Solutions to the multiple-component $1/r$ Hubbard model

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In this work we introduce a one-dimensional multiple-component Hubbard model with $1/r$ hopping and on-site energy U. The wave functions, the spectrum, and the thermodynamics are studied for this model in the strong-interaction limit $U\rightarrow\infty$. In this limit, the system is a special example of SU(N) Luttinger liquids, exhibiting spin-charge separation in the full Hilbert space. Speculations on the physical properties of the model at finite on-site energy are also discussed.

Recent studies on low-dimensional systems have renewed great interest in the Gutzwiller-Jastrow wave functions. The wave functions are useful in the sense that they may serve as good variational wave functions or they may be exact solutions of the Hamiltonians.[1-19] Most recently, intense research in the field of integrable systems has shown the wave functions to be exact solutions of some quantum many-particle systems. These integrable systems are characterized by the fact that the full spectrum may even be written in terms of more generalized Jastrow wave functions, as in the cases of $1/r^2$ Fermi or Bose gases, $1/r^2$ Haldane-Shastry spin chain, and the 1D supersymmetric t-J model with $1/r^2$ hopping and exchange.

The Hubbard model has been of great interest since the discovery of high-T_c superconductivity. About two years ago, Gebhard and Ruckenstein introduced the one-dimensional SU(2) Hubbard model with $1/r$ hopping and on-site energy U (Ref. 11). The model is completely integrable for arbitrary on-site energy. In the strong-interaction limit $U\rightarrow\infty$, it has been discovered recently that a set of Gutzwiller-Jastrow wave functions is an exact set of eigenfunctions of the Hamiltonian,[16] and that the system exhibits spin-charge separation in the full Hilbert space.[11,16]

In this work, we introduce an integrable model, the one-dimensional $1/r$ multiple-component Hubbard model. In the following we only discuss the strong-interaction limit $U\rightarrow\infty$. Generalizing our previous work, we show that a set of SU(N) Gutzwiller-Jastrow wave functions is a set of eigenstates of the system. The full excitation spectrum and the thermodynamics are also given explicitly in this strong-interaction case. Spin and charge are decoupled in the full Hilbert space and the system is a special example of SU(N) Luttinger liquids. At the end of the work, we also discuss speculations of further investigation of the system of finite on-site energy U.

The Hamiltonian for the one-dimensional Hubbard model is given by

$$H=\sum_{\sigma}\sum_{i\neq j}t_{ij}c_{i\sigma}^{\dagger}c_{j\sigma}+U\sum_{i}\sum_{\sigma\neq\sigma'}n_{i\sigma}n_{i\sigma'}\ , \tag{1}$$

where $c_{i\sigma}^{\dagger}$ and $c_{i\sigma}$ are creation and annihilation operators at site i with spin component σ. The sum over σ runs from 1 to N, where N is the number of flavors of the fermions. We take the hopping matrix $t_{ij}=it(-1)^{(i-j)}/d(i-j)$, where $d(n)=(L/\pi)\sin(n\pi/L)$, and U is the on-site energy. Here, because of the special form of the hopping matrix for the wave functions of the system, we assume periodic boundary conditions for odd L, or antiperiodic boundary conditions for even L.

In the strong-interaction limit $U\rightarrow\infty$, each site can be occupied at most by one particle. In this limit, we work in the Hilbert space of no double occupancy and no multiple-occupancy. The Hamiltonian can be written in terms of the Hubbard operators,

$$H=\sum_{\sigma=1,2,\dots,N}\sum_{i\neq j}t_{ij}X_i^{\sigma 0}X_j^{0\sigma}\ . \tag{2}$$

Let us denote the number of holes by Q, that of the fermions of the first flavor by M_1, that of the second flavor by $M_2,\dots,$ that of the Nth flavor by M_N. Following notations used in previous literature, states in the Hilbert space can be represented by spin and hole excitations from the state full of fermions with the Nth flavor $|P\rangle$, as

$$|\Phi\rangle=\sum_{(\alpha,i),j}\Phi(\{x_i^{\alpha}\}\ ,\{y_j\})\prod_{\alpha,i}b_{i\alpha}^{\dagger}\prod_{j}h_j^{\dagger}|P\rangle\ . \tag{3}$$

Here $b_{i\alpha}^{\dagger}=c_{i\alpha}^{\dagger}c_{iN}$ annihilates one Nth flavored fermion at site i and creates one αth flavored fermion at site i for $\alpha=1,2,\dots,(N-1)$, while $h_j^{\dagger}=c_{jN}$ creates a hole at site j. Here or in the following we always implicitly assume that we work in the space of no double occupancy and no multiple occupancy in the discussion of the strong-interaction limit. The amplitude $\Phi(\{x_i^{\alpha}\},\{y_j\})$ is symmetric when exchanging the fermions at positions x_i^{α} and x_j^{α}, and antisymmetric in the hole positions $\{y_i\}$.

Let us consider the following generalized SU(N) Gutzwiller-Jastrow wave functions corresponding to uniform motion and magnetization,[9]

$$\Phi_G(\{x_i^{\alpha}\},\{y_m\})=\exp\frac{2\pi i}{L}\left[\sum_{\alpha}J_{\alpha}\sum_{i}x_i^{\alpha}+J_h\sum_{i}y_i\right]\Phi_0\ ,$$

$$\Phi_0=\prod_{\alpha;i<j}d^2(x_i^{\alpha}-x_j^{\alpha})\prod_{\alpha<\beta;i,j}d(x_i^{\alpha}-x_j^{\beta})$$

$$\times\prod_{\alpha,i,m}d(x_i^{\alpha}-y_m)\prod_{m<n}d(y_m-y_n)\ , \tag{4}$$

where $\alpha,\beta=1,2,\ldots,N-1$. The quantum numbers J_α and J_h govern the momenta of the fermions and the holes. They can be integers or half integers, such that the wave functions are periodic (or antiperiodic) for odd L (for even L) under the translations $x_i^\alpha \to x_i^\alpha + L$, and $y_m \to y_m + L$. For the wave functions to be eigenstates of the Hamiltonian, the quantum numbers must be chosen from some restricted regions, which will be specified below.

To demonstrate that the wave functions are eigenstates of the Hamiltonian, we have to consider the effect of the hopping operator very carefully. The hopping operator can be broken into N parts, each corresponding to the hopping of fermions of different flavors. Let us first consider the hopping operator of the Nth flavor, $\hat{T}(N)=\sum_{i\neq j} t_{ij} c_{iN}^\dagger c_{jN}$. When it operates on the wave functions, the hopping of the fermions of Nth flavor is equivalent to the hopping of holes

$$\frac{\hat{T}(N)\Phi_G}{\Phi_G} = -it \sum_{\bar{n}=1}^{L-1} \frac{(-1)^{\bar{n}}}{d(\bar{n})} z^{\bar{n}J_h} \times \sum_n \prod_{m\neq n} F_{nm}(\bar{n}) \prod_{(\alpha,i)} F_{n(\alpha,i)}(\bar{n}) , \quad (5)$$

where

$$F_{nm}(\bar{n}) = \cos\frac{\bar{n}\pi}{L} + \sin\frac{\bar{n}\pi}{L}\cot\Theta_{nm} ,$$

$$F_{n(\alpha_i)}(\bar{n}) = \cos\frac{\bar{n}\pi}{L} + \sin\frac{\bar{n}\pi}{L}\cot\Theta_{n(\alpha i)} ;$$

$$\Theta_{nm} = \pi(y_n - y_m)/L , \Theta_{n(\alpha i)} = \pi(y_n - x_i^\alpha)/L .$$

The sum can be carried out after expanding the products and classifying terms by the number of particles involved. In the end, only the zero-particle term and two-particle terms are left. Many particle terms vanish, yielding the following result:

$$\frac{\hat{T}(N)\Phi_G}{\Phi_G} = -\frac{2\pi t}{L} Q J_h + (2\pi t/L) i \sum_n \sum_{\alpha,i} \cot\Theta_{n(\alpha i)} . \quad (6)$$

This result is valid under the condition

$$|J_h| \le L/2 - [(Q) + (M_1 + M_2 + \cdots + M_{N-1})]/2 .$$

To consider the effects of other parts of the hopping operators, we cannot use the wave functions directly, since the hopping will involve the fermions and holes simultaneously when they operate on the wave functions. We can generalize the idea of the spin-rotated version developed in the recent work of the $1/r^2t$-J model to this SU(N) case. For example, to deal with the hopping operator $T(N-1)=\sum_{i\neq j} c_{i(N-1)}^\dagger c_{j(N-1)}$, we can write the Gutzwiller-Jastrow wave functions in terms of the hole positions and the positions of the fermions of flavors excluding the $(N-1)$th flavor. In terms of these coordinates, the wave functions can be found to be still in a similar product form, and thus the effect of the operator $T(N-1)$ can be calculated in the same way as for $T(N)$.

For the other hopping operators $T(N-2)$, $T(N-3),\ldots,T(2),T(1)$, similar procedures can be carried out. After adding all the effects of the hopping operators together, the two-particle terms vanish since positions of all the fermions and the positions of the holes span the entire lattice. Thus the Gutzwiller-Jastrow wave functions are found to be exact eigenstates of the Hamiltonian, with eigenenergies given by

$$E(J_h;J_1,J_2,\ldots,J_{N-1}) = -(2\pi t/L) Q[J_h + \tilde{J}_h^{(1)} + \tilde{J}_h^{(2)} + \cdots + \tilde{J}_h^{(N-1)}] , \quad (7)$$

where we have

$$\tilde{J}_h^{(1)} = J_h - J_{N-1} + L/2 ,$$

$$\tilde{J}_h^{(2)} = J_h - J_{N-2} + L/2 , \ldots ,$$

$$\tilde{J}_h^{(N-1)} = J_h - J_1 + L/2 .$$

The many-particle terms vanish, and thus our result holds, under the conditions:

$$\begin{aligned} &|J_h| \le (M_N)/2 , \\ &|\tilde{J}_h^{(1)}| \le (M_{N-1})/2 , \\ &|\tilde{J}_h^{(2)}| \le (M_{N-2})/2 , \\ &\cdots \\ &|\tilde{J}_h^{(N-1)}| \le (M_1)/2 . \end{aligned} \quad (8)$$

Here the ground-state energy is given by $E_0 = -(2\pi|t|/L)[L/2 - Q/2]Q$.

For this multiple-component system, the spectrum can also be written in terms of more generalized Jastrow functions. Here, we just write down the spectrum without getting into the detailed algebra as follows:

$$E = -(2\pi t/L) \sum_{i=1}^{Q} K_i + (\pi t Q/L)(L+1) , \quad (9)$$

where K_i takes values from the region $(1,2,\ldots,L)$. Each energy level is determined by a charge configuration such as (101010) for $Q=3$ and $L=6$, where the 1's represent the values occupied by the charge momenta K_i. In this system, the spin and charge degrees are decoupled from each other in the entire Hilbert space. On these physical grounds, we see that for each charge configuration, the degeneracy of the corresponding energy level is given by the number of the ways to distribute the free spins among the $L-Q$ empty values. With this result, we find the free energy per lattice site given by

$$F(T,\mu)/L = -\mu - \frac{T}{2\pi} \int_{-\pi}^{\pi} dq \ln[N + e^{\beta(qt-\mu)}] , \quad (10)$$

where μ is the chemical potential of the fermions. This free energy has also been found to correctly reproduce the first three terms in the high-temperature perturbation expansion.

In summary, we have solved the multiple-component Hubbard model in the strong-interaction limit. In this limit, the spin degrees of freedom decouple from the charge degrees of freedom in the *entire* excitation spectrum, and the system is a special example of the SU(N)

Luttinger liquids in the sense of Haldane. We have shown that the SU(N) Gutzwiller-Jastrow wave functions are eigenstates of the Hamiltonian.

In the end, we notice that in the half filling and large U limit, our model reduces to the SU(N) Haldane-Shastry spin model with $1/r^2$ exchange interaction. We suspect that our multiple-component Hubbard model of the $1/r$ hopping is also completely integrable for *arbitrary* on-site energy U at *arbitrary* filling numbers. However, we have not found any elegant way to obtain the wave functions and the energy spectrum for the finite on-site energy case. It is very likely that the SU(N) system also exhibits a metal-insulator phase transition at half filling when changing the bandwidth and the on-site energy, as in the SU(2) case discovered by Gebhard and Ruckenstein about two years ago. It is also of great interest to study the ground-state properties of the system as a function of the interaction strength, such as the spin and charge susceptibilities and various ground-state correlators. It also remains to find the integrability condition for the model at the finite on-site energy.

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