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Electron lifetime and transport time for inverse-power-law electron-impurity scattering potentials

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We examine, in weakly disordered metals, the conditions governing the existence of the electron lifetime and the transport time. We show that the potential does not need to be integrable for these two quantities to exist. We also study scattering potentials of the form $r^{-(D+\sigma)}$ and show that the problem should shift from the short-range universality class to the long-range universality class when the parameter σ becomes smaller than $-(D-1)/2$, where D is the dimensionality.

I. INTRODUCTION

We recently showed[1] that the electrical conductivity σ of a weakly disordered metal (the Drude part plus the weak-localization correction calculated to first order in perturbation) can be expressed in terms of only two parameters, the elastic lifetime of the electrons τ and the transport time τ_{tr} (in the absence of inelastic or spin-orbit scattering). These lifetimes were calculated within the Born approximation. It was pointed out that σ can thus be obtained for an arbitrary scattering potential $V(r)$, as long as this potential is integrable, more precisely as long as the calculation of τ and τ_{tr} is feasible. Under these conditions, it was shown that the problem belongs to the same universality class as the problem involving a short-range potential, although the precise structure of the potential does not appear in the problem.

In the present paper, we examine this point more closely and we show that a supposedly long-range potential may still yield a result belonging to the short-range universality class.

II. GENERAL CONDITIONS ON THE SCATTERING POTENTIAL

Within the Born approximation, τ and τ_{tr} are expressed as angular integrals of the square of the Fourier transform of the potential:

$$\tau^{-1}=2\pi N(0)n_I\int V^2(q)d\Omega \Big/ \int d\Omega \ , \tag{1a}$$

$$\tau_{tr}^{-1}=2\pi N(0)n_I\int V^2(q)(1-\cos\theta)d\Omega \Big/ \int d\Omega \ . \tag{1b}$$

$N(0)$ is the density of states at the Fermi level, n_I the impurity density, $q=|\mathbf{k}'-\mathbf{k}|$ with $\mathbf{k}'$ and $\mathbf{k}$ the electronic momenta before and after scattering on the impurity, θ is the scattering angle between $\mathbf{k}'$ and $\mathbf{k}$, and $d\Omega$ is the angular element of integration [$\int d\Omega=2\pi^{D/2}/\Gamma(D/2)$ as recalled in Ref. 1]; D is the dimensionality restricted here to $D=3$ and $D=2$. Since $|\mathbf{k}|\approx|\mathbf{k}'|\approx k_F$, k_F being the Fermi momentum, one has $q\approx 2k_F\sin(\theta/2)$, so that

$$0\le q\le 2k_F \ . \tag{2}$$

Therefore, in principle, $V^2(q)$ and thus $V(q)$ must be well defined in the above q range. In particular, $V(q=0)$ should exist, i.e., $V(r)$ must be integrable since $V(q=0)=\int V(r)d^D\mathbf{r}$. However if we are only interested in the existence of τ and τ_{tr}, $V(q=0)$ may well be singular although integrals involving $V(q)$ relevant for the existence of τ and τ_{tr} are well defined. More precisely, Eqs. (1) impose that

$$\int_0^{2k_F} V^2(q)q\ dq=\text{finite quantity, in 3D} \ , \tag{3a}$$

$$\int_0^{2k_F} V^2(q)\frac{dq}{\sqrt{1-(q^2/4k_F^2)}}=\text{finite quantity, in 2D} \ . \tag{3b}$$

These two conditions insure the existence of τ^{-1} in 3D and 2D. If (3a) and (3b) are satisfied, then the corresponding integrals for $\tau_{tr}^{-1}[\sim\int V^2q^3dq$ in 3D, and $\sim\int V^2(q)q^2dq/\sqrt{1-(q^2/4k_F^2)}$ in 2D] are also finite quantities. In other words $V(q\to 0)$ may be singular but τ and τ_r still exist.

Potentials which are themselves well behaved are of no interest here since they will yield well-behaved expressions for τ and τ_{tr}. Instead, in the following, we will examine potential forms $V(r)$ which are singular under certain conditions.

III. YUKAWA TYPE OF POTENTIALS

We examine here two Yukawa types of potentials

$$V_1(r)\sim e^{-\varepsilon k_F r}/r \ , \tag{4}$$

$$V_2(r)\sim \varepsilon^{D-1}V_1(r) \ , \tag{5}$$

where ε is a parameter which may be varied between 0 and ∞. The Fourier transforms are straightforward:

$$V_1(q)\sim(q^2+\varepsilon^2k_F^2)^{(1-D)/2} \ , \tag{6}$$

$$V_2(q)\sim\varepsilon^{D-1}V_1(q) \ . \tag{7}$$

Then we have the following.

(i) When $\varepsilon\to\infty$, $V_2(q)\to$const for any q values (in particular for $q=0$), and one recovers for $V_2(r)$ a contact potential for which τ and τ_{tr} are well defined (and $\tau\equiv\tau_{tr}$ in this particular case). In contrast, $V_1(q)\to 0$ and $V_1(r)\to 0$.

(ii) When $\varepsilon \to 0$, V_2 $(q=0) \to 0$; $V_2(r)$ is integrable and τ and $\tau_{\rm tr}$ $(\neq \tau)$ are well behaved. In contrast, $V_1(r) \to 1/r$; one recovers the pure Coulomb potential which is known to be pathological.[2] Moreover the use of the Born approximation becomes questionable.[3] In that case $V(q) \sim q^{-2}$ in 3D and $\sim q^{-1}$ in 2D and one easily checks that formulas (3) diverge.

Therefore $V_2(r)$ is integrable and yields nonsingular results for τ and $\tau_{\rm tr}$ for all values of ε, $0 \le \varepsilon \le \infty$; one easily finds that $\tau^{-1} \equiv \tau_{\rm tr}^{-1} = $const for $\varepsilon \to \infty$ and $\tau^{-1} = \tau_{\rm tr}^{-1} = 0$ for $\varepsilon \to 0$. In contrast, $V_1(r)$ is integrable and yields well-behaved formulas for τ and $\tau_{\rm tr}$ only for finite values of $\varepsilon, \varepsilon > 0$.

Note that potentials like (4) and (5), containing an exponential, are usually called "short ranged" and yield results for transport phenomena which belong to the short-range universality class, even (4), as long as ε remains finite. But when ε vanishes the physical properties of problems involving (4) most likely suddenly changes, via a plausible "Hopf bifurcation,"[4] towards long-range universality class behavior. In the following section the borderline will become more precise.

IV. INVERSE-POWER-LAW POTENTIALS

We now turn to examine scattering potentials of the form

$$V(r) \sim r^{-(D+\sigma)} , \tag{8}$$

where σ is a parameter $\sigma \gtrless 0$. We choose this form because in the different problem of interacting spin systems,[5] for positive σ, spin interactions of the type (8) are called "short range" for $\sigma \ge 2$ and "long range" for $0 < \sigma < 2$.

Here we study what happens in transport phenomena for electron-impurity scattering potentials of the form (8) and examine what the pathological ranges (if any) are for σ. Let us first note the conditions insuring integrability of (8) in order to later remark that the conditions of existence of τ and $\tau_{\rm tr}$ are not necessarily the same. If we write that $V(r)$ is integrable we must have $\int r^{-(D+\sigma)} d^D \mathbf{r} =$ finite quantity. This yields $\sigma^{-1}[r_{\rm min}^{-\sigma} - r_{\rm max}^{-\sigma}] =$ finite. There is indeed always a minimum value of r, $r_{\rm min}$, since the impurity always has a finite radius. However, the maximum value of r is $r_{\rm max} = \infty$ in infinite systems. Therefore the conditions of integrability of (8) are the following: $\sigma > 0$ in infinite systems, while $\sigma \gtrless$ in finite systems.

Now we turn to the calculation of the Fourier transform of (8),

$$V(q) \sim \int \frac{r^{D-1} dr}{r^{D+\sigma}} I(r) , \tag{9}$$

$$\begin{aligned} I(r) &= \int_0^{\pi} \sin^{D-2}\theta e^{iqr\cos\theta} d\theta , \\ &= 2\int_0^1 (1-x)^{(D-3)/2} \cos(qrx) dx , \quad x = \cos\theta . \end{aligned} \tag{10}$$

Standard tables[6] tell us that

$$\int_0^u (u^2 - x^2)^{\nu-(1/2)} \cos(ax) dx = \frac{\sqrt{\pi}}{2} \left[\frac{2u}{a} \right]^{\nu} \Gamma(\nu + \tfrac{1}{2}) J_\nu(au) ,$$

with $a > 0$, $u > 0$, $\mathrm{Re}\nu > -\frac{1}{2}$. We note that the condition over ν is necessary in order that $\Gamma(\nu+\frac{1}{2})$ be finite. However, the condition over a does not look necessary: for small values of (au) indeed, $J_\nu(au) \sim (au/2)^\nu 1/\Gamma(\nu+1)$, so that the limit, in the right-hand side of the above expression, of $(1/a)^\nu J_\nu(au)$ is perfectly well defined when $a \to 0$.

Back to (10) we thus get

$$I(r) = \sqrt{\pi} \left[\frac{2}{qr} \right]^{(D/2)-1} \Gamma \left[\frac{D-1}{2} \right] J_{(D/2)-1}(qr) , \quad D > 1 . \tag{11}$$

Therefore, (9) reads

$$V(q) \sim \sqrt{\pi} \Gamma \left[\frac{D-1}{2} \right] \left[\frac{2}{q} \right]^{(D/2)-1} \times \int J_{(D/2)-1}(qr) \frac{dr}{r^{\sigma+(D/2)}} . \tag{12}$$

The following integral is known:[6]

$$\int_0^\infty x^\mu J_\nu(ax) dx = \frac{2^\mu}{a^{\mu+1}} \frac{\Gamma \left[\frac{1+\nu+\mu}{2} \right]}{\Gamma \left[\frac{1+\nu-\mu}{2} \right]} , \tag{13}$$

with $-\mathrm{Re}\nu - 1 < \mathrm{Re}\mu < \frac{1}{2}, a > 0$. The condition $\mathrm{Re}\mu < \frac{1}{2}$ insures convergence of the integral when $x \to \infty$. Applied to (12) it imposes that $\sigma > -(D+1)/2$ for infinite systems where the upper bound on r, $r_{\rm max} = \infty$, while that condition is not necessary in finite systems where $r_{\rm max}$ is finite. In infinite systems, had we required the integrability of $V(r)$, the condition $\sigma > 0$ would have rendered $\sigma > -(D+1)/2$ automatically satisfied. But choosing to only require τ^{-1} and $\tau_{\rm tr}^{-1}$ to exist rather than $V(r)$ be integrable, leaves us with the condition

$$\begin{aligned} &\sigma > -(D+1)/2 \quad \text{for infinite systems} , \\ &\sigma \text{ arbitrary for finite systems} . \end{aligned} \tag{14}$$

On the other hand the condition $-\mathrm{Re}\nu - 1 < \mathrm{Re}\mu$ insures the convergence of the integral $\int_0^\infty x^\mu J_\nu(ax) dx$ for $x \to 0$. This last condition is not necessary when applied to (12) since we noted that there is always a minimum value $r_{\rm min}$.

Then $V(q)$ will thus be approximately given by

$$V(q) \sim \sqrt{\pi} \Gamma \left[\frac{D-1}{2} \right] \frac{1}{2^{\sigma+1}} \frac{\Gamma \left[-\frac{\sigma}{2} \right]}{\Gamma \left[\frac{D+\sigma}{2} \right]} q^\sigma . \tag{15}$$

We now have to insert (15) in formulas (3). Therefore we must insure that the coefficient of q^σ in $V(q)$ is not infinite, i.e.,

$$\Gamma\left[-\frac{\sigma}{2}\right]\neq\infty\ ,\quad \text{i.e.}\ ,\ \sigma\neq 0,2,4\ldots,$$
$$\Gamma\left[\frac{D+\sigma}{2}\right]\neq 0\ . \tag{16}$$

The cases where (16) is not fulfilled are actually not unphysical. Indeed one has then to calculate the integral in (12) with more care taking the lower cutoff (r_0 on r) precisely into account. Then switching to the new variable R given by $r=r_0R$ the integral in (12) is taken between 1 and ∞ and can be expressed in terms of the difference between two known integrals:[6] $\int_0^\infty - \int_0^1$. The result involves, instead of (15), q^σ times a combination of Bessel and Lommel functions of (qr_0). To simplify, in the following, we stick to the approximate form (15).

Putting (15) into (3) implies straightforwardly that

$$\sigma > -\frac{D-1}{2}\ . \tag{17}$$

Note that (17) renders (14) automatically satisfied in infinite systems. Equation (17) evidently excludes the pure Coulomb potential corresponding to $\sigma=-2$ in 3D and $\sigma=-1$ in 2D.

Finally the conditions for τ and $\tau_{\rm tr}$ to exist in the case of the scattering potential of the form (8) are given by Eqs. (17) and (16). We note, in particular, that for $-(D-1)/2<\sigma<0$, the potential (8) is not integrable in infinite systems although it still yields finite values of τ and $\tau_{\rm tr}$. On the other hand, the range $0<\sigma<2$ which yields "long-range" interaction in spin systems, plays no particular role here for the scattering potential (8). Equation (8) behaves as a short-range potential since it amounts to give τ^{-1} and $\tau_{\rm tr}^{-1}$ of the same universality class as that short-range potentials.

Most likely bifurcation to long-range behavior would start when (17) is no longer satisfied, i.e., when $\sigma<-(D-1)/2$ (as pointed out above, the Coulomb potentials belong to such a range). For completeness, we give the expressions for τ and $\tau_{\rm tr}$ using (1) and (15), with (16) and (17) fulfilled. In 3D,

$$\frac{1}{\tau}\sim 2\pi N(0)n_I C^2\frac{2}{\sigma+1}(2k_F)^{2\sigma}\ ,\quad \sigma>-1 \tag{18}$$

$$\frac{1}{\tau_r}\sim 2\pi N(0)n_I C^2\frac{4}{\sigma+2}(2k_F)^{2\sigma}\ ,\quad \sigma>-1\ , \tag{19}$$

with

$$C=\sqrt{\pi}\,\Gamma\left[\frac{D-1}{2}\right]\frac{1}{2^{\sigma+1}}\frac{\Gamma\left[-\frac{\sigma}{2}\right]}{\Gamma\left[\frac{D+\sigma}{2}\right]}\ . \tag{20}$$

In 2D, we need the known[6] integral

$$\int_0^u x^{\nu-1}(u-x)^{\mu-1}dx = u^{\mu+\nu-1}\frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)}\ ,$$

with $\mathrm{Re}\mu>0$, $\mathrm{Re}\nu>0$.

We thus get

$$\frac{1}{\tau}\sim 2\pi N(0)n_I C^2(2k_F)^{2\sigma}\frac{\Gamma(\sigma+\frac12)\Gamma(\frac12)}{\Gamma(\sigma+1)}\ ,\quad \sigma>-\tfrac12\ , \tag{21}$$

$$\frac{1}{\tau_r}\sim 2\pi N(0)n_I C^2(2k_F)^{2\sigma}\frac{\Gamma(\sigma+\frac12)\Gamma(\frac12)}{\Gamma(\sigma+1)}\frac{2\sigma+1}{\sigma+1}\ ,\quad \sigma>-\tfrac12\ . \tag{22}$$

C is given by (20).

V. CONCLUSION

We have shown here some important points relevant to transport properties: (a) It is not necessary that the scattering potential be integrable for the elastic lifetime of the electrons and the transport time to exist, and (b) for scattering potentials of the form $r^{-D-\sigma}$ the potential is integrable for $\sigma>0$ and τ and τ_r are well defined; the potential is not integrable for $\sigma<0$ in infinite systems; however, for $-(D-1)/2<\sigma<0$, τ and $\tau_{\rm tr}$ are well defined and belong to the same universality class as that those corresponding to short-range potentials; finally, for $\sigma<-(D-1)/2$, τ and $\tau_{\rm tr}$ diverge; the problem then shifts from a short-range universality class to a long-range one. It has also been emphasized that the range of σ for which, in ordering spin problems, spin-spin interactions behaving like $r^{-D-\sigma}$ are called long range $(0<\sigma<2)$, is different from the one where, in transport phenomena, the electron-impurity scattering potential behaves as a long-range one $[\sigma<-(D-1)/2]$. In other words, the behavior of an interaction potential being "short range" or "long range" depends very much on the problem which is considered.

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