Bose-Einstein condensation of charged bosons in a magnetic field

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A critical magnetic field in which charged bosons are condensed is derived, taking into account their localization in a random potential. Positive curvature $(T_c - T)^{3/2}$ near T_c and a divergence or reentry effect, depending on the impurity concentration at low temperatures, are found.

The charged Bose liquid has been studied by several authors and recently became of particular interest in the context of some theories of high-temperature superconductors. For a Coulomb Bose gas the excitation spectrum at $T = 0$,[1] the condensation temperature T_c,[2] and the dielectric properties[3] were studied in the high-density limit, including low-dimensional [two-dimensional (2D)] systems.[4–7] As noted by Schafroth[8] an ideal charged Bose gas in a magnetic field cannot be condensed because of the one-dimensional character of particle motion within the lowest Landau level. We have shown that the interacting charged Bose gas is condensed in a field lower than a certain critical value H^* because the interaction with impurities,[9] or between bosons[10] broadens the Landau levels and thereby eliminates the one-dimensional singularity of the density of states. The critical field of Bose-Einstein condensation has an unusual positive curvature near T_{c0}, $H^*(T) \sim (T_{c0} - T)^{3/2}$ and diverges at $T \to 0$, where $T_{c0} \simeq 3.31 n^{2/3}/m$ is the critical temperature of Bose-Einstein condensation of an ideal gas in zero field, n is the concentration of bosons, and m is their effective mass ($\hbar = k_B = c = 1$).

In this paper I derive the temperature dependence of H^* taking into account the localization of bosons in a random potential. The localization drastically changes the low-temperature behavior of the critical field. H^* saturates with the temperature lowering at some value of the impurity concentration and at higher concentrations the reentrance into the normal state takes place. I also discuss a possibility for an experimental observation of these features.

H^* is determined as the field in which the first nonzero solution of the linearized stationary Ginzburg-Pitaevskii equation[11] for the macroscopic condensate wave function $\psi_0(\mathbf{r}) = \langle N|\hat{\psi}(\mathbf{r},\tau)|N+1\rangle$, $(N \to \infty, N/V = n = \text{const})$ appears:

$$\left(-\frac{1}{2m}[\nabla - 2ie\mathbf{A}(\mathbf{r})]^2 + U_{\rm imp}(\mathbf{r})\right)\psi_0(\mathbf{r}) = \mu\psi_0(\mathbf{r}), \quad (1)$$

where $2e$ is the charge of a boson, $\mathbf{A}(\mathbf{r})$, $U_{\rm imp}(\mathbf{r})$, and μ are the vector, random, and chemical potentials, respectively. I assume that the particle-particle interaction is taken into account within the Hartree approximation and included in the chemical potential, so the main origin of the broadening of the Landau levels lies in the impurity scattering ("dirty" limit). A weak particle-particle scattering ("clean" limit) has been discussed in Ref. 10. It does not change the major features of the temperature behavior of H^*. The Coulomb scattering is weak if the canonical RPA parameter $r_s = me^2/\epsilon_0 n^{1/3}\sqrt{\pi}$ with ϵ_0 the static dielectric constant of the host material is small. The definition of H^*, Eq. (1) is identical to that of the upper critical field H_{c2} of BCS superconductors of the second kind. Therefore, H^* determines the upper critical field of bipolaronic or any "bosonic" superconductor.

In general, the energy spectrum of the Hamiltonian Eq. (1) contains discrete levels (localized states) and a continuous part (delocalized states). The density of delocalized states $\tilde{N}(\epsilon, H^*) \sim \Im\Sigma(\epsilon)$ and the lowest delocalized energy E_c [the mobility edge,[15] $\tilde{N}(E_c, H^*) = 0$] can be found with the random phase ("ladder") approximation for the one-particle self-energy (for details see, e.g., Refs. 12–14):

$$\Sigma(\epsilon) = \frac{4\pi^2 n_{\rm im} f^2}{m}\int \frac{N(\epsilon', H^*)d\epsilon'}{[\epsilon - \epsilon' - \Sigma(\epsilon)]}, \quad (2)$$

where $n_{\rm im}$ is the impurity concentration, f is the scattering amplitude in zero field, and

$$N(\epsilon, H) = \frac{\sqrt{2}m^{3/2}\omega}{4\pi^2}\Re\sum_{N=0}^{\infty}\frac{1}{\sqrt{\epsilon - \omega(N+1/2)}} \quad (3)$$

is the density of states for a noninteracting system with $\omega = 2eH^*/m$. The solution of Eq. (2) yields

$$\tilde{N}_0(\epsilon, H^*) = \frac{\sqrt{6}m^{3/2}\omega}{8\pi^2\sqrt{\Gamma_0}}\left[\left(\frac{\tilde{\epsilon}^3}{27} + \frac{1}{2} + \sqrt{\frac{\tilde{\epsilon}^3}{27} + \frac{1}{4}}\right)^{1/3} - \left(\frac{\tilde{\epsilon}^3}{27} + \frac{1}{2} - \sqrt{\frac{\tilde{\epsilon}^3}{27} + \frac{1}{4}}\right)^{1/3}\right] \quad (4)$$

and

$$E_c = \omega/2 - 3\Gamma_0/2^{2/3} \quad (5)$$

with $\Gamma_0 = (n_{\rm im} 8\pi f^2 eH^*)^{2/3}/2m$ and $\tilde{\epsilon} = (\epsilon - \omega/2)/\Gamma_0$. Eq. (4) describes the energy dependence of the density of states of the lowest Landau level (N=0) near the mobility edge. Since the square-root singularity of the density of states of upper levels is integrated out (see below) one can neglect their quantization using the zero-field density of states for $\epsilon > \omega$:

$$N(\epsilon) \simeq m^{3/2}\sqrt{\epsilon}/\sqrt{2}\pi^2. \quad (6)$$

The first nontrivial *delocalized* solution of Eq. (1) appears at $\mu = E_c$. Thus the critical curve $H^*(T)$ is determined from the conservation of the number of particles n under the condition that the chemical potential coincides with the mobility edge:

$$\int_{E_c}^{\infty} \frac{\tilde{N}_0(\epsilon, H^*)d\epsilon}{\exp[(\epsilon - E_c)/T] - 1} = n\left(1 - (T/T_{c0})^{3/2} - \frac{n_L(T)}{n}\right), \quad (7)$$

where $n_L(T)$ is the number of localized bosons. The left-hand side of Eq. (1) is the number of bosons on the lowest Landau level, while the second term of the right-hand side is the number of bosons on all upper Landau levels, calculated with the classical density of states, Eq. (6).

The intuitive picture of interacting bosons with short-range interaction filling up all localized single-particle states in random potential and Bose-condensing onto the first extended state is well known in the literature.[16,17] To calculate $n_L(T)$ one should take into account repulsion between localized bosons. One cannot ignore the fact that localization length ξ generally varies with energy and diverges at the mobility edge. One would expect that the number of *hard-core* bosons in a localized state near the mobility edge diverges in a similar way as the localization length does. However, in the case of *charged* bosons their number in a single-potential well is determined by the competition between their long-range Coulomb repulsion $\simeq 4e^2/\xi$ and the binding energy $E_c - \epsilon$. If localization length diverges with the critical exponent $\nu < 1$:[18] $[\xi \sim (E_c - \epsilon)^{-\nu}]$, one can apply a "single-well–single-particle" approximation to calculate $n_L(T)$ assuming that one can place only one boson in each potential well (see also Ref. 20). The gross features of $H^*(T)$ as the $(T_c - T)^{3/2}$ behavior at a sufficiently high temperature and the reentrance behavior at a low temperature are not influenced by this approximation if the number of bosons in a potential well is finite.[18] Thus localized *charged* bosons obey the Fermi-Dirac statistics:

$$n_L(T) = \int_{-\infty}^{E_c} \frac{N_L(\epsilon)d\epsilon}{\exp[(\epsilon - E_c)/T] + 1}, \quad (8)$$

where the density of localized states $N_L(\epsilon)$ may be approximated in many cases by the exponential tail:

$$N_L(\epsilon) = (n_L/\gamma)\exp[(\epsilon - E_c)/\gamma], \quad (9)$$

with γ of the order of a binding energy of a single random potential well and n_L the concentration of localized states. Substitution of Eqs. (4) and (9) into Eq. (7) yields the final expression for the critical field of Bose-Einstein condensation:

$$H^*(T) = H_d(T_{c0}/T)^{3/2} \times \left(1 - (T/T_{c0})^{3/2} - \frac{Tn_L}{\gamma n}\beta(T/\gamma)\right)^{3/2}, \quad (10)$$

with

$$\beta(x) = \sum_{k=0}^{\infty}[(-1)^k/x + k], \quad (11)$$

and temperature independent $H_d = \phi_0/2\pi\xi_0^2$. The "coherence" length ξ_0 is determined by both the mean free path $l = (4\pi n_{\rm im} f^2)^{-1}$ and the interparticle distance:

$$\xi_0 \simeq 0.8(l/n)^{1/4}, \quad (12)$$

$\phi_0 = \pi/e$ is the flux quantum. Using the asymptotics $\beta(x) \simeq (2x)^{-1}$ at temperature $T > \gamma$ one obtains:

$$H^* = H_d\sqrt{1 - n_L/2n}[1 - (T/T_c)^{3/2}]^{3/2}(T_c/T)^{3/2}, \quad (13)$$

where

$$T_c = T_{c0}(1 - n_L/2n)^{2/3} \quad (14)$$

is the critical temperature in zero magnetic field renormalized by the localization. Thus the localization does not change the positive $(T_c - T)^{3/2}$ curvature of the critical magnetic field near T_c.[9] I believe that this curvature is a universal feature of a charged Bose gas, which does not depend on a particular scattering mechanism and on approximations made. The number of bosons at the lowest Landau level is proportional to the density of states near the mobility edge $\tilde{N}_0 \sim H/\sqrt{\Gamma(H)}$, where the "width" of the Landau level is also proportional to the same density of states $\Gamma(H) \sim H/\sqrt{\Gamma(H)}$. Hence, $\Gamma(H) \sim H^{2/3}$ and the number of condensed bosons is proportional to $H^{2/3}$. On the other hand this number in the vicinity of T_c should be proportional to $T_c - T$ (the total number minus the number of thermally excited bosons). That gives the $(T_c - T)^{3/2}$ law for H^*, Eq. (13). At low temperature $T \ll \gamma$ the temperature dependence of H^* turns to be drastically different for different impurity concentration. If $0 < n_L < n$ the critical field diverges at $T \to 0$:

$$H^* \simeq H_d(T_{c0}/T)^{3/2}(1 - n_L/n)^{3/2} \quad (15)$$

because the number of localized states is smaller than the number of bosons. In this case only the paramagnetic limit restricts the value of H^* if bosons are composed from two fermions with the opposite spins. If $n_L = n$ the critical field reaches its maximum at $T = 0$:

$$H^* \simeq H_d(T_{c0}\ln 2/\gamma)^{3/2}\left(1 - \pi^2 T/8\gamma \ln 2\right). \quad (16)$$

And finally, if $n < n_L < 2n$ there is a reentrance effect to the normal state at temperature below some T^*, so $H^* = 0$ for $T < T^*$, Fig. 1. If $n_L - n \ll n$

$$T^* = \gamma\left[(n_L - n)/n_L \ln 2\right]. \quad (17)$$

If the number of localized states is large, $n_L > 2n$ Bose condensation is impossible: $T_c = 0$ according to Eq. (14). Deriving Eq. (10) for H^* I expand the exponent in the left-hand side of Eq. (7) and assume that the scattering amplitude f in Eq. (2) is energy independent. Minor features of the ultralow $(T \ll \Gamma_0)$ temperature behavior of H^* depend on these assumptions. In this temperature region one can expect $T^{-9/2}$ behavior of H^* instead of $T^{-3/2}$ as in Eq. (15). These features depend also on the shape of the localized levels distribution $N_L(\epsilon)$ and on the vertex corrections to the ladder approximation. However, the major features like the divergent behavior at $T \to 0$

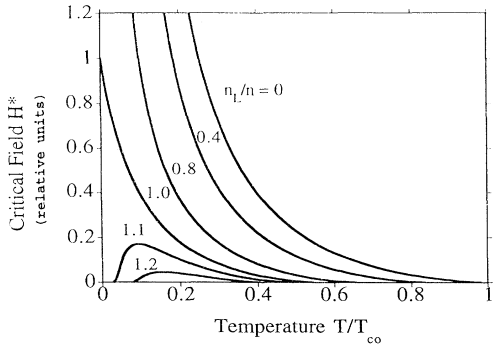

FIG. 1. Temperature dependence of the critical magnetic field of Bose-Einstein condensation [in units of $H_d(T_{c0}\ln 2/\gamma)^{3/2}$] for the different relative number of localized states n_L/n and $\gamma/T_{c0} = 0.2$.

for the low impurity concentration and the reentry effect for the sufficiently large concentration ($n_L > n$) are independent of the model and approximation made. My consideration is self-consistent if the localization length $\xi(\epsilon)$ and the localization exponent ν are field independent. In general, the magnetic field tends to change the localized state. So ν might increase with H and the assumption of $\nu < 1$ might fail before H^* is reached. However, this is not the case under consideration because the localization length $\xi(\epsilon)$ in the relevant energy region $E_c - \epsilon \sim \Gamma_0$ remains smaller than the magnetic length $(2eH^*)^{-1/2} \simeq \xi_0$. Taking for simplicity $n_{\rm im} = n$ and the scattering amplitude in zero field f equal to the screening radius, one obtains using the definition of the coherence length ξ_0, Eq. (12), and Γ_0, Eq. (5):

$$\xi/\xi_0 \simeq r_s^{7/4} < 1 \tag{18}$$

for the marginal case $\nu = 1$. In metal oxides the smallness of r_s is guaranteed by the large value of the static dielectric constant[19] $\epsilon_0 \simeq 10^2 - 10^3$ assuring that ξ is field independent.

The hypothesis of charged bosons is very attractive for high-T_c oxides in view of an extremely short coherence length ($\simeq$ 10 Å) in these materials and of many other anomalous properties (for different manifestations of the Bose-liquid behavior of superconducting oxides see recent review papers[20,21]). Some microscopic models show that charged bosons, formed by strong electron-phonon or (and) electron-electron exchange interaction might be responsible for the puzzling thermodynamic and kinetic properties of high-T_c oxides: small[22] and large[23] bipolarons, spin bipolarons,[24] local pairs,[25] and holon pairs[26] have been suggested. An upward curvature of H_{c2} near T_c has been observed in practically all superconducting oxides, including cubic ones. Reversible dc magnetization measurements on single crystals of YBaCuO (Ref. 27) proved that this curvature is a truly thermodynamic property. Magnetization measurements on single untwinned crystals[28] indicate that this upward curvature depends on the quality of the sample. However, while some authors apply our theory to explain the observed anomalous temperature dependence of H_{c2} [see for example Ref. 29 ($YBa_2Cu_4O_8$) and a more recent paper[30] ($YBa_2Cu_3O_7$)] the restricted temperature and magnetic-field interval, available for measurements, and also the significant broadening of the resistivity transition in high-T_c copper-based oxides induced by the magnetic field make any definite conclusion practically impossible. One should add the reversibility problem and the uncertainty due to the vortex lattice melting. It seems one can overcome all these difficulties with "low-T_c" oxides measuring the transition in a magnetic field for a wide temperature range starting from mK level up to T_c. Such a study has recently been reported in Ref. 31. Resistively determined H_{c2} values from $T/T_c = 0.0025$ to $T/T_c = 1$ in a $T_c = 20$ K single crystal of $Tl_2Ba_2CuO_6$ follow a temperature dependence that is in good qualitative agreement with the type of curve shown in Fig. 1 for $n_L/n \simeq 1$. For low enough T_c the melting seems to be impossible and the transition is relatively sharp. If the system is a "bosonic" superconductor one should see pronounced unusual features of H_{c2}, Fig. 1.

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