Three-dimensional superconducting networks in a magnetic field

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The phase boundary of a three-dimensional cubic superconducting network made of thin wires is not only sensitive to the amplitude of the magnetic field but also to its orientation. This sensitivity reflects the three-dimensional arrangement of the vortex lattice on the network.

I. INTRODUCTION

It has been known for about a decade that the phase boundary of superconducting networks reflects the configuration of the magnetic flux lattice on the network.[1–3] When the magnetic field is such that there is an integral number of flux quanta per unit cell, the critical temperature recovers its zero field value. When the field varies, the critical line $T_c(H)$ exhibits a cusp at every rational value of the flux. This corresponds to a commensurate arrangement of the vortex lattice. In a mean field approach of thin networks, the critical temperature $T_c(H)$ is derived from the Ginzburg-Landau equations on each link, with conservation of the current at the nodes of the lattice.[1] A set of coupled linear equations results, whose smallest eigenvalues gives $T_c(H)$. It turns out that, when all the links have the same size, the structure of the equations is that of an isotropic problem of tight-binding electrons in a magnetic field. The spectrum of such a problem is known to exhibit a fractal structure, the famous "Azbel-Hofstadter" butterfly.[4,5] The phase boundary is given by the lower edge of this spectrum. This frustration effect has been observed experimentally in different types of networks[6,7] including quasicrystalline[8] or anisotropic.[9]

All these studies dealt with two-dimensional (2D) networks and we now consider the case of three-dimensional (3D) networks. Indeed, we argue in this paper that in such a case interesting effects appear due to additional periodicities introduced by the magnetic field, when it is tilted. In such a case, the vortex lattice has to arrange in a 3D way and accommodate the geometry of the network.

The spectrum of a 3D tight-binding model in a uniform magnetic field has been studied.[10–13] A tilted magnetic field is then characterized by three fluxes through the three elementary plaquettes of the lattice (instead of one in 2D). Consider a commensurate flux case in which the three flux quanta through three elementary plaquettes are rationals, $\phi_a = p_a/q_a$, $\phi_b = p_b/q_b$, and $\phi_c = p_c/q_c$ (p_i and q_i are integers). Then the spectrum has Q subbands where Q is the least common multiple of q_a, q_b, and q_c. Some of the subbands may overlap or may have finite gaps between them. By rotation of the field, one expects a dramatic change in the spectrum since the commensurability of the three fluxes changes drastically. This behavior is contrasted with the 2D case where there is only one flux ϕ. If ϕ is rational p/q, the spectrum consists of q subbands. All the gaps are open except when q is even, in which case the two subbands at the center touch.[14–16]

In this paper, we focus on the lower edge of the spectrum of 3D tight-binding electrons, which gives directly the phase boundary of the corresponding superconducting network. It is expected to vary in a nontrivial way when the field is varied or tilted. A similar problem has been studied recently which is the behavior of a cubic superconducting circuit made of 12 identical wires[17] and up to $6\times6\times6$ cubes.[18] A rich behavior was obtained versus field variation or orientation, which confirmed our first predictions. But these papers only considered a small number of cubes. One can expect that the translation invariance of an infinite 3D cubic network may add interesting new physics because the flux lattice has now to accommodate an infinite structure.

II. EQUATIONS FOR THE TRANSITION TEMPERATURE IN A THREE-DIMENSIONAL SUPERCONDUCTING NETWORK

The superconducting transition temperature for a network of thin superconducting wires in an external magnetic field can be determined by the linearized Gimzburg-Landau equations. At each node i, the equation is given by[2,3]

$$\Delta_i \sum_j s_{ij} \cot\left[\frac{L_{ij}}{\xi(T)}\right] = \sum_j s_{ij} \Delta_j \frac{\exp[i\gamma_{ij}]}{\sin[L_{ij}/\xi(T)]} , \qquad (1)$$

where Δ_i is the superconducting order parameter at node i, L_{ij} is the length of the link connecting node i and node

j, s_{ij} is the cross section of the link, $\xi(T)$ is the coherence length, and γ_{ij} is the phase factor due to the external field;

$$\gamma_{ij} \equiv \frac{2\pi}{\Phi_0} \int_i^j \mathbf{A} \cdot dl \ , \tag{2}$$

with the vector potential $\mathbf{A}$ and the flux quantum $\Phi_0 = hc/2e$.

Here we consider a network of a simple cubic lattice. In this case $L_{ij} = L$ is independent of i and j, and the equation is written as

$$-\sum_j t_{ij} e^{i\gamma_{ij}} \Delta_j = \lambda \Delta_i \ , \tag{3}$$

where

$$t_{ij} = \frac{s_{ij}}{\sum_j s_{ij}} \ , \tag{4}$$

and

$$\lambda = -\cos[L/\xi(T)] \ . \tag{5}$$

Equation (3) is the same as that obtained from the tight-binding Hamiltonian for electrons (charge $-2e$) in an external magnetic field with the hopping matrix element t_{ij}. The cross sections of the links may be different for each direction, so we take $t_{ij} = t_\alpha$ $(\alpha = a, b, c)$ for the nearest-neighbor sites i and j in the α direction. If all links have the same cross section, $t_a = t_b = t_c = \frac{1}{6}$. Since the coherence length depends on temperature as $\xi(0)/\xi(T) = \sqrt{1 - T/T_{c0}}$, the transition temperature is determined by the smallest eigenvalue of the tight-binding Hamiltonian $\lambda_{\min}$ as

$$1 - \frac{T_c}{T_{c0}} = \left[\frac{\xi(0)}{L} \arccos(-\lambda_{\min}) \right]^2 . \tag{6}$$

We consider the commensurate case where the flux through each plaquette is a rational number, i.e., the magnetic flux is given by

$$(\phi_a, \phi_b, \phi_c) = \left[\frac{p_a}{q_a}, \frac{p_b}{q_b}, \frac{p_c}{q_c} \right] , \tag{7}$$

where $\phi_\alpha = 1/(2\pi) \sum_\alpha \gamma_{ij}$ is the flux through a plaquette perpendicular to α axis $(\alpha = a, b, c)$, and p_α and q_α are mutually prime integers. Even if the flux is irrational, the smallest eigenvalue will be a continuous function of the flux which can be approximated by a rational number, although the minimum energy is not smooth. Let Q be the least common multiple of q_a, q_b, and q_c. Then we can define an integer p such that

$$\phi_a = \frac{p}{Q} m_1 \ , \quad \phi_b = \frac{p}{Q} m_2 \ , \quad \phi_c = \frac{p}{Q} m_3 \ , \tag{8}$$

where the integers m_1, m_2, and m_3 have no common factor. Let $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$, and $\hat{\mathbf{c}}$ be the original primitive lattice vectors which correspond to the superconducting links. Since the choice of primitive lattice vectors is not unique, we consider a new set of primitive lattice vectors $\hat{\mathbf{a}}'$, $\hat{\mathbf{b}}'$, and $\hat{\mathbf{c}}'$. As shown by Halperin,[19] one can define a new set of primitive lattice vectors, where the magnetic field is parallel to one of the lattice vectors, say $\hat{\mathbf{c}}'$. Then a vector can be represented in two ways as

$$x\hat{\mathbf{a}} + y\hat{\mathbf{b}} + z\hat{\mathbf{c}} = x'\hat{\mathbf{a}}' + y'\hat{\mathbf{b}}' + z'\hat{\mathbf{c}}' \tag{9}$$

and (x', y', z') is related to (x, y, z) by a matrix R as

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = R \begin{bmatrix} x \\ y \\ z \end{bmatrix} . \tag{10}$$

We now take

$$R = \begin{bmatrix} \frac{m_3}{q} & 0 & -\frac{m_1}{q} \\ -s_1 m_2 & q & -s_3 m_2 \\ s_1 s_4 & s_2 & s_3 s_4 \end{bmatrix} , \tag{11}$$

where q is the greatest common factor of m_1 and m_3 and integers s_i $(i = 1-4)$ are the solution of

$$\begin{aligned} s_1 m_1 + s_3 m_3 &= q \ , \\ s_2 m_2 + s_4 q &= 1 \ . \end{aligned} \tag{12}$$

Note that all the matrix elements of both R and its inverse R^{-1} are integers and $\det(R) = 1$. These properties guarantee that $\hat{\mathbf{a}}'$, $\hat{\mathbf{b}}'$, and $\hat{\mathbf{c}}'$ are primitive lattice vectors. Now consider a vector $\phi_a \hat{\mathbf{a}} + \phi_b \hat{\mathbf{b}} + \phi_c \hat{\mathbf{c}}$ which is parallel to the magnetic field. The new representation of the vector is

$$\begin{bmatrix} \phi_a' \\ \phi_b' \\ \phi_c' \end{bmatrix} = R \begin{bmatrix} \phi_a \\ \phi_b \\ \phi_c \end{bmatrix} = \frac{p}{Q} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} , \tag{13}$$

i.e., magnetic field is along the $\hat{\mathbf{c}}'$ axis and the flux in a plaquette formed by $\hat{\mathbf{a}}'$ and $\hat{\mathbf{b}}'$ is p/Q.

We can take the Landau gauge in the new axes, i.e., the gauge potential depends only on x'. Since $x' = (m_3/q)x - (m_1/q)z$, we can write the gauge as

$$\mathbf{A} = \pi \frac{p}{Q} \left[\frac{m_3}{q} x - \frac{m_1}{q} z \right] (l_1 \hat{\mathbf{a}} + l_2 \hat{\mathbf{b}} + l_3 \hat{\mathbf{c}}) \ , \tag{14}$$

where l_1, l_2, and l_3 are integers which give

$$\begin{aligned} \mathbf{H} = \nabla \times \mathbf{A} &= \pi(\phi_a \hat{\mathbf{a}} + \phi_b \hat{\mathbf{b}} + \phi_c \hat{\mathbf{c}}) \\ &= \pi \frac{p}{Q} (m_1 \hat{\mathbf{a}} + m_2 \hat{\mathbf{b}} + m_3 \hat{\mathbf{c}}) \ . \end{aligned} \tag{15}$$

(The length of links L is set to unity for simplicity. Also the unit $\hbar = c = e = 1$ is taken, so the flux quanta $\Phi_0 = \pi$.) Thus $l_2 = q$ and l_1 and l_3 are the solution of the equation

$$\frac{m_1}{q} l_1 + \frac{m_3}{q} l_3 + m_2 = 0 \ . \tag{16}$$

Since m_1/q and m_3/q are integers and prime each other by definition, l_1 and l_3 are integers. Inserting Eq. (14) into Eq. (2) we get for γ_{ij}

$$\gamma_{ij}=\begin{cases} 2\pi\frac{p}{Q}l_1\left[\frac{m_3}{q}i_a-\frac{m_1}{q}i_c+\frac{m_3}{2q}\right] & \text{for } \mathbf{j}=(i_a+1,i_b,i_c)\ , \\ 2\pi\frac{p}{Q}l_2\left[\frac{m_3}{q}i_a-\frac{m_1}{q}i_c\right] & \text{for } \mathbf{j}=(i_a,i_b+1,i_c)\ , \\ 2\pi\frac{p}{Q}l_3\left[\frac{m_3}{q}i_a-\frac{m_1}{q}i_c-\frac{m_1}{2q}\right] & \text{for } \mathbf{j}=(i_a,i_b,i_c+1)\ , \end{cases} \tag{17}$$

where $\mathbf{i}=(i_a,i_b,i_c)$ and $\mathbf{j}$ label the neighboring sites. Note that γ_{ij} depends on $\mathbf{i}$ only through the combination $m_3i_a-m_1i_c$.

Now we perform Fourier transformation of Eq. (1). Write

$$\Delta_i=\sum_{\mathbf{k}}\exp[i\mathbf{k}\cdot\mathbf{r}_i]\Delta(\mathbf{k})\ . \tag{18}$$

Substituting Eq. (18) into Eq. (3), we can easily show that $\Delta(\mathbf{k})$ couples only with $\Delta(k_x\mp2\pi(p/Q)(m_3/q)l_1,k_y,k_z\pm2\pi(p/Q)(m_1/q)l_1)$, $\Delta(k_x\mp2\pi(p/Q)(m_3/q)l_2,k_y,\ k_z\pm2\pi(p/Q)(m_1/q)l_2)$, and $\Delta(k_x\mp2\pi(p/Q)(m_3/q)l_3,k_y,k_z\pm2\pi(p/Q)(m_1/q)l_3)$. Then Eq. (3) is written

$$H(\mathbf{k})\Psi(\mathbf{k})=\lambda\Psi(\mathbf{k})\ , \tag{19}$$

where

$$\Psi(\mathbf{k})=\begin{bmatrix}\varphi_1(\mathbf{k})\\ \varphi_2(\mathbf{k})\\ \vdots\\ \varphi_Q(\mathbf{k})\end{bmatrix}, \tag{20}$$

$$\varphi_n(\mathbf{k})=\Delta(k_x-2\pi(p/Q)(m_3/q)n,k_y,k_z+2\pi(p/Q)(m_1/q)n)\ , \tag{21}$$

and $H(\mathbf{k})$ is a $Q\times Q$ matrix with the elements

$$(H)_{nm}=\begin{cases} M_a(n) & \text{for } m=n+l_1(\text{mod}Q)\ ,\\ M_a^*(n-l_1) & \text{for } m=n-l_1(\text{mod}Q)\ ,\\ M_b(n) & \text{for } m=n+l_2(\text{mod}Q)\ ,\\ M_b^*(n-l_2) & \text{for } m=n-l_2(\text{mod}Q)\ ,\\ M_c(n) & \text{for } m=n+l_3(\text{mod}Q)\ ,\\ M_c^*(n-l_3) & \text{for } m=n-l_3(\text{mod}Q)\ ,\\ 0 & \text{otherwise}\end{cases} \tag{22}$$

with

$$\begin{aligned} M_a(n)&=-t_a\exp\left\{i\left[k_x-2\pi\frac{p}{Q}\frac{m_3}{q}\left[n+\frac{l_1}{2}\right]\right]\right\}\ ,\\ M_b(n)&=-t_b\exp[ik_y]\ ,\\ M_c(n)&=-t_c\exp\left\{i\left[k_z+2\pi\frac{p}{Q}\frac{m_1}{q}\left[n+\frac{l_3}{2}\right]\right]\right\}\ . \end{aligned} \tag{23}$$

It is seen from (19) that the spectrum consists of Q magnetic subbands. These subbands may or may not overlap. The structure of the matrix elements (22) shows that Eq. (19) can be regarded as a one-dimensional tight-binding equation with long-range hoppings. Thus the original problem of 3D tight-binding electrons in a magnetic field is reduced to a 1D problem. Equation (19) is the generalization of that studied previously for the cases $p_c/q_c=0$ (Ref. 10) and $p_a/q_a=p_b/q_b=p_c/q_c$.[11,13] Kunszt and Zee[12] have studied the case that l_3 or l_1 can be taken zero. In the case $p_a/q_a=p_b/q_b=p_c/q_c$, we get $Q=q_a$, $p=p_a$, $m_1=m_2=m_3=1$, $q=1$, $l_1=1$, and $l_3=0$, i.e., H has nonzero component only for $(n,n\pm1)$.

III. NUMERICAL RESULT

The smallest eigenvalue of Eq. (19), $\lambda_{\min}$, is obtained numerically by scanning momentum $\mathbf{k}$ in the Brillouin zone. We plot $[\arccos(-\lambda_{\min})]^2$, which is proportional to $1-T_c/T_{c0}$ [see Eq. (6)], as a function of ϕ for $(\phi_a,\phi_b,\phi_c)=(\phi,\phi,\phi)$ in Fig. 1 for the isotropic case (all the cross sections of the links are the same). The anisotropic networks corresponding to $t_c=0.3t_a=0.3t_b$ are also considered with the magnetic field $(\phi_a,\phi_b,\phi_c)=(\phi,\phi,\phi)$ in Fig. 2. In Figs. 3 and 4 one or two components of the flux are fixed and the transition temperature is plotted as a function of the other components, $(\frac{1}{3},\phi,\phi)$ and $(\frac{1}{3},\phi,\frac{1}{3})$. In these calculations we take $\phi=n/100$ in Fig. 1 and $\phi=n/60$ in Figs. 2–4 with integer n. As seen in Figs. 1–4, the overall structure is not changed by the anisotropy. Since the lattice is bipartite, $\lambda_{\min}$ for (ϕ_a,ϕ_b,ϕ_c) is the same as that for

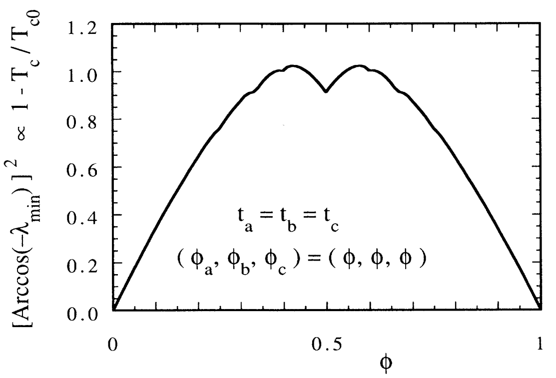

FIG. 1. Transition temperature of the 3D superconducting network as a function of external magnetic field. The lattice is isotropic and the field direction is fixed to (111).

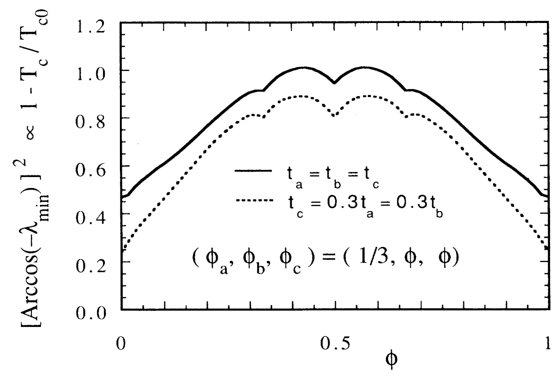

FIG. 3. Transition temperature of the 3D superconducting network as a function of $\phi=\phi_b=\phi_c$ with $\phi_a=\frac{1}{3}$ for isotropic (solid line) and anisotropic (dashed line) cases.

$(n_a\pm\phi_a, n_b\pm\phi_b, n_c\pm\phi_c)$ with any integers n_a, n_b, and n_c. The transition temperature has a sharp cusplike local maximum at $\phi=\frac{1}{2}$ in each case. For the other values of ϕ the peaky structure of the transition temperature as a function of ϕ is not very visible but we can see small anomalies at $\phi=\frac{1}{3}, \frac{1}{4}, \frac{2}{5}$, and $\frac{1}{6}$. In these cases the frustration is caused by only one value ϕ. Thus it is not surprising that the dependence of $1-T_c/T_{c0}$ on ϕ is similar to that obtained in the two-dimensional case.[1–6] A new feature appears when the magnetic field is tilted along a direction characterized by three different fluxes. In Fig. 5 we plot $[\arccos(-\lambda_{\min})]^2$ for the isotropic case with $(\phi_a,\phi_b,\phi_c)=(n/10,n/20,n/30)\|(\frac{6}{7},\frac{3}{7},\frac{2}{7})$ as a function of n. This choice of the field direction is the same as calculated by Yi and Hu for a finite size lattice.[18] As discussed before, Yi and Hu have studied the transition temperature of several cubes of superconducting links, whereas we calculate that for the infinite lattice. Here we can see several cusplike minima corresponding to $\phi_\alpha=1$ $(\alpha=a,b,c)$. The complicated structure is due to the fact that the three components of the flux cause the frustration in a complicated way. If we look in more detail, we see small cusps corresponding to $\phi_\alpha=\frac{1}{2}, \frac{1}{3}$, etc. In actual calculations it is difficult to rotate the magnetic field with fixed amplitude, since we can calculate only the case that

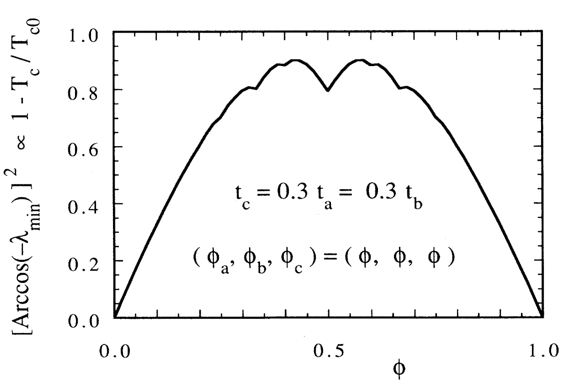

FIG. 2. The same as Fig. 1 but the lattice is anisotropic.

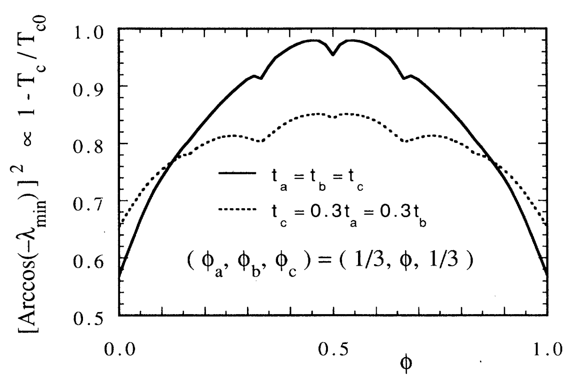

FIG. 4. Transition temperature of the 3D superconducting network as a function of external magnetic field for isotropic (solid line) and anisotropic (dashed line) cases. The a and c components of the field are fixed as $\phi_a=\phi_c=\frac{1}{3}$ and ϕ_b is changed.

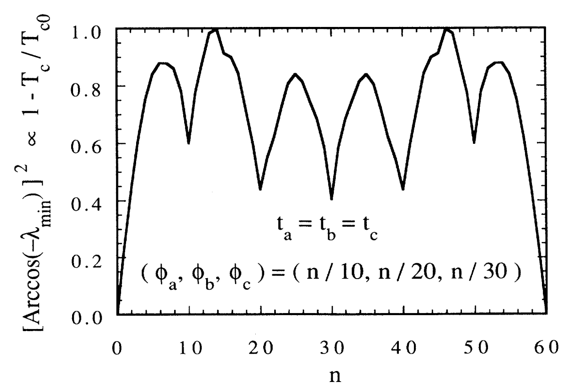

FIG. 5. Transition temperature of the 3D superconducting network as a function of external magnetic field. The direction of the field direction is fixed to $(\frac{1}{10},\frac{1}{20},\frac{1}{30})$ and the magnitude of the field is changed.

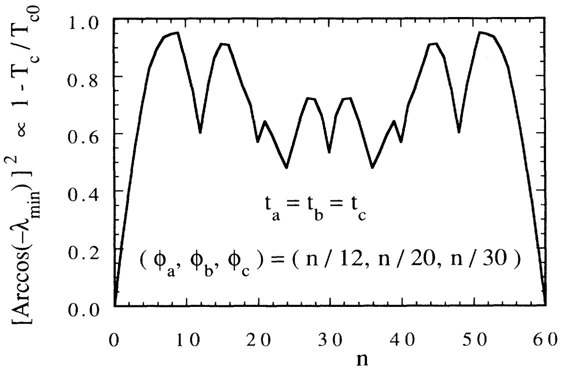

FIG. 6. Transition temperature of the 3D superconducting network as a function of external magnetic field. The direction of the field is fixed to $(\frac{1}{12},\frac{1}{20},\frac{1}{30})$ and the magnitude of the field is changed.

every component of the flux is a rational number with small denominator. Instead of rotating magnetic field for fixed amplitude, we calculate the transition temperature for a slightly different direction of the field, $(\phi_a,\phi_b,\phi_c)=(n/12,n/20,n/30)$ as shown in Fig. 6. Drastic differences are seen between Figs. 5 and 6. Therefore we can conclude that if we rotate the magnetic field, the transition temperature of the superconducting network has cusplike maximum whenever one component of the flux divided by the flux quantum is an integer or a rational number with a small denominator. As a result the transition temperature depends in a complicated way on the direction of the magnetic field.

IV. CONCLUSION

We have shown that the critical temperature of the 3D superconducting network in a uniform magnetic field can be calculated by solving the eigenvalue problem which is a generalization of the 2D Hofstadter problem. When the magnetic field is fixed in the (1,1,1) direction and the magnitude is changed, the critical temperature has a similar dependence on the magnitude of the field as in the 2D network case. However, if the direction of the magnetic field is tilted, T_c depends drastically on the direction and magnitude of the field. This is because in 3D networks there are three components of the flux and T_c has a cusplike maximum when one component of the flux is an integer or a rational number with a small denominator.

It is known that the transition temperature for the Josephson junction networks has the same properties if the mean field approximation is adopted.[20] Since the mean field approximation is thought to be a better approximation in 3D than in 2D, we expect that the sensitive dependence of the transition temperature on the magnitude and direction of the magnetic field is also observed in the 3D Josephson junction networks.

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