Numerical algorithm for Ginzburg-Landau equations with multiplicative noise: Application to domain growth

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We consider stochastic partial differential equations with multiplicative noise. We derive an algorithm for the computer simulation of these equations. The algorithm is applied to study domain growth of a model with a conserved order parameter. The numerical results corroborate previous analytical predictions obtained by linear analysis.

I. INTRODUCTION

In a previous paper[1] we began a general theoretical study of the role of multiplicative noise in the dynamic evolution of spatially extended systems far from equilibrium. We proposed a Ginzburg-Landau equation with multiplicative noise that could be used in the study of domain growth at intermediate temperatures. In general, the study of this type of equation would have relevance in phase-separation dynamics,[2] pattern formation,[3] polymers,[4] etc.

In Ref. 1, we made analytical predictions regarding the linear regime. However, nonlinear effects of such models are difficult to study analytically. Here, in order to obtain further insight into the main effects of the multiplicative noise, we derive an algorithm to simulate this type of noise and we present the numerical results for the model obtained in Ref. 1. Although computer simulations of field equations with additive noise are standard, by using, for example, first-order Euler algorithms,[5–7] there is no systematic method to deal with multiplicative noise. The algorithm for the one-variable simulation[8] can easily be generalized to multivariable systems with nonconserved order parameters.[9] However, the case of conserved order parameters could not be easily implemented, since, in this last case, the standard algorithm involves not only the simulation of simple Gaussian processes but also the appearance of non-Gaussian processes that could not be simulated in an exact way. Here, we resolve this general problem by taking a different point of view. It is well known that a prescription is needed to interpret the stochastic integrals that appear in the formal integration of the stochastic differential equations with multiplicative noise. Some useful prescriptions for this integration has been proposed, and the stochastic processes theory establishes the form of the Langevin equation once the prescription is chosen, as well as the way of changing from one prescription to another.[10] Here, our approach is to use, in the derivation of the algorithm, the prescription of the stochastic integrals that was employed in the formulation of the model. In this way, and in the corresponding approximation, we will obtain a closed algorithm in terms of only Gaussian processes. We will apply these ideas to propose a very simple algorithm of simulation of a multiplicative Langevin equation with a conserved variable, which constitutes one of the main results of this paper.

Section II is devoted to the detailed derivation of a general algorithm. In Sec. III, we construct a simplified version of this algorithm. In Sec. IV, we present an application to the results of the previous sections to the study of domain growth in a spinodal decomposition process. In Sec. V, we give a summary of conclusions. The explicit algorithm for a nonconserved field is presented in the Appendix.

II. GENERAL ALGORITHM

We start by writing a stochastic partial differential Langevin equation with multiplicative noise for a field variable $\Psi(\mathbf{r},t)$ in the following generic form:

$$\frac{\partial\Psi(\mathbf{r},t)}{\partial t}=V(\Psi(\mathbf{r},t),\nabla\Psi(\mathbf{r},t)) + G(\Psi(\mathbf{r},t),\nabla\Psi(\mathbf{r},t))\xi(\mathbf{r},t)\ , \tag{2.1}$$

where V and G are nonlinear functions and the noise $\xi(\mathbf{r},t)$ is a Gaussian uncorrelated process in space and time of intensity β^{-1}. Equation (2.1) can be expressed in a discrete form by

$$\dot{\Psi}_\mu(t)=v_\mu(\Psi)+g_{\mu\alpha}(\Psi)\xi_\alpha(t)\ , \tag{2.2}$$

where we have now used the notation $\Psi_\mu(t)=\Psi(\mathbf{r},t)$. Equation (2.2) gives a set of coupled ordinary stochastic differential equations for the variables $\Psi_\mu(t)$, defined in a d-dimensional lattice of total volume V and cubic cells of volume $\Delta V=(\Delta x)^d$. The noise $\xi_\alpha(t)$ has the Gaussian white-noise correlation:

$$\langle \xi_\alpha(t)\xi_\beta(t')\rangle = 2D\delta_{\alpha\beta}\delta(t-t') , \tag{2.3}$$

where the intensity of the discrete noise is now $D=\beta^{-1}\Delta x^{-d}$. $g_{\mu\nu}(\Psi)$ is the multiplicative function that couples the variable to the noise. This discretization requires, as usual, that the relevant scales in the system are larger than Δx and smaller than $V^{1/d}$. Now, the problem is to integrate a finite number of coupled multiplicative Langevin equations, which we interpret in the Stratonovich sense.

The algorithm for one variable Langevin equations with multiplicative noise can easily be generalized to the multivariable case, Eq. (2.1), when $g_{\mu\alpha}$ is either constant (additive noise) or is diagonal,[8,9] as in the case of a nonconserved variable as, described in the Appendix. When $g_{\mu\alpha}$ is neither constant nor diagonal, as for the case of a conserved variable, some problems appear which are related with the interpretation of stochastic integrals. An example of this last case is presented in Sec. IV.

The numerical algorithm is obtained from the formal integration of Eq. (2.2) during a time step Δ:

$$\Psi_\mu(t+\Delta)-\Psi_\mu(t)=\int_t^{t+\Delta}\{v_\mu(\Psi(t')) + g_{\mu\alpha}(\Psi(t'))\xi_\alpha(t')\}dt' . \tag{2.4}$$

Previous approaches start by expanding the arguments of the integral at a time t,[11–14] as it is done in the case of higher-order algorithms for deterministic equations. Here, we take a different approach and we use the interpretation of the stochastic integral associated with Eq. (2.2) from the very beginning.[15] We start by expanding the deterministic term in Eq. (2.4) as

$$v_\mu(t')=v_\mu(t)+\left.\frac{\partial v_\mu}{\partial\Psi_\nu}\right|_{\Psi(t)}[\Psi_\nu(t')-\Psi_\nu(t)]+\ldots . \tag{2.5}$$

For the stochastic term, we use the Stratonovich prescription, which establishes that a stochastic integral, like that appearing in Eq. (2.4), should be interpreted as[10,15]

$$\int_t^{t+\Delta}dt' g_{\mu\alpha}(\Psi(t'))\xi_\alpha(t')=g_{\mu\alpha}(\tfrac{1}{2}[\Psi(t)+\Psi(t+\Delta)]) \times\int_t^{t+\Delta}dt'\xi_\alpha(t')+O(\Delta^{3/2}) . \tag{2.6}$$

Now, $g_{\mu\alpha}$ is expanded as

$$g_{\mu\alpha}\left[\frac{\Psi(t)+\Psi(t+\Delta)}{2}\right]=g_{\mu\alpha}(\Psi(t)) +\frac{1}{2}\frac{\partial g_{\mu\alpha}(\Psi(t))}{\partial\Psi_\nu(t)} \times g_{\nu\beta}X_\beta(t)+O(\Delta) . \tag{2.7}$$

By substituting Eqs. (2.5)–(2.7) into Eq. (2.4), we get up to first order in Δ:

$$\Psi_\mu(t+\Delta)=\Psi_\mu(t)+v_\mu(\Psi(t))\Delta+g_{\mu\alpha}(\Psi(t))X_\alpha(t) +\frac{1}{2}\frac{\partial g_{\mu\alpha}(\Psi(t))}{\partial\Psi_\nu}g_{\nu\beta}X_\alpha(t)X_\beta(t)+O(\Delta^{3/2}) , \tag{2.8}$$

where

$$X_\alpha(t)=\int_t^{t+\Delta}\xi_\alpha(t')dt' \tag{2.9}$$

is a Gaussian random variable of order $\Delta^{1/2}$ with zero mean and variance $2D\Delta$. It can easily be simulated as

$$X_\alpha(t)=\sqrt{2D\Delta}\eta_\alpha , \tag{2.10}$$

where η_α are Gaussian random numbers of zero mean and variance equal to 1.

It is worth commenting on the possibility of an algorithm of higher order (Δ^n, $n>1$). In our scheme this would also imply a higher-order interpretation of the stochastic integral in Eq. (2.6), which is not available. A systematic expansion like the one used in Eq. (2.5) would allow, in principle, a derivation of an algorithm order by order, but each term would contain multiple time integrals of the noise, with increasing difficulty in their simulation because they are not Gaussian. Apart from some particular cases, these integrals cannot be computed exactly and need additional approximations.

III. MINIMUM ALGORITHM FOR MULTIPLICATIVE LANGEVIN EQUATIONS (MAMLE)

The algorithm (2.8) can present some technical difficulties when one tries to write the corresponding computer program. In this section, we present a simplified scheme. To this end, we take into account that the statistical properties associated with the Langevin Eq. (2.2) are described by the probability density, $P[\Psi]$. If our aim is to substitute the algorithm given by Eq. (2.8) by a simpler one, this procedure should maintain the same dynamics for $P[\Psi]$. The well-known Fokker-Planck equation obeyed by $P[\Psi]$ corresponding to the stochastic differential Eqs. (2.2) and (2.3) would be written, in the Stratonovich interpretation,[10] as

$$\frac{\partial P}{\partial t}=-\frac{\partial}{\partial\Psi_\mu}\left[v_\mu+D\frac{\partial g_{\mu\alpha}}{\partial\Psi_\nu}g_{\nu\alpha}\right]P+D\frac{\partial}{\partial\Psi_\mu}\frac{\partial}{\partial\Psi_\nu}g_{\mu\alpha}g_{\nu\alpha}P. \tag{3.1}$$

It is easy to check that this is the Fokker-Planck equation associated with the algorithm Eq. (2.8) when the following limits are considered.[10]

$$\lim_{\Delta\to0}\frac{\langle\Psi_\alpha(t+\Delta)-\Psi_\alpha(t)\rangle}{\Delta}=v_\mu(\Psi(t)) +D\left.\frac{\partial g_{\mu\alpha}}{\partial\Psi_\nu}\right|_{\Psi(t)}g_{\nu\alpha}(\Psi(t)) , \tag{3.2}$$

$$\lim_{\Delta\to0}\frac{\langle[\Psi_\mu(t+\Delta)-\Psi_\mu(t)][\Psi_\nu(t+\Delta)-\Psi_\nu(t)]\rangle}{\Delta} =Dg_{\mu\alpha}g_{\nu\alpha}(\Psi(t)) , \tag{3.3}$$

$$\lim_{\Delta\to 0}\frac{\left\langle\prod_{i=1}^{n}[\Psi_{\mu_i}(t+\Delta)-\Psi_{\mu_i}(t)]\right\rangle}{\Delta}=0 \quad n\geq 3 , \tag{3.4}$$

where the brackets indicate average over the realization of the noise. It is interesting to notice that the same Fokker-Planck Eq. (3.1) is obtained if Eq. (2.8) is substituted by the more simple algorithm

$$\Psi_\mu(t+\Delta)=\Psi_\mu(t)+\left[v_\mu+D\frac{\partial g_{\mu\alpha}}{\partial\Psi_\nu}g_{\nu\alpha}\right]\Delta +g_{\mu\alpha}(\Psi(t))X_\alpha(t) , \tag{3.5}$$

where we have replaced the last term in Eq. (2.8) by its mean value. Then, both algorithms, of first order in Δ, are stochastically equivalent in the sense that both are associated with the same Fokker-Planck Eq. (3.1). We call Eq. (3.5) the minimum algorithm for multiplicative Langevin equations (MAMLE) and it is one of the main results of this paper.

IV. APPLICATION TO DOMAIN GROWTH

In this section, the algorithm (3.5) derived previously is applied to a particular example. In Ref. 1, we introduced fluctuations in a model of phase separation with a general assumption of a concentration-dependent diffusion coefficient, $M(c)$. This assumption has been considered to model deep quenching or to take into account the presence of an external field such as gravity.[16] We have derived the associated Ginzburg-Landau equations and we have found that this assumption gives rise to multiplicative thermal fluctuations. The resulting equation for the conserved field variable $c(\mathbf{r},t)$ is

$$\frac{\partial c(\mathbf{r},\tau)}{\partial\tau}=\nabla M\cdot\nabla\frac{\delta F}{\delta c}-\frac{\beta^{-1}}{2}\nabla\cdot\left[\nabla\frac{\delta}{\delta c}\right]M+\nabla^i m\xi^i(\mathbf{r},\tau) , \tag{4.1}$$

where $F[c]$ is a Ginzburg-Landau free-energy functional:

$$F[c]=\frac{1}{2}\int dr\left[-\frac{c^2}{2}+\frac{c^4}{4}+\frac{(\nabla c)^2}{2}\right] . \tag{4.2}$$

$M(c)=m^2(c)$ is the variable-dependent diffusion coefficient:

$$M(c)=1-ac^2 . \tag{4.3}$$

The expression of $M(c)$ has been proposed from phenomenological arguments.[16] The constant a is related to the temperature. The noise $\xi(\mathbf{r},\tau)$ is a Gaussian uncorrelated (white) process in time and space, of intensity β^{-1}. Equation (4.1) is interpreted in the Stratonovich sense. For $a=0$ we obtain the usual model B of phase-separation dynamics with additive noise.[11] For $a\neq 0$, apart from the multiplicative term, we find a spurious term, the second term on the right-hand side of Eq. (4.1), of stochastic origin. To understand the origin of this term, we only need to realize that we are considering noise of thermal origin. Then, although the noise is multiplicative, it could not affect the equilibrium properties, and for example, the fluctuation-dissipation theorem has to be fulfilled. Then, the spurious term is such that together with the multiplicative term, it ensures the evolution of the system to the correct equilibrium solution determined by the free energy, Eq. (4.2).

The simulation will take place in a two dimensional lattice of $L\times L$ cells of volume Δx^2. Then Eqs. (4.1)–(4.3) have to be expressed in terms of the discrete variables. The discretization of Eq. (4.1) is specified by the following discrete Langevin equation:

$$\dot{c}_\alpha=(\nabla_L^i)_{\alpha\beta}M_{\beta i}(\nabla_R^i)_{\beta\sigma}\frac{\partial F}{\partial c_\sigma} -\frac{\beta^{-1}}{2}\Delta x^{-d}(\nabla_L^i)_{\alpha\beta}(\nabla_R^i)_{\beta\sigma}\frac{\partial M_{\beta i}}{\partial c_\sigma}+(\nabla_L^i)_{\alpha\beta}m_{\beta i}\xi_\beta^i(t) . \tag{4.4}$$

The noise correlation is given by

$$\langle\xi_\alpha^i(\tau)\xi_\beta^j(\tau')\rangle=2\frac{\beta^{-1}}{\Delta x^d}\delta_{ij}\delta_{\alpha\beta}\delta(\tau-\tau') , \tag{4.5}$$

where ∇_L^i and ∇_R^i are the left and right discrete versions of the gradient operators

$$(\nabla_R^i)_{\alpha\beta}=\frac{1}{\Delta x}(\delta_{\alpha+i\beta}-\delta_{\alpha\beta}) ,$$
$$(\nabla_L^i)_{\alpha\beta}=\frac{1}{\Delta x}(\delta_{\alpha\beta}-\delta_{\alpha-i\beta}) . \tag{4.6}$$

Since there is no derivation from first principles of M in terms of the discrete variables, in Ref. 1 we proposed a family of mesoscopic models. In terms of $m_{\alpha i}(\{c\})$, the models are given by

$$m_{\alpha i}=\sum_\beta Q_{\alpha\beta}^i h(c_\beta) ; \quad \sum_\beta Q_{\alpha\beta}^i=1 , \tag{4.7}$$

where $h(c_\beta)=(1-ac_\beta)^{1/2}$.

The matrix elements $Q_{\alpha\beta}^i$ are different from zero only when the indices α and β correspond to the lattice points α, $\alpha+i$ or the $\mathbf{n}$ lattice points in the vicinity of these two points α and $\alpha+i$. From Eq. (4.7) we find that $m(c)=h(c)$ in the continuous limit. To simplify the model, we take that $Q_{\alpha,\alpha}^i=Q_{\alpha,\alpha+i}^i\equiv Q_0$. These models have a mesoscopic length R that gives the size of the region that contains all the cell points involved in an interchange of matter at each time step. From Eq. (4.7), we know that Q_0 is of order $\mathbf{n}^{-1}$ and then, we define the parameter R by

$$R=\Delta x Q_0^{-(1/d)} . \tag{4.8}$$

Here, in order to consider a simple version for the computer simulation, we have considered a model in which only the dependence on one couple α, $\alpha+i$ is taken into account. Then, we take the following expression of $m_{\alpha i}(c)$:

$$m_{\alpha i}(\{c\})=\tfrac{1}{2}[h(c_\alpha)+h(c_{\alpha+i})] . \tag{4.9}$$

For this choice, Eq. (4.8) gives $R=\sqrt{2}\Delta x$.

Now, the MAMLE is obtained by substituting the particular expressions of Eqs. (4.4) and (4.9) into the general Eq. (3.5). The explicit calculation of the third term in the right-hand side of Eq. (3.5) is

$$\frac{\partial g_{\mu\alpha}}{\partial c_\nu} g_{\nu\alpha} = (\nabla_L^i)_{\mu\alpha} \tfrac{1}{2} [h'(c_\alpha)\delta_{\alpha\nu} + h'(c_{\alpha+i})\delta_{\alpha+i\nu}](\nabla_L^i)_{\nu\alpha} m_{\alpha i} = -\tfrac{1}{2}(\nabla_L^i)_{\mu\alpha} m_{\alpha i} (\nabla_R^i)_{\alpha\nu} h'(c_\nu) \ . \tag{4.10}$$

With this result, the MAMLE is finally

$$c_\mu(\tau+\Delta) = c_\mu(\tau) + \{ -\tfrac{1}{2}(\nabla_L^i)_{\mu\nu} m_{\nu i}^2 (\nabla_R^i)_{\nu\sigma} [c - c^3 + (\nabla)^2 c]_\sigma - 2\beta^{-1} (\nabla_L^i)_{\mu\nu} m_{\nu i} (\nabla_R^i)_{\nu\sigma} h'(c_\sigma) \} \Delta + (\nabla_L^i)_{\alpha\beta} m_\beta X_\beta^i(\tau) \ , \tag{4.11}$$

where the Gaussian numbers $X_\mu(\tau)$ are computed from Eq. (2.10).

Before commenting on the results obtained from the computer simulation of Eq. (4.11), we discuss the result of a linear analysis for the evolution of the structure function $S(k,t)$. This quantity is the Fourier transform of the pair-correlation function[5–7,17]

$$G(\mathbf{r},t) = \frac{1}{V} \int d\mathbf{r}' \langle c(\mathbf{r}+\mathbf{r}',t) c(\mathbf{r}',t) \rangle \ , \tag{4.12}$$

where the brackets indicate the average over the realizations of the noise. The evolution of the structure function in the linear regime is obtained from Eqs. (4.1) and (4.2):[1]

$$\frac{d}{dt} S(k,t) = -k^2 \left[k^2 - 1 + \frac{4a\beta^{-1}}{R^d} \right] S(k,t) + 2\beta^{-1} k^2 - 2\beta^{-1} a k^2 \frac{1}{(2\pi)^d} \int d\mathbf{q} \, S(q,t) \ . \tag{4.13}$$

For $a = 0$ we obtain the equation valid for model B.[2,6] Furthermore, the two terms proportional to a are the contributions given by the multiplicative noise. From the first term of Eq. (4.13), we find that modes with $k < k_c \equiv 1 - 4a\beta^{-1}/R^d$ are unstable, and they will grow in the early stages of the evolution. In contrast, modes with $k > k_c$ remain stable during the linear regime. So for initial and intermediate times, with a smaller number of modes growing, the domain growth would be different from that in the case of additive noise, $a = 0$. This simple theoretical prediction is confirmed in Fig. 1, where we compare the results obtained by a numerical integration of Eq. (4.13) with the numerical simulation results obtained from Eq. (4.11) without the nonlinear term c.[3]

In order to obtain further information on the multiplicative noise effects, we simulate this system for different values of the parameters at longer times. We take different values of the intensity of the noise below the critical point, and two values of the constant a. We start from a homogeneous initial state $c = 0$, and the system is allowed to evolve until a time $\tau = 3000$ in order to explore intermediate stages of the evolution far from the linear regime. The lattice spacing used in the simulation was $\Delta x = 1$, the integration step was $\Delta = 0.025$ and the size of the system was 120×120. Each datum results from the statistical average of ten independent runs.

In Figs. 2(a) and 2(b) ($\beta^{-1} = 0.3$), we show two configurations of the system for $a = 0$, which corresponds to the white-noise case, and for $a = 0.8$, for which we expect to see the new aspects coming from the multiplicative noise. It is clear that in the multiplicative noise case the pattern has more diffuse interfaces than in the additive one. In order to obtain a more quantitative characterization of the pattern, we have studied the evolution of the structure function.

In Figs. 3(a) and 3(b), we show the evolution of the structure function for two values of a and a fixed value of $\beta^{-1} = 0.2$. The peaks of the structure function are less pronounced in the multiplicative noise case, Fig. 3(b), than that corresponding to the additive case, Fig. 3(a). This is in agreement with our comment of Figs. 2(a) and 2(b). Moreover, we observe that, for the same times, the peaks in Figs. 3(a) and 3(b) are located at similar values of k for different values of a. This result is different from those obtained for the deterministic case,[17], in which the peak is located at higher values of k by increasing a. This indicates that, when a multiplicative noise is considered, not only is the dynamics slowed by increasing a, but this effect could not be simply included in a temporal rescaling, as in the deterministic case.[17] Furthermore, this result is in accordance with the linear-stability analysis, Eq. (4.13), for which the growth of modes with large k are suppressed by the multiplicative noise, implying a translation of the position of the peak to smaller values of k. This effect seems partially to balance the displacement of the peaks to larger k when $a \neq 0$, which is characteris-

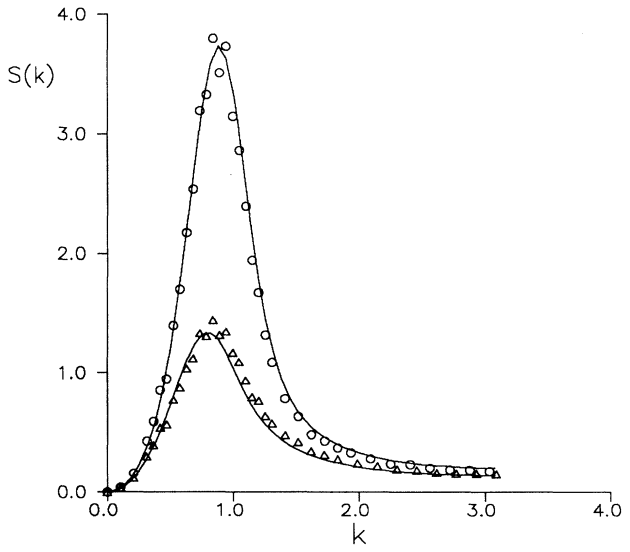

FIG. 1. Evolution of the structure function for a very small time $\tau = 5$. Circles and triangles correspond to numerical simulation of Eq. (4.11) with $c^3 = 0$, for $\mathbf{a} = 0$ and $\mathbf{a} = 0.8$, respectively. The lines correspond to the numerical integration of Eq. (4.13) for the same values of $\mathbf{a}$ ($\beta^{-1} = 0.3$).

tic of the deterministic evolution. This point needs to be studied in more detail. Nevertheless, the qualitative picture of the domain growth remains valid for multiplicative noise. The peak of $S(k,t)$ moves to smaller values of k and increases its height with time.

Figures 4(a) and 4(b) clarify the differences introduced by the multiplicative noise. For a fixed time and two values of a, we observe the effects of an increase in the intensity β^{-1}. Figure 4(a) corresponds to additive noise and it is plotted as a reference. First, the peaks are located at the same position but the heights depend on β^{-1}. Then, the patterns have the same characteristic length but they have more diffuse interfaces by increasing β^{-1}, as expected.[6] In Fig. 4(b), for the same values of the parameters as in Fig. 4(a) but with $a=0.8$, we observe important differences, especially for larger values of β^{-1}. In particular, the position of the peaks depends strongly on β^{-1}. They are located at smaller values of k and have drastically reduced their height. This effect can also be understood from our analysis of Eq. (4.13). The fact that more modes are now stable makes the peak of $S(k,t)$ grow at smaller values of k.

The possibility of scaling properties on the late stage and the characterization of this regime by means of the structure function and the time behavior of the characteristic domain size remain an open question.

V. CONCLUSIONS

We have derived a simulation algorithm for general multivariable Langevin equations with multiplicative

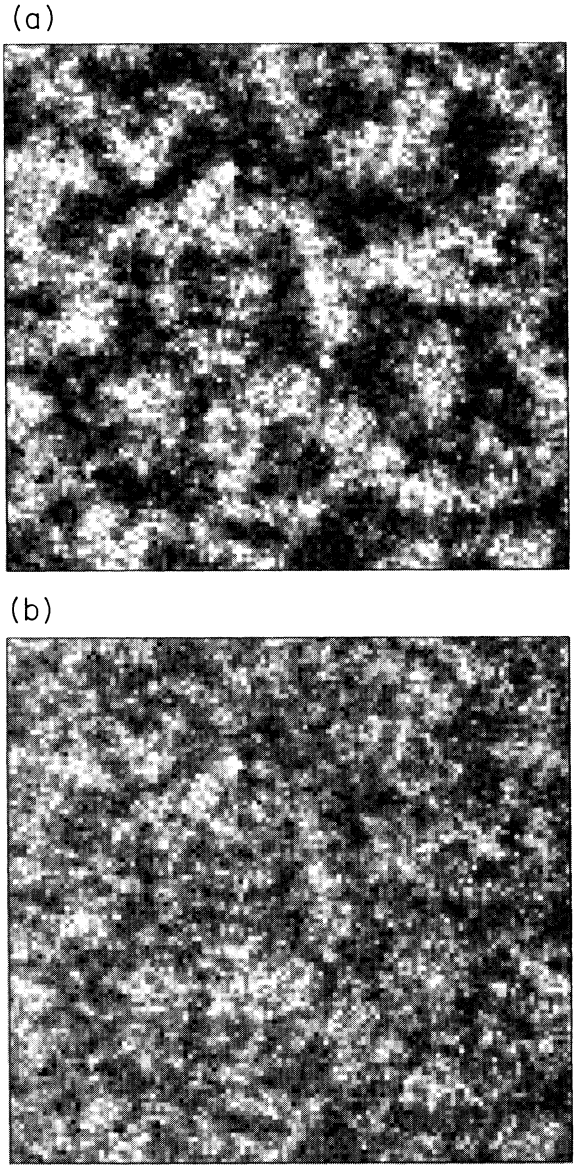

FIG. 2. Typical configurations obtained from Eq. (4.11) at $\tau=500$, starting from an homogeneous situation, $c=0$, for $\beta^{-1}=0.3$ and (a) $a=0$ and (b) $a=0.8$.

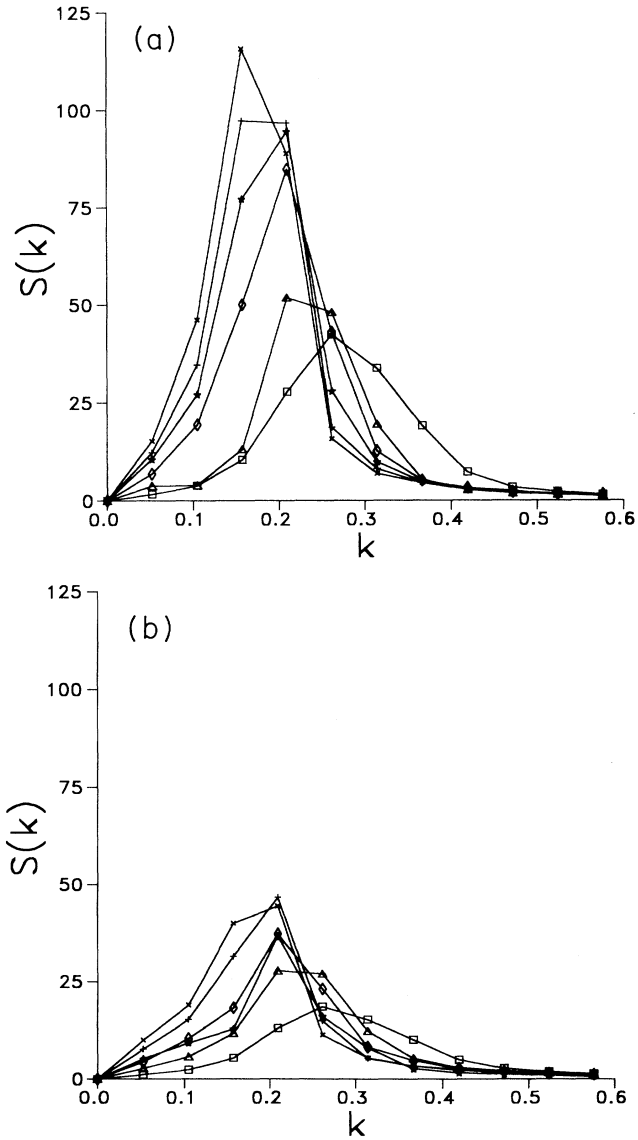

FIG. 3. Evolution of the structure function obtained from Eq. (4.11) for $\beta^{-1}=0.2$ and (a) $a=0$ and (b) $a=0.8$. The times represented are $\tau=500, 1000, \ldots, 3000$.

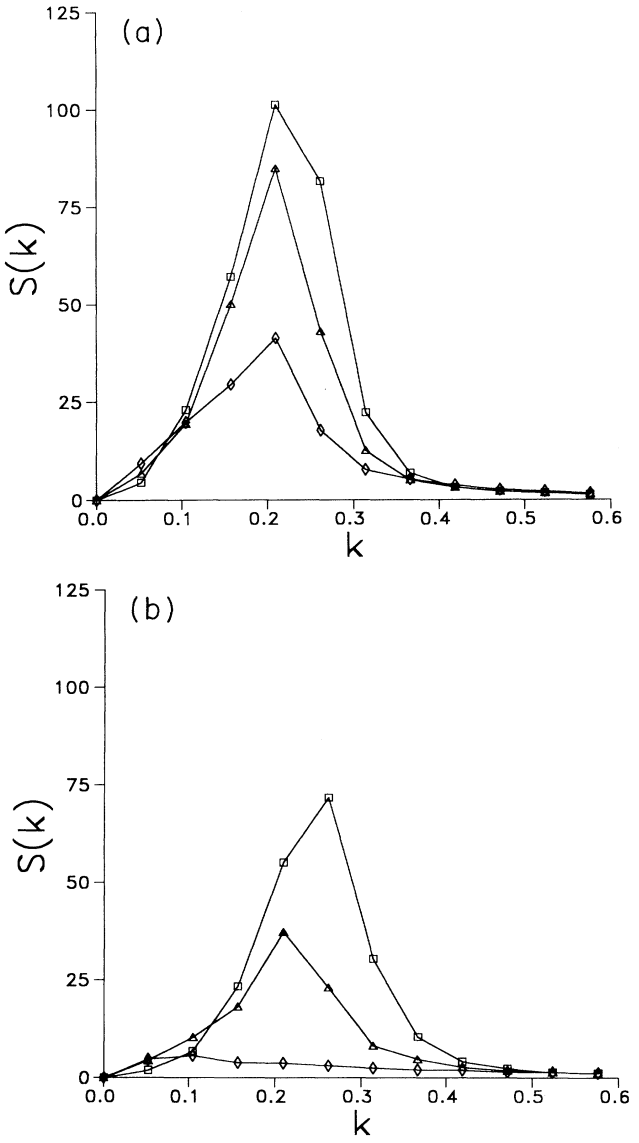

FIG. 4. Structure function obtained from Eq. (4.11) at $\tau=1500$ for (a) $a=0$ and (b) $a=0.8$. Squares, $\beta^{-1}=0.1$; triangles, $\beta^{-1}=0.2$ and rhombus, $\beta^{-1}=0.3$.

noise. Furthermore, we have also introduced a simplified version of the algorithm which reproduces the statistical properties of the original Langevin equation. We call it the minimum algorithm for multiplicative Langevin equations (MAMLE).

As an application, we have made use of the MAMLE in the integration of stochastic equations of a concentration field in the context of domain growth. The simulation of a Ginzburg-Landau equation with multiplicative noise proposed in Ref. 1 to study spinodal decomposition has confirmed the role of the multiplicative noise that was predicted from the linear analysis of the model.[1] In particular, we observe a suppression of the growth of modes with large k. This implies that the dynamics is not only slower by reducing temperature but this effect could not be considered as a simple temporal rescaling.

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APPENDIX: EXPLICIT GENERAL ALGORITHM FOR THE NONCONSERVED CASE

In this appendix, we consider the case of a Ginzburg-Landau with multiplicative noise and a nonconserved field variable:

$$\frac{\partial c(\mathbf{r},\tau)}{\partial \tau}=-M_{\mathrm{NC}}\frac{\delta F}{\delta c}+\frac{\beta^{-1}}{2}\frac{\delta M_{\mathrm{NC}}}{\delta c}+m_{\mathrm{NC}}\xi(\mathbf{r},\tau)\ , \tag{A1}$$

where $M_{\mathrm{NC}}(c)=m_{\mathrm{NC}}^2(c)$ is the corresponding concentration-dependent diffusion coefficient. This is a very simple case, and it does not seem necessary to use the minimum algorithm (3.5) to obtain a compact code. Then, we present a general algorithm analogous to Eq. (2.8).

The functions v_μ and $g_{\mu\nu}$ are obtained by writing the Langevin Eq. (A1) in the lattice variables:

$$v_\mu(\{c\})=\tfrac{1}{2}(c-c^3+\nabla^2 c)_\mu+\frac{\beta^{-1}}{2\Delta x^d}\frac{\partial M_\mu}{\partial c_\mu}\ , \tag{A2}$$

$$g_{\mu\nu}(\{c\})=m_\mu\delta_{\mu\nu}\ . \tag{A3}$$

Let us suppose that the mobility m_μ depends only on the concentration value of one site $m_\mu=m(c_\mu)$. Then the explicit algorithm can be obtained straightforwardly from Eqs. (A1)–(A3) and (2.8).

$$\begin{aligned}c_\mu(\tau+\Delta)=&c_\mu(\tau)\\&+\left[\frac{1}{2}(c-c^3+\nabla^2 c)_\mu+\frac{\beta^{-1}}{2\Delta x^d}M'(c_\mu)\right]\Delta\\&+\frac{1}{4\Delta x^d}M'(c_\mu)X_\mu^2(\tau)+m(c_\mu)X_\mu(\tau)\ ,\end{aligned} \tag{A4}$$

where $X_\mu(\tau)$ can be obtained by using Eq. (2.10).

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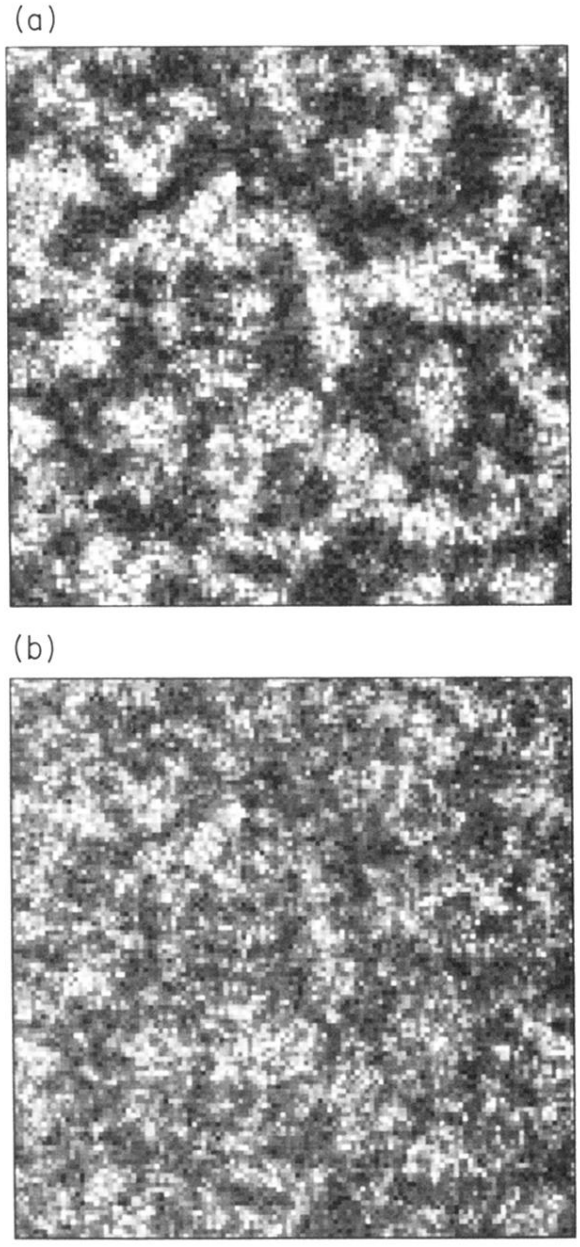

FIG. 2. Typical configurations obtained from Eq. (4.11) at $\tau=500$, starting from an homogeneous situation, $c=0$, for $\beta^{-1}=0.3$ and (a) $a=0$ and (b) $a=0.8$.