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Bose condensation and relaxation explosion in magnetically trapped atomic hydrogen

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We predict and analyze nontrivial relaxational behavior of magnetically trapped gases near the Bose-condensation temperature T_c. Due to strong compression of the condensate by the inhomogeneous trapping field, particularly at low densities, the relaxation rate shows a strong, almost jumplike, increase below T_c. As a consequence the maximum fraction of condensate particles is limited to a few percent. This phenomenon can be called a "relaxation explosion." We discuss its implication for the detectability of Bose-Einstein condensation in atomic hydrogen.

I. INTRODUCTION

The problem of Bose-Einstein condensation (BEC) in weakly interacting systems, such as low-temperature gases and metastable excitons in crystals, attracts a great deal of interest in view of the possibility to observe new quantum phenomena which allow a transparent physical description. In this respect, dilute gases of spin-polarized atoms in static magnetic traps are of particular interest as examples of purely self-interacting systems. In these traps, proposed for atomic hydrogen (H) by Hess,[1] the effective elimination of physical boundaries is accomplished by creating a magnetic field minimum in free space. This minimum forms a potential well for electron spin-up polarized atoms (H$\uparrow$), called low-field seekers. Due to spin relaxation in pair collisions, and subsequent ejection of the spin flipped (H$\downarrow$) atoms (high-field seekers) from the trap,[2] the densities attainable in H$\uparrow$ are limited to $n \lesssim 10^{14} - 10^{15}$ cm^{-3}, when the characteristic time of relaxational decay is of the order of seconds. Such densities are considerably higher than those typically encountered in cold gases of neutral atoms as studied in, e.g., optical molasses, but at the same time low compared to densities reached with H$\downarrow$ in conventional environments (confinement with walls).[3,4]

The critical BEC temperature is expressed by the relation:

$$T_c = 3.31 \frac{\hbar^2}{m} n^{2/3}. \tag{1}$$

Here n is the density of the gas at the trap center and m is the mass of the atom. In H$\uparrow$ the maximum achievable densities are of the order of 10^{15} cm^{-3} corresponding to $T_c \approx 100$ μK. In other trapped gases this temperature should be much lower. For H, the technique which has proved most proficient in producing these low temperatures is evaporative cooling,[5–8] which is based on the preferential removal of the most energetic atoms. Currently the lowest temperature reported using this method is 100 μK at $n = 8 \times 10^{13}$ cm^{-3} ($T/T_c \approx 3$).[7] Recently, optical cooling techniques were applied to trapped H, using a narrow-band pulsed Lyman-α source.[9] In these experiments both Doppler cooling, as well as light-induced evaporative cooling were demonstrated. In the latter method, an optical analogue of evaporative cooling, the preferential removal of energetic atoms is accomplished by optical pumping to nontrapped states (H$\downarrow$).

In spite of the promising results of evaporative and optical cooling experiments, one may expect to encounter many underwater stones when attempting to achieve BEC in trapped gases. In this paper, we predict and analyze the nontrivial relaxational behavior of the trapped gas near the BEC point and address the question of how the appearance of the condensate will manifest itself. In this context, two features of the system are of prime importance; namely, the inhomogeneity of the magnetic field, and the low density of the gas. Due to strong compression by the inhomogeneous field, the condensate will be confined to a very small spatial region near the center of the trap,[10,11] the condensate density being much higher than the density of above condensate particles. Similarly to the effect predicted for three body recombination in high density H$\downarrow$ gas,[12] this should lead to a strong increase of the relaxation rate ν_r, as ν_r is proportional to n^2. Moreover, due to the low density of the gas, the compression of the condensate will be so strong, that ν_r can exhibit a strong, almost jumplike, increase below T_c, a "relaxation explosion," even when the fraction of particles in the condensate is small. As we will show, this relaxation explosion prevents one from penetrating deeply into the BEC regime. In the case of H, the maximum fraction of particles in the condensate will remain very small. Concomitantly, the temperature will never be lower than a few percent below T_c. We will argue that even in such a situation the relaxation explosion may manifest itself through increased production of H$\downarrow$ atoms which may provide a method to detect the presence of the condensate in atomic hydrogen.

II. SPIN RELAXATION RATE: RELAXATION EXPLOSION BELOW T_C

We first consider collisional spin relaxation to nontrapped states in a trapped gas of N spin-up polarized atoms at temperature T. We assume that the decay rate due to this process, as well as the cooling or heating rate, is much smaller than the rate of thermalizing elastic collisions. At sufficiently low T the atoms are concentrated near the center of the trap, and their potential energy is found by expanding the trapping potential near the field minimum. For many types of traps we have a parabolic expansion. Without loss of generality, we can restrict ourselves to a spherical trap:

$$V(r) = \mu_B B_0 + \frac{m}{2}\omega^2 r^2, \tag{2}$$

where r is the distance to the trap center, and ω is oscillation frequency. The case where the radial and axial oscillation frequencies are different is obtained by applying a simple scaling. The first term in Eq. (2), which represents the Zeeman energy of atoms located at $r = 0$, will henceforth be omitted.

We first discuss relaxation at temperatures above T_c, considering the motion of the atoms in the potential $V(r)$ as classical ($T \gg \hbar\omega$). The loss of atoms from the trap occurs in two channels:

$$A\uparrow + A\uparrow \rightarrow A\uparrow + A\downarrow \tag{3}$$

$$A\uparrow + A\uparrow \rightarrow A\downarrow + A\downarrow, \tag{4}$$

where the symbol $A\uparrow$ stands for atoms in trapped states and $A\downarrow$ for atoms in nontrapped states. It should be emphasized that the process described by Eq. (3) can lead to heating of the gas, as the particle which remains in the trap suffers a recoil which at low incident energy equals the energy difference between $A\uparrow$ and $A\downarrow$. This recoil can be substantial and depends on the value of B_0 in Eq. (2). However, often the process leading to evaporation is such that these recoiling particles are rapidly removed, before they thermalize with the remaining atoms. We shall assume this to be the case and, hence, the total rate ν_r of atom loss can be written in the form:

$$\nu_r = 2\alpha \int d\mathbf{r}\, n^2(r), \tag{5}$$

where $n(r)$ is the density distribution and α is the sum of the rate constants governing the processes in Eqs. (3) and (4). For hydrogen $\alpha \approx 10^{-15}$ cm^3/s.[13,14]

The density distribution above T_c is determined by the following expression:

$$n(r) = \int \frac{d^3k}{(2\pi)^3}\left[\exp\left(\frac{\varepsilon_k + V(r) + 2n(r)\tilde{U} - \mu}{T}\right) - 1\right]^{-1}$$

$$\equiv n_c(T)\frac{1}{g_{3/2}(1)} g_{3/2}\left[\exp\left(-\frac{V(r) + 2n(r)\tilde{U} - \mu}{T}\right)\right]. \tag{6}$$

Here $\varepsilon_k = \hbar^2k^2/2m$, μ is the chemical potential, $2n\tilde{U}$ is the mean field potential energy originating from elastic pair interaction between atoms, $\tilde{U} = 4\pi\hbar^2 a/m$ where a is the scattering length (for spin polarized H $\tilde{U} = 4.3 \times 10^{-22}$ K cm^3). $n_c = g_{3/2}(1)(\frac{mT}{2\pi\hbar^2})^{3/2}$ is the critical BEC density for temperature T, and[15]

$$g_\ell(y) = \sum_{n=1}^{\infty} \frac{y^n}{n^\ell}. \tag{7}$$

In the density range under consideration we have $n_c\tilde{U} \ll T_c$, and since in the classical regime ($T \gg T_c$) the maximum density $n_{\max} \ll n_c$, we can neglect the interaction term $2n(r)\tilde{U}$ in Eq. (6) in this region. Near the BEC point we can replace $n(r)$ by n_c in the exponent of Eq. (6).

The chemical potential can be obtained from Eq. (6) by using the expression

$$\int d\mathbf{r}\, n(r) = N. \tag{8}$$

In the classical regime we find the well-known expression:

$$\mu = -T \ln\left[\left(\frac{mT}{2\pi\hbar^2}\right)^{3/2}\frac{V_e}{N}\right], \tag{9}$$

where

$$V_e = \left(\frac{2\pi T}{m}\right)^{3/2}\frac{1}{\omega^3}, \tag{10}$$

is the classical characteristic effective volume of the trapped gas: $n(r) = (N/V_e)\exp[-V(r)/T]$. From Eq. (5) we find

$$\nu_r = \nu_{\text{cl}} \equiv \frac{\alpha N^2}{\sqrt{2}V_e} \quad ; \quad T \gg T_c. \tag{11}$$

The BEC point (the first appearance of the condensate at $r = 0$) is determined by the condition:

$$\mu = 2n_c\tilde{U}. \tag{12}$$

The critical BEC temperature was calculated for various power law potentials by Bagnato, Pritchard, and Kleppner.[16] For the potential given by Eq. (2) the result is

$$T_c = [N/g_3(1)]^{1/3}\hbar\omega. \tag{13}$$

This result, following from Eqs. (6), (8), and (12), clearly shows that the inequality $T \gg \hbar\omega$ is satisfied for $T > T_c$, and the motion of the particles is classical.

At temperatures close to but still above T_c, Eqs. (5), (6), and (8) yield

$$\nu_r = \nu_{\text{cl}} G(\xi) \quad ; \quad T \geq T_c, \tag{14}$$

where $\xi = \exp(\mu - 2n_c\tilde{U})/T$ with the chemical potential determined by the equation

$$g_3(\xi) = \left(\frac{T_c}{T}\right)^3 g_3(1) \tag{15}$$

and

$$G(y) = \frac{2^{3/2}}{g_3^2(y)} \sum_{m,n=1}^{\infty} \frac{y^{m+n}}{n^{3/2}m^{3/2}(m+n)^{3/2}}. \tag{16}$$

The coefficient G slowly changes from 1 in the classical regime to 1.3 at $T = T_c$. Thus, the quantity $\nu_r T^{3/2}/N^2$ remains practically unchanged for temperatures down to T_c.

Let us now turn to the relaxation below T_c. The appearance of the condensate leads to a fundamental change in the symmetry and correlation properties of the system, which also manifests itself in the rate of inelastic processes.[17,18] Therefore, we should replace Eq. (5) by a more general expression:[18,19]

$$\nu_r = 2\alpha \int d\mathbf{r}\, \frac{1}{2} < \hat{\psi}^\dagger(\mathbf{r})\hat{\psi}^\dagger(\mathbf{r})\hat{\psi}(\mathbf{r})\hat{\psi}(\mathbf{r}) >, \tag{17}$$

where $\hat{\psi}$ is the field operator of the atoms and $< \hat{\psi}^\dagger\hat{\psi}^\dagger\hat{\psi}\hat{\psi} >$ is the local two-particle correlator. We assume to be outside the region of critical fluctuations, which is the case when

$$\Delta T = T_c - T \gg n_c\tilde{U}. \tag{18}$$

Confining ourselves to temperatures

$$T \gg n_{\max}\tilde{U}, \tag{19}$$

we may neglect the phonon branch of the excitation spectrum and, hence, the anomalous averages in Eq. (17). This equation then reduces to the form

$$\nu_r = 2\alpha \int d\mathbf{r}\, [\tfrac{1}{2}n_0^2(r) + 2n_0 n'(r) + n'^2(r)], \tag{20}$$

where n_0 and n' are the densities of condensate and above-condensate particles, respectively.

The problem of the spatial distribution of condensate and above-condensate particles in the inhomogeneous field, addressed by Walraven and Silvera,[20] was investigated by Goldman, Silvera, and Leggett,[10] and by Huse and Siggia.[11] We assume that the space occupied by the condensate is much larger than the correlation radius of the condensate wave function, which is equivalent to the condition $n_0\tilde{U} \gg \hbar\omega$ (this will be valid everywhere in the BEC spatial region with the exception of a very small neighborhood of boundary of this region). In this case the kinetic energy term in the equation for the condensate wave function can be neglected and the equation for the condensate density takes the form (see Refs. 10 and 11):

$$n_0(r)\tilde{U} + 2n'(r)\tilde{U} + V(r) = \mu, \tag{21}$$

where

$$n'(r) = n_c(T)\frac{1}{g_{3/2}(1)} g_{3/2}(\exp[-n_0\tilde{U}/T]). \tag{22}$$

The condensate will exist in a finite region of space determined by the condition

$$V(r) \leq \mu - 2n_c\tilde{U}. \tag{23}$$

At the boundary of the condensate region $n_0 = 0$ and $n' = n_c$. Moreover, under condition Eq. (19) we obtain $n'(r) \approx n_c$ everywhere inside this region.

As will be shown below, in cases of practical interest we are necessarily restricted to small condensate fractions and, hence, when considering the relaxation below T_c we may confine ourselves to the case where

$$\Delta T \ll T_c. \tag{24}$$

The condition Eq. (18) for the absence of critical fluctuations leads to the inequality

$$\mu - 2n_c\tilde{U} \gg n_c\tilde{U}, \tag{25}$$

which ensures a pronounced condensate peak in the density distribution since we have: $n_{0\max} = (\mu - 2n_c\tilde{U})/\tilde{U} \gg n_c \gtrsim n'(r)$. In this case we may neglect above-condensate particles inside the spatial BEC region. Then the number of condensate particles $N_0 = N - N'(T)$, where $N'(T) = (T/T_c)^3 N$ is the number of above-condensate particles outside the BEC region, following from the density distribution Eq. (6). Hence, under condition Eq. (24) we obtain

$$\frac{N_0}{N} = 3\frac{\Delta T}{T_c}. \tag{26}$$

On the other hand, from the condensate density profile, given by Eq. (21), and taking into account Eq. (13), we find

$$\begin{aligned} N_0 &= \frac{8\pi}{15}\frac{(\mu - 2n_c\tilde{U})^{5/2}}{(m/2)^{3/2}\omega^3\tilde{U}} \\ &= N\frac{8}{15\sqrt{\pi}}\frac{g_{3/2}(1)}{g_3(1)}\left(\frac{\mu - 2n_c\tilde{U}}{T_c}\right)^{5/2}\frac{T_c}{n_c\tilde{U}}. \end{aligned} \tag{27}$$

Equations (26) and (27) allow us to obtain μ as a function of ΔT. The volume occupied by the condensate is determined by the expression

$$\begin{aligned} V_0 &= \frac{4}{3\sqrt{(\pi)}}\left(\frac{\mu - 2n_c\tilde{U}}{T_c}\right)^{3/2} V_e \\ &\approx 2.0\left(\frac{n_c\tilde{U}}{T_c}\right)^{3/5}\left(\frac{\Delta T}{T_c}\right)^{3/5} V_e. \end{aligned} \tag{28}$$

Clearly, we have $V_0 \ll V_e$.

Under the conditions just described there are two dominant contributions to the relaxation rate of the gas: the process in which two condensate particles are involved, and the mechanism involving two above-condensate particles outside the BEC spatial region. The processes involving above-condensate particles inside the spatial BEC region are negligible compared to the first process as $n_0 \gg n_c \approx n'$. Let us denote the loss rate due to the process involving two condensate particles as ν_0 and that due to above-condensate particles as ν'. From Eqs. (21) and (20) we find

$$\nu_0 = \frac{32\pi\alpha}{105\tilde{U}^2}\frac{(\mu - 2n_c\tilde{U})^{7/2}}{(m/2)^{3/2}\omega^3}. \tag{29}$$

Since $\Delta T \gg n_c\tilde{U}$, the difference $\nu' - \nu_r(T_c) =$

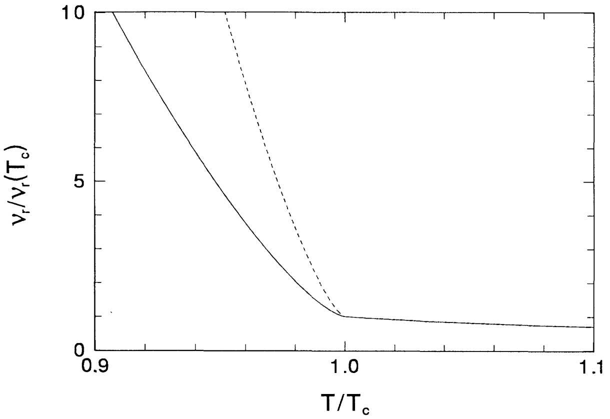

FIG. 1. Number of relaxation events per unit time for a fixed number of atoms, normalized to the value at T_c. The solid line corresponds to $n_c = 10^{15}$ cm^{-3} and the dashed line to $n_c = 10^{13}$ cm^{-3}.

$\Delta T[\mathrm{d}\nu(T)/\mathrm{d}T]$ is small compared to ν_0. Thus, $\nu_r \approx \nu_r(T_c) + \nu_0(T)$. Hence, substituting the chemical potential μ from Eqs. (27) and (26) into Eq. (29), and using Eq. (14) with Eq. (13) in mind, we obtain

$$\nu_r(T) = \nu_r(T_c)\left[1 + 7.5\left(\frac{T_c}{n_c\tilde{U}}\right)^{3/5}\left(\frac{\Delta T}{T_c}\right)^{7/5}\right] \quad ; \quad T \le T_c. \tag{30}$$

In Fig. 1 the value of $\nu_r(T)/\nu_r(T_c)$ is given for hydrogen atoms for $n_c = 10^{15}$ cm^{-3} ($T_c = 150$ μK, and $n_c = 10^{13}$ cm^{-3} ($T_c = 8$ μK). Clearly a large, almost jumplike, increase in ν_r can be seen, inspite of the fact that $N_0 \ll N$. The occurrence of this phenomenon, which can be called a relaxation explosion, is related to the large value of $(T_c/n_c\tilde{U}) \sim n_c^{-1/3}$. It is this parameter, especially large for the small densities characteristic of the trapped gas, that determines the large compression of the condensate by the inhomogeneous field. The ratio of the maximum condensate density to the characteristic density of above-condensate particles is

$$\frac{n_{0\max}}{n_c} = \frac{(\mu - 2n_c\tilde{U})}{n_c\tilde{U}} \approx 2.0\left(\frac{T_c}{n_c\tilde{U}}\right)^{3/5}\left(\frac{\Delta T}{T_c}\right)^{2/5}. \tag{31}$$

Although the condensate fraction is small, the quadratic dependence of ν_r on density, in particular, $\nu_0 \sim n_{0\max}^2 V_0$ and $\nu' \sim n_c^2 V_e$, ensures that the condensate particles dominate the relaxation when

$$\frac{\Delta T}{T_c} > 0.2\left(\frac{n_c\tilde{U}}{T_c}\right)^{3/7}. \tag{32}$$

For trapped atomic H, with $n_c \sim 10^{13} - 10^{15}$ cm^{-3} and $T_c \sim 10 - 100$ μK, the inequality in Eq. (32) is already satisfied for $\Delta T/T_c \gtrsim 10^{-2}$.

III. KINETICS OF COOLING: PENETRATION INTO THE BEC REGIME

In this section, we first address the question whether, in view of the relaxation explosion phenomenon discussed above, it is at all possible to cool the gas to temperatures $T < T_c$ and get a measurable fraction of the condensate. The discussion will be rather general and in principle applicable to a variety of cooling methods. In Sec. IV we will apply the concepts developed in this section to atomic H and discuss whether it is possible to use the relaxation explosion for detecting the onset of BEC.

In order to obtain a conceptually clear picture, we assume that the cooling of the gas proceeds sufficiently slow to ensure that the system can be described as evolving in quasiequilibrium, i.e., the cooling should proceed much slower than the event rate of elastic collisions per particle, which for $T \gg T_c$ is given by

$$R_c = \tfrac{1}{4}\sigma v_T N/V_e. \tag{33}$$

Here σ is the cross section of elastic collisions, which is independent of T in the temperature range considered (for H$\uparrow$ $\sigma = 8\pi a^2 = 1.3\times 10^{-15}$ cm^2), and $v_T = \sqrt{8T/\pi m}$ is the thermal velocity.

Using Eq. (6) and neglecting the interaction between atoms, we find the following expressions for the internal energy of the trapped gas at temperatures above and below T_c:

$$U = 3NT\frac{g_4(u)}{g_3(u)} \quad ; \quad T \ge T_c, \tag{34}$$

$$U = 3NT\left(\frac{T}{T_c}\right)^3\frac{g_4(1)}{g_3(1)} = 3g_4(1)\frac{T^4}{(\hbar\omega)^3} \quad ; \quad T \le T_c, \tag{35}$$

where $u = \exp(\mu/T)$. From this we obtain relations between the time derivatives of T, U, and N:

$$\left(4\frac{g_4(u)g_2(u)}{g_3^2(u)} - 3\right)\frac{\dot{T}}{T} = \frac{\dot{U}}{U}\frac{g_2(u)g_4(u)}{g_3^2(u)} - \frac{\dot{N}}{N} \quad ; \quad T \ge T_c. \tag{36}$$

$$\frac{\dot{T}}{T} = \frac{1}{4}\frac{\dot{U}}{U} \quad ; \quad T \le T_c. \tag{37}$$

If we parametrize the cooling process by an energy removal rate $R_U \equiv -\dot{U}/U$, and an accompanying rate R_N of particle removal, we can write the rate of change of particle number as

$$\frac{\dot{N}}{N} = -R_N - \frac{\nu_r}{N}, \tag{38}$$

where ν_r is given by Eqs. (14) or (30). For the rate of energy change we have

$$\frac{\dot{U}}{U} = -R_U + \frac{\dot{U}_r}{U}. \tag{39}$$

Here $\dot{U}_r$ is the rate of energy change associated with relaxation. As the relaxing particles have a finite energy, the internal energy decreases with decreasing particle number. Hence, $\dot{U}_r$ is always negative. From Eq. (37) one thus finds that when $T \le T_c$, because the energy does not explicitly depend on N, relaxation does not lead to heating. In contrast, from Eq. (36) we see that for

$T > T_c$ removal of particles only leads to cooling if the energy of these particles is sufficiently high. This is the principle of evaporative cooling. In this sense, any loss of (above-condensate) particles leads to evaporative cooling for $T \leq T_c$. Note, moreover, that the ratio $\dot{T}/\dot{N}$ changes discontinuously at T_c. This behavior has the same origin as the discontinuity in the derivative of specific heat at the BEC point.

Neglecting again the interaction energy between atoms we find for the rate of energy change due to relaxation,

$$\dot{U}_r = -2\alpha \int d\mathbf{r} \int \frac{d^3k_1 d^3k_2}{(2\pi)^6}\, n(\varepsilon_{k_1}, r) \times n(\varepsilon_{k_2}, r)[\varepsilon_{k_1} + V(r)]. \tag{40}$$

Here $\varepsilon_{k_{1,2}}$ and $\mathbf{k}_{1,2}$ are the kinetic energies and momenta of the colliding atoms. A straightforward calculation shows that in the classical limit

$$\dot{U}_r(T) = -\frac{9}{4} T\nu_r \ ; \ T \gg T_c. \tag{41}$$

When considering $T < T_c$ (with $\Delta T \ll T_c$) we can neglect the contribution to $\dot{U}_r$ from the relaxation of condensate particles as their characteristic energy $n_{0\max}\tilde{U} \ll T_c$. In this case Eq. (40) gives

$$\dot{U}_r(T) = -\frac{9}{4} T \frac{1.05}{G(1)} \nu_r(T_c) \ ; \ T < T_c. \tag{42}$$

The maximum penetration into the BEC region, $N_{0\max}/N$ or $\Delta T_{\max}/T_c$ can be found from the condition $\dot{T}/T = (1/3)\dot{N}/N$ (the trajectories in the N-T plane are parallel to the BEC line). Then, using Eqs. (35), (37), (38), (39), (42), and the explicit expression Eq. (30) for the relaxation rate, we obtain

$$\frac{N_{0\max}}{N} = \frac{3\Delta T_{\max}}{T_c} \approx 0.6\left(\frac{n_c\tilde{U}}{T_c}\right)^{3/7}\left[\frac{N(R_U - (4/3)R_N)}{\nu_r(T_c)} - 0.67\right]^{5/7}, \tag{43}$$

where the first equality follows from Eq. (26).

Equation (43) in principle applies to a variety of cooling methods. In order to simplify the discussion we confine ourselves to the case where R_U and R_N are proportional to the interatomic collision rate R_c. In this case it is convenient to define $\tilde{R}_U \equiv R_U/R_c$ and $\tilde{R}_N \equiv R_N/R_c$. By assuming $1 \gg \tilde{R}_U$, we ensure that the system remains close to equilibrium. The ratio $\tilde{R}_U/\tilde{R}_N$ will depend on the particular cooling method selected, but typically efficient cooling corresponds to low values of $\tilde{R}_N$. Hence, we may simplify somewhat by restricting ourselves to $\tilde{R}_N = 0$, in which case we may rewrite Eq. (43) as

$$\frac{N_{0\max}}{N} \approx 0.2\left(\frac{n_c\tilde{U}}{T_c}\right)^{3/7}\left(\tilde{R}_U\frac{\sigma v_T}{\alpha} - 2.5\right)^{5/7}. \tag{44}$$

For sufficiently small ratio $n_c\tilde{U}/T_c$ the quantity $N_{0\max}/N$ ($= 3\Delta T_{\max}/T_c$) is much smaller than unity, expressing the difficulty to penetrate into the BEC regime.

Some care is however warranted when applying Eq. (43), since a significant point was omitted in its derivation. We implicitly assumed that the condensate, once it appears, is in thermal equilibrium with the above condensate particles. The condition of slow cooling, $R_U \ll R_c$, surely leads to equilibrium only between above condensate particles. As for the condensate, the situation is more complicated and the answer depends on the characteristic time τ_0 in which the condensate is formed.

The problem of the characteristic time scale of condensate formation has been discussed in the literature[21–24] (for a discussion see Ref. 21). In a weakly interacting Bose gas the characteristic time τ_0 is determined by the kinetic stage of the evolution[21] of the particle distribution in the momentum space. In the homogeneous gas, if the condensate is initially absent, τ_0 is of the order of the characteristic collisional time of above-condensate particles, $\sim R_c^{-1}$. Once a condensate is present, the characteristic time associated with accumulation of particles by the condensate, turns out to be of the same order. One may expect a similar situation in the inhomogeneous gas. Then the condition for equilibrium is met when the characteristic decay time of the condensate due to relaxation is much larger than τ_0.

Since the cooling proceeds much slower than the rate of elastic collisions ($1 \gg \tilde{R}_U \gg \tilde{R}_N$), we may consider a quasistationary solution for the number of condensate particles. If we are close to equilibrium, i.e., $\Delta N_0 \equiv \bar{N}_0 - N_0 \ll \bar{N}_0$, where $\bar{N}_0$ is the equilibrium number of condensate particles, we obtain

$$\Delta\dot{N}_0 = -\frac{\Delta N_0}{\tau_0} + \nu_0 \approx 0. \tag{45}$$

Here the term $\Delta N_0/\tau_0$ describes the accumulation of the condensate from above-condensate particles, and the characteristic time τ_0 can be written as $\tau_0 = (\gamma R_c)^{-1}$. The term ν_0 represents the relaxational decay of the condensate. The preliminary analysis of the kinetic equation in the parabolic potential Equation (2), with the condensate included explicitly, gives γ close to 4. With Eqs. (26) and (30), the condition of thermal equilibrium between the condensate and above-condensate particles, $\Delta N_0/\bar{N}_0 \approx \nu_0\tau_0/\bar{N}_0 \ll 1$, implies

$$\frac{N_{0\max}}{N} \ll 0.01\gamma^{5/2}\left(\frac{n_c\tilde{U}}{T_c}\right)^{3/2}\left(\frac{\sigma v_T}{\alpha}\right)^{5/2} \equiv \gamma^{5/2}\left(\frac{T_c}{T_{\text{kin}}}\right)^{2}. \tag{46}$$

For H, the characteristic temperature $T_{\text{kin}} \approx 135\ \mu$K.

The maximum achievable condensate fraction and ΔT, are determined by Eqs. (43) and (44) only when Eq. (46) is satisfied. If this is not the case, the maximum achievable N_0/N will be determined by the balance between the formation rate of the condensate and the spin relaxation rate. In contrast to the equilibrium case, now a further increase of ΔT becomes possible, but it will not lead to an essential increase of N_0/N and, hence, to an increased relaxation rate. In other words, the relaxation explosion will be bottlenecked by the "slow" formation rate of the condensate.

IV. APPLICATION TO H

In this section we will illustrate the results of the previous sections, using atomic H as an example. To date the most proficient cooling method applied to H is evaporative cooling, first discussed by Hess[1] and Tommila[25] in the context of this atomic species. This method is based on the preferential removal of atoms with a total energy exceeding a certain threshold E_t, which is essentially larger than the average energy per particle. If E_t is kept constant, T will reach a steady state value determined by the balance between evaporation and relaxation.[6] The cooling process can be kept going by continuously reducing E_t. This can be accomplished in various manners, e.g., by ramping down a field barrier, demonstrated at MIT,[6,7] insertion of a cold absorbing surface, or removal through optical pumping, as shown by Setija *et al.*[9] In the first two methods, essentially all particles with total energy exceeding E_t are instantaneously removed. The resulting energy distribution is truncated at E_t and essentially nonthermal which makes the problem difficult to analyze exactly.

To keep the picture transparent, we retain the assumptions that the system remains close to equilibrium and $1 \gg \tilde{R}_U \gg \tilde{R}_N$. Then, using Eq. (44), the value of $N_{0\max}/N$ can be calculated for H$\uparrow$. The result is shown in Fig. 2 for three values of $\tilde{R}_U$. Also shown in Fig. 2 is $\gamma^{5/2}(T_c/T_{\rm kin})^2$ with $\gamma = 1$ and $\gamma = 4$. Realistically, the temperature at which one can hope to reach BEC is restricted to values below approximately 200 μK. For higher values of T_c the concomitantly higher density makes the lifetime of the sample prohibitively short. In view of this restriction we see from Fig. 2 that $N_{0\max}/N$ (and $\Delta T_{\max}/T_c$) is at most of the order of a few percent. Moreover, it can be seen from Fig. 2 that the curves tend to a vertical tangent, reflecting the existence of a minimum temperature $T_{\min}$ below which BEC cannot be reached. The value of $T_{\min}$ is higher for smaller cooling rates R_U. Finally, as is clear from Fig. 2, the formation rate of the condensate is the dominating limitation only at relatively high values of R_U and comparatively low temperatures.

We can use the same simple model to calculate the trajectories in the N-T plane, using Eqs. (36) and (37). In order to present the trajectories in a form which is independent of ω, and which resembles the common way of representation as curves in temperature-density space, we have divided N by the classical effective volume V_e given in Eq. (10). We emphasize, however, that the quantity N/V_e represents the true density at the trap center $n(r = 0)$, only in the Boltzmann regime ($T \gg T_c$). The results are shown in Fig. 3 for $\tilde{R}_U = 0.1$ and $\tilde{R}_U = 0.025$. The initial values $T = 1$ mK and $N/V_e = 10^{14}$ cm^{-3} are chosen within the experimentally accessible range. When the BEC line is crossed a clear change in the trajectory is noticeable. As mentioned before, the temperature remains very close to T_c and the trajectory essentially follows the BEC line. Observation of the change in the derivative in these trajectories may be a way of detecting whether BEC has been reached. Note that the relaxation explosion does not drive the system out of the BEC region. The attraction of the trajectories to the BEC line is particularly clear when comparing with the corresponding trajectories obtained by assuming the system to obey Boltzmann statistics at any temperature and density.

It is interesting to speculate whether the increase of ν_r can be measured directly. In principle this is possible. First, ν_r can be inferred if the flux of particles escaping from the trap can be monitored. Another option is the detection of the optical fluorescence of the recoiling H$\uparrow$ atoms produced in the proces described by Eq. (3). The production rate of these atoms is directly proportional to ν_r. As the recoil energy acquired by these atoms is much larger than T, and essentially monochromatic, they can be spectrally distinguished from the thermal atoms. Al-

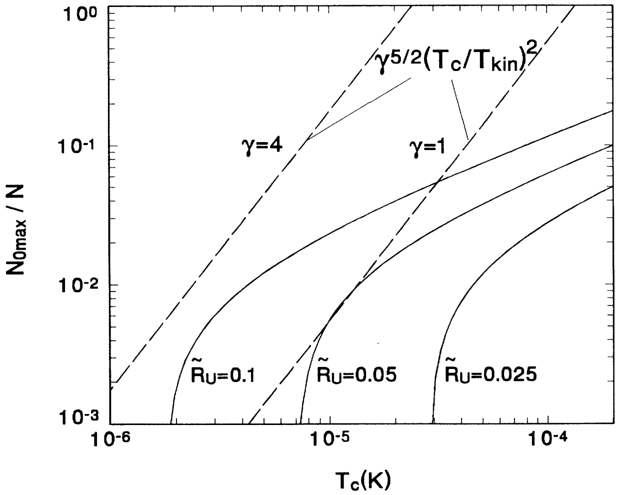

FIG. 2. $N_{0\max}/N = 3\Delta T_{\max}/T_c$ vs T_c for H$\uparrow$ at $\tilde{R}_U = 0.1$, 0.05, and 0.025. Also shown the kinetic limit for the condensate formation $\gamma^{5/2}(T_c/T_{\rm kin})^2$ for $\gamma = 4$ and $\gamma = 1$.

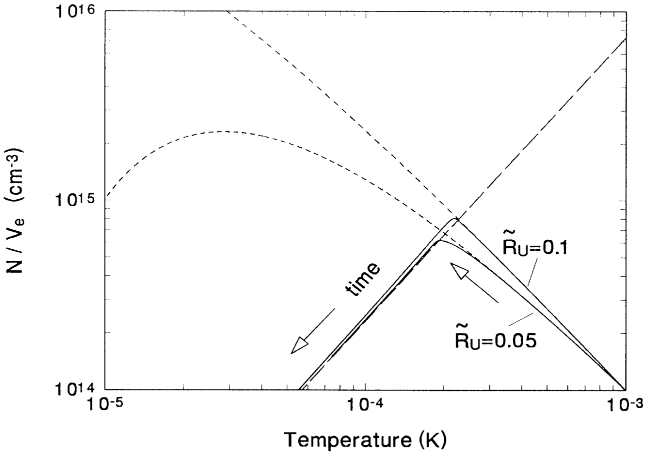

FIG. 3. Cooling trajectories for $\tilde{R}_U = 0.1$ and 0.05 (solid lines) plotted as N/Ve vs T. The long-dashed curve is the BEC line. The short-dashed curves represent the trajectories corresponding to a system obeying Boltzmann statistics.

though in evaporative cooling the recoiling atoms are continuously being removed, our analysis shows that their density can be sufficient to observe the fluorescence.

V. CONCLUDING REMARKS

We have shown that the compression of the condensate in a weakly interacting Bose gas in an external potential, leads to a strongly enhanced relaxation. This "relaxation explosion," prevents the system from penetrating deeply into the condensate region. More precisely, the temperature always remains close to T_c. This behavior is a consequence only of the nonuniform formation of the condensate and can occur in any weakly interacting Bose gas which decays through two- or three-body inelastic collisions. For a given value of T_c the penetration into the condensate region $N_{0\max}/N$ (or $\Delta T_{\max}/T_c$) remains small and independent of ω. Because any real system is bound to have inhomogeneities, this may prove an important feature of the weakly interacting gas.

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