad (2.2)$$

In this way the bosonization is simply a procedure to take off the phase factors. Obviously, it is only possible in two spatial dimensions.

The second quantization is also trivially made, which leads to a CS gauge theory. After second quantization, (2.2) yields

$$\varepsilon_{ij}\partial_i a_j = 2\pi m\psi^\dagger\psi, \quad (2.3)$$

which defines the statistical gauge field a_k in terms of the bosonized electron field ψ, satisfying $[\psi(x),\psi^\dagger(y)] = \delta(x-y)$. The bosonized electron is a hard-core boson. The statistical gauge field a_k, representing solely the phase degree of freedom, has no independent dynamics.

The Hamiltonian is simply given by

$$\mathcal{H} = \frac{1}{2M}\int d^2x|D_k\psi|^2 + \mathcal{V}[\psi], \quad (2.4)$$

with

$$iD_k = i\partial_k + a_k - eA_k, \quad (2.5)$$

in terms of the bosonized electron field ψ and the statistical gauge field a_k subject to (2.3); M is the effective mass of electrons; N is the total number of electrons. The external magnetic field B is applied perpendicular to the layers with $A_k = -\frac{1}{2}B\varepsilon_{kj}x^j$. Term $\mathcal{V}[\psi]$ represents the intralayer Coulomb interactions that drive the planar electron system into the QH liquid:

$$\mathcal{V}[\psi] = \frac{e^2}{2\varepsilon}\int d^2x\, d^2y : \{\psi^\dagger\psi(x) - \rho\} \times\frac{1}{|x-y|}\{\psi^\dagger\psi(y) - \rho\} :, \quad (2.6)$$

where ρ stands for the constant neutralizing background charge and ε for the dielectric constant. In these expressions the ordering of the operator $|\mathcal{O}|^2$ is $\mathcal{O}^\dagger\mathcal{O}$, and the dots $:\mathcal{O}:$ denote the normal ordering of the operator $\mathcal{O}$. With these operator orderings it can be proved[13] that the Hamiltonian (2.4) leads to exactly the N-body Schrödinger equation of motion of electrons.

The statistical gauge field a_k is also called the CS gauge field since the Hamiltonian (2.4) together with the constraint equation (2.3) follow from the so-called CS Lagrangian;[9] see the Appendix for the case of the double-layer system.

It is convenient to rewrite the Hamiltonian by using the Bogomol'nyi decomposition:[14]

$$\int d^2x|D_k\psi|^2 = \int d^2x\Big(|(D_1 - iD_2)\psi|^2 + M\omega_c : |\psi|^2 : -2m\pi : |\psi|^4 :\Big), \quad (2.7)$$

where $\omega_c = eB/M$ is the cyclotron frequency. Here, the last term represents the contact interaction, which can be discarded for the hard-core bosons. Hence, we obtain[9]

$$\mathcal{H} = \frac{1}{2M}\int d^2x|(D_1 - iD_2)\psi|^2 + \frac{1}{2}\omega_c N + \mathcal{V}[\psi], \quad (2.8)$$

which defines a microscopic field theory of the planar system of N electrons.

III. THE LLL PROJECTION

If the Coulomb interaction is switched off the Hamiltonian (2.8) simply describes cyclotron motions of electrons. When all N electrons occupy the LLL, the total kinetic energy is $\frac{1}{2}\omega_c N$. In this case it is obvious from this Hamiltonian that such states $|f\rangle$ satisfy the condition

$$(D_1 - iD_2)\psi|f\rangle = 0, \tag{3.1}$$

and have the energy $\frac{1}{2}\omega_c N$ of the LLL. We call this equation the LLL constraint. As we shall see, in order to implement the LLL projection in the presence of the Coulomb interaction, we can start with this LLL constraint as we do in this paper, or we can start without it and take the limit $M \to 0$ afterwards by using the results of our previous paper.[9] Both of the methods give the same results.

First, let us prove that the LLL constraint (3.1) defines surely the states in the LLL. For this purpose we analyze the wave function

$$\Psi_f \equiv \langle 0|\psi(x_1)\psi(x_2)\cdots\psi(x_N)|f\rangle, \tag{3.2}$$

with $\psi(x)|0\rangle = 0$. Introducing the complex coordinate $z = x+iy$ for each electron, we rewrite the LLL constraint (3.1) as

$$\left(\frac{\partial}{\partial z_r} + \frac{eB}{4}\bar{z}_r - \frac{m}{2}\sum_{s\neq r}\frac{1}{z_r - z_s}\right)\Psi_f = 0. \tag{3.3}$$

This equation can be exactly solved.[9,15] Recovering the phase factor given by (2.1) we obtain

$$\widehat{\Psi}_f = F(\bar{z})\prod(\bar{z}_r - \bar{z}_s)^m \exp\left\{-\frac{1}{4}eB\sum|z_r|^2\right\}, \tag{3.4}$$

where $F(\bar{z})$ is an arbitrary analytic function. The arbitrariness in $F(\bar{z})$ implies the degeneracy of the states in the LLL. As is well known, this is the most general form of the wave function of the electrons in the LLL. Hence, the condition (3.1) selects the LLL states in the second quantized formalism of bosonized electrons.

The real ground state is determined by minimizing the Coulomb energy (2.6) in the LLL states. Namely, we diagonalize the Coulomb interaction $\mathcal{V}$, i.e., the matrix $\langle g|\mathcal{V}|f\rangle$ with $|f\rangle$ and $|g\rangle$ being LLL states obeying (3.1). An exact treatment is practically impossible. We use the mean-field approximation and then take into account Gaussian fluctuations. As we shall see, the Laughlin wave function is obtained for the QH state in this approximation.

In the mean-field approximation, the LLL states are given by solving the classical equation

$$(D_1 - iD_2)\psi = 0, \tag{3.5}$$

which is found[9] to contain ensembles of vortices in general, implying the degeneracy of the LLL states. One vortex carries a Coulomb energy of the order of e^2/ℓ_B with $\ell_B \equiv 1/\sqrt{eB}$ being the magnetic radius. Among these solutions the Coulomb energy (2.6) is minimized by the solution

$$a_k = eA_k, \qquad \psi = \sqrt{\rho}e^{i\theta}, \tag{3.6}$$

with constant phase θ, as corresponds to a uniform distribution of electrons. It is realized only at the magic filling factor $\nu = 1/m$ due to the constraint equation (2.3).

This mean-field ground state illustrates our physical picture of the condensation of bosonized electrons, as we have promised. Although the decomposition of an electron into a bosonized electron and a CS flux is just a mathematical trick, the CS flux is traded with the external magnetic flux in this mean-field ground state. Then, the bosonized electrons acquire a physical reality as composites of electrons and magnetic flux. They are condensed in the zero-momentum state; the QH state is a condensed phase of bosonized electrons. Note that this condensation occurs only at the magic filling factor; in its vicinity the ground states contain ensembles of topological vortices (quasiparticles) which are excited but trapped by impurities on this condensate.

We next study the Gaussian fluctuations around the mean-field ground state (3.6) at the magic filling factor. Choosing $\theta = 0$ for simplicity, we set

$$\psi = \sqrt{\rho} + \eta, \tag{3.7a}$$

with

$$\eta = \frac{1}{\sqrt{V}}\sum_{\mathbf{p}} a_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}}, \tag{3.7b}$$

where V is the volume of the system and $a_{\mathbf{p}}$ is an annihilation operator satisfying $[a_{\mathbf{p}}, a^\dagger_{\mathbf{q}}] = \delta_{\mathbf{p},\mathbf{q}}$. Substituting (3.7a) into the LLL constraint (3.1) and the constraint equation (2.3), we linearize them as

$$\{(\partial_1 - i\partial_2)\eta - i\sqrt{\rho}[(a_1 - eA_1) - i(a_2 - eA_2)]\}|f\rangle = 0, \tag{3.8}$$

and

$$\varepsilon_{ij}\partial_i a_j = 2\pi m[\rho + \sqrt{\rho}(\eta + \eta^\dagger)], \tag{3.9}$$

respectively. Now, using a scalar function a, we may set

$$a_i - eA_i = -\varepsilon_{ij}\partial_j a. \tag{3.10}$$

Then, from (3.8) and (3.9) we obtain

$$\partial_z(\eta - \sqrt{\rho}a)|f\rangle = 0 \tag{3.11}$$

and

$$\partial^2 a = 2\sqrt{\rho}m\pi(\eta^\dagger + \eta), \tag{3.12}$$

where we have used $eB = 2\pi\rho m$. Using the mode expansion (3.7b) for η this equation is solved as

$$\eta - \sqrt{\rho}a = \frac{1}{\sqrt{V}}\sum_{\mathbf{p}}\sqrt{1 + 2/x_{\mathbf{p}}}T_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}}, \tag{3.13}$$

where $x_{\mathbf{p}} = \mathbf{p}^2/2\pi m\rho$ and

$$T_{\mathbf{p}} = \frac{a^{\dagger}_{-\mathbf{p}} + (1+x_{\mathbf{p}})a_{\mathbf{p}}}{\sqrt{x_{\mathbf{p}}^2 + 2x_{\mathbf{p}}}}. \tag{3.14}$$

We have normalized the operator $T_{\mathbf{p}}$ so that $[T_{\mathbf{p}}, T^{\dagger}_{\mathbf{q}}] = \delta_{\mathbf{pq}}$ and $[T_{\mathbf{p}}, T_{\mathbf{q}}] = 0$. Therefore, the LLL constraint (3.11) is equivalent to the condition

$$T_{\mathbf{p}}|f\rangle = 0. \tag{3.15}$$

This condition implies that the LLL states have no excitations of the $T_{\mathbf{p}}$ modes.

We now diagonalize the Coulomb term within the LLL states. Substituting the mode expansion (3.7b) into the Coulomb term (2.6) we obtain

$$\mathcal{V} = \frac{\pi e^2 \rho}{\varepsilon}\sum_{\mathbf{p}} \frac{1}{|\mathbf{p}|}(a^{\dagger}_{\mathbf{p}}a^{\dagger}_{-\mathbf{p}} + 2a^{\dagger}_{\mathbf{p}}a_{\mathbf{p}} + a_{\mathbf{p}}a_{-\mathbf{p}}). \tag{3.16}$$

Using (3.14) we can express this in terms of operators $T_{\mathbf{p}}$ and $T^{\dagger}_{\mathbf{p}}$. Since $T_{\mathbf{p}}|f\rangle = 0$, when $\mathcal{V}$ acts on the states in the LLL, it reads

$$\mathcal{V}|f\rangle = \left(E_{\eta} + \frac{\pi e^2\rho}{\varepsilon}\sum_{\mathbf{p}}\frac{1}{|\mathbf{p}|}\frac{x_{\mathbf{p}}}{x_{\mathbf{p}}+2}T^{\dagger}_{\mathbf{p}}T^{\dagger}_{-\mathbf{p}}\right)|f\rangle, \tag{3.17}$$

where

$$E_{\eta} = -\frac{2\pi e^2\rho}{\varepsilon}\sum_{\mathbf{p}}\frac{1}{|\mathbf{p}|}\frac{1}{x_{\mathbf{p}}+2} = -\frac{e^2 N\pi}{2\sqrt{2}\varepsilon\ell_B} \tag{3.18}$$

is the Coulomb energy of the state. Hence, we obtain $\langle g|\mathcal{V}|f\rangle = E_{\eta}\delta_{f,g}$, which is the result of the diagonalization of the Coulomb term. The QH state is incompressible since there are no degeneracies in the ground state.

In the previous paper without imposing the LLL constraint we have diagonalized the Hamiltonian (2.8) by way of the Bogoljubov transformation. As we have derived in the first paper of Ref. 9, the result is

$$H = \frac{1}{2}\omega_c N + E_{\eta} + \sum_{\mathbf{p}\neq 0} E_{\mathbf{p}} b^{\dagger}_{\mathbf{p}} b_{\mathbf{p}}, \tag{3.19}$$

where the Coulomb energy E_{η} is given by (3.18), and

$$E_{\mathbf{p}}^2 = \left(\frac{\mathbf{p}^2}{2M} + \omega_c\right)^2 + \frac{2\pi e^2\rho}{\varepsilon M}|\mathbf{p}|. \tag{3.20}$$

The operator $b_{\mathbf{p}}$ is defined by Eq. (4.7) therein, which is the one obtained by the Bogoljubov transformation of $a_{\mathbf{p}}$. It is easy to see that the mode operator $T_{\mathbf{p}}$ is identical to the mode operator $b_{\mathbf{p}}$ when the limit $M \to 0$ is taken. Note that the gap energy $E_{\mathbf{p}}$ of the mode $b_{\mathbf{p}}$ is given by the cyclotron energy. It is natural that the excitation of this mode is suppressed when the LLL projection is made.

We conclude that the LLL constraint (3.1) specifies uniquely the LLL state in this approximation: namely, no perturbative excitations associated with the mode expansion (3.7b) are allowed around the mean-field ground state in the LLL. To go beyond the approximation and to recover other LLL states with higher Coulomb energy we need to take into account vortex excitations (quasiparticles) which arise as soliton solutions in (3.5). We wish to discuss this problem in a future work.

We proceed to calculate the wave function (3.2) of the mean-field ground state (3.6) in the Gaussian approximation. It is expressed in terms of two-point wave functions as

$$\begin{aligned}\Psi_f &= \langle 0|\prod_r\{\sqrt{\rho} + \eta(x_r)\}|f\rangle \\ &\approx \rho^{N/2}\langle 0|\left\{1 + \frac{1}{\rho}\sum_{r<s}\eta(x_r)\eta(x_s)\right\}|f\rangle \\ &\approx \rho^{N/2}\langle 0|f\rangle \prod_{r<s}\exp\left(\frac{\langle 0|\eta(x_r)\eta(x_s)|f\rangle}{\rho\langle 0|f\rangle}\right).\end{aligned} \tag{3.21}$$

Now, using $a_{\mathbf{p}}|0\rangle = 0$, $T_{\mathbf{p}}|f\rangle = 0$ and (3.14), it is easy to verify that

$$\langle 0|a_{\mathbf{p}}a_{\mathbf{q}}|f\rangle = -\frac{1}{x_{\mathbf{p}}+1}\delta_{\mathbf{p},-\mathbf{q}}\langle 0|f\rangle, \tag{3.22}$$

which yields

$$\langle 0|\eta(x)\eta(y)|f\rangle = -m\rho K_0(\sqrt{2m\pi\rho}|x-y|)\langle 0|f\rangle, \tag{3.23}$$

with K_0 being the modified Bessel function. As is explained in Ref. 9, the wave function (3.21) together with (3.23) leads to the characteristic short-range correlation of the Laughlin wave function, that is (3.4) with $F(\bar{z}) = 1$, for $|x_r - x_s| < \ell_B$. (Note that $\ell_B \approx 1/\sqrt{\rho}$ in the QH state.) It turns out that the statistics parameter m determines the short-range correlations of the electrons in the QH state at $\nu = 1/m$.

It is remarkable that the Laughlin wave function is obtained without using the explicit form of the Coulomb interaction once the mean-field solution is determined as in (3.6). Thus, the actual form of the repulsive interaction is irrelevant when the LLL projection is made. To make this observation clear let us recall our previous result on the wave function[9] which is obtained without imposing the LLL projection. In this case there is a correction to (3.22) and hence also to (3.23), which depends on the actual form of the interaction. As we have derived in the second paper of Ref. 9, the ground-state wave function is given by the Laughlin wave function multiplied by the factor

$$1 + \prod_{r>s} J(x_r - x_s), \tag{3.24}$$

where

$$J(x) = M\int \frac{d^2p}{(2\pi)^2}\frac{\mathbf{p}^4}{(\mathbf{p}^2 + 2eB)(\mathbf{p}^2 + eB)^2}V(p)e^{ipx}, \tag{3.25}$$

with

$$V(p) \equiv \int dx\, V(x) e^{-ipx} = \frac{2\pi e^2}{\varepsilon|\mathbf{p}|} \tag{3.26}$$

for the Coulomb potential $V(x) = e^2/(\varepsilon|x|)$. It is clear that, for the interaction $V(x)$ for which $J(x)$ is well defined, the correction term vanishes as $M \to 0$. In this way the Laughlin wave function is obtained regardless of the form of the repulsive interaction $V(x)$ when the LLL projection ($M \to 0$) is made.

We have found that the present analysis together with the LLL constraint is consistent with the previous one[9] based on the Hamiltonian (2.8) together with the kinetic term. Namely, all the present results are reproduced from the corresponding ones in Ref. 9 by taking the limit $M \to 0$.

IV. DOUBLE-LAYER QH SYSTEM

The CS gauge theory of the double-layer electron system is similarly constructed. The wave function of the bosonized electrons Ψ is defined by taking off the phase factors from that of the electrons $\widehat{\Psi}$ as in (2.1):

$$\Psi \equiv e^{im\sum_{r<s}\theta(x_r^1-x_s^1)} e^{im\sum_{p<q}\theta(x_p^2-x_q^2)} e^{in\sum\theta(x_r^1-x_p^2)} \widehat{\Psi}, \tag{4.1}$$

with the statistics parameters m and n; here, m is an odd integer but n is any integer. In second-quantized theory, the phase degrees of freedom are extracted as the CS gauge fields a_k^α, which are defined by

$$\begin{aligned} \varepsilon_{ij}\partial_i a_j^1 &= 2\pi(m\psi_1^\dagger\psi_1 + n\psi_2^\dagger\psi_2), \\ \varepsilon_{ij}\partial_i a_j^2 &= 2\pi(n\psi_1^\dagger\psi_1 + m\psi_2^\dagger\psi_2), \end{aligned} \tag{4.2}$$

in terms of the bosonized electron fields ψ_α, where α is the layer index: $\alpha = 1, 2$. In this way we need two CS gauge fields in general.

The Hamiltonian is given by[7]

$$\mathcal{H} = \sum_\alpha \frac{1}{2M}\int d^2x |(D_1^\alpha - iD_2^\alpha)\psi_\alpha|^2 + \frac{1}{2}\omega_c N + H_E[\psi] + \mathcal{V}[\psi], \tag{4.3}$$

with $iD_k^\alpha = i\partial_k + a_k^\alpha - eA_k$. The potential term H_E is given by

$$H_E[\psi] = \sum_\alpha \int d^2x\, eA_0^\alpha \psi_\alpha^\dagger\psi_\alpha \tag{4.4}$$

with A_0^α the electric potential at the layer α. The Coulomb term $\mathcal{V}$ is given by

$$\mathcal{V}[\psi] = \mathcal{V}_{11}[\psi] + \mathcal{V}_{22}[\psi] + \mathcal{V}_{12}[\psi], \tag{4.5}$$

with

$$\mathcal{V}_{\alpha\beta}[\psi] = \frac{e^2}{2\varepsilon}C_{\alpha\beta}\int d^2x\, d^2y : \{\psi_\alpha^\dagger\psi_\alpha(x) - \rho_0\} \times \frac{1}{\sqrt{(x-y)^2 + d_{\alpha\beta}^2}}\{\psi_\beta^\dagger\psi_\beta(y) - \rho_0\} :, \tag{4.6}$$

where $d_{11} = d_{22} = 0$ and $d_{12} = d$ is the interlayer distance ($d \approx \ell_B$); $C_{11} = C_{22} = 1$ and $C_{12} = 2$; ρ_0 is the background charge in each layer.

We have already analyzed[7] the double-layer electron system in detail for the statistics parameters $m \neq n$, and we have found that the QH state is realized at the magic filling factor

$$\nu \equiv \frac{2\pi\rho}{eB} = \frac{2}{m+n}, \tag{4.7}$$

and that this state is described by the Halperin wave function[16]

$$\prod(\bar{z}_r^1 - \bar{z}_s^1)^m \prod(\bar{z}_p^2 - \bar{z}_q^2)^m \prod(\bar{z}_r^1 - \bar{z}_p^2)^n \times \exp\left\{-\frac{1}{4}eB\left(\sum|z_r^1|^2 + \sum|z_p^2|^2\right)\right\}. \tag{4.8}$$

The role of the statistics parameters m and n is manifest in this wave function: m describes the intralayer correlation, while n is the interlayer correlation. For instance, the QH state at $\nu = \frac{1}{2}$ recently observed experimentally[17] is explained by choosing $m = 3$ and $n = 1$. The double-layer QH state is incompressible when $m \neq n$, and the Josephson effect is hardly expected. Although our analysis[7] was made without making the LLL projection, all the results are correct; the results with the LLL projection are produced simply by taking the limit $M \to 0$ in the corresponding formulas in Ref. 7.

When $m = n$, it has been argued[5] that the system contains a gapless mode, which we have related to a Goldstone mode associated with the spontaneous breakdown of a global phase symmetry in the double-layer QH system.[7] However, our previous analysis does not seem to be fully satisfactory from the point of view of the LLL projection. In what follows we analyze this mode carefully by making the LLL projection, and make clear the microscopic mechanism of the Josephson effect induced by this mode.

It should be remarked that, when $m = n$, the QH state exhibits the same intralayer and interlayer correlations as in (4.8). Hence, in order to realize such a QH state it is necessary to choose the interlayer distance d to be the order of the magnetic radius ℓ_B.

In this choice of the statistics parameters ($m = n$), only the combination of the CS gauge fields $a_k = \frac{1}{2}(a_k^1 + a_k^2)$ is relevant, since the combination $(a_k^1 - a_k^2)$ decouples from the system; see the Appendix. Namely, we need only one CS gauge field to extract the phase degrees of freedom from the electron system. This is the crucial difference from the general double-layer system with $m \neq n$. Indeed, the constraint equation (4.2) is reduced to a single equation:

$$\varepsilon_{ij}\partial_i a_j = 2\pi m(\psi_1^\dagger\psi_1 + \psi_2^\dagger\psi_2). \tag{4.9}$$

The Hamiltonian is given by[7]

$$\mathcal{H} = \sum_\alpha \frac{1}{2M}\int d^2x |(D_1 - iD_2)\psi_\alpha|^2 + \frac{1}{2}\omega_c N + H_E[\psi] + H_T[\psi] + \mathcal{V}[\psi], \tag{4.10}$$

with $iD_k = i\partial_k + a_k - eA_k$, where the potential term $H_E[\psi]$ and the Coulomb term $\mathcal{V}[\psi]$ is given by (4.4) and (4.5). Here, we have introduced an interlayer tunneling term with strength λ:

$$H_T[\psi] = -\lambda \int d^2x(\psi_1^\dagger\psi_2 + \psi_2^\dagger\psi_1), \tag{4.11}$$

where λ should be much smaller than the Coulomb energy ($\lambda \ll e^2/\ell_B$) so that the formation of the double-layer QH state is not ruined by the interlayer tunneling.

Except for the Goldstone mode which we discuss later, all the analysis proceeds in parallel to the case of the single-layer Hall system. First of all, the LLL constraints read

$$(D_1 - iD_2)\psi_\alpha|f\rangle = 0, \tag{4.12}$$

which are exactly solvable:[7] The wave function is given by (4.8) with $m = n$ and multiplied by an arbitrary analytic function $F(\bar{z}_1, \bar{z}_2)$. This arbitrariness is fixed in the real ground state by minimizing the Coulomb energy (4.5), as has been explicitly shown in the single-layer case.

In the absence of the interlayer tunneling ($\lambda = 0$) the mean-field ground state is given by

$$a_k = eA_k, \qquad \psi_\alpha = \sqrt{\rho_0}e^{i\theta_\alpha}, \tag{4.13}$$

with constant phase θ_α. This solution exists only at the magic filling factor

$$\nu \equiv \frac{2\pi\rho}{eB} = \frac{1}{m} \tag{4.14}$$

with $\rho = 2\rho_0$. There is an essential observation. The LLL condition (4.12) and the constraint equation (4.9) allow an infinite number of solutions given by $a_k = eA_k$ and $\psi_\alpha = \sqrt{\rho_\alpha}e^{i\theta_\alpha}$ with $\rho_1 \neq \rho_2$ at the same filling factor provided that $\rho_1 + \rho_2 = \rho$ is fixed. They are also condensed states of bosonized electrons, although they have higher Coulomb energy. Therefore, retaining the coherent phases in each layer, electrons in one layer can move to the other layer. This fact leads to the Josephson effect in the presence of the tunneling ($\lambda \neq 0$). Recall that this is not the case for the general QH states with statistics parameters $m \neq n$, where the constraint conditions (4.2) fix the electron numbers in each layer uniquely as $\rho_1 = \rho_2 = \rho_0$. Then, a movement of electrons from one layer to the other breaks the coherent phases in each layer.

We study the Gaussian fluctuations around the mean-field ground state (4.13) by setting

$$\psi_\alpha = \sqrt{\rho_0} + \eta_\alpha, \tag{4.15}$$

where we have chosen $\theta_\alpha = 0$ for simplicity. Linearizing the LLL constraint (4.12) as in the case of the single-layer case, we obtain that

$$\partial_z(\eta_\alpha - \sqrt{\rho_0}a)|f\rangle = 0, \tag{4.16}$$

where a is defined by (3.10). Now, it is trivial to derive that

$$\partial_z(\eta_+ - \sqrt{\rho}a)|f\rangle = 0, \tag{4.17a}$$

$$\partial_z\eta_-|f\rangle = 0, \tag{4.17b}$$

with $\eta_\pm = (\eta_1 \pm \eta_2)/\sqrt{2}$, and that a is determined by

$$\partial^2 a = 2\sqrt{\rho}m\pi(\eta_+^\dagger + \eta_+). \tag{4.18}$$

Let us make the mode expansion of $\eta_\pm$ as in (3.7b); we denote the corresponding modes by $a_{(\pm)\mathbf{p}}$.

The condition (4.17a) with (4.18) is formally identical to (3.11) with (3.12) in the single-layer case; hence, the analysis is identical. Thus, this LLL condition yields

$$T_{(+)\mathbf{p}}|f\rangle = 0, \tag{4.19}$$

where $T_{(+)\mathbf{p}}$ is defined by the equation identical to (3.14) with the replacement of $a_\mathbf{p}$ by $a_{(+)\mathbf{p}}$. On the other hand, the condition (4.17b) simply means

$$a_{(-)\mathbf{p}}|f\rangle = 0 \quad \text{for} \quad \mathbf{p} \neq 0. \tag{4.20}$$

It is important to recognize that this condition does not restrict the number of the zero-momentum mode, $a_{(-)\mathbf{p}}$ with $\mathbf{p} = 0$, in the LLL states, although it requires the number of the nonzero-momentum modes to be zero. Hence, even in the mean-field approximation there are many LLL states in this double-layer case.

As in the single-layer case, we can derive the ground-state wave function by evaluating the two-point wave functions $\langle 0|\eta_\alpha(x)\eta_\beta(y)|f\rangle$. The result is the Halperin wave function (4.8) with $m = n$; see Ref. 7 for more details where the LLL constraint has not been imposed.

We go on to diagonalize the Coulomb term (4.5) within the LLL states, which can be expressed in terms of $\eta_\pm$ as $\mathcal{V} = \mathcal{V}_+ + \mathcal{V}_-$ with

$$\mathcal{V}_\pm[\eta_\pm] = \frac{e^2\rho_0}{2\varepsilon}\int d^2x\, d^2y : [\eta_\pm^\dagger(x) + \eta_\pm(x)] \times\left(\frac{1}{|x-y|} \pm \frac{1}{\sqrt{(x-y)^2+d^2}}\right) \times[\eta_\pm^\dagger(y) + \eta_\pm(y)] : . \tag{4.21}$$

The Coulomb term $\mathcal{V}_+[\eta_+]$ is calculated precisely as in the single-layer case, and we have a similar expression to (3.17). Hence, the diagonalization is trivial in the LLL states. On the other hand, with respect to the Coulomb term $\mathcal{V}_-[\eta_-]$, using the mode expansion for η_- we obtain

$$\mathcal{V}_-[\eta_-]|f\rangle = \frac{\pi e^2\rho_0}{\varepsilon}\sum_{\mathbf{p}\neq 0}\frac{1}{|\mathbf{p}|}\left(1 - e^{-|\mathbf{p}|d}\right)a_{(-)\mathbf{p}}^\dagger a_{(-)\mathbf{p}}^\dagger|f\rangle + \frac{\pi e^2\rho_0 d}{\varepsilon}(c^\dagger c^\dagger + cc + 2c^\dagger c)|g\rangle, \tag{4.22}$$

where we have set $c \equiv a_{(-)0}$ for simplicity. Hence, the diagonalization of $\mathcal{V}_-[\eta_-]$ is also trivial in the LLL states obeying (4.20), as far as the nonzero-momentum modes ($a_{(-)\mathbf{p}}$ with $\mathbf{p} \neq 0$) are concerned.

V. GOLDSTONE MODE AND JOSEPHSON EFFECT

With the LLL constraints (4.12) imposed we have found that the only dynamical mode is the zero-momentum mode $c \equiv a_{(-)0}$. We first show that this is a Goldstone mode related with a global phase symmetry in the double-layer QH system.

In the absence of the interlayer tunneling ($\lambda = 0$), the Hamiltonian (4.10) is invariant under two global phase transformations:

$$\psi_\alpha \to e^{i\Lambda_\alpha}\psi_\alpha. \tag{5.1}$$

One of the generators (conserved quantities) is the total electron number $N_{\text{total}} \equiv N_1 + N_2$ for the choice of $\Lambda_1 = \Lambda_2$, while the other is the electron number difference $\Delta N \equiv N_1 - N_2$ for the choice of $\Lambda_1 = -\Lambda_2$, where

$$N_\alpha \equiv \int \psi_\alpha^\dagger \psi_\alpha d^2x. \tag{5.2}$$

Both of these symmetries are broken by the mean-field solution (4.13), and as a result two Goldstone modes are generated. However, one of the Goldstone modes associated with N_{total} is absorbed by the CS gauge field a_k via the Anderson-Higgs mechanism and disappears. The other Goldstone mode, which remains to be unabsorbed, is associated with the electron number difference

$$\Delta N = N_1 - N_2 = \sqrt{2N_0}(c^\dagger + c), \tag{5.3}$$

whose canonical conjugate is the phase difference of the condensates

$$\Delta\theta = \theta_2 - \theta_1 = \frac{i}{\sqrt{2N_0}}(c - c^\dagger), \tag{5.4}$$

together with

$$\left[\Delta\theta, \frac{1}{2}\Delta N\right] = i, \tag{5.5}$$

where $N_0 = \rho_0 V$ and

$$\theta_\alpha = \frac{1}{V}\int d^2x\, \theta_\alpha(x) \tag{5.6}$$

with $\psi_\alpha = |\psi_\alpha| \exp[i\theta_\alpha(x)]$. Here, we have made the following identification:

$$\psi_\alpha = \sqrt{\rho_0 + \delta_\alpha} e^{i\theta_\alpha} = \sqrt{\rho_0} + \eta_\alpha, \tag{5.7}$$

with $\eta_\alpha \approx \delta_\alpha/(2\sqrt{\rho_0}) + i\sqrt{\rho_0}\theta_\alpha$.

Using these variables the Coulomb term $\mathcal{V}_-$ given by (4.22) is diagonalized within the LLL states as

$$\mathcal{V}_- = \frac{\pi e^2 d}{\varepsilon}\left(\frac{(\Delta N)^2}{2V} - \rho_0\right). \tag{5.8}$$

This implies that the double-layer system has an electric capacity

$$C = \frac{V}{4\pi d}. \tag{5.9}$$

It is seen that the states with $\Delta N \neq 0$ degenerate with the ground state in the limit $V \to \infty$, which demonstrates that ΔN is indeed the Goldstone mode.

We assume that the tunneling strength λ is much smaller than the Coulomb energy $e^2\rho d \approx O(e^2/\ell_B)$ so that the double-layer QH states are realized. Substituting (5.7) into the term (4.11) and taking only the Goldstone mode we find

$$H_T = -2\lambda\rho_0 V \cos(\Delta\theta) + \frac{\lambda(\Delta N)^2}{\rho_0 V}\cos(\Delta\theta). \tag{5.10}$$

The second term in this equation may be regarded as a chemical potential term induced by the tunneling interaction. This term vanishes in the limit $V \to \infty$ just as the electric potential term does for a fixed value of ΔN. Hence, the energy of the system does not change even when some electrons tunnel from one layer to the other.

We now analyze the mechanism of Josephson tunneling by using ΔN and $\Delta\theta$. The term (4.4) reads

$$H_E = \tfrac{1}{2}e(A_0^1 - A_0^2)\Delta N\ . \tag{5.11}$$

After the LLL projection, by combining (5.8), (5.10), and (5.11), the total Hamiltonian (4.10) amounts to

$$\begin{aligned} H_{\text{GM}} = {} & \frac{\pi e^2 d}{2\varepsilon V}(\Delta N)^2 + \frac{\lambda}{\rho_0 V}(\Delta N)^2\cos(\Delta\theta) \\ & -2\lambda N_0\cos(\Delta\theta) + \tfrac{1}{2}e(A_0^1 - A_0^2)\Delta N\ , \end{aligned} \tag{5.12}$$

up to an irrelevant constant term.

The Heisenberg equations of motion follow:

$$J \equiv \partial_t\Delta\rho = -2\lambda\rho_0\left[1 - \frac{1}{2}\left(\frac{\Delta\rho}{\rho_0}\right)^2\right]\sin\Delta\theta\ , \tag{5.13a}$$

$$\begin{aligned} \partial_t\Delta\theta &= e(A_0^1 - A_0^2) + \frac{4\pi e^2 d\Delta\rho}{\varepsilon} + \frac{2\lambda\Delta\rho}{\rho_0}\cos(\Delta\theta) \\ &\equiv eV_{\text{ext}}(t) \end{aligned} \tag{5.13b}$$

with $\delta\rho \equiv \Delta N/2V$. Except for the chemical potential term these are the famous equations governing the Josephson current familiar in a superconductor.[18] Indeed, in a superconductor the chemical potential term is negligible compared with the electric potential term. However, this is not so in the QH state. The chemical potential difference $(2\lambda\Delta\rho/\rho_0)\cos(\Delta\theta)$ is actually of the same order as the induced electric voltage $(4\pi e^2 d\Delta\rho/\varepsilon)$ associated with the electric capacity (5.9). This is a peculiar feature to the QH-state Josephson junction where $\rho_0 \sim 10^{11}/\text{cm}^2$, $\lambda \sim 1$ K, $d \sim 100$ Å, and $\varepsilon \sim 10$.

By solving (5.13a) and (5.13b) for small fluctuations of $\Delta\theta$ and $\Delta\rho$, it is easy to see that the Goldstone mode acquires a gap energy:

$$E_0 = \sqrt{\omega_J^2 + (\Delta_{\text{SAS}})^2}\ , \tag{5.14a}$$

with

$$\omega_J \equiv \sqrt{\frac{4\pi e^2 d\rho_0\Delta_{\text{SAS}}}{\varepsilon}}\ , \tag{5.14b}$$

and $\Delta_{\rm SAS} = 2\lambda$, where ω_J is several times larger than $\Delta_{\rm SAS}$ in a typical junction.

As is well known,[19] it is the total electrochemical potential difference $V_{\rm ext}$ that is measurable experimentally by a "voltmeter." Therefore, the phase difference $\Delta\theta$ is controlled by (5.13b) and develops as in the well-known way:

$$\Delta\theta(t) = e\int_0^t dt' V_{\rm ext}(t') + \Delta\theta_0 \ , \qquad (5.15)$$

with $\Delta\theta_0$ the initial phase difference and $V_{\rm ext}$ the voltage measured. Here, the appearance of the unit charge e is the consequence of the bosonized electron condensation. Next, we examine the correction term $(\Delta\rho/\rho_0)^2$ in the Josephson current (5.13a). Obviously, there is no correction when the density imbalance $\Delta\rho$ is immediately compensated by the external supply (as in the dc Josephson circuit). Otherwise, we need to estimate the correction. It is easy to see that $(\Delta\rho/\rho_0)^2 \approx (2\lambda/eV_{\rm ext})^2 \sim 1/100$ with $2\lambda = 1$ K and $V_{\rm ext} = 1$ mV. Therefore, it should be possible to observe the Josephson effect in the QH state when parameters are adjusted as $(2\lambda/eV_{\rm ext})^2 \ll 1$. A further analysis of the QH-state Josephson effect is found in Ref. 4, where the Meissner effect of the parallel magnetic field is studied: In this reference the above chemical potential term is missed but its existence does not modify the topological property of vortices nor numerical estimations.

A comment is in order. It seems that the phase symmetry (5.1) with $\Lambda_1 = -\Lambda_2$ is broken explicitly by the tunneling interaction (4.11). However, this is not true because the phase symmetry is a part of the electromagnetic gauge symmetry. In this paper we have neglected the electromagnetic fields except for a background magnetic field. When we recover them, the manifestly gauge invariant tunneling Hamiltonian is given by

$$\mathcal{H}_T = -\lambda\Bigg\{\psi_1^\dagger(x)\exp\left[ie\int_{z_2}^{z_1} dz\, A_z(x,z)\right]\psi_2(x) + \psi_2^\dagger(x)\exp\left[ie\int_{z_1}^{z_2} dz\, A_z(x,z)\right]\psi_1(x)\Bigg\}, \qquad (5.16)$$

where the z axis is taken perpendicular to the layers and the coordinates z_α are assigned to each layer. Note that the tunneling interaction (4.11) is obtained simply by making a gauge choice $A_z = 0$. Now, it is obvious that the tunneling Hamiltonian is invariant under the local gauge transformation

$$\psi_\alpha \to e^{if(x,z_\alpha)}\psi_\alpha \ , \qquad (5.17a)$$

$$A_z \to A_z + \frac{1}{e}\partial_z f(x,z) \ . \qquad (5.17b)$$

The global phase symmetry (5.1) is a special case with $f(x,z_\alpha) = \Lambda_\alpha$. In the covariant gauge with (5.16), the Goldstone boson is absorbed into the electromagnetic gauge potential, as is the Anderson-Higgs mechanism familiar in the case of a superconductor. Therefore, we may say that the Josephson effect is a natural consequence of the spontaneous symmetry breakdown of the electromagnetic gauge symmetry.

VI. DISCUSSION

The Josephson effect is described by the mutual conjugate variables ΔN and $\Delta\theta$. Their existence is assured by the Goldstone theorem. Therefore, it does not depend on the detail of the dynamics. The essential point is that the movement of electrons in one layer to the other layer (namely $\Delta N \neq 0$) does not destroy the coherent phases of the QH states. Thus, the mode ΔN exists in the QH states as the Goldstone mode. In the general QH states described with statistics parameters $m \neq n$ there exist no such Goldstone modes.

Our Goldstone mode is rather peculiar since it is an isolated mode without the nonzero-momentum component. This does not contradict the Goldstone theorem. In the derivation of the theorem we analyze a spectral function $\sigma(\mathbf{p},E)$. In the present case it follows that $\sigma(\mathbf{p},E) \propto \delta(\mathbf{p})\delta(E)$ due to the LLL projection. It is instructive to examine the problem by making use of our previous formalism,[7] where without imposing the LLL projection we have derived the "full" Goldstone mode such that

$$E(\mathbf{p}) \simeq \sqrt{\frac{\pi e^2\rho d}{\varepsilon M}}|\mathbf{p}|. \qquad (6.1)$$

When the LLL projection is made ($M \to 0$), the nonzero-momentum component disappears from the dispersion relation except for the zero-momentum component, just as the nonzero-momentum component disappears from the spectral function. The isolated mode is indeed the Goldstone mode since the states with $\Delta N \neq 0$ degenerate with the ground state in the limit of large volume ($V \to \infty$).

Let us remark that a gapless mode with a linear dispersion relation has been found in other microscopic approaches[5] even when the LLL projection is taken. (This gapless mode acquires a gap in the presence of the interlayer tunneling as in our case.) A possible reason why we do not have such a gapless mode with a linear dispersion relation is because nonperturbative excitations involving quasiparticles are missed in our analysis. However, the fact that such a nonperturbative excitation becomes gapless seems to be accidental. We are currently investigating this puzzle. In any case, the Josephson effect is induced by the zero-momentum mode, that is, the global mode ΔN and its conjugate $\Delta\theta$.

In this paper we have made clear the dynamical mechanism of the QH-state Josephson effect in the mean-field approximation. Our analysis is based on the CS gauge theory along with the LLL projection. Observation of the effect constitutes an experimental verification of a possible statistical transmutation on the plane.

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APPENDIX

As we have emphasized in the text, when the statistics parameters are equal ($m = n$), there is only one CS gauge field. This is why the system possesses the Goldstone mode driving the Josephson effect. Let us show explicitly how the combination of the CS fields $(a_k^1 - a_k^2)$ decouples from the system when $m = n$ in the Lagrangian formalism.

When two statistics parameters are not equal ($m \neq n$), the Lagrangian for the double-layer system is given by[7]

$$\mathcal{L} = \sum_\alpha \left(\psi_\alpha^\dagger i D_0^\alpha \psi_\alpha - \frac{1}{2M} |(D_1^\alpha - i D_2^\alpha)\psi_\alpha|^2 \right) + \mathcal{L}_{\rm CS} - \frac{1}{2}\omega_c N - \mathcal{V}[\psi], \tag{A1}$$

together with the CS term

$$\mathcal{L}_{\rm CS} = -\frac{m}{4(m^2 - n^2)\pi} \varepsilon^{\mu\nu\lambda} \left(a_\mu^1 \partial_\nu a_\lambda^1 + a_\mu^2 \partial_\nu a_\lambda^2 \right) + \frac{n}{4(m^2 - n^2)\pi} \varepsilon^{\mu\nu\lambda} \left(a_\mu^1 \partial_\nu a_\lambda^2 + a_\mu^2 \partial_\nu a_\lambda^1 \right) \tag{A2}$$

and

$$i D_\mu^\alpha = i \partial_\mu + a_\mu^\alpha - e A_\mu. \tag{A3}$$

It is straightforward to see that the Hamiltonian (4.3) and the constraint equation (4.2) are reproduced from this CS Lagrangian. In particular, the constraint equation follows from the variation of the Lagrangian with respect to a_0^α.

The CS term $\mathcal{L}_{\rm CS}$ is diagonalized when we introduce

$$a_\mu = \frac{1}{2}(a_\mu^1 + a_\mu^2), \qquad \widehat{a}_\mu = \frac{1}{2}(a_\mu^1 - a_\mu^2), \tag{A4}$$

or

$$a_\mu^1 = a_\mu + \widehat{a}_\mu, \qquad a_\mu^2 = a_\mu - \widehat{a}_\mu. \tag{A5}$$

Substituting (A4) into the CS term (A2), we obtain

$$\mathcal{L}_{\rm CS} = -\frac{1}{2(m+n)\pi} \varepsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda - \frac{1}{2(m-n)\pi} \varepsilon^{\mu\nu\lambda} \widehat{a}_\mu \partial_\nu \widehat{a}_\lambda. \tag{A6}$$

After rescaling $\widehat{a}_\mu$ as $\widehat{a}_\mu = \sqrt{|m-n|} b_\mu$, we take the limit $m \to n + 0$. Then, the Lagrangian (A1) reads

$$\mathcal{L} = \sum_\alpha \left(\psi_\alpha^\dagger i D_0 \psi_\alpha - \frac{1}{2M} |(D_1 - i D_2)\psi_\alpha|^2 \right) + \mathcal{L}_{\rm CS} - \frac{1}{2}\omega_c N - \mathcal{V}[\psi], \tag{A7}$$

together with

$$\mathcal{L}_{\rm CS} = -\frac{1}{4m\pi} \varepsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda - \frac{1}{2\pi} \varepsilon^{\mu\nu\lambda} b_\mu \partial_\nu b_\lambda \tag{A8}$$

and

$$i D_\mu = i \partial_\mu + a_\mu - e A_\mu. \tag{A9}$$

It is clear that the CS field b_μ decouples from the system since it does not couple with the field ψ_α. Hence, a_μ is the only relevant CS gauge field in the system with $m = n$. It is easy to see that the Hamiltonian (4.10) and the constraint equation (4.9) follow from the Lagrangian (A7) with (A8) and (A9), except for the tunneling term H_T neglected in this appendix.

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