Kinetic theory of flux-line hydrodynamics: Liquid phase with disorder

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We study the Langevin dynamics of flux lines of high-T_c superconductors in the presence of random quenched pinning. The hydrodynamic theory for the densities is derived by starting with the microscopic model for the flux-line liquid. The dynamic functional is expressed as an expansion in the dynamic order parameter and the corresponding response field. We treat the model within the Gaussian approximation and calculate the dynamic structure function in the presence of pinning disorder. The disorder leads to an additive static peak proportional to the disorder strength. On length scales larger than the line static transverse wandering length and at long times, we recover the hydrodynamic results of simple frictional diffusion, with interactions additively renormalizing the relaxational rate. On shorter length and time scales line internal degrees of freedom significantly modify the dynamics by generating wave-vector-dependent corrections to the density-relaxation rate.

I. INTRODUCTION

Unlike conventional, low-temperature superconductors high-T_c superconductors exhibit strong fluctuations due to the combined effect of the small coherence length ξ, anisotropic layered structure, and high temperatures. It has been argued that the Abrikosov vortex lattice melts as a consequence of these enhanced thermal fluctuations.[1-7] Quite recently there has been experimental evidence[8-10] that in clean crystal samples (in the absence of twin boundary pinning) the Abrikosov flux lattice melts via a first-order phase transition.[11] In the flux-liquid state the vortex lines are free to move through the sample (except for their mutual repulsion) and will collectively drift in the presence of an external transverse current. This flux-line motion will then in turn generate a finite voltage and lead to a nonzero linear resistivity.

The phase diagram is changed if one includes the effects of disorder. Depending on the type and strength of the disorder, the vortex liquid state will persist down to an irreversibility line associated with a possible second-order phase transition to a vortex-glass,[12] or Bose-glass state.[13] The melting of the flux lattice has clearly very important consequences as most vividly illustrated in Fig. 1 where the mean-field phase diagram is contrasted with the phase diagram which includes effects of thermal fluctuations and disorder.

Because the flux-line liquid phase occupies a large portion of the H-T phase diagram, much of the efforts have been directed toward a better understanding of the properties of this phase in the presence of disorder. There has been considerable progress in understanding the static properties of flux lines in the liquid phase.[1] The resulting phase is well described by a collection of directed flexible lines with a line tension related to H_{c1}. The lines traverse the sample in the direction of the applied magnetic field and at finite temperatures wander throughout the sample analogously to the Brownian motion executed by atoms or small molecules in conventional isotropic liquids. This flux-line liquid in the presence of point disorder is depicted in Fig. 2. The flux-line states depicted in Fig. 1 have been extensively studied in many experiments. Early experiments used the Bitter technique in which the location of the flux-line ends emerging from the sample is resolved

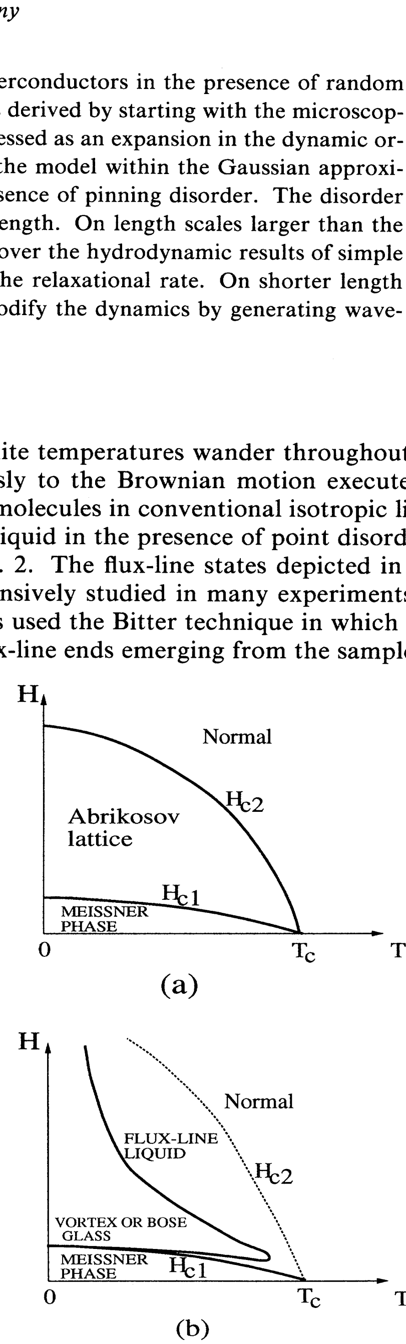

FIG. 1. (a) Mean-field phase diagram of type-II superconductors. (b) Schematic picture of a phase diagram of high-T_c superconductors, which includes effects of thermal fluctuations and disorder.

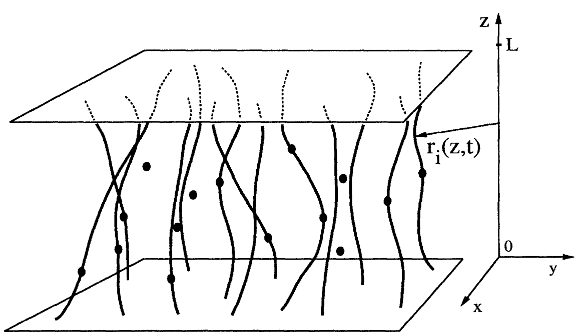

FIG. 2. Schematic picture of flux lines in high-T_c superconductors in the presence of pinning disorder (indicated by black circles). Conformation of the ith line is described by a two-dimensional vector $\mathbf{r}_i(z,t)$.

by sprinkling magnetic powder on the sample surface.[5,14] The disappearance of the regular, hexagonal pattern as the field or temperature are increased is suggestive of melting of the Abrikosov flux lattice. These experiments only directly probe the surface configuration of flux lines and therefore do not exclude a possibility of melting confined to the surface. Indeed theoretical analysis shows that the surface interaction always dominates in determining the decay of translational correlations in the asymptotic long-wavelength limit.[15] However, such large length scales have not been probed by the decoration experiments. Later vibrating-reed experiments by Gammel *et al.*[3] have found a signal suggestive of the melting transition. Although very difficult, more direct measurements of the bulk properties of the flux-line liquid are possible. In principal, the structure function of the interacting line liquid can be measured using neutron-scattering techniques. These experiments can directly probe the density correlations in the line liquid, which are governed by very different physics than ordinary liquids of point particles with an isotropic structure function.[16] Also, recently, revolutionary electron holographic techniques have been used to image the motion of flux lines in thin Pb films. This probe, which can image flux lines in real time, can be used to study mixed states also in high-T_c superconductors, and can provide invaluable information about the dynamics of flux lines.[17]

The interactions between lines, thermal fluctuations and the effects of disorder are clearly very important and are the main physical features that must be taken into account by the theory. Significant progress has been made by mapping the problem of flux lines onto the quantum statistical mechanics of interacting $2D$ bosons, where the roles of $\hbar$, temperature, and mass of the bosons are played by the temperature, the inverse sample thickness and line tension, respectively.[1] With this mapping much of the insight gained from the study of systems like helium was taken over to the problem of line liquids. When the lines are long, (i.e., the sample along the applied field direction is thick), the temperature is high, and the $2D$ density is large, such that a typical line wandering distance is larger than the average interline spacing, a highly entangled line liquid results. However, the entanglement should persist only if the line crossing barriers are significantly larger than the thermal energy. Although to date no detailed analysis of line crossing barriers exists, simple estimates give $U_x/k_B T_{\text{melt}} \sim 10-30$, which translates into very slow relaxational rates. The entangled phase corresponds to the Bose-condensed phase in the boson picture.

The static structure function for the interacting flux lines has been previously computed within the Bogoliubov approximation taking advantage of the boson mapping.[1] The contours of constant scattering intensity form a butterfly pattern with two peaks, and is quite different from the structure function of isotropic fluids of particles. At long wavelengths, away from any critical transitions, the theory of the vortex liquid can very well be described by a "hydrodynamic" model in terms of density fields, with phenomenological nonlocal coefficients. The advantage of starting with a microscopic description, however, is that the long-wavelength description of the liquid can be understood in terms of a more basic, microscopic model, thereby providing a more detailed understanding.

Although the boson mapping has been instrumental for understanding the static long-wavelength behavior of line liquids, unfortunately, there does not appear to be an obvious extension of this mapping to study the real hydrodynamics. Some progress has been made through phenomenological approaches in which the dynamic equations of motion are written directly for the coarse-grained density fields, using the static free energy, with phenomenological nonlocal coefficients determined by the static structure function.[18]

As for the statics it is useful to obtain the description of the dynamics, starting with equations of motion for individual interacting flux lines, and to derive the dynamics for the observable hydrodynamic quantities like density. The goal is to construct a kinetic theory of flux-line hydrodynamics analogous to that of point liquids, which was useful in understanding hydrodynamics of simple liquids many years ago. With this approach it should be possible to calculate the hydrodynamic parameters such as (line-liquid viscosity, if it exists) which arise completely from the flux-line interaction and entanglement effects and from the single-line microscopic friction. The bare diffusion parameter will be an input to the theory, and is related to the real microscopic coupling of flux lines to the underlying crystal lattice, the Bardeen-Stephen friction coefficient.[19]

A question of renormalization of the bare diffusion coefficient of a tagged line by the presence of the flux-line liquid and the interaction with this liquid through excluded volume interaction is also of interest and is related to the line-fluid viscosity. It is expected that if disorder is strong enough and the system is in an entangled regime a localization phenomena will take place, driving the renormalized diffusion constant to zero.[20]

In this paper we take a step toward a description of the flux-line liquid in terms of a kinetic theory of line liquids. We introduce a formalism that is useful for microscopic calculations of the dynamics in the flux-line liquid phase. The hydrodynamics of the flux-line liquid is studied by starting with the microscopic description of the interacting flux lines in terms of the Hamiltonian that includes the repulsive interline interactions and in the presence of quenched pinning disorder that couples to the density of lines. We expect that entanglement effects are in principle automatically incorporated in the full theory derived with this kinetic approach. The repulsive interaction will inhibit the lines from passing through each other and for high line densities will result in slowing down of their dynamics due to these constraints. It is not clear, however, what simplest approximation to the resulting interacting field theory will retain these effects.

This paper is organized as follows. In Sec. II we introduce our microscopic model for the dynamics of N interacting lines in the presence of quenched disorder, and in Sec. III formulate the dynamics in terms of a more convenient Martin-Siggia-Rose (MSR) description. In Sec. IV the hydrodynamic description is discussed and the microscopic model is partially recast in terms of the density fields. In Sec. V, by using the method of auxiliary fields, we integrate out the microscopic degrees of freedom and derive the effective MSR dynamic functional thereby obtaining a hydrodynamic description of the interacting flux-line liquid. In Sec. VI we approximate this theory by truncating the expansion of the hydrodynamic functional at quadratic order, and within this approximation calculate the interacting dynamic structure function in the presence of disorder. We analyze this dynamic structure function in various regimes in Sec. VII, and derive the corresponding static structure function demonstrating that it agrees with the result obtained via the boson mapping method. We find that on time scales longer than Rouse time (time required for the single-line excitation of size L to relax elastically) or equivalently on length scales larger than the transverse line wandering length, the noninteracting (single flux line) dynamics is dominated by the center-of-mass mode with a k^2 relaxational rate. In this regime we find that the *interacting* structure function reduces to that of hydrodynamic frictional diffusion, consistent with the phenomenological model of Marchetti and Nelson.[18] The interactions between the lines additively renormalize the relaxational rate generating a crossover between the noninteracting and interacting dynamics. This crossover occurs at a flux-line length $L=L_I$, and is physically related to the entanglement length defined in Ref. 1. On wavelengths smaller than the transverse wandering length and for times shorter than the Rouse time we find that the noninteracting dynamics is controlled by the internal modes. In this regime we obtain a complicated wave-vector-dependent renormalization of the dynamics summarized by the interacting structure function. In Sec. VIII we take the phenomenological approach to the hydrodynamic description of the flux-line liquids and compare with the results of the kinetic approach derived in Sec. VII. Appendix A describes an independent derivation of the static structure function using the methods of auxiliary random fields. In Appendixes B and C we derive the nonlinear terms in the expansion of the hydrodynamic functional and hydrodynamic Hamiltonian, and analyze the single-line dynamics in various regimes.

II. DYNAMICAL MODEL

The statistical mechanics of flux-line liquids is very different from that of a liquid of point vortices because the lines are long and connected. Compared to these important topological properties the detailed internal structure of an individual flux line is relatively unimportant. The essential physics of the flux-line liquid can therefore be described by the conformation and position of each line. These configurations of vortex lines are characterized by a set of N functions $\vec{R}_i(z)=(\mathbf{r}_i(z),z)$, where $\mathbf{r}_i(z)$ specifies the position of the ith line in the (x,y) plane as it wanders along the direction of the applied magnetic field $\mathbf{H}\|\hat{z}$ through the sample of thickness L (see Fig. 2). The probability of an equilibrium configuration of N interacting lines in the presence of disorder is given by the Boltzmann weight $\exp(-\mathcal{H}/k_BT)$ with

$$\mathcal{H}=\frac{\epsilon}{2}\sum_{i=1}^{N}\int_0^L dz(\partial_z\mathbf{r}_i)^2+\frac{1}{2}\sum_{i\neq j=1}^{N}\int_0^L dz\, V[\mathbf{r}_i(z)-\mathbf{r}_j(z)] + \sum_{i=1}^{N}\int_0^L dz\, U(\mathbf{r}_i(z),z)\ . \tag{2.1}$$

The first term describes the elastic energy of N noninteracting lines with the line tension ϵ, which for isotropic superconductors is given in terms of the London penetration length λ and the ratio $\kappa=\lambda/\xi$ by $\epsilon=(\phi_0/4\pi\lambda)^2\ln\kappa$, with the flux quantum $\phi_0=hc/2e=2\times10^{-7}$ G cm^2 and ξ the superconducting coherence length and the vortex-line core thickness. Here we are working in the regime for which ϵ can be approximated by a constant, although with our formalism we can easily treat the case of a nonlocal elastic energy.

The anisotropic superconductors can be well described by an effective-mass tensor diagonal in the coordinate system with the z axis aligned with the $\hat{c}$ axis of the crystal

$$M_{nm}=\begin{pmatrix} M_1 & 0 & 0\\ 0 & M_1 & 0\\ 0 & 0 & M_3\end{pmatrix}\ . \tag{2.2}$$

For these anisotropic layered compounds the line tension is considerably smaller $\tilde{\epsilon}=\epsilon M_1/M_3$,[1] where M_1 is the in-plane anisotropic mass, and $M_3\approx10^2M_1$ is the much larger effective mass describing the weak Josephson coupling between the planes.[21] The above formula for $\tilde{\epsilon}$ applies when the flux lines are dense ($n_0\lambda_\perp^2\gg1$, where $\lambda_\perp$ is the in-plane London penetration depth, and n_0 the average density of vortex lines per unit area). In the opposite limit $n_0\lambda_\perp^2\ll1$ the electromagnetic coupling between the planes is important and one gets $\tilde{\epsilon}=\epsilon/\ln\kappa$.[22,12]

The second term in Eq. (2.1) incorporates the flux-line interactions. We treat the regime in which the line coordinates vary slowly with z, although with our formalism we can easily extend our treatment beyond this regime. This leads to the interaction energy which can be expressed in terms of a pair potential which is local in z, and in the London limit is given by

$$V(\mathbf{r})=\frac{\phi_0^2}{8\pi^2\lambda^2}[K_0(|\mathbf{r}|/\lambda)-K_0(|\mathbf{r}|/\xi)]\ , \tag{2.3}$$

where $K_0(x)$ is the modified Bessel function with the asymptotics,

$$K_0(x)\approx\begin{cases}(\pi/2x)^{1/2}e^{-x} & \text{for } x\to\infty\ ,\\ -\ln(x) & \text{for } x\to 0\ .\end{cases} \tag{2.4}$$

In Eq. (2.3) we have introduced a short-distance cutoff, the superconducting coherence length ξ.

The final term in Eq. (2.1) is the contribution of the pinning impurities to the free energy of the vortex line liquid. For simplicity we will take the quenched disorder interaction $U(\mathbf{r})$ to be Gaussian with zero mean,

$$\overline{U(\mathbf{r},z)}=0\ , \tag{2.5a}$$

$$\overline{U(\mathbf{r}_1,z_1)U(\mathbf{r}_2,z_2)}=F(\mathbf{r}_1-\mathbf{r}_2,z_1-z_2)\ , \tag{2.5b}$$

where $F(\mathbf{r}_1-\mathbf{r}_2,z_1-z_2)$ encodes the strength and range of disorder correlations. We will later specialize to point, line, and plane disorder. These three cases appear to be the most relevant experimentally as we discuss below, with oxygen vacancies and interstitials playing the role of point disorder, columnar defects and grain or twin boundaries as the line and plane disorders.

The model introduced above has been used, in somewhat more specialized form to describe the static features of the vortex liquid.[1] Nelson and co-workers used the boson mapping to compute the interacting static structure function in the dense phase where the density fluctuations are small and mean-field theory is a good description. Using renormalization group methods, they were also able to treat the dilute line liquid near H_{c1}, where the fluctuations are strong. By matching to the dense phase, where the mean-field theory is accurate they computed the fluctuation-corrected constitutive relation $B(H)$ and the static structure function near H_{c1}. This work was further extended to include the effects of point pinning disorder on the static structure function in this phase.[23]

Since we are seeking a hydrodynamic description of the flux-line dynamics we rewrite the interaction and disorder terms in a convenient form,

$$\begin{aligned}\sum_{i\neq j}^{N}\int_0^L dz\, V[\mathbf{r}_i(z,t)-\mathbf{r}_j(z,t)]&=\int_{\mathbf{r}_1,\mathbf{r}_2,z}V(\mathbf{r}_1-\mathbf{r}_2)\sum_{i\neq j}^{N}\delta^{(2)}[\mathbf{r}_1-\mathbf{r}_i(z,t)]\delta^{(2)}[\mathbf{r}_2-\mathbf{r}_j(z,t)]\ ,\\ &=\int_{\mathbf{r}_1,\mathbf{r}_2,z}V(\mathbf{r}_1-\mathbf{r}_2)n(\mathbf{r}_1,z,t)n(\mathbf{r}_2,z,t)-NLV(0)\ ,\end{aligned} \tag{2.6a}$$

$$\begin{aligned}\sum_{i=1}^{N}\int_0^L dz\, U(\mathbf{r}_i(z,t))&=\int d^2r\, dz\, U(\mathbf{r})\sum_{i=1}^{N}\delta^{(2)}[\mathbf{r}-\mathbf{r}_i(z,t)]\ ,\\ &=\int d^2r\, dz\, U(\mathbf{r})n(\mathbf{r},z,t)\ .\end{aligned} \tag{2.6b}$$

In above we defined $\int_{\mathbf{r},z}\equiv\int d^2r\,dz$. The self-energy term $NLV(0)$ in Eq. (2.6a), appropriately cutoff by ξ, can be absorbed into the line tension energy, and we will therefore ignore it in the following analysis. The form of Eqs. (2.6) suggests that interactions in the liquid phase are naturally described in terms of the line density

$$n(\mathbf{r},z,t)=\sum_{i=1}^{N}\delta^{(2)}[\mathbf{r}-\mathbf{r}_i(z,t)]\ .$$

Here we use the model defined by Eqs. (2.1) and (2.5) to study the dynamics of the flux-line liquid. We take the kinetic theory approach and write down the microscopic dynamic equations for each of the interacting lines. We assume that at this basic level the line fluctuations are overdamped and are therefore governed simply by Model-A-type Langevin dynamics[24]

$$\partial_t\mathbf{r}_i(z,t)=-D\frac{\delta\mathcal{H}}{\delta\mathbf{r}_i(z,t)}+\zeta_i(z,t)\ . \tag{2.7}$$

For simplicity we assume that the noise ζ_i is Gaussian with zero average, and covariance

$$\begin{aligned}\langle\zeta_i^a(z,t)\zeta_j^b(z',t')\rangle=&\,2Dk_BT\delta_{ij}\delta_{ab}\\ &\times\delta(t-t')\delta(z-z'),\quad a,b=1,2\ .\end{aligned} \tag{2.8}$$

The parameter D is the microscopic kinetic coefficient proportional to the inverse of the Bardeen-Stephen friction coefficient,[19]

$$\gamma_0=\frac{n_0\pi\hbar^2}{2e^2\xi_\perp^2}\sigma_n\ , \tag{2.9}$$

with $\xi_\perp$ as the superconducting coherence length in the copper-oxide planes and σ_n is the normal-state conductivity. D describes the effective drag on a flux line due to the interactions of the normal core electrons with the underlying solid. In the simplest case of center-of-mass-dominated diffusion of noninteracting flux lines D is proportional to the macroscopic diffusion coefficient (see Sec. VII). The above dynamics of flux lines is quite similar to the dynamics of polymer melts, with the important difference that for flux lines there is no solvent. Since the vortex lines are not moving in any fluid solution, Rouse rather than Zimm dynamics[25] applies, and D is a constant. However, we expect significant changes in the hydrodynamic parameters of the line liquid coming from the pinning disorder and flux-line entanglement, when there are significant barriers to line crossing.

In Eq. (2.7) we have for simplicity ignored the component of the line drag normal to the flux velocity $\mathbf{v}$, which can be accounted for by adding $\Delta\mathbf{z}\times\partial_t\mathbf{r}_i(z,t)$ to the left-hand side of Eq. (2.7). This term determines the Hall angle θ_H, according to $\tan\theta_H=\Delta$,[19] and is generally quite small.

III. MARTIN-SIGGIA-ROSE FORMULATION OF THE MODEL

We would like to derive a hydrodynamic description of the line liquid in terms of coarse-grained variables like the density of the flux lines, starting with the microscopic model presented in Sec. II. This procedure is most conveniently implemented using Martin-Siggia-Rose (MSR) formalism[26,27] which allows the solution to the Langevin equation to be formulated in terms of a constrained path integral.

The idea is that instead of solving the Langevin equation for the conformation variables $\mathbf{r}_i(z,t)$ in terms of the random forces $\zeta_i(z,t)$ and then computing the correlations functions by averaging over the noise with the Gaussian weight

$$\omega[\zeta_i(z,t)] \sim \exp\left[-\frac{1}{4Dk_BT}\int dt \int dz |\zeta_i(z,t)|^2\right], \quad (3.1)$$

one can consider $\mathbf{r}_i(z,t)$ as the basic stochastic field with a path probability density $W[\mathbf{r}_i(z,t)]$ defined by

$$W[\mathbf{r}_i(z,t)]\mathcal{D}\mathbf{r}_i(z,t) = \omega[\zeta_i(z,t)]\mathcal{D}\zeta_i(z,t) \quad (3.2)$$

and eliminate the random forces in favor of the conformation variables. This is accomplished via a constrained path integral over the noise with the Langevin equation as the constraint.[26,28,29]

To implement the procedure of MSR we note that the noise average of any observable $\mathcal{O}[\mathbf{r}_i(z,t)]$, with flux-line conformational variables $\mathbf{r}_i(z,t)$, as the solution of the Eq. (2.7), can be expressed in terms of a constrained path integral,

$$\langle \mathcal{O}(\mathbf{r}_i)\rangle = \int \mathcal{D}\zeta_i(z,t)\mathcal{D}\mathbf{r}_i(z,t) J[\mathbf{r}_i] \prod_{a,i,z,t} \delta\left[\partial_t r_i^a(z,t) + D\frac{\delta\mathcal{H}}{\delta r_i^a(z,t)} - \zeta_i^a(z,t)\right]\mathcal{O}(\mathbf{r}_i)\exp\left[-\frac{1}{4Dk_BT}\int dz\, dt |\zeta_i(z,t)|^2\right]. \quad (3.3)$$

Here $\int \mathcal{D}\zeta_i(z,t)\mathcal{D}\mathbf{r}_i(z,t)$ denotes a path integral over the noise and the conformation of the flux lines with the implied discretization of z and t to define the path integral. The quantity $J[\mathbf{r}_i]$ is the Jacobian of the transformation from $\zeta_i(z,t)$ to $\mathbf{r}_i(z,t)$ imposed by the functional δ function. It ensures that the path probability density $W[\mathbf{r}_i(z,t)]$, with which the averages are computed, is still normalized to 1, i.e., $\langle 1 \rangle = 1$.

We eliminate the functional δ function by performing the integral over $\zeta_i(z,t)$ and obtain

$$\langle \mathcal{O}(\mathbf{r}_i)\rangle = \int \mathcal{D}\mathbf{r}_i(z,t) J[\mathbf{r}_i]\mathcal{O}(\mathbf{r}_i)\exp\left[-\frac{1}{4Dk_BT}\int dz\, dt \left[\partial_t r_i^a(z,t) + D\frac{\delta\mathcal{H}}{\delta r_i^a(z,t)}\right]^2\right]. \quad (3.4)$$

Further, it is convenient to perform a Gaussian transformation in order to "linearize" the argument in the exponential, the dynamic functional. This is accomplished by introducing auxiliary fields $\tilde{\mathbf{r}}_i(z,t)$, usually called the response fields due their utility in computation of response functions as will become clear below (see Appendix C).

$$\langle \mathcal{O}(\mathbf{r}_i)\rangle = \int \mathcal{D}\tilde{\mathbf{r}}_i(z,t) \int \mathcal{D}\mathbf{r}_i(z,t) J[\mathbf{r}_i]\mathcal{O}(\mathbf{r}_i)\exp(-\mathcal{J}_{id}[\tilde{\mathbf{r}}_i,\mathbf{r}_i]), \quad (3.5)$$

where $\mathcal{J}_{id}[\tilde{\mathbf{r}}_i,\mathbf{r}_i]$ is the dynamic functional for a particular realization of disorder,

$$\mathcal{J}_{id}[\tilde{\mathbf{r}}_i,\mathbf{r}_i] = \int dt \int_0^L dz \left[\tilde{r}_i^a(z,t) Dk_BT\, \tilde{r}_i^a(z,t) + i\, \tilde{r}_i^a(z,t)\left[\partial_t r_i^a(z,t) + D\frac{\delta\mathcal{H}}{\delta r_i^a(z,t)}\right]\right]. \quad (3.6)$$

It is convenient to work in Fourier representation. We reserve q for the wave vector in the z direction, $\mathbf{k}$ for the transverse wave vector, and ω for the frequency variable (see below). Since the flux-line ends are freely fluctuating, these boundary conditions are naturally satisfied by the discrete cosine-Fourier transform with $\tilde{\mathbf{r}}_i(z,t)$ and $\mathbf{r}_i(z,t)$ given by

$$\begin{aligned}\mathbf{r}(z,t) &= \int \frac{d\omega}{2\pi}\left[\mathbf{r}_0(\omega) + 2\sum_{q_n>0}\cos(q_n z)\,\mathbf{r}(q_n,\omega)\right]e^{i\omega t} \\ &= \int_{q_n,\omega} e^{i\omega t}\cos(q_n z) 2^{(\delta_{n,0}-1)}\mathbf{r}(q_n,\omega),\end{aligned} \quad (3.7)$$

where $q_n = n\pi/L$ and we have been careful to separately treat the center-of-mass, $q_n=0$, mode. This separation will turn out to be essential in order to recover the correct hydrodynamic result at long wavelengths (see Appendix C).

To examine the explicit form of $\mathcal{J}_{id}[\tilde{\mathbf{r}}_i,\mathbf{r}_i]$, we split it up into three contributions corresponding to the three terms of the Hamiltonian in Eq. (2.1),

$$\mathcal{J}_{id}[\tilde{\mathbf{r}}_i,\mathbf{r}_i] = \mathcal{J}_0[\tilde{\mathbf{r}}_i,\mathbf{r}_i] + \mathcal{J}_i[\tilde{\mathbf{r}}_i,\mathbf{r}_i] + \mathcal{J}_d[\tilde{\mathbf{r}}_i,\mathbf{r}_i]. \quad (3.8)$$

(i) $\mathcal{A}_0[\tilde{\mathbf{r}}_i,\mathbf{r}_i]$ is the dynamic functional for the noninteracting flux lines,

$$\mathcal{A}_0[\tilde{\mathbf{r}}_i,\mathbf{r}_i]=L\int_{q_n,\omega}\sum_{a,i}[\tilde{r}_i^a(q_n,\omega)Dk_BT\,\tilde{r}_i^a(q_n,-\omega)+\tilde{r}_i^a(q_n,\omega)(\omega+iD\epsilon q_n^2)r_i^a(q_n,-\omega)]2^{\delta_{n,0}-1}$$
$$=\frac{1}{2}\int \mathbf{R}_i(q_n,\omega)\cdot\overleftrightarrow{G}(q_n,\omega)\cdot\mathbf{R}_i^T(q_n,-\omega)\ , \tag{3.9}$$

where

$$\int_{q_n,\mathbf{k},\omega}\equiv\sum_n\int d^2k/(2\pi)^2\int d\omega/(2\pi)\ ,$$

and we defined,

$$\mathbf{R}_i(q_n,\omega)=[\tilde{\mathbf{r}}_i(q_n,\omega),\mathbf{r}_i(q_n,\omega)]\ , \tag{3.10a}$$

$$\overleftrightarrow{G}(q_n,\omega)=L2^{\delta_{n,0}-1}\begin{bmatrix}2Dk_BT & \omega+iD\epsilon q_n^2\\ -\omega+iD\epsilon q_n^2 & 0\end{bmatrix}. \tag{3.10b}$$

(ii) $\mathcal{A}_i[\tilde{\mathbf{r}}_i,\mathbf{r}_i]$ is the contribution to the total dynamic functional due to the interaction between flux lines,

$$\mathcal{A}_i[\tilde{\mathbf{r}}_i,\mathbf{r}_i]=-D\sum_{i=1}^{N}\sum_{j\neq i}\int dt\int_0^L dz\int_k(\mathbf{k}\cdot\tilde{\mathbf{r}}_i)V(\mathbf{k})e^{i\mathbf{k}\cdot(\mathbf{r}_i-\mathbf{r}_j)}\ , \tag{3.11}$$

where $V(\mathbf{k})$ is the Fourier transformed flux-line interaction, Eq. (2.3)

$$V(\mathbf{k})=\int d^2rV(\mathbf{r})e^{-i\mathbf{k}\cdot\mathbf{r}} \tag{3.12a}$$
$$=\frac{\phi_0^2}{8\pi^2\lambda^2}\frac{1}{k^2+\lambda^{-2}}\ . \tag{3.12b}$$

(iii) $\mathcal{A}_d[\tilde{\mathbf{r}}_i,\mathbf{r}_i]$ is the disorder contribution to the total dynamic functional, in which we again transformed to Fourier $\mathbf{k}$ space,

$$\mathcal{A}_d[\tilde{\mathbf{r}}_i,\mathbf{r}_i]=-D\sum_{i=1}^{N}\int dt\int_0^L dz\int_k(\mathbf{k}\cdot\tilde{\mathbf{r}}_i)U(\mathbf{k},z)e^{i\mathbf{k}\cdot\mathbf{r}_i}\ . \tag{3.13}$$

The Jacobian function $J[\mathbf{r}_i]$ in Eq. (3.3) depends on the discretization scheme of the path integral. It is simplest to adopt the causal discretization procedure in which the Jacobian is a constant, independent of $\mathbf{r}_i$ and can be omitted, provided that simultaneously the ambiguous equal-time correlator is defined to vanish, (see Refs. 26–29 for the details).

$$\langle\tilde{r}_i^a(z,t)r_j^b(z',t)\rangle=0\ . \tag{3.14}$$

The averages of physical observables as well as the correlation functions of these observables can now be expressed in terms of functional derivatives of the generating functional with respect to external fields that couple to these observables. For example, n-point correlation function of an observable $\mathcal{O}(\mathbf{r}_i,\tilde{\mathbf{r}}_i)$ can be obtained from the generating functional $\mathcal{Z}[h(z,t)]$,

$$\mathcal{Z}[h(z,t)]=\int\mathcal{D}\tilde{\mathbf{r}}_i(z,t)\int\mathcal{D}\mathbf{r}_i(z,t)\exp\left[-\mathcal{A}_{id}[\tilde{\mathbf{r}}_i,\mathbf{r}_i]+\int dt\int_0^L dz\,\mathcal{O}[\mathbf{r}_i(z,t),\tilde{\mathbf{r}}_i(z,t)]h(z,t)\right]\ , \tag{3.15}$$

by functionally differentiating n times with respect to $h(z,t)$,

$$\langle\mathcal{O}[\mathbf{r}_i(z_1,t_1),\tilde{\mathbf{r}}_i(z_1,t_1)]\cdots\mathcal{O}[\mathbf{r}_i(z_n,t_n),\tilde{\mathbf{r}}_i(z_n,t_n)]\rangle=\left.\frac{\delta^n\mathcal{Z}[h(z,t)]}{\delta h(z_1,t_1)\cdots\delta h(z_n,t_n)}\right|_{\{h(z_i,t_i)\}=0}\ . \tag{3.16}$$

Hence, all the information about the dynamics of the flux-line liquid is encoded in the generating functional which we will study in the hydrodynamic limit, below.

IV. DERIVATION OF HYDRODYNAMIC DESCRIPTION

In neutron-scattering experiments, the neutrons interact with the magnetic field of the flux lines. The resulting scattering intensity is therefore proportional to the Fourier transform of the two-point correlation function of the magnetic field.[30,31] The London equation relates the Fourier transform of the magnetic-field components, along and perpendicular to $\hat{\mathbf{z}}$, to the flux-line number and tangent density $n(k,q,t)$, $\tau(k,q,t)$, respectively,[23,18]

$$B_z(\mathbf{k},q,t)=\frac{\phi_0}{1+\lambda_\perp^2M_1(\mathbf{k}^2+q^2)}n(\mathbf{k},q,t)\ , \tag{4.1a}$$

$$B_{\perp a}(\mathbf{k},q,t)=\frac{\phi_0}{1+\lambda_\perp^2(M_3\mathbf{k}^2+M_1q^2)}P_{ab}^T(\mathbf{k})\tau_b(\mathbf{k},q,t)+\frac{\phi_0}{1+\lambda_\perp^2M_1(\mathbf{k}^2+q^2)}P_{ab}^L(\mathbf{k})\tau_b(\mathbf{k},q,t)\ , \tag{4.1b}$$

where $P_{ab}^T(\mathbf{k})=\delta_{ab}-k_ak_b/k^2$, $P_{ab}^L(\mathbf{k})=k_ak_b/k^2$, and with the densities related to the flux-line position,

$$n(\mathbf{r},z,t)=\sum_{i=1}^{N}\delta^{(2)}[\mathbf{r}-\mathbf{r}_i(z,t)]\ , \tag{4.2a}$$

$$\tau(\mathbf{r},z,t)=\sum_{i=1}^{N}\frac{\partial\mathbf{r}_i}{\partial z}\delta^{(2)}[\mathbf{r}-\mathbf{r}_i(z,t)]\ . \tag{4.2b}$$

Hence, inelastic neutron-scattering experiments are a direct probe of the dynamic structure function of interacting flux lines, i.e., of the flux-line density-density correlation function.

The transport coefficients like flux-line liquid viscosity and friction can also be extracted from the dynamic structure function of these line liquids, and hence can be compared with the experiments that measure resistance.[18] The interacting dynamic structure function also enters into the perturbative calculation of disorder corrected flux-flow velocity in the high velocity regime. It has been argued that this relation holds beyond its expected regime of validity, down to low flow velocities. The dynamic structure function is therefore directly related to the current-voltage curves.[33,32]

Because it is the flux-line density $n(\mathbf{r},z,t)$ rather than the microscopic conformation field $\mathbf{r}_i(z,t)$ that is measured in most of the experiments, it is more natural to work with the hydrodynamic density variables. By starting with the microscopic theory defined by $\mathcal{J}_{id}[\tilde{\mathbf{r}}_i,\mathbf{r}_i]$, Eqs. (3.8)–(3.13), and integrating out the microscopic conformational degrees of freedom $\mathbf{r}_i(z,t)$ and $\tilde{\mathbf{r}}_i(z,t)$, with the constraint that the densities $n(\mathbf{r},z,t)$ and corresponding response field $\tilde{n}(\mathbf{r},z,t)$ remain fixed, we derive the dynamic functional of the density fields. With this form of the dynamic theory the density correlation and response functions are easily computable and the approximations that are necessary can be more easily physically motivated, because the density and its correlations are directly observable.

We begin by introducing a density response field $\tilde{n}(\mathbf{r},z,t)$,

$$\tilde{n}(\mathbf{k},z,t)=k_BTD\sum_{i=1}^{N}[\mathbf{k}\cdot\tilde{\mathbf{r}}_i(z,t)]e^{-i\mathbf{k}\cdot\mathbf{r}_i(z,t)}, \tag{4.3}$$

in addition to the physical density field $n(\mathbf{r},z,t)$ already introduced in Eq. (4.2a). The response field $\tilde{n}(\mathbf{r},z,t)$ will earn its name by generating dynamic density response functions (see below and Appendix C). Here we will only study correlation and response functions of the number density field and therefore will trace over the tangent density field $\tau(\mathbf{r},z,t)$ that is related to the fluctuations of the magnetic field in the ab plane, Eq. (4.2b).

We reexpress the interaction parts of the dynamic functional, Eqs. (3.11) and (3.13), in terms of the density fields,

$$\mathcal{J}_i[\tilde{\mathbf{r}}_i,\mathbf{r}_i]=iD\int_{t,z}\int_{\mathbf{r},\mathbf{r}'}\sum_{i=1}^{N}\delta^{(2)}[\mathbf{r}-\mathbf{r}_i(z,t)]\sum_{j=1}^{N}\tilde{\mathbf{r}}_j\cdot\frac{\partial}{\partial\mathbf{r}_j}\delta^{(2)}[\mathbf{r}'-\mathbf{r}_j(z,t)]V(\mathbf{r}-\mathbf{r}'), \tag{4.4a}$$

$$=\frac{1}{k_BT}\int_{t,z}\int_{\mathbf{r}}\int_{\mathbf{r}'}n(\mathbf{r},z,t)\tilde{n}(\mathbf{r}',z,t)V(\mathbf{r}-\mathbf{r}'), \tag{4.4b}$$

and the disorder contribution,

$$\mathcal{J}_d[\tilde{\mathbf{r}}_i,\mathbf{r}_i]=iD\int_{t,z,\mathbf{r}}\sum_{i=1}^{N}\tilde{\mathbf{r}}_i\cdot\frac{\partial}{\partial\mathbf{r}_i}\delta^{(2)}[\mathbf{r}-\mathbf{r}_i(z,t)]U(\mathbf{r},z), \tag{4.5a}$$

$$=\frac{1}{k_BT}\int_{t,z,\mathbf{r}}\tilde{n}(\mathbf{r},z,t)U(\mathbf{r},z). \tag{4.5b}$$

It is important to remember that in the above equations $\tilde{n}(\mathbf{r},z,t)$ and $n(\mathbf{r},z,t)$ are not fields independent of the fundamental microscopic fields $\tilde{\mathbf{r}}_i(z,t)$ and $\mathbf{r}_i(z,t)$ and at this point are only a notational convenience.

Although the mean-field theory of the flux-line liquid hydrodynamics is described by the average densities $n_0=\langle n(\mathbf{r},z,t)\rangle$ and $\tilde{n}_0=\langle\tilde{n}(\mathbf{r},z,t)\rangle$, it is the fluctuations about the mean density that are nontrivial and are probed by the scattering experiments. We therefore introduce fields,

$$\rho(\mathbf{r},z,t)=n(\mathbf{r},z,t)-n_0, \tag{4.6a}$$

$$\tilde{\rho}(\mathbf{r},z,t)=\tilde{n}(\mathbf{r},z,t)-\tilde{n}_0, \tag{4.6b}$$

describing the deviations from the average uniform densities $n_0=\langle n(\mathbf{r},z,t)\rangle$ and $\tilde{n}_0=\langle\tilde{n}(\mathbf{r},z,t)\rangle$.

It is convenient to work in the canonical ensemble in which the number of flux lines N is fixed (i.e., total magnetic field through the sample is constant), and for fixed sample area A, $n_0=N/A$. Substituting Eqs. (4.6a) and (4.6b) into Eqs. (4.4) and (4.5), using the conservation of flux-line identities,

$$\int_{\mathbf{r}}\rho(\mathbf{r},z,t)=\rho(\mathbf{k}=\mathbf{0},z,t)=0, \tag{4.7a}$$

$$\int_{\mathbf{r}}\tilde{\rho}(\mathbf{r},z,t)=\tilde{\rho}(\mathbf{k}=\mathbf{0},z,t)=0, \tag{4.7b}$$

and ignoring the remaining constant terms that do not effect the dynamics, we obtain the interaction part of dynamic functional in Fourier space,

$$\mathcal{J}_i[\tilde{\mathbf{r}}_i,\mathbf{r}_i]=\frac{1}{k_BT}\int_{\mathbf{k},q,\omega}\tilde{\rho}(\mathbf{k},q,\omega)V(\mathbf{k})\rho(-\mathbf{k},-q,-\omega) \tag{4.8a}$$

$$=\frac{1}{2k_BT}\int_{\mathbf{k},q,\omega}\rho_\alpha(\mathbf{k},q,\omega)\times V_{\alpha\beta}(\mathbf{k})\rho_\beta(-\mathbf{k},-q,-\omega), \tag{4.8b}$$

where we define,

$$\boldsymbol{\rho}=(\tilde{\rho},\rho), \tag{4.9a}$$

$$\overleftrightarrow{V}=\begin{bmatrix}0 & V(\mathbf{k})\\ V(\mathbf{k}) & 0\end{bmatrix}. \tag{4.9b}$$

For the disorder contribution to the dynamic functional we obtain,

$$\mathcal{A}_d[\tilde{\mathbf{r}}_i,\mathbf{r}_i]=\frac{1}{k_BT}\int_{\mathbf{k},q,\omega}\tilde{\rho}(\mathbf{k},q,\omega)U(-\mathbf{k},-q)\delta(\omega) \tag{4.10a}$$

$$=\frac{1}{k_BT}\int_{\mathbf{k},q,\omega}\rho_\alpha(\mathbf{k},q,\omega)u_\alpha(-\mathbf{k},-q,-\omega)\ , \tag{4.10b}$$

where we defined disorder vector,

$$\mathbf{u}(\mathbf{k},q,\omega)=[U(\mathbf{k},q,\omega)\delta(\omega),\,0]\ . \tag{4.11}$$

As discussed in Sec. III, we construct the dynamic generating functional for the computation of density correlation and response function, by choosing the operator $\mathcal{O}_\alpha(\tilde{\mathbf{r}}_i,\mathbf{r}_i)=\rho_\alpha(\mathbf{r},z,t)$ and coupling an external field $h_\alpha(\mathbf{r},z,t)$ to this density observable,

$$\mathcal{Z}_d[\mathbf{h}(\mathbf{r},z,t)]=\int\mathcal{D}\mathbf{R}\exp\left[-\mathcal{A}_{id}[\mathbf{R}(z,t)]+\int_{\mathbf{r},z,t}\rho(\mathbf{r},z,t)\cdot\mathbf{h}(\mathbf{r},z,t)\right]. \tag{4.12}$$

In the dynamic functional, expressed in terms of densities, the disorder field, that couples linearly to the density field, can be integrated out exactly. It is important to note that in the dynamic formulation presented here we can integrate over the quenched disorder directly at the level of the dynamic generating functional $\mathcal{Z}_d[h]$. This is to be contrasted with static calculations where one must first compute the physical observables such as free energy or correlation functions for particular realization of disorder and then perform the average over the disorder (see Appendix A). This leads to the usual problems of disorder averaging $\ln Z_d$ or Z_d^{-1} (since the static averages have to be normalized by the partition function Z_d), problem usually handled using the "replica trick." In the dynamic calculations presented here no such complications arise because the "dynamic partition function" $\mathcal{Z}_d[h=0]=1$, a constraint enforced by the MSR Jacobian in Eq. (3.3). We then integrate out the quenched disorder exactly, using the assumption of Gaussian disorder, Eq. (2.5). Upon averaging $\mathcal{Z}_d[h]$ over the quenched pinning potential $\mathbf{u}$, we obtain

$$\mathcal{Z}[\mathbf{h}(\mathbf{r},z,t)]=\overline{\mathcal{Z}_d[\mathbf{h}]}\ , \tag{4.13a}$$

$$=\int\mathcal{D}\mathbf{R}\exp\left[-\mathcal{A}[\mathbf{R}(z,t)]+\int_{\mathbf{r},z,t}\rho(\mathbf{r},z,t)\cdot\mathbf{h}(\mathbf{r},q,t)\right], \tag{4.13b}$$

where,

$$\mathcal{A}[\mathbf{R}(z,t)]=\mathcal{A}_0[\mathbf{R}]+\mathcal{A}_{\rm int}[\boldsymbol{\rho}]\ , \tag{4.14}$$

and

$$\mathcal{A}_{\rm int}[\boldsymbol{\rho}]=\frac{1}{2k_BT}\int_{\mathbf{k},q,\omega}\rho_\alpha(\mathbf{k},q,\omega)K_{\alpha\beta}(\mathbf{k},q,\omega)\times\rho_\beta(-\mathbf{k},-q,-\omega)\ , \tag{4.15}$$

with

$$\overleftrightarrow{K}(\mathbf{k},q,\omega)=\begin{bmatrix}-F(\mathbf{k},q)\delta(\omega)/k_BT & V(\mathbf{k})\\ V(\mathbf{k}) & 0\end{bmatrix}. \tag{4.16}$$

The interaction and disorder contributions have now been expressed as a quadratic function of the density fields ρ_α. However, the computation of the dynamic generating function $\mathcal{Z}[\mathbf{h}]$ is still nontrivial because (i) the independent variables are $\mathbf{R}$ and not ρ_α, and (ii) the noninteracting part of the dynamic functional $\mathcal{A}_0[\mathbf{R}]$ cannot be trivially rewritten in terms of the density fields. To make progress however we can proceed with an uncontrolled variational approximation. We replace $\mathcal{A}_0[\mathbf{R}]$ by an ansatz that is Gaussian in the density fields

$$\mathcal{A}_0[\mathbf{R}]\to\mathcal{A}_0[\rho_\alpha]=\frac{1}{2n_0}\int_{k,q,\omega}\rho_\alpha(\mathbf{k},q,\omega)\Gamma^{-1}_{\alpha\beta}(\mathbf{k},q,\omega)\times\rho_\beta(-\mathbf{k},-q,-\omega)\ , \tag{4.17}$$

and use a measure $\mathcal{D}\rho_\alpha$ instead of $\mathcal{D}\mathbf{R}$ in the functional integral. The matrix $\Gamma_{\alpha\beta}$ is determined by requiring that the correlation functions of the Gaussian ansatz agree with the original noninteracting theory where the averages are performed with measure $\mathcal{D}\mathbf{R}\exp(-\mathcal{A}_0[\mathbf{R}])$. The simplest requirement is that the structure functions in two theories agree. This completely fixes $\Gamma_{\alpha\beta}=S^0_{\alpha\beta}$, where $S^0_{\alpha\beta}$ is the structure function for a single flux line. Combining with $\mathcal{A}_{\rm int}$ this approximation then gives the full dynamic functional,

$$\mathcal{A}[\rho_\alpha]=\frac{1}{2k_BTn_0}\int_{\mathbf{k},q,\omega}\rho_\alpha(\mathbf{k},q,\omega)[n_0K_{\alpha\beta}(\mathbf{k},q,\omega)+k_BTS^{0\ -1}_{\alpha\beta}(\mathbf{k},q,\omega)]\rho_\beta(-\mathbf{k},-q,-\omega)\ , \tag{4.18}$$

from which all the dynamic correlation and response functions of density fields can be computed.

In the next section we will treat the full dynamic functional more rigorously. We will show how the above *ad hoc* approximation emerges as a result of truncation of a systematic expansion of the dynamic functional in the density fields.

V. DECOUPLING FLUX-LINE DYNAMICS (METHOD OF AUXILIARY RANDOM FIELDS)

As we have already noted the density fields $\tilde{\rho}(\mathbf{r},z,t)$ and $\rho(\mathbf{r},z,t)$ are not independent of the fundamental microscopic fields $\tilde{\mathbf{r}}_i(z,t)$ and $\mathbf{r}_i(z,t)$ and at this point are only a notational convenience. The computation of the generating function in the above equation is made nontrivial by the highly nonlinear flux-line interactions $V(\mathbf{r})$, when expressed in terms of the kinetic fields $\tilde{\mathbf{r}}_i(z,t)$ and $\mathbf{r}_i(z,t)$. To obtain interaction corrections to the noninteracting lines dynamics via a straight perturbation theory in powers of $V(\mathbf{r})$ is possible but is a nontrivial exercise.[34] We note however that this interaction is simply quadratic when expressed in terms of the density fields. The idea is then to transform the functional integral from the microscopic variables to independent density fields. This can be accomplished using the method of auxiliary random fields that has been previously applied to treat both statics and dynamics of polymers and is described below.[35]

We introduce a set of transformations that transform the functional of $\mathbf{R}(\mathbf{r},z,t)$ into a functional for the density fields. We accomplish this by introducing an independent auxiliary density field $\boldsymbol{\psi}(\mathbf{r},z,t)$ constrained by the functional δ-function to equal the physical density fields $\boldsymbol{\rho}(\mathbf{r},z,t)$ for all $\mathbf{r},z,t$,

$$\mathcal{Z}[h(\mathbf{r},z,t)]=\int \mathcal{D}\mathbf{R}\, e^{-\mathcal{J}_0[\mathbf{R}]-\mathcal{J}_{\rm int}[\rho]+\int \rho\cdot\mathbf{h}} \times\int \mathcal{D}\psi \prod_{\mathbf{r},z,t}\delta[\boldsymbol{\psi}(\mathbf{r},z,t)-\boldsymbol{\rho}(\mathbf{r},z,t)] \quad (5.1a)$$

$$=\int \mathcal{D}\boldsymbol{\psi} e^{-\mathcal{J}_{\rm int}[\psi]+\int \psi\cdot\mathbf{h}}\langle \delta(\boldsymbol{\psi}-\boldsymbol{\rho})\rangle_0\ , \quad (5.1b)$$

where the average is performed with the Gaussian dynamic measure $e^{-\mathcal{J}_0[\mathbf{R}]}$ of N noninteracting flux lines, with $\mathcal{J}_0[\mathbf{R}]$ given by Eq. (3.9). We observe that with this transformation the troublesome interaction is indeed quadratic in $\boldsymbol{\psi}(\mathbf{r},z,t)$ and therefore in this sense can be treated nonperturbatively. The nontrivial part of the calculation reduces to the computation of the average of the functional δ function with the Gaussian measure $e^{-\mathcal{J}_0[\mathbf{R}]}$.

Using the functional representation of the δ function we obtain,

$$\langle \delta(\boldsymbol{\psi}-\boldsymbol{\rho})\rangle_0=\left\langle \int \mathcal{D}\boldsymbol{\phi}\, e^{i\int \phi\cdot(\psi-\rho)}\right\rangle_0 \quad (5.2a)$$

$$=\int \mathcal{D}\boldsymbol{\phi}\, e^{i\int \phi\cdot\psi}\langle e^{-i\int \phi\cdot\rho}\rangle_0\ . \quad (5.2b)$$

Noting that $\boldsymbol{\rho}(\mathbf{r},z,t)=\sum_{i=1}^{N}\boldsymbol{\rho}_i(\mathbf{r},z,t)$, with $\boldsymbol{\rho}_i(\mathbf{r},z,t)$ as the single-line density, and since the average for each line is identical, Eq. (5.2) further reduces to,

$$\langle \delta(\boldsymbol{\psi}-\boldsymbol{\rho})\rangle_0=\int \mathcal{D}\boldsymbol{\phi}\, e^{i\int \phi\cdot\psi}[\langle e^{-i\int \phi\cdot\rho_1}\rangle_0]^N\ . \quad (5.3)$$

Combining this result with interaction contribution to the hydrodynamic functional in Eq. (5.1) we obtain the dynamic generating function

$$\mathcal{Z}[\mathbf{h}]=\int \mathcal{D}\boldsymbol{\psi}\int \mathcal{D}\boldsymbol{\phi}\exp\left[-\mathcal{J}[\boldsymbol{\psi},\boldsymbol{\phi}] +\int \boldsymbol{\psi}(\mathbf{r},z,t)\cdot\mathbf{h}(\mathbf{r},z,t)\right]\ , \quad (5.4)$$

in terms of the hydrodynamic functional $\mathcal{J}[\boldsymbol{\psi},\boldsymbol{\phi}]$,

$$\mathcal{J}[\boldsymbol{\psi},\boldsymbol{\phi}]=\mathcal{J}_{\rm int}[\boldsymbol{\psi}]+n_0\Gamma[\boldsymbol{\phi}]-i\int \boldsymbol{\phi}\cdot\boldsymbol{\psi}\ , \quad (5.5)$$

with the contribution $n_0\Gamma[\boldsymbol{\phi}]$ coming from the average of the δ function in Eq. (5.3) and is expressed in terms of a single flux-line cumulant expansion,

$$-\Gamma[\boldsymbol{\phi}]=\frac{(-i)^2}{2!}\int \phi_\alpha\phi_\beta\Gamma^{(2)}_{\alpha\beta}+\frac{(-i)^3}{3!}\int \phi_\alpha\phi_\beta\phi_\gamma\Gamma^{(3)}_{\alpha\beta\gamma} +\frac{(-i)^4}{4!}\int \phi_\alpha\phi_\beta\phi_\gamma\phi_\delta\Gamma^{(4)}_{\alpha\beta\gamma\delta}+\cdots\ , \quad (5.6)$$

where $\Gamma^{(m)}_{\alpha_1\alpha_2\cdots\alpha_m}$ is a connected correlation function of a single-line density fluctuations $\boldsymbol{\rho}_1[\mathbf{R}_1]$, and in coordinate space $x=(\mathbf{r},z,t)$ is

$$\Gamma^{(m)}_{\alpha_1\alpha_2\cdots\alpha_m}(x_1,x_2,\ldots,x_m) =A\langle \rho_{\alpha_1}(x_1)\rho_{\alpha_2}(x_2)\cdots\rho_{\alpha_m}(x_m)\rangle_0^c\ , \quad (5.7)$$

where the area factor A was inserted for convenience. In the above we have also dropped the single-line label 1 on $\boldsymbol{\rho}_1$, since for a while, we will be dealing with a single-line density and no confusion should arise. The cumulant functions $\Gamma^{(m)}_{\alpha_1\alpha_2\cdots\alpha_m}(x_1,x_2,\ldots,x_m)$ can be systematically computed for any m (see Appendixes B and C).

We note that the density field $\boldsymbol{\psi}$ appears only linearly and quadratically in Eq. (5.4) and therefore can be integrated out exactly. This leads to the dynamic functional that is expressed not in terms of the physical densities $\boldsymbol{\psi}$ but in terms of the auxiliary fields $\boldsymbol{\phi}$, conjugate to $\boldsymbol{\psi}$. Since the hydrodynamic functional is derived in the presence of sources $\mathbf{h}$ the density-correlation functions and response functions of $\boldsymbol{\psi}$ can still be easily computed as functional derivatives with respect to these sources [see Eq. (3.16)]. Upon integrating out $\boldsymbol{\psi}$ we obtain,

$$\mathcal{Z}[\mathbf{h}]=\int \mathcal{D}\boldsymbol{\phi}\, e^{-\mathcal{J}[\boldsymbol{\phi},\mathbf{h}]}\ , \quad (5.8)$$

where

$$\mathcal{J}[\boldsymbol{\phi},\mathbf{h}]=n_0\Gamma[\boldsymbol{\phi}]-\frac{k_BT}{2}\int (i\phi_\alpha+h_\alpha)K^{-1}_{\alpha\beta}(i\phi_\beta+h_\beta)\ . \quad (5.9)$$

The above equations define the hydrodynamic field theory of flux lines derived directly from the microscopic interacting single-line dynamics. No approximations have been made up to now, and both the flux-line interactions and interaction with disorder have been treated exactly. To fully solve the hydrodynamic theory we need to compute the generating function $\mathcal{Z}[\mathbf{h}]$ by integrating over $\boldsymbol{\phi}$. Since the resulting theory has a form of an interacting field theory, it cannot be solved exactly. However, one could try to treat the nonlinear interactions perturbatively with various techniques. In the long-wavelength limit (which can, in principle, be studied with sufficiently low angle neutron-scattering experiments) the $\boldsymbol{\phi}$ fluctuations can be treated with renormalization-group methods as was done for the statics of polymer solutions by Ohta and Nakanishi.[35] This procedure would lead to renormalized vertex functions $\Gamma^{(m)}$ corrected by the thermal fluctuations of $\boldsymbol{\phi}$. As will be shown in Sec. VI, some of the flux-line fluctuations and interactions are nontrivially taken into account even if we truncate the hydrodynamic functional at the quadratic order. The resulting truncated theory can be solved exactly, allowing calculations of dynamic and static properties of the interacting flux-line liquid.

VI. GAUSSIAN APPROXIMATION TO THE DYNAMIC FUNCTIONAL

In Sec. V we derived the expression for the dynamic generating functional. It is expressed in terms of an infinite series of interactions in the auxiliary field $\boldsymbol{\phi}$ and, in principle, allows for calculation of any hydrodynamic correlation function of the flux-line liquid. Near H_{c1}, where the magnetic field is weak and the flux-line liquid is dilute, the fluctuations in the density fields are large, i.e., comparable to the average line density. In this critical region the nonlinear interactions must be carefully taken into account. Using renormalization-group methods the dynamics of dilute line liquids can be studied, as was done for the statics.[1] Similarly, near H_{c2}, where the flux-line liquid is dense, the vortex cores begin to overlap, the effects of nonlinear interactions are large and renormalization-group treatment is again needed to take the nonlinearities into account.[36] Here, however, we study the dynamics of a semidilute flux-line liquid, away from such critical regions, i.e., in the regime where $H_{c1} \ll H \ll H_{c2}$. In this regime n_0 is large, hence the method of steepest descents applied to Eqs. (5.8) and (5.9) allows us to treat $\mathcal{Z}[\mathbf{h}]$ in mean-field theory. The nonlinear interactions can then be treated in perturbation theory, but in this semidilute regime we do not expect qualitative corrections to the results of our mean-field approximation. In this section we will therefore use a Gaussian approximation for computation of $\mathcal{Z}[\mathbf{h}]$ by truncating expansion of the dynamic functional Eq. (5.6), at quadratic order in $\boldsymbol{\phi}$.

The remaining $\Gamma^{(2)}_{\alpha\beta}(\mathbf{k},q,\omega)$ vertex function has a clear physical interpretation of a matrix whose (2,2) and (2,1) components are the noninteracting dynamic structure function $S^0(\mathbf{k},q,\omega)$ and response function $\tilde{S}^0(\mathbf{k},q,\omega)$ of the flux-line liquid, as can be seen from Eq. (5.7). $\Gamma^{(2)}_{\alpha\beta}(\mathbf{k},q,\omega)$ is computed and analyzed in various regimes in Appendix C. With the truncation at quadratic order we obtain the interacting structure and response functions $S_{\alpha\beta}(\mathbf{k},q,\omega)$ in terms of the noninteracting ones $S^0_{\alpha\beta}(\mathbf{k},q\omega)=\Gamma^{(2)}_{\alpha\beta}(\mathbf{k},q,\omega)$ and therefore will be able to study the effects of flux-line interactions and quenched disorder on the dynamics of the flux-line liquid.

In the Gaussian approximation it is more convenient to return to Eq. (5.4) and integrate out the $\boldsymbol{\phi}$ field, thereby producing a hydrodynamic functional $\mathcal{J}[\boldsymbol{\psi}]$ expressed directly in terms of physical density fields $\boldsymbol{\psi}$. Upon performing all the calculations in Fourier space (q and $\mathbf{k}$ are reserved for wave vectors in the z and the transverse directions, respectively) we obtain,

$$\mathcal{Z}[\mathbf{h}]=\int \mathcal{D}\boldsymbol{\psi} \exp\left[\int_{\mathbf{k},q,\omega}[-\tfrac{1}{2}\psi_\alpha(\mathbf{k},q,\omega)\hat{K}_{\alpha\beta}(\mathbf{k},q,\omega)\psi_\beta(-\mathbf{k},-q,-\omega)+\psi_\alpha(\mathbf{k},q,\omega)h_\alpha(-\mathbf{k},-q,-\omega)]\right], \tag{6.1}$$

where

$$\hat{K}_{\alpha\beta}(\mathbf{k},q,\omega)=\frac{1}{k_BT}K_{\alpha\beta}(\mathbf{k},q,\omega)+\frac{1}{n_0}(S^0)^{-1}_{\alpha\beta}(\mathbf{k},q,\omega)\ . \tag{6.2}$$

An integration over $\boldsymbol{\psi}$ leads to the dynamic generating function within the Gaussian approximation,

$$\mathcal{Z}[\mathbf{h}]=\exp\left\{\frac{1}{2}\int_{\mathbf{k},q,\omega}h_\alpha(\mathbf{k},q,\omega)\,\hat{K}^{-1}_{\alpha\beta}(\mathbf{k},q,\omega)\,h_\beta(-\mathbf{k},-q,-\omega)\right\}. \tag{6.3}$$

Functionally differentiating twice with respect to external field $\mathbf{h}(\mathbf{k},q,\omega)$, we obtain the interacting correlation/response function matrix $S_{\alpha\beta}(\mathbf{k},q,\omega)=\hat{K}^{-1}_{\alpha\beta}(\mathbf{k},q,\omega)$,

$$S_{\alpha\beta}(\mathbf{k},q,\omega)=\begin{bmatrix} 0 & \tilde{S}(\mathbf{k},q,\omega) \\ \tilde{S}(-\mathbf{k},-q,-\omega) & S(\mathbf{k},q,\omega)\end{bmatrix}, \tag{6.4}$$

where $S_{22}=S(\mathbf{k},q,\omega)$ and $S_{12}=\tilde{S}(\mathbf{k},q,\omega)$ are the interacting hydrodynamic structure and response functions, respectively, for the flux-line liquid in the presence of disorder,

$$S(\mathbf{k},q,\omega)=\frac{F(\mathbf{k},q)\delta(\omega)|\tilde{S}^0(\mathbf{k},q,\omega)|^2n_0^2/(k_BT)^2+n_0S^0(\mathbf{k},q,\omega)}{|V(\mathbf{k})\tilde{S}^0(\mathbf{k},q,\omega)n_0/k_BT+1|^2}, \tag{6.5a}$$

$$\tilde{S}(\mathbf{k},q,\omega)=\frac{V(\mathbf{k})n_0^2|\tilde{S}^0(\mathbf{k},q,\omega)|^2/k_BT+n_0\tilde{S}^0(\mathbf{k},q,\omega)}{|V(\mathbf{k})\tilde{S}^0(\mathbf{k},q,\omega)n_0/k_BT+1|^2}\ . \tag{6.5b}$$

Note that in Eq. (6.4) the S_{11} component vanishes, $\langle \tilde{\psi}(\mathbf{r},z,t)\tilde{\psi}(\mathbf{r}',z',t')\rangle = 0$. This is a consequence of the fluctuation dissipation theorem and causality, and implies that dynamics of the density field ψ_2 encoded in the effective dynamic functional in Eq. (6.3) can equivalently be described by a linear differential equation for the density field ψ.

VII. DETAILS OF THE DYNAMIC STRUCTURE FUNCTION

In this section we will analyze the dynamic structure function $S(\mathbf{k},q,\omega)$. Information about the behavior of $\tilde{S}(\mathbf{k},q,\omega)$ can be obtained from $S(\mathbf{k},q,\omega)$ by using fluctuation dissipation theorem (FDT),

$$\partial_t \Gamma_{22}^{(2)}(\mathbf{k},z,t) = \Gamma_{12}^{(2)}(\mathbf{k},z,t) - \Gamma_{12}^{(2)}(\mathbf{k},z,-t) \; . \qquad (7.1)$$

The interacting dynamic structure function in Eq. (6.5a) is expressed in terms of the noninteracting structure and response functions, $S^0(\mathbf{k},q,\omega), \tilde{S}^0(\mathbf{k},q,\omega)$ of the vortex liquid and depends on the correlation function of the disorder $F(\mathbf{k},q)$ as well as the interline interaction $V(\mathbf{k})$.

The details of a single-line dynamics are presented in Appendix C. There are two regimes with very different dynamic behavior that are separated by a characteristic time, $t_{\text{Rouse}} = L^2/(D\epsilon)$. The Rouse time is the time required for the center of mass of the flux line to diffuse its transverse radius of gyration. Equivalently, for times larger than t_{Rouse} the diffusion of the center of mass dominates over the internal mode dynamics. The dynamics of a single flux line for $t \gg t_{\text{Rouse}}$ and $t \ll t_{\text{Rouse}}$ is depicted in Fig. 3.

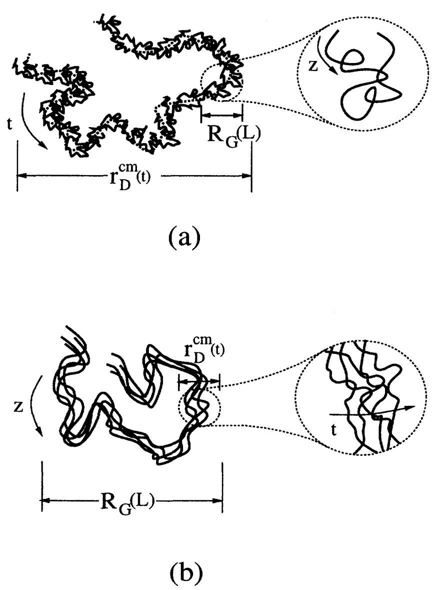

FIG. 3. 2D-projected configuration of a diffusing flux line for (a) $t \gg t_{\text{Rouse}}$ and (b) $t \ll t_{\text{Rouse}}$. In (a) the progression in time is illustrated, with the inset showing a snap-shot configuration, "diffusion" in z direction. For $t \gg t_{\text{Rouse}}$ the flux-line diffusion is dominated by the center-of-mass mode and the dynamics is that of a rigid rod. The dashed line shows the trajectory of the center of mass. In (b) the flux line appears frozen with dynamic fluctuations (due to the center-of-mass and internal modes) small relative to its transverse size. The inset shows evolution in time.

We find that for times larger than the Rouse time, $t_{\text{Rouse}} = L^2/(D\epsilon)$ the diffusion is dominated by the center-of-mass mode. In this case the average transverse distance diffused is simply

$$r_D^{\text{cm}}(t) = \sqrt{1/2\langle[\mathbf{r}_0(t)-\mathbf{r}_0(0)]^2\rangle} = \sqrt{2Dk_BTt/L} \propto t^{1/2} \; . \qquad (7.2)$$

We note that the flux-line diffusion coefficient Dk_BT/L is scaled down by L with respect to the point vortex diffusion constant Dk_BT in the ab plane. This $1/L$ behavior has been previously derived in the appendix of Ref. 18 in terms of a simple model of point vortices coupled in the z direction.

In the opposite limit, $t < t_{\text{Rouse}}$, the flux-line diffusion is dominated by the internal modes. For two points on the flux line separated by a distance $z \ll (D\epsilon t)^{1/2}$ the autocorrelation function becomes independent of z. This allows us again to extract a transverse diffusion distance

$$r_D^i(t) = \sqrt{(1/2)\langle[\mathbf{r}(z,t)-\mathbf{r}(0,0)]^2\rangle} \approx \sqrt{2k_BT[(D/\pi\epsilon)t]^{1/2}} \propto t^{1/4} \; , \qquad (7.3)$$

but now with an anomalous t dependence. This behavior can be understood as two "random walks" on top of each other; the flux-line segment executes a random walk on the flux-line conformation, which can be thought of being generated by a random walk (fictitious dynamics along the z axis). For $z \gg (D\epsilon t)^{1/2}$ one gets "diffusion" in the timelike variable z. The corresponding square root of the mean-square displacement $\sqrt{2k_BT|z|/\epsilon}$ corresponds to the projected two-dimensional (2D) radius of gyration of a flux line of length $|z|$. At intermediate scales $z \geq (D\epsilon t)^{1/2}$ the z,t dependence of the segment-correlation function has to be taken into account, and one finds

$$\langle[\mathbf{r}(z,t)-\mathbf{r}(0,0)]^2\rangle = \frac{4k_BT}{\epsilon}|z| f\left[\frac{D\epsilon|t|}{z^2}\right] , \qquad (7.4)$$

with $f(x)$ shown in Fig. 4.

The dynamic crossover described above can be equivalently extracted from the behavior of the ω pole in the noninteracting structure function. For $(k_BT/\epsilon)k^2L \ll 1$ the center-of-mass diffusion dominates the hydrodynamics and the noninteracting relaxation rate is $\Gamma_{\text{cm}}(\mathbf{k}) = Dk_BTk^2/L$. In the opposite limit of $(k_BT/\epsilon)k^2L \gg 1$ the internal modes dominate the dynamics and lead to the relaxation rate $\Gamma_{\mathbf{k}} = (k_BT)^2Dk^4/(4\epsilon)$. This internal-mode relaxational rate can be equivalently rewritten in the form of the center-of-mass relaxational rate $\Gamma_{\mathbf{k}} = Dk_BTk^2/L(\mathbf{k})$, with an effective flux-line length $L(\mathbf{k}) = 4\epsilon/k_BTk^2$. Not surprisingly, $L(\mathbf{k})$ is approximately the length in the z direction corresponding to the xy length scale of k^{-1}. The crossover from a dynamics dominated by the center-of-mass motion to a dynamics governed by the internal

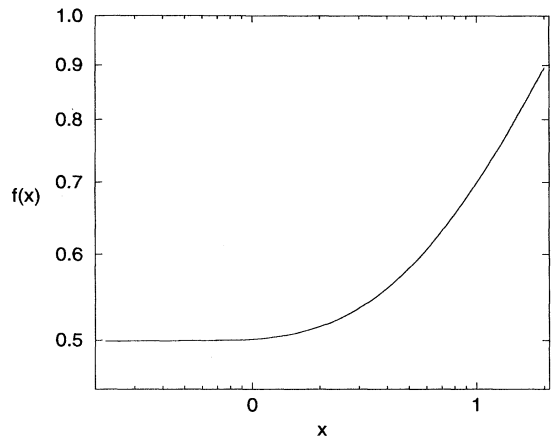

FIG. 4. Scaling function $f(x)$ describing single flux-line diffusion due to internal modes, displayed on a double logarithmic scale.

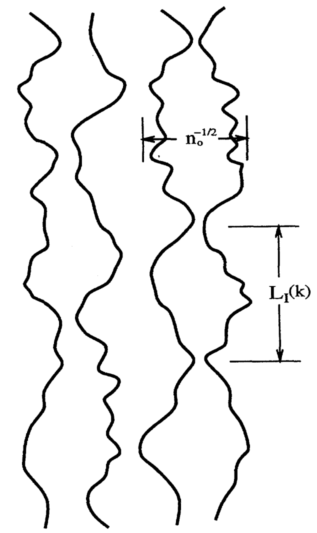

FIG. 5. Illustration of the interaction length $L_I(\mathbf{k})$ defined in the text.

modes [or equivalently from L to $L(\mathbf{k})$] occurs when the wavelength (transverse length scale) k^{-1} becomes smaller than the *static* transverse "diffusion" length $\sqrt{2k_BTL/\epsilon}$ (the 2D projected radius of gyration).

Substituting the expressions for the noninteracting correlation and response functions in these two regimes from Appendix C into Eq. (6.5a), we obtain the dynamic interacting structure function in the Gaussian approximation.

A. Center-of-mass dominated regime

In the wave vector regime, $(k_BT/\epsilon)k^2L \ll 1$, where the center-of-mass mode dominates we obtain

$$S(\mathbf{k},q=0,\omega)\approx\frac{2n_0Dk_BTk^2}{\omega^2+[Dk_BTk^2(1/L+n_0V(\mathbf{k})/k_BT)]^2}+\frac{\delta(\omega)F(\mathbf{k},0)n_0^2/(k_BT)^2}{(1/L+n_0V(\mathbf{k})/k_BT)^2}\ . \tag{7.5}$$

Concentrating on the first term which describes the dynamics in the absence of disorder we observe that the center of mass, noninteracting relaxation rate $\Gamma_{\rm cm}(\mathbf{k})$ has been additively renormalized by the interaction between the flux lines. We define an interaction length scale in the z direction, $L_I(\mathbf{k})=(n_0V(\mathbf{k})/k_BT)^{-1}$, and note that the crossover between the noninteracting and interacting dynamics occurs when $L>L_I(\mathbf{k})$, see Fig. 5.

Physically, $L_I(\mathbf{k})$ is the flux-line length (or equivalently sample thickness) beyond which there are multiple interactions between the flux lines. For samples thicker than $L_I(\mathbf{k})$ (if the crossing barriers are large) significant line entanglement will take place. This length is therefore analogous to the entanglement length discussed in Ref. 1.

The expression for $L_I(\mathbf{k})$ derived in the Gaussian approximation will be corrected by the higher-order interactions appearing in the dynamic functional expansion, Eq. (5.6). The flux-line dilute regime can be treated using renormalization group as was done for the statics in Refs. 1 and 23. It is found that in the dilute limit the interacting theory becomes asymptotically free, with the flux-line interactions renormalizing to zero, logarithmically with length scale. Following the renormalization-group equations down into the dense regime and matching, gives the effective interaction $V\approx 4\pi(k_BT)^2/\epsilon/\ln(1/n_0\lambda^2)$, which is independent of the original bare interaction. In the dilute regime (dropping the unimportant logarithmic factor and constants of order unity) we therefore obtain $L_I\approx\epsilon/(2k_BTn_0)$. This length is exactly the entanglement length defined by the static transverse wandering being on the order of the average line spacing.

For three-dimensional samples ($L\to\infty$), the flux-line density relaxations are dominated by the interactions, with the relaxation rate given by the rate for flux lines with length $L_I(\mathbf{k})$,

$$\Gamma_{\rm cm}^R=\frac{Dk_BTk^2}{L_I(\mathbf{k})} \tag{7.6a}$$

$$\approx n_0V(\mathbf{k})Dk^2,\quad \text{in the dense regime} \tag{7.6b}$$

$$\approx\frac{n_0(k_BT)^2Dk^2}{\epsilon},\quad \text{in the dilute regime}\ . \tag{7.6c}$$

We thus find that, while for the noninteracting flux line the center of mass diffuses very slowly, with the rate vanishing as $1/L$, the relaxations in the interacting line liquid are independent of the flux-line length L. In the interacting theory the $1/L$ dependence of the diffusion coefficient gets cutoff by the interaction (entanglement) length L_I. Within the Gaussian approximation we therefore find that the flux-line interactions speed up the dynamics of the line liquid, probably because they lead to a stiffer response to density inhomogeneities.

By comparing with the results of the static structure function obtained with the Gaussian approximation (see Appendix A and the second part of this section) we find that flux-line liquid kinetic coefficients are not modified and the renormalization of the statics is solely responsible for the increase in the relaxational rate Γ_{cm}. Therefore, within the Gaussian approximation no viscosity is generated. We expect, however, that this will be corrected by higher-order interactions, which also become important in the dense limit, near H_{c_2}. It should be possible to control these interactions via renormalization-group or mode coupling theory. If the line crossing barriers remain high, the additional effects of flux-line entanglement, not taken into account by the Gaussian approximation, will play a major role in the dynamics and probably can be described by these higher-order nonlinearities. We expect that entanglement and caging effects to become important in the regime of high line densities and large flux-line crossing barriers, which is possible at intermediate field strengths. These effects should have a slowing down effect on the dynamics, and will compete and eventually swamp the relaxational rate increase found in Eq. (7.5).

B. Internal modes dominated regime

In the regime $(k_BT/\epsilon)k^2L \gg 1$,[37] where the internal modes dominate we find

$$S(\mathbf{k},q,\omega)=\frac{2n_0\alpha(\mathbf{k},q)}{\omega^2\beta^2(\mathbf{k},q)+\gamma^2(\mathbf{k},q)}+\left[\frac{n_0}{k_BT}\right]^2\delta(\omega)F(k,q)\times\frac{[A(\mathbf{k})b_0(\mathbf{k},q)]^2}{[1+(n_0V(\mathbf{k})/k_BT)A(\mathbf{k})b_0(\mathbf{k},q)]^2}, \tag{7.7}$$

where we have introduced

$$\alpha(\mathbf{k},q)=A(\mathbf{k})\frac{b_1^2}{b_3}\Gamma_\mathbf{k}, \tag{7.8a}$$

$$\gamma(\mathbf{k},q)=\sqrt{b_1/b_3}\,\Gamma_\mathbf{k}\left[1+\frac{n_0V(\mathbf{k})}{k_BT}A(\mathbf{k})b_0\right], \tag{7.8b}$$

$$\beta^2(\mathbf{k},q)=\frac{b_3}{b_1}\frac{\gamma^2(\mathbf{k},q)}{\Gamma_\mathbf{k}^2}-\frac{b_1}{b_3}\frac{n_0V(\mathbf{k})A(\mathbf{k})}{k_BT}\times\left[2b_2+\frac{n_0V(\mathbf{k})A(\mathbf{k})}{k_BT}(2b_0b_2-b_1^2)\right], \tag{7.8c}$$

and $A(\mathbf{k})=4\epsilon/(k_BT\sqrt{\pi}k^2)$ and $\Gamma_\mathbf{k}=D(k_BT)^2k^4/(4\epsilon)$. The coefficients b_n are calculated in Appendix C and for $2\epsilon q/(k_BTk^2)\to 0$ are constants. The interacting dynamic structure function above consists of two parts. The first term is the thermal contribution to the density-density correlation function and has the standard Lorentzian shape. We note that the coefficient in front of the ω^2 term depends nonanalytically on wave vectors $\mathbf{k}$ and q and is therefore a breakdown of traditional hydrodynamics, here due to the effects of the internal modes.

The second term in Eq. (7.7) is the elastic contribution to the dynamic structure function arising from quenched pinning disorder. This contribution leads to a time-independent persistent contribution to the structure function. While the density correlations of the unpinned fraction of lines decay in time as the lines move around, the pinned density fraction has correlations that are time independent and have spatial correlations of the random potential.

The linewidth $\gamma(\mathbf{k})$ has the explicit form

$$\gamma(\mathbf{k})=\sqrt{b_1/b_3}\left[\frac{b_0}{\sqrt{\pi}}n_0V(\mathbf{k})Dk^2+\frac{(k_BT)^2}{4\epsilon}Dk^4\right]. \tag{7.9}$$

Note that the k^4 term is independent of the interaction between the flux lines and is therefore just related to the internal dynamics of a single flux line. We find that no interaction-generated viscosity appears. However, flux-line interaction modifies the relaxation rate of a single line in an important way. $\gamma(\mathbf{k})$ describes the crossover of the relaxation rate from the $(k_BT)^2Dk^4/(4\epsilon)$ behavior of a noninteracting line liquid to $n_0V(\mathbf{k})Dk^2$ relaxation rate of an interacting liquid. This crossover is similar to the Bogoliubov crossover in statics.[1] The interacting dynamics is again diffusive and aside from some constants is similar to the center-of-mass dominated regime considered in the beginning of this section.

C. Static structure function

The static structure function $S_s(\mathbf{k},q)$ is an equal-time density-correlation function, and therefore can be obtained from the dynamic structure function,

$$S_s(\mathbf{k},q)=S(\mathbf{k},q,t=0) \tag{7.10a}$$

$$=\int\frac{d\omega}{2\pi}S(\mathbf{k},q,\omega). \tag{7.10b}$$

Applying these equations we obtain the interacting static structure function of the vortex liquid in the presence of disorder (see also Appendix A),

$$S_s(\mathbf{k},q)=\frac{n_0S_s^0(\mathbf{k},q)}{1+n_0V(\mathbf{k})S_s^0(\mathbf{k},q)/k_BT}+\frac{F(\mathbf{k},q)}{(k_BT)^2}\left[\frac{n_0S_s^0(\mathbf{k},q)}{1+n_0V(\mathbf{k})S_s^0(\mathbf{k},q)/k_BT}\right]^2$$

$$=\frac{n_0k_BTk^2/\epsilon}{q^2+(q_B(\mathbf{k})/k_BT)^2} \tag{7.11a}$$

$$+F(\mathbf{k},q)\left[\frac{n_0k^2/\epsilon}{q^2+(q_B(\mathbf{k})/k_BT)^2}\right]^2, \tag{7.11b}$$

where $q_B(\mathbf{k})$ is the Bogoliubov spectrum of the corresponding bosons,[1]

$$\frac{q_B(\mathbf{k})}{k_BT}=\left[\left(\frac{k_BTk^2}{2\epsilon}\right)^2+\frac{n_0V(\mathbf{k})k^2}{\epsilon}\right]^{1/2} . \tag{7.12}$$

The equations for the structure functions that we derived here are valid for general type of disorder, characterized by disorder correlation function $F(\mathbf{k},q)$. The physically relevant cases are as follows:

(i) Point disorder due to oxygen vacancies (uncorrelated);

$$F(\mathbf{r},z)=\Delta_0\delta^{(2)}(\mathbf{r})\delta(z) , \tag{7.13a}$$

$$F(\mathbf{k},q)=\Delta_0 . \tag{7.13b}$$

(ii) Columnar defects (line-correlated along z axis);

$$F(\mathbf{r},z)=\Delta_1\delta^{(2)}(\mathbf{r}) , \tag{7.14a}$$

$$F(\mathbf{k},q)=\Delta_1\delta(q) . \tag{7.14b}$$

(iii) Grain or twin boundaries (plane-correlated, with normal $\hat{\mathbf{n}}$ in xy plane);

$$F(\mathbf{r},z)=\Delta_2\delta(\mathbf{r}\cdot\hat{\mathbf{n}}) , \tag{7.15a}$$

$$F(\mathbf{k},q)=\Delta_2\delta[\hat{\mathbf{z}}\cdot(\mathbf{k}\times\hat{\mathbf{n}})]\delta(q) . \tag{7.15b}$$

Equation (7.7) also depends on the interline interaction $V(\mathbf{k})$ and is valid for a general range of interactions or equivalently for arbitrary line density. In the limits of low and high line densities $V(k)$ reduces, respectively, to

$$V(k)\approx\frac{\phi_0^2}{8\pi^2},\quad k\lambda\ll 1 , \tag{7.16a}$$

$$V(k)\approx\frac{\phi_0^2}{8\pi^2\lambda^2k^2}=V_0,\quad k\lambda\gg 1 . \tag{7.16b}$$

Equation (7.16a) is valid for $H\approx H_{c1}$ where the lines are much farther apart than λ, and Eq. (7.16b) holds for $H\gg H_{c1}$ where the average interline distance is smaller than λ.

Specializing our general result for the interacting static structure function, Eq. (7.11) to point disorder and to short-range line interaction, we recover the result of Nelson and LeDoussal,[23]

$$S_s^{\rm SR}(\mathbf{k},q)=\frac{n_0k_BTk^2/\epsilon}{q^2+(q_B(\mathbf{k})/k_BT)^2}+\Delta_0\left[\frac{n_0k^2/\epsilon}{q^2+(q_B(\mathbf{k})/k_BT)^2}\right]^2 . \tag{7.17}$$

They utilized the boson mapping to obtain this static result. For pure superconductors this result reduces to the original result of Nelson and Seung,[1] also obtained via the boson mapping.

VIII. PHENOMENOLOGICAL HYDRODYNAMICS

As explained in the introduction, one can also take a more macroscopic, phenomenological approach to derive equations of motion for the hydrodynamic density fields.[18] We take this approach in this section with the intent to subsequently compare the results of our kinetic theory derived in previous sections with the hydrodynamic approach.

As already emphasized in Sec. IV, on long time and space scales the important, slow degrees of freedom that characterize the flux-line liquid are the number and tangent densities of the vortex liquid,

$$n(\mathbf{r},z,t)=\sum_{i=1}^{N}\delta^{(2)}[\mathbf{r}-\mathbf{r}_i(z,t)] , \tag{8.1a}$$

$$\mathbf{t}(\mathbf{r},z,t)=\sum_{i=1}^{N}\partial_z\mathbf{r}_i(z,t)\delta^{(2)}[\mathbf{r}-\mathbf{r}_i(z,t)] , \tag{8.1b}$$

together with the current fields $j_a^n(\mathbf{r},z,t)=n_0v_a(\mathbf{r},z,t)$ and $j_{ab}^t(\mathbf{r},z,t)$

$$j_a^n(\mathbf{r},z,t)=\sum_{i=1}^{N}\partial_tr_{ia}(z,t)\delta^{(2)}[\mathbf{r}-\mathbf{r}_i(z,t)] , \tag{8.2a}$$

$$j_{ab}^t(\mathbf{r},z,t)=\sum_{i=1}^{N}[\partial_tr_{ia}(z,t)\partial_zr_{ib}(z,t)-\partial_tr_{ib}(z,t)\partial_zr_{ia}(z,t)]\delta^{(2)}[\mathbf{r}-\mathbf{r}_i(z,t)] , \tag{8.2b}$$

which transport the number and tangent densities $n(\mathbf{r},z,t)$, $t_a(\mathbf{r},z,t)$, respectively. As for the hydrodynamics of a liquid of point particles, the dynamics of a line liquid is constrained by dynamic continuity equations arising from local conservation of the density fields,

$$\partial_tn(\mathbf{r},z,t)+\partial_aj_a^n(\mathbf{r},z,t)=0 , \tag{8.3a}$$

$$\partial_tt_a(\mathbf{r},z,t)+\partial_bj_{ab}^t(\mathbf{r},z,t)-\partial_zj_a^n(\mathbf{r},z,t)=0 . \tag{8.3b}$$

Furthermore, $n(\mathbf{r},z,t)$ and $t_a(\mathbf{r},z,t)$ are not completely independent. Continuity of flux lines introduces a spatial constraint between the number density and the longitudinal part of the tangent density,

$$\partial_zn(\mathbf{r},z,t)+\partial_at_a(\mathbf{r},z,t)=0 , \tag{8.4}$$

which in the language of bosons plays the role of a temporal continuity equation for the conservation of the boson density along imaginary "time" iz. Equations (8.3) and (8.4) can be easily verified by substitution using the microscopic definitions in Eqs. (8.1) and (8.2). The dynamics of a flux-line liquid is therefore governed by only two (as opposed to three) hydrodynamic variables, the number density $n(\mathbf{r},z,t)$ and the transverse part of the tangent density $\mathbf{t}^T(\mathbf{r},z,t)$.

It is also enlightening to interpret the above continuity equations in terms of the underlying electromagnetic fields. At length scales much larger than the London penetration lengths, $k^{-1}\gg\lambda_\perp$, $q^{-1}\gg\lambda$, Eqs. (4.1) (or equivalently, the long-wavelength limit of London equation) lead to a simple relation between the magnetic field and the line densities,

$$\mathbf{B}(\mathbf{r},z,t)=\hat{\mathbf{z}}\phi_0n(\mathbf{r},z,t)+\phi_0\mathbf{t}(\mathbf{r},z,t) . \tag{8.5}$$

It is easy to see that Eqs. (8.3) are just the z and xy components of Maxwell equation $\partial_t\mathbf{B}/c+\nabla\times\mathbf{E}=0$, with the electric field $\mathbf{E}$, which results from motion of flux lines, related to the current fields of Eq. (8.2).[18] The spatial continuity equation, Eq. (8.4), is equivalent to $\nabla\cdot\mathbf{B}=0$.

A closed set of hydrodynamic equations is obtained when we supplement the continuity equations with the constitutive equations for the currents. The constitutive equations are a statement of Newton's second law, cast in terms of the hydrodynamic variables. The relations equate the rate of change of the velocity to the forces acting on the flux-line liquid. These forces include the frictional forces, due to flux lines' interaction with the underlying lattice and weak disorder, and the pressure gradients due to nonuniformity in the density of the vortex liquid. For simplicity we will treat the case of small Hall angle and therefore neglect the component of the velocity response perpendicular to the forces. At long times, and for large frictional forces the velocity fields quickly decay to their steady-state value and the inertia term can be ignored. The resulting time-independent constitutive relation equates the frictional forces, proportional to velocity fields, to the pressure forces, expressed in terms of the density fields. A simplified equation for the velocity of the number density current is given by

$$(\gamma-\eta\nabla_\perp^2-\eta_z\partial_z^2)v_a(\mathbf{r},z,t)=-n_0\partial_a\frac{\delta\mathcal{H}}{\delta n(\mathbf{r},z,t)}+n_0\partial_z\frac{\delta\mathcal{H}}{\delta t_a(\mathbf{r},z,t)}+f_a^{\text{ext}}(\mathbf{r},z,t)+\nu_a(\mathbf{r},z,t)\ . \tag{8.6}$$

The terms on the left-hand side of the above equation represent the frictional and viscous forces acting on the flux lines, which are balanced by the pressure gradient forces, the external forces $\mathbf{f}_{\text{ext}}$ and a random noise force appearing on the right-hand side of the equation. $\mathbf{f}_{\text{ext}}$ might include the Lorentz force, $\mathbf{f}_{\text{ext}}=-n_0\phi_0\hat{z}\times\mathbf{j}_e$ due to the charge current $\mathbf{j}_e$ coupling to the magnetic field of the flux-line liquid. Here we are interested in the equilibrium regime, $\mathbf{f}_{\text{ext}}=0$. We take $\nu_a(\mathbf{r},z,t)$ to be a Gaussian zero-mean noise with the correlations determined by the fluctuation dissipation theorem,

$$\langle \nu_a(\mathbf{k},q,\omega)\nu_b(\mathbf{k}',q',\omega')\rangle=k_BT(\gamma+\eta\mathbf{k}^2+\eta_zq^2)(2\pi)^4\delta^{(2)}(\mathbf{k}+\mathbf{k}')\times\delta(q+q')\delta(\omega+\omega')\delta_{ab}\ , \tag{8.7a}$$

$$\langle \nu_a(\mathbf{k},q,\omega)\rangle=0\ . \tag{8.7b}$$

For the hydrodynamic description it is sufficient to assume that the effective hamiltonian $\mathcal{H}[n,\mathbf{t}]$ is an expansion in powers of hydrodynamic variables $n(\mathbf{r},z,t)$ and $\mathbf{t}(\mathbf{r},z,t)$, which for simplicity we truncate at the quadratic order. It is more convenient to expand in terms of fluctuations $\rho(\mathbf{r},z)=n(\mathbf{r},z)-n_0$ and $\tau(\mathbf{r},z)=\mathbf{t}(\mathbf{r},z)-\mathbf{t}_0$ around the average values $n_0,\mathbf{t}_0$, so that the linear terms can be eliminated. Ignoring disorder (because its effects can be easily included) and assuming translational invariance, we obtain in Fourier space,

$$\mathcal{H}[\rho(\mathbf{k},q),\tau(\mathbf{k},q)]=\int_{\mathbf{k},q}\frac{1}{2}[K_1(\mathbf{k},q)\rho^2+K_2(\mathbf{k},q)\tau^2]\ , \tag{8.8}$$

where the functions $K_1(\mathbf{k},q)$ and $K_2(\mathbf{k},q)$ are related to the static density-correlation functions. Imposing the constraint of Eq. (8.4) in Fourier space, $q\rho(\mathbf{k},q)=-\mathbf{k}\cdot\tau(\mathbf{k},q)$ to reexpress $\mathcal{H}$ in terms of independent hydrodynamic variables, we easily compute the static structure function, $\langle\rho(\mathbf{k},q)\rho(-\mathbf{k},-q)\rangle$, and the tangent correlation function $\langle\tau_a(\mathbf{k},q)\tau_b(-\mathbf{k},-q)\rangle$

$$S^s(\mathbf{k},q)=\frac{k_BTk^2}{k^2K_1(\mathbf{k},q)+q^2K_2(\mathbf{k},q)}\ , \tag{8.9a}$$

$$T^s_{ab}(\mathbf{k},q)=P^T_{ab}(k)\frac{k_BT}{K_2(\mathbf{k},q)}+P^L_{ab}(k)\frac{q^2}{k^2}S^s(\mathbf{k},q)\ . \tag{8.9b}$$

This allows us to reexpress Eq. (8.8) in terms of the static structure function $S^s(\mathbf{k},q)$ and the transverse part of the tangent density-correlation function $T_T(\mathbf{k},q)$,

$$\mathcal{H}[\rho(\mathbf{k},q),\tau(\mathbf{k},q)]=\int_{\mathbf{k},q}\frac{k_BT}{2}\left[\left[S_s^{-1}(\mathbf{k},q)-\frac{q^2}{k^2}T_T^{-1}(\mathbf{k},q)\right]\rho^2+T_T^{-1}(\mathbf{k},q)\tau^2\right]\ . \tag{8.10}$$

Combining the above equation with Eqs. (8.6), (8.4), and (8.3a) we find that the dynamic equations for $n(\mathbf{k},q,t)$ and $\mathbf{t}(\mathbf{k},q,t)$ decouple. Here we will only concentrate on the hydrodynamics of $n(\mathbf{k},q,t)$, which is governed by a simple Langevin equation,

$$\partial_t n+[n_0^2k_BTD(\mathbf{k},q)k^2S_s^{-1}(\mathbf{k},q)]n=\hat{\zeta}(\mathbf{k},q,t)\ , \tag{8.11}$$

where we defined the hydrodynamic diffusion coefficient $D^{-1}(\mathbf{k},q)=\gamma+\eta\mathbf{k}^2+\eta_zq^2$ and the noise $\hat{\zeta}(\mathbf{k},q,t)=in_0D(\mathbf{k},q)\mathbf{k}\cdot\boldsymbol{\nu}(\mathbf{k},q)$, which is also Gaussian with zero-mean and the correlations determined by those of $\boldsymbol{\nu}$, Eq. (8.7),

$$\langle\hat{\zeta}(\mathbf{k},q,\omega)\hat{\zeta}(\mathbf{k}',q',\omega')\rangle=k_BTD(\mathbf{k},q)k^2n_0^2(2\pi)^4\times\delta^{(2)}(\mathbf{k}+\mathbf{k}')\delta(q+q')\delta(\omega+\omega')\ , \tag{8.12a}$$

$$\langle\hat{\zeta}(\mathbf{k},q,\omega)\rangle=0\ . \tag{8.12b}$$

These equations then lead to the hydrodynamic structure and response functions previously derived in Ref. 20:

$$S^{\text{phenom}}(\mathbf{k},q,\omega)=\frac{2n_0^2k_BTk^2D(\mathbf{k},q)}{\omega^2+[n_0^2k_BTk^2D(\mathbf{k},q)/S_s(\mathbf{k},q)]^2}\ , \tag{8.13a}$$

$$\tilde{S}^{\text{phenom}}(\mathbf{k},q,\omega)=\frac{n_0^2k_BTk^2D(\mathbf{k},q)}{i\omega+n_0^2k_BTk^2D(\mathbf{k},q)/S_s(\mathbf{k},q)}\ . \tag{8.13b}$$

We are now in a position to compare the results of the phenomenological model of hydrodynamic with our kinetic theory of the flux-line liquid. Making the comparison in the physically most relevant regime dominated by center-of-mass motion, Eq. (7.5), we find the following identifications,

$$D \leftrightarrow n_0 D(k,q=0) \ , \tag{8.14a}$$

$$D\left[\frac{1}{L}+\frac{n_0 V(\mathbf{k})}{k_B T}\right] \leftrightarrow \frac{n_0 D(k,q=0)}{S_s(\mathbf{k},q=0)} \ . \tag{8.14b}$$

As we already observed in Sec. VII, no viscosity is generated within the Gaussian approximation employed here and we are only able to establish a simple relations between the parameters of our kinetic model and the phenomenological theory,

$$\gamma \leftrightarrow \frac{n_0}{D} \ , \tag{8.15a}$$

$$S_s \leftrightarrow n_0\left[\frac{1}{L}+\frac{1}{L_I(\mathbf{k})}\right]^{-1} \ . \tag{8.15b}$$

Equation (8.15a) is physically appealing in its identification of the phenomenological friction coefficient γ with the inverse of the kinetic diffusion coefficient D. Equation (8.15b) leads to the static structure function, in agreement with the result obtained from boson analogy. The above equations show that at least within the Gaussian approximation, in the center-of-mass dominated regime the dynamics is modified only through the statics, i.e., the kinetic coefficients are not renormalized in this order of approximation.

IX. CONCLUSIONS

In this paper we have formulated the dynamic theory of the flux-line liquid phase directly from the kinetic theory of individual, interacting flux lines. We have used the resulting theory to study the dynamic structure and response functions of the line liquid in the presence of various types of pinning disorder. In order to solve the theory we employed a Gaussian approximation to the dynamic functional, which should be valid at intermediate flux-line densities or fields $H_{c_1} \ll H \ll H_{c_2}$, where the effects of large fluctuations are not as important.

We expressed the interacting dynamic structure function in terms of the single-line structure functions. In the long-time limit, $t \gg t_{\mathrm{Rouse}}$ and/or for transverse wavelengths larger than the line wandering, the center-of-mass mode dominates over the internal mode, and we recover the hydrodynamics of rigid rods. While for noninteracting lines the diffusion coefficient vanishes as $L \rightarrow \infty$, the interactions between the lines lead to an increase in the relaxation rate of the line liquid. The diffusion rate remains finite for any L, with L cutoff by the interaction length $L_I(\mathbf{k})$. In the dense limit L_I is determined by the interactions, with the two-body interaction giving only a first-order estimate to this length. A detailed calculation that takes into account higher-order interactions is required in this dense limit. In the dilute regime, the renormalization-group calculations lead to a line interaction that vanishes as an inverse of a logarithm of the length scale and the microscopic interaction drops out. In this case L_I becomes just the entanglement length defined in Ref. 1. In either case for $t \gg t_{\mathrm{Rouse}}$, our results are therefore in agreement with the phenomenological model of Marchetti and Nelson, but unfortunately the Gaussian approximation is not accurate enough to generate the viscosity of their model from our kinetic theory.

In the opposite limit of short times and/or $k_B T k^2 L/\epsilon \gg 1$, we find that the internal modes dominate. In the presence of interactions we find that the k^4 relaxation rate of a noninteracting flexible line liquid crosses over to a diffusive k^2 relaxation rate. In contrast with the phenomenological model, however, a wave-vector-dependent coefficient of the ω^2 term is generated. This coefficient $\sim k^{-4}$ and by this constitutes a break down of conventional hydrodynamics due to the internal mode fluctuations.

We find that the Gaussian approximation successfully reproduces the static Bogoliubov "spectrum" previously obtained via boson analogy. For the statics this approximation is therefore equivalent to an infinite summation of all ladder graphs in the single-line picture, and then treating the resulting field theory in the mean-field approximation. The dynamic results of this approximation have the structure of the dynamic random-phase approximation. In addition to the effects discussed above this approximation leads to a disorder-generated perturbative correction to the dynamic structure function, which agrees with the phenomenological approach.

It is well known that preserving the discreteness of the flux lines is crucial for the correct treatment of low-temperature ordered phases such as the vortex glass and Bose glass. It is likely that in the liquid phase, however, much of the dynamics can be described in terms of the density fields which coarse-grain over the discrete line coordinates as we have done here. In this description slow dynamics for high line crossing barriers and entanglement can, in principle, be incorporated by higher-order nonlocal interactions in the densities, although only in some average sense. Also line crossing and recombination are present in the theory and can be controlled in an average sense by the strength of the repulsive flux-line interaction. This description of a flux-line liquid can be improved by also introducing a local tangent field that keeps track of the local orientation of the flux lines as we do for the statics in Appendix A.

We expect that the results derived in this paper will be corrected by the higher-order interaction which can be taken into account using renormalization-group or mode coupling methods. These corrections will undoubtedly lead to renormalization of the kinetic coefficients. We also expect that the slow dynamics resulting from the lines being caged by its neighbors and from flux-line entanglement effects, can be reproduced by these higher-order interactions. It is quite likely, however, that these effects will turn out to be nonperturbative. It is also possible that the k^2 correction to the diffusion coefficient will be absent even beyond the Gaussian approximation. The interactions might generate only nonanalytic terms in k and therefore the flux-line liquid viscosity will be strictly zero, leading to a breakdown of the phenomenological model described in Sec. VIII. In any case we believe that the formulation of hydrodynamics in terms of kinetic theory of lines introduced in this paper will be useful for a more detailed understanding of flux-line liquid phase at and possibly away from equilibrium.

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APPENDIX A: CALCULATION OF THE INTERACTING STATIC STRUCTURE FUNCTION

In this appendix we derive the interacting static structure function of Eq. (7.11) within the static equilibrium formulation, directly from the microscopic Hamiltonian Eq. (2.1), using methods similar to the ones used in the main text to obtain the dynamic structure function.

We begin with the microscopic Hamiltonian,

$$\mathcal{H}=\frac{\epsilon}{2}\sum_{i=1}^{N}\int_0^L dz(\partial_z \mathbf{r}_i)^2+\frac{1}{2}\sum_{i\neq j=1}^{N}\int_0^L dz\, V[\mathbf{r}_i(z)-\mathbf{r}_j(z)] + \sum_{i=1}^{N}\int_0^L dz\, U(\mathbf{r}_i(z),z) \ . \quad \text{(A1)}$$

Analogously to the dynamics calculation we want to derive a macroscopic Hamiltonian in terms of number and tangent densities $\rho(\mathbf{r},z)=n(\mathbf{r},z)-n_0$ and $\boldsymbol{\tau}(\mathbf{r},z)=\mathbf{t}(\mathbf{r},z)-\mathbf{t}_0$, where,

$$n(\mathbf{r},z)=\sum_{i=1}^{N}\delta^{(2)}[\mathbf{r}-\mathbf{r}_i(z)] \ , \quad \text{(A2a)}$$

$$\mathbf{t}(\mathbf{r},z)=\sum_{i=1}^{N}\frac{\partial \mathbf{r}_i}{\partial z}\delta^{(2)}[\mathbf{r}-\mathbf{r}_i(z)] \ . \quad \text{(A2b)}$$

The tangent density field can be written as a sum of its longitudinal and transverse parts, $\boldsymbol{\tau}(\mathbf{r},z)=\boldsymbol{\tau}^L(\mathbf{r},z)+\boldsymbol{\tau}^T(\mathbf{r},z)$ with $\tau_a^L(\mathbf{r},z)=P_{ab}^L\tau_b(\mathbf{r},z)$ and $\tau_a^T(\mathbf{r},z)=P_{ab}^T\tau_b(\mathbf{r},z)$, where $P_{ab}^T(k)=\delta_{ab}-k_ak_b/k^2$ and $P_{ab}^L(k)=k_ak_b/k^2$, in Fourier space.

As already mentioned in the derivation of dynamics, the longitudinal part of the tangent density field $\boldsymbol{\tau}^L(\mathbf{r},z)$ is related to the number density $n(\mathbf{r},z)$ by the "continuity equation,"

$$\frac{\partial}{\partial z}\rho(\mathbf{r},z)+\nabla_r\cdot\boldsymbol{\tau}(\mathbf{r},z)=0 \ , \quad \text{(A3)}$$

which is a statement that the flux lines do not end within the sample, or equivalently in terms of the magnetic field is the Maxwell equation $\nabla\cdot\mathbf{B}(\mathbf{r},z)=0$. This equation is clearly trivial for $\boldsymbol{\tau}^T(\mathbf{r},z)$, but gives a relation between $\rho(\mathbf{r},z)$ and $\boldsymbol{\tau}^L(\mathbf{r},z)$. Clearly then it is not necessary to keep track of the longitudinal part of the tangent density, since its correlations can be easily obtained from those of the number density using Eq. (A3). We derive an effective Hamiltonian in terms of static density fields $\rho(\mathbf{r},z)$ and $\boldsymbol{\tau}^T(\mathbf{r},z)$ by starting with the microscopic Hamiltonian Eq. (A1) and integrating out microscopic degrees of freedom $\mathbf{r}_i(z)$ as we did for the dynamic calculation.

The microscopic Hamiltonian can be rewritten (aside from irrelevant constants) in the suggestive form in terms of the relevant density fields,

$$\mathcal{H}[\mathbf{r}_i]=\mathcal{H}_0[\mathbf{r}_i]+\mathcal{H}_{\text{int}}[\rho,\boldsymbol{\tau}] \ , \quad \text{(A4)}$$

where $\mathcal{H}_0[\mathbf{r}_i]$ is the Hamiltonian of N noninteracting lines, the first term in Eq. (A1) and,

$$\mathcal{H}_{\text{int}}[\rho,\boldsymbol{\tau}]=\int_{\mathbf{r},\mathbf{r}',z}\rho(\mathbf{r},z)\rho(\mathbf{r}',z)V(\mathbf{r}-\mathbf{r}') + \int_{\mathbf{r},z}\rho(\mathbf{r},z)U(\mathbf{r},z) \ . \quad \text{(A5)}$$

Since we are after the description in terms of the density fields, we construct a generating functional by introducing external fields $h(\mathbf{r},z)$ and $\mathbf{h}_\tau(\mathbf{r},z)$ that couple to $\rho(\mathbf{r},z)$ and $\boldsymbol{\tau}^T(\mathbf{r},z)$, respectively,

$$Z_d[h,\mathbf{h}_\tau]=\int\mathcal{D}\mathbf{r}_i(z)\exp\left[-\frac{1}{k_BT}\mathcal{H}[\mathbf{r}_i(z)]+\int_{\mathbf{r},z}[\rho(\mathbf{r},z)h(\mathbf{r},z)+\boldsymbol{\tau}(\mathbf{r},z)\cdot\mathbf{h}_\tau(\mathbf{r},z)]\right] \ . \quad \text{(A6)}$$

We introduce unity inside above expression in the form of functional δ-functions, constraining the auxiliary fields $\psi(\mathbf{r},z)$ and $\boldsymbol{\psi}_\tau(\mathbf{r},z)$ to equal the physical densities fluctuations, respectively,

$$\begin{aligned} Z_d[h,\mathbf{h}_\tau]=&\int\mathcal{D}\mathbf{r}_i(z)\exp\left[-\beta\mathcal{H}_0[\mathbf{r}]-\beta\mathcal{H}_{\text{int}}[\rho,\boldsymbol{\tau}]+\int_{\mathbf{r},z}[\rho(\mathbf{r},z)h(\mathbf{r},z)+\boldsymbol{\tau}(\mathbf{r},z)\cdot\mathbf{h}_\tau(\mathbf{r},z)]\right] \\ &\times\int\mathcal{D}\psi\mathcal{D}\boldsymbol{\psi}_\tau\prod_{\mathbf{r},z}\delta[\psi(\mathbf{r},z)-\rho(\mathbf{r},z)]\delta[\boldsymbol{\psi}_\tau(\mathbf{r},z)-\boldsymbol{\tau}^T(\mathbf{r},z)] \\ =&\int\mathcal{D}\psi\mathcal{D}\boldsymbol{\psi}_\tau\exp\left[-\beta\mathcal{H}_{\text{int}}[\psi,\boldsymbol{\psi}_\tau]+\int\psi h+\int\boldsymbol{\psi}_\tau\cdot\mathbf{h}_\tau\right]\langle\delta[\psi(\mathbf{r},z)-\rho(\mathbf{r},z)]\delta[\boldsymbol{\psi}_\tau(\mathbf{r},z)-\boldsymbol{\tau}^T(\mathbf{r},z)]\rangle_0 \ . \end{aligned} \quad \text{(A7)}$$

The interactions and disorder are quadratic and linear functions of auxiliary density fields and the nontrivial part of the calculation reduces to averaging the functional δ functions with the Boltzmann weight $\exp[-\beta\mathcal{H}_0]$ of N noninteracting lines. These averages decouple to single line averages,

$$\langle\delta[\psi(\mathbf{r},z)-\rho(\mathbf{r},z)]\delta[\boldsymbol{\psi}_\tau(\mathbf{r},z)-\boldsymbol{\tau}^T(\mathbf{r},z)]\rangle_0=\int\mathcal{D}\phi\mathcal{D}\boldsymbol{\phi}_\tau\exp\left[i\int(\phi\psi+\boldsymbol{\phi}_\tau\cdot\boldsymbol{\psi}_\tau)\right]\left[\langle\exp[-i\int(\phi\rho_1+\boldsymbol{\phi}_\tau\cdot\boldsymbol{\tau}_1^T)]\rangle_0\right]^N \quad \text{(A8)}$$

Inserting the above equation into Eq. (A7) we obtain the partition function (for a fixed realization of disorder) in terms of an effective Hamiltonian expressed as an expansion in macroscopic density fields,

$$Z_d[h,\mathbf{h}_\tau]=\int \mathcal{D}\psi\mathcal{D}\boldsymbol{\psi}_\tau\int \mathcal{D}\phi\mathcal{D}\boldsymbol{\phi}_\tau \exp\left[-\mathcal{H}[\psi,\boldsymbol{\psi}_\tau,\phi,\boldsymbol{\phi}_\tau]+\int(\psi h+\boldsymbol{\psi}_\tau\cdot\mathbf{h}_\tau)\right] , \tag{A9}$$

where the effective Hamiltonian is given by

$$\mathcal{H}[\psi,\boldsymbol{\psi}_\tau,\phi,\boldsymbol{\phi}_\tau]=\mathcal{H}_{\rm int}[\psi]+n_0\Gamma[\phi,\boldsymbol{\phi}_\tau]+i\int(\phi\psi+\boldsymbol{\phi}_\tau\cdot\boldsymbol{\psi}_\tau) , \tag{A10}$$

with the new contribution $n_0\Gamma[\phi,\boldsymbol{\phi}_\tau]$ coming from the average of the δ functions in Eq. (A8) and is expressed in terms of a single flux-line cumulant expansion of ρ_1 and τ_1^T, in Fourier space,

$$\begin{aligned}-\Gamma[\phi,\boldsymbol{\phi}_\tau]=\sum_{l,m}^{\infty}\frac{(-i)^{(l+m)}}{l!m!}\int_{q_1,\mathbf{k}_1}\cdots\int_{q_{l+m},\mathbf{k}_{l+m}}\Gamma^{(l,m)}_{a_1\cdots a_m}(\mathbf{k}_1,q_1\cdots\mathbf{k}_{l+m},q_{l+m})(2\pi)^3\delta^{(2)}\left[\sum_{i=1}^{l+m}\mathbf{k}_i\right]\delta\left[\sum_{i=1}^{l+m}q_i\right]\\ \times\phi(-\mathbf{k}_1,-q_1)\cdots\phi(-\mathbf{k}_l,-q_l)\\ \times\phi_{\tau a_1}(-\mathbf{k}_{l+1},-q_{l+1})\cdots\phi_{\tau a_m}(-\mathbf{k}_{l+m},-q_{l+m}) ,\end{aligned} \tag{A11}$$

where the sum over a_i is implied. $\Gamma^{(l,m)}_{a_1\cdots a_m}$ is a connected correlation function of a single-line density fluctuations ρ_1 and τ_1^T, and in Fourier space is

$$\begin{aligned}\Gamma^{(l,m)}_{a_1\cdots a_m}(\mathbf{k}_1,q_1\cdots\mathbf{k}_{l+m},q_{l+m})(2\pi)^3\delta^{(2)}\left[\sum_{i=1}^{l+m}\mathbf{k}_i\right]\delta\left[\sum_{i=1}^{l+m}q_i\right]\\ =A\langle\rho(\mathbf{k}_1,q_1)\cdots\rho(\mathbf{k}_l,q_l)\tau^T_{a_1}(\mathbf{k}_{l+1},q_{l+1})\cdots\tau^T_{a_m}(\mathbf{k}_{l+m},q_{l+m})\rangle_0 .\end{aligned} \tag{A12}$$

As in dynamics we have inserted an area factor A for convenience and dropped the single-line label 1. The vertex functions $\Gamma^{(l,m)}_{a_1\cdots a_m}$ can be systematically computed for any l,m. We calculate some of the vertex functions for both statics and dynamics in Appendixes B and C.

Within the Gaussian approximation we truncate $\Gamma[\phi,\boldsymbol{\phi}_\tau]$ at quadratic order in the densities. We observe that $\Gamma^{(2,0)}(\mathbf{k},q)=S_s^0(\mathbf{k},q)$, $\Gamma^{(0,2)}_{ab}(\mathbf{k},q)=T^0_{s\,ab}(\mathbf{k},q)$, and $\Gamma^{(1,1)}_a(\mathbf{k},q)$ are single-line number and tangent static structure functions [with notation $S_s^0(\mathbf{k},q)$ and $T^0_{s\,ab}(\mathbf{k},q)$ from Ref. 1], derived in Appendix B and discussed in the main text,

$$\Gamma^{(2,0)}(\mathbf{k},q)=\frac{k^2k_BT/\epsilon}{q^2+(k^2k_BT/2\epsilon)^2} , \tag{A13a}$$

$$\Gamma^{(0,2)}_{ab}(\mathbf{k},q)=\frac{k_BT}{\epsilon}P^T_{ab} , \tag{A13b}$$

$$\Gamma^{(1,1)}_a(\mathbf{k},q)=0 . \tag{A13c}$$

The fact that $\Gamma^{(1,1)}_a(\mathbf{k},q)$ and all the cross-correlation functions vanish for the transverse part of the tangent density field τ_a^T is a result of the independence of τ_a^T from ρ. Mathematically it is true to all orders because any cross-correlation function of τ_a is a tensor with at least one index a which must be carried by k_a. This means that all the cross-correlation functions are purely longitudinal and when contracted with P^T_{ab} (for every τ_a^T) automatically vanish.

Integrating over the auxiliary fields $\phi(\mathbf{k},q)$ we obtain an effective Hamiltonian expressed solely in terms of the physical number and transverse tangent density fields,

$$\begin{aligned}\mathcal{H}[\psi,\boldsymbol{\psi}^\tau]=\int_{\mathbf{k},q}[&\tfrac{1}{2}\psi(\mathbf{k},q)A^{-1}(\mathbf{k},q)\psi(-\mathbf{k},-q)\\ &+\tfrac{1}{2}\psi_a^\tau(\mathbf{k},q)B^{-1}_{ab}(\mathbf{k},q)\psi_b^\tau(-\mathbf{k},-q)\\ &+\psi(\mathbf{k},q)U(-\mathbf{k},-q)] ,\end{aligned} \tag{A14}$$

where $A(\mathbf{k},q)$ and $B_{ab}(\mathbf{k},q)$ are the static interacting but disorder-free two-point correlation functions of ρ and τ_a^T,

$$A(\mathbf{k},q)=\frac{n_0k_BTk^2/\epsilon}{q^2+(q_B(\mathbf{k})/k_BT)^2} , \tag{A15a}$$

$$B_{ab}(\mathbf{k},q)=P^T_{ab}\frac{k_BTn_0}{\epsilon} , \tag{A15b}$$

and $q_B(\mathbf{k})$ is the Bogoliubov spectrum given by Eq. (7.12) in the main text.

We use above Hamiltonian to derive the interacting static functions in the presence of disorder $S^s(\mathbf{k},q)=\overline{\langle\rho(\mathbf{k},q)\rho(-\mathbf{k},-q)\rangle_0}$ and

$$\begin{aligned}T^s_{ab}(\mathbf{k},q)&=\overline{\langle\tau_a(\mathbf{k},q)\tau_b(-\mathbf{k},-q)\rangle_0}\\ &=T_s^T(\mathbf{k},q)P^T_{ab}+T_s^L(\mathbf{k},q)P^L_{ab} .\end{aligned}$$

The longitudinal part of T^s_{ab} can be obtained from $S^s(\mathbf{k},q)$ by using flux-line continuity Eq. (A3). Averaging over the annealed density fields, followed by the average over the quenched disorder we obtain

$$\begin{aligned}S_s(\mathbf{k},q)=&\frac{n_0k_BTk^2/\epsilon}{q^2+(q_B(\mathbf{k})/k_BT)^2}\\ &+F(\mathbf{k},q)\left[\frac{n_0k^2/\epsilon}{q^2+(q_B(\mathbf{k})/k_BT)^2}\right]^2 ,\end{aligned} \tag{A16a}$$

$$T_s^T(\mathbf{k},q)=\frac{k_B T n_0}{\epsilon}\ , \tag{A16b}$$

$$T_s^L(\mathbf{k},q)=\frac{q^2}{k^2}S_s(\mathbf{k},q)\ . \tag{A16c}$$

Equation (A16a) is in complete agreement with static limit of the interacting dynamic structure function obtained in the main text, Eq.(7.11b). For short-range disorder and flux-line interaction, $F(\mathbf{k},q)=\Delta_0$ and $V(k)=\phi_0^2/8\pi^2$, respectively, and the above equations reduce to the results of Ref. 23 obtained using the boson mapping.

APPENDIX B: CALCULATION OF THE NONLINEARITIES IN THE STATIC CUMULANT EXPANSION

In this appendix we calculate the coefficients of the nonlinear interactions which appear in the cumulant expansion of the Hamiltonian in Appendix A.

We begin with static problem and calculate the static single-line correlation functions of the density fields ρ and τ_α^T, defined by Eq. (A12). It is convenient to work in the space of $(\mathbf{k},z)$. We evaluate the following static correlation function,

$$\Gamma^{(l,m)}_{a_1\cdots a_m}(\mathbf{k}_1,z_1\cdots\mathbf{k}_{m+l},z_{m+l})\frac{(2\pi)^2}{A}\delta^{(2)}\left[\sum_{i=1}^{l+m}\mathbf{k}_i\right]$$

$$=\langle \tau^T_{a_1}(\mathbf{k}_1,z_1)\cdots\tau^T_{a_m}(\mathbf{k}_m,z_m)\rho(\mathbf{k}_{m+1},z_{m+1})\cdots\rho(\mathbf{k}_{m+l},z_{m+l})\rangle_0 \tag{B1a}$$

$$=\langle \partial_{z_1}r_{a_1}(z_1)\cdots\partial_{z_m}r_{a_m}(z_m)\exp\{-i[\mathbf{k}_1\cdot\mathbf{r}(z_1)+\cdots+\mathbf{k}_{m+l}\cdot\mathbf{r}(z_{m+l})]\}\rangle_0^T\ , \tag{B1b}$$

where the averages are performed with a Boltzmann weight of a single line and the superscript T extracts the transverse part of the average.

The above average can be easily computed by using the following Lemma that holds for Gaussian averages,

$$\langle X_1\cdots X_l e^Y\rangle_0=\left[\sum_{i\neq j}^{l}\langle X_iX_j\rangle_0+\langle YX_1\rangle_0\cdots\langle YX_l\rangle_0\right]e^{\langle Y^2\rangle_0/2}\ , \tag{B2}$$

where X_i and Y are Gaussian random variables with respect to the measure used. The above equation can be easily proved by rewriting the left-hand side as a multiderivative with respect to l parameters of an exponential generating function,

$$\langle X_1\cdots X_l e^Y\rangle_0=\partial_{\mu_1}\cdots\partial_{\mu_l}|_{\{\mu_i\}=0}\langle e^{Y+\mu_iX_i}\rangle_0 \tag{B3a}$$

$$=\partial_{\mu_1}\cdots\partial_{\mu_l}|_{\{u_i\}=0}\exp[\langle Y^2\rangle_0/2+\langle Y\mu_iX_i\rangle_0+\langle(\mu_iX_i)^2\rangle_0/2]\ , \tag{B3b}$$

where we used a property of averages of Gaussian variables $\langle\exp(\phi)\rangle_0=\exp(\langle\phi^2\rangle_0/2)$. Performing the differentiation and evaluating the result at $\{\mu_i\}=0$ we obtain Eq. (B2).

Returning to the original problem, Eq. (B1), we make the identification $X_i=\partial_{z_i}r_a(z_i)$ and $Y=-i\sum_{i=1}^{l+m}\mathbf{k}_i\cdot\mathbf{r}(z_i)$, obtaining,

$$\langle Y^2\rangle_0=-\left\langle\left[\sum_{i=1}^{l+m}\mathbf{k}_i\cdot\mathbf{r}(z_i)\right]^2\right\rangle_0 \tag{B4a}$$

$$=-\frac{1}{2}\sum_{i=1}^{l+m}k_i^2\langle r^2(z_i)\rangle_0-\sum_{i>j}^{l+m}\mathbf{k}_i\cdot\mathbf{k}_j\langle\mathbf{r}(z_i)\cdot\mathbf{r}(z_j)\rangle_0 \tag{B4b}$$

$$=-\left[\sum_{i=1}^{l+m}k_i\right]^2C(0)$$

$$-2\sum_{i>j}^{l+m}\mathbf{k}_i\cdot\mathbf{k}_j[C(z_i-z_j)-C(0)]\ , \tag{B4c}$$

where the $\mathbf{r}(z_i)$ averages are isotropic,

$$\tfrac{1}{2}\langle\mathbf{r}(z_i)\cdot\mathbf{r}(z_j)\rangle_0=C(z_i-z_j) \tag{B5a}$$

$$=\int_q\frac{e^{iq(z_i-z_j)}}{\epsilon q^2}\ . \tag{B5b}$$

Using this expression we find,

$$C(z_i-z_j)-C(0)=\int_q\frac{e^{iq(z_i-z_j)}-1}{\epsilon q^2} \tag{B6a}$$

$$=-\frac{|z_i-z_j|}{2\epsilon}\ . \tag{B6b}$$

From Eq. (B5b) we observe that $C(0)\approx L\approx A\to\infty$ and therefore this first factor leads to a δ function as $L,A\to\infty$ and imposes momentum conservation in the xy plane for all the correlation functions.

The $\langle X_iY\rangle$ and $\langle X_iX_j\rangle$ averages can be similarly evaluated,

$$\langle X_iY\rangle_0=-i\partial_{z_i}\sum_{j=1}^{l+m}\langle\mathbf{r}(z_i)\mathbf{k}_j\cdot\mathbf{r}(z_j)\rangle_0 \tag{B7a}$$

$$=-i\sum_{j=1}^{l+m}\mathbf{k}_jg(z_i-z_j)\ , \tag{B7b}$$

where

$$g(z_i-z_j)=\tfrac{1}{2}\partial_{z_i}\langle \mathbf{r}(z_i)\cdot\mathbf{r}(z_j)\rangle_0 \tag{B8a}$$

$$=-\frac{2}{\epsilon}\int_0^\infty \frac{dq}{2\pi}\frac{\sin q(z_i-z_j)}{q} \tag{B8b}$$

$$=-\frac{1}{2\epsilon}\,\mathrm{sgn}(z_i-z_j)\ , \tag{B8c}$$

with $\mathrm{sgn}(z)=1$ for $z>0$ and $\mathrm{sgn}(z)=-1$ for $z<0$:

$$\langle X_i X_j\rangle_0=\partial_{z_i}\partial_{z_j}\langle \mathbf{r}(z_i)\cdot\mathbf{r}(z_j)\rangle_0 \tag{B9a}$$

$$=2\int_q e^{iq(z_i-z_j)} \tag{B9b}$$

$$=2\delta(z_i-z_j)\ . \tag{B9c}$$

Combining the above equations inside Eq. (B1) we find for odd m the cumulants vanish and for even m we obtain,

$$\Gamma^{(l,m)}_{a_1\cdots a_m}(\mathbf{k}_1,z_1\cdots\mathbf{k}_{l+m},z_{l+m})$$
$$=\left[\frac{1}{\epsilon}\right]^{m/2}[P^T_{a_1a_2}\delta(z_1-z_2)P^T_{a_3a_4}\delta(z_3-z_4)\cdots P^T_{a_{m/2-1}a_{m/2}}\delta(z_{m/2-1}-z_{m/2})+\text{ All pair combinations}]$$
$$\times\exp\left[\frac{k_BT}{2\epsilon}\sum_{i>j}^{l+m}\mathbf{k}_i\cdot\mathbf{k}_j|z_i-z_j|\right]\ . \tag{B10}$$

For Gaussian approximation that we are concerned with in the main text we obtain,

$$\Gamma^{(2,0)}(\mathbf{k},z_1,-\mathbf{k},z_2)=\exp\left[-\frac{k_BT}{2\epsilon}k^2|z_1-z_2|\right]\ , \tag{B11a}$$

$$\Gamma^{(0,2)}_{a_1a_2}(\mathbf{k},z_1,-\mathbf{k},z_2)=\frac{k_BT}{\epsilon}P^T_{a_1a_2}\delta(z_1-z_2)\times\exp\left[-\frac{k_BT}{2\epsilon}k^2|z_1-z_2|\right]\ , \tag{B11b}$$

$$\Gamma^{(1,1)}_{a_1}(\mathbf{k},z_1,-\mathbf{k},z_2)=0\ . \tag{B11c}$$

These expressions when Fourier transformed to q space give Eqs. (A13) of Appendix A.

APPENDIX C: NONINTERACTING FLUX-LINE LIQUID DYNAMICS

In this appendix we apply methods similar to Appendix B to calculate the dynamic cumulants

$$\Gamma^{(2)}_{ab}(x_1,x_2)=\langle\rho_a(x_1)\rho_b(x_2)\rangle_0\ , \tag{C1}$$

where $x=(\mathbf{k},z,t)$ and $a,b=1,2$. The procedure is similar to the one used for statics. The averages are performed with the exponential of the single flux-line dynamic functional $\mathcal{J}_0[\tilde{\mathbf{r}},\mathbf{r}]$, Eqs. (3.9) and (3.10), instead of with the static Boltzmann factor. As explained in the main text in the derivation of hydrodynamics we traced over the tangent density field and its corresponding response field. The longitudinal part of the dynamic tangent correlation function can be extracted from those of the number density field by using flux-line continuity Eq. (A3), while as for the statics, the transverse part of the tangent correlation function is probably not affected by the flux-line interaction or disorder.

In order to calculate the averages in Eq. (C1) we introduce "Rouse modes,"

$$\mathbf{r}(q,t)=\frac{1}{L}\int_0^L dz\cos(p_qz)\mathbf{r}(z,t)\ , \tag{C2a}$$

$$\mathbf{r}(z,t)=\mathbf{r}_0(t)+2\sum_{q\geq 1}\cos(p_qz)\mathbf{r}(q,t)\ , \tag{C2b}$$

where $p_q=\pi q/L$ and L is the length of the flux line. $\mathbf{r}_0(t)$ denotes the center-of-mass mode. From the single-line dynamic functional $\mathcal{J}_0[\tilde{\mathbf{r}},\mathbf{r}]$, Eqs. (3.9) and (3.10), one can derive the following correlation functions for the Rouse modes,

$$\langle r^a(q',0)r^b(q,t)\rangle_0=\frac{k_BT}{2L\epsilon p_q^2}e^{-D\epsilon p_q^2|t|}\delta_{qq'}\delta_{ab}\ , \tag{C3a}$$

$$\langle[r_0^a(t)-r_0^a(0)][r_0^b(t)-r_0^b(0)]\rangle_0=\frac{2Dk_BT}{L}|t|\delta_{ab}\ , \tag{C3b}$$

$$\langle\tilde{r}^a(q',0)r^b(q,t)\rangle_0=-\frac{i}{2L}\theta(t)e^{-D\epsilon p_q^2|t|}\delta_{qq'}\delta_{ab}\ , \tag{C3c}$$

$$\langle\tilde{r}_0^a(0)r_0^b(t)\rangle_0=-\frac{i}{L}\theta(t)\delta_{ab}\ , \tag{C3d}$$

$$\langle\tilde{r}_0^a(0)\tilde{r}_0^b(t)\rangle_0=0\ . \tag{C3e}$$

Using the above equations we deduce for the segment-correlation function of the internal modes

$$\langle[r^a(z,t)-r^a(z',0)][r^b(z,t)-r^b(z',0)]\rangle_0$$
$$=\delta_{ab}\left[\frac{2Dk_BT}{L}|t|+\frac{2L}{\pi^2}\sum_{q\geq 1}\frac{k_BT}{\epsilon q^2}\{\tfrac{1}{2}[\cos(2p_qz)+\cos(2p_qz')]+\cos[p_q(z+z')]e^{-D\epsilon p_q^2|t|}\}\right.$$
$$\left.+\frac{2L}{\pi^2}\sum_{q\geq 1}\frac{k_BT}{\epsilon q^2}\{1-\cos[p_q(z-z')]e^{-D\epsilon p_q^2|t|}\}\right]\ . \tag{C4}$$

This has to be inserted in the corresponding expression for the single-line correlation function $S^0(\mathbf{k},z,z',t) = \Gamma_{22}^{(2)}(\mathbf{k},z,z',t)$

$$\Gamma_{22}^{(2)}(\mathbf{k},z,z',t)=\langle \rho(-\mathbf{k},z,t)\rho(\mathbf{k},z',0)\rangle_0 =\exp\{-\tfrac{1}{4}k^2\langle[\mathbf{r}(z,t)-\mathbf{r}(z',0)]^2\rangle_0\} . \quad \text{(C5)}$$

Since we are dealing with a finite-size system there is no translational invariance with respect to the z axis, and therefore correlation functions like Eq. (C5) depend explicitly on z and z'. But, except for the small region where z and/or z' are close to one of the ends of a flux line the second term in Eq. (C4) can be neglected. This is due to the rapid oscillations of cosine terms such as $\cos(2p_q z)$. In the rest of this appendix we will neglect those contributions and hence get expressions which depend only on the relative coordinate $(z-z')$.

For $t \geq t_{\text{Rouse}} = L^2/D\epsilon$ the summation $\sum_{q\geq 1}$ is rapidly converging. If $t \ll t_{\text{Rouse}}$ the sums can be replaced by integrals

$$\frac{L}{\pi^2}\sum_{q\geq 1}\frac{1}{q^2}\{1-\cos[p_q(z-z')]e^{-D\epsilon p_q^2|t|}\} \rightarrow |z-z'| f\left[\frac{D\epsilon|t|}{(z-z')^2}\right] , \quad \text{(C6)}$$

where

$$f(y)=\frac{1}{\pi}\int_0^\infty \frac{dx}{x^2}[1-e^{-yx^2}\cos x] . \quad \text{(C7)}$$

The function $f(y)$ can be written in terms of the incomplete gamma function $\Gamma(a,x)$ as

$$f(y)=\frac{1}{2}+\frac{1}{4\sqrt{\pi}}\Gamma\left[-\frac{1}{2},\frac{1}{4y}\right] , \quad \text{(C8)}$$

with the asymptotic representations

$$f(y)=\begin{cases} \dfrac{\sqrt{y}}{\pi}\left[1+\dfrac{1}{4y}+\cdots\right] & \text{for } y \gg 1 , \\ \dfrac{1}{2}+\dfrac{2y^{3/2}}{\sqrt{\pi}}e^{-1/4y}(1-6y+\cdots) & \text{for } y \ll 1 . \end{cases} \quad \text{(C9)}$$

With the latter definitions we find for the two-point function

$$\Gamma_{22}^{(2)}(\mathbf{k},z,t)=\exp\left[-\frac{Dk_BTk^2}{L}|t| -k^2\frac{k_BT}{\epsilon}|z| f\left[\frac{D\epsilon|t|}{z^2}\right]\right] . \quad \text{(C10)}$$

Since $f(0)=\frac{1}{2}$, we recover the static result of Appendix B as we must for the equal time, $t_i-t_j=0$, correlation function.

Next we consider the single-line response function

$$\tilde{S}^{(0)}(\mathbf{k},z,z',t)=\Gamma_{21}^{(2)}(\mathbf{k},z,z',t) =\langle \rho(-\mathbf{k},z,t)\tilde{\rho}(\mathbf{k},z',0)\rangle_0 ,$$

again in a regime where z and z' are not too close to one of the ends of the flux line. We find

$$\Gamma_{21}^{(2)}(\mathbf{k},z,t)=iDk_BTk^2\tfrac{1}{2}\langle \tilde{\mathbf{r}}(0,0)\cdot\mathbf{r}(z,t)\rangle_0 \exp\left\{-\frac{Dk_BTk^2}{L}|t|-k^2\frac{k_BT}{\epsilon}|z| f\left[\frac{D\epsilon|t|}{z^2}\right]\right\}$$

$$=\theta(t)k^2Dk_BT\left[\frac{1}{L}+\left[\frac{1}{4\pi D\epsilon|t|}\right]^{1/2}e^{-z^2/(4D\epsilon|t|)}\right]\exp\left\{-\frac{Dk_BTk^2}{L}|t|-k^2\frac{k_BT}{\epsilon}|z| f\left[\frac{D\epsilon|t|}{z^2}\right]\right\} , \quad \text{(C11a)}$$

$$\Gamma_{12}^{(2)}(\mathbf{k},z,t)=\Gamma_{21}^{(2)}(\mathbf{k},z,-t) . \quad \text{(C11b)}$$

For the correlation function $\Gamma_{11}^{(2)}(\mathbf{k},z,t)$ one gets

$$\Gamma_{11}^{(2)}(\mathbf{k},z,t)=-(Dk_BT\mathbf{k})^2\exp\left\{-\frac{Dk_BTk^2}{L}|t|-k^2\frac{k_BT}{\epsilon}|z| f\left[\frac{D\epsilon|t|}{z^2}\right]\right\}$$
$$\times\left[\frac{1}{2}\langle\tilde{\mathbf{r}}(z,t)\cdot\tilde{\mathbf{r}}(0,0)\rangle_0-\frac{k^2}{4}[\langle\tilde{\mathbf{r}}(z,t)\cdot\mathbf{r}(z,t)\rangle_0-\langle\tilde{\mathbf{r}}(z,t)\cdot\mathbf{r}(0,0)\rangle_0]\right.$$
$$\left.\times[\langle\tilde{\mathbf{r}}(0,0)\cdot\mathbf{r}(z,t)\rangle_0-\langle\tilde{\mathbf{r}}(0,0)\cdot\mathbf{r}(0,0)\rangle_0]\right] . \quad \text{(C12)}$$

All the terms inside the large parentheses in Eq. (C12) vanish. $\langle\tilde{\mathbf{r}}(z,t)\cdot\tilde{\mathbf{r}}(0,0)\rangle_0=0$ as can be seen from Eq. (3.10b), and the equal-time correlators $\langle\tilde{\mathbf{r}}(z,t)\cdot\mathbf{r}(z,t)\rangle_0$ and $\langle\tilde{\mathbf{r}}(0,0)\cdot\mathbf{r}(0,0)\rangle_0$ vanish by the causality together with our choice of causal-time discretization, Eq. (3.14). Finally, the remaining term

$$\langle\tilde{\mathbf{r}}(z,t)\cdot\mathbf{r}(0,0)\rangle_0\langle\tilde{\mathbf{r}}(0,0)\cdot\mathbf{r}(z,t)\rangle_0=0 ,$$

because it is proportional to $\theta(t)\theta(-t)$, which vanishes by the definition of $\theta(t)$ and by the choice of causal-time discretization. Therefore,

$$\Gamma_{11}^{(2)}(\mathbf{k},z,t)=0 . \quad \text{(C13)}$$

We now observe that $\Gamma_{22}^{(2)}(\mathbf{k},z,t)$ and $\Gamma_{12}^{(2)}(\mathbf{k},z,t)$ satisfy the standard fluctuation dissipation theorem (FDT),

$$\partial_t \Gamma_{22}^{(2)}(\mathbf{k},z,t)=\Gamma_{12}^{(2)}(\mathbf{k},z,t)-\Gamma_{12}^{(2)}(\mathbf{k},z,-t)\ . \tag{C14}$$

This is the reason why in the main text we called $\rho_1=\tilde{\rho}$ the response field of the physical density field $\rho_2=\rho$ and called $\Gamma_{22}^{(2)}=S^0$ and $\Gamma_{21}^{(2)}=\tilde{S}^0$ the correlation and response functions, respectively.

Now we discuss the correlation functions in various limiting cases. For $k_BTk^2L/\epsilon \ll 1$, where the wavelength is much larger than the static transverse "radius of gyration" $\sqrt{2k_BTL/\epsilon}$ of the flux line, the dominant term in Eq. (C10) is the first term. This term describes the center-of-mass motion of a single flux line. We find exponentially decaying correlation functions

$$S^0(\mathbf{k},z,t)|_{k_BTk^2L/\epsilon\ll 1}\approx \exp\left[-\frac{Dk_BTk^2}{L}|t|\right], \tag{C15a}$$

$$\tilde{S}^0(\mathbf{k},z,t)|_{k_BTk^2L/\epsilon\ll 1}\approx \theta(t)\frac{Dk_BTk^2}{L}\times\exp\left[-\frac{Dk_BTk^2}{L}|t|\right]. \tag{C15b}$$

All the other terms are of order $O(k^2L)$ and can therefore be neglected.

For $k_BTk^2L/\epsilon \gg 1$, where the wavelength is much smaller than the radius of gyration, the internal modes, i.e., the second term in Eq. (C10), dominate provided t is not too large. For times $t\ge t_{\text{Rouse}}$ the center-of-mass motion becomes the dominant contribution again, since the contribution from the internal modes scales as $\sqrt{t}$ for large times [note $f(y\gg 1)\sim\sqrt{y}$]. Hence we get for $t\ll t_{\text{Rouse}}$

$$S^0(\mathbf{k},z,t)|_{k_BTk^2L/\epsilon\gg 1}\approx \exp\left\{-k^2\frac{k_BT}{\epsilon}|z|f\left[\frac{D\epsilon|t|}{z^2}\right]\right\}, \tag{C16a}$$

$$\tilde{S}^0(\mathbf{k},z,t)|_{k_BTk^2L/\epsilon\gg 1}\approx\theta(t)k^2\frac{k_BT}{2}\sqrt{D/\pi\epsilon t}\ \exp\left[-\frac{z^2}{4D\epsilon t}\right]\exp\left\{-k^2\frac{k_BT}{\epsilon}|z|f\left[\frac{D\epsilon t}{z^2}\right]\right\}. \tag{C16b}$$

Finally, we study the Fourier transforms of these noninteracting structure and response functions to be used in the main text to calculate the interacting counterparts. In the center-of-mass limit one gets a Lorentzian shape for the correlation function ($f(q=0)=\int_0^L dz\, f(z)$),

$$S^0(\mathbf{k},q=0,\omega)|_{k_BTk^2L/\epsilon\ll 1}\approx\frac{2L\Gamma_{\text{cm}}(\mathbf{k})}{\omega^2+\Gamma_{\text{cm}}^2(\mathbf{k})}, \tag{C17a}$$

$$\tilde{S}^0(\mathbf{k},q=0,\omega)|_{k_BTk^2L/\epsilon\ll 1}\approx\frac{L}{-i\omega+\Gamma_{\text{cm}}(\mathbf{k})} \tag{C17b}$$

with the line width

$$\Gamma_{\text{cm}}(\mathbf{k})=\frac{Dk_BT}{L}k^2\ . \tag{C18}$$

In the limit where the internal modes dominate we find

$$\tilde{S}^0(\mathbf{k},q,\omega)=\frac{\epsilon}{k_BT}\frac{4}{\sqrt{\pi}k^2}\int_0^\infty d\hat{t}\, e^{i\hat{\omega}\hat{t}}\int_0^\infty dx\,\cos(\hat{q}\sqrt{\hat{t}}x)e^{-x^2/4-\sqrt{\hat{t}}m(x)}, \tag{C19}$$

where we have introduced the scaling variables $\hat{\omega}=\omega/\Gamma_{\mathbf{k}}$, $\hat{q}=2q\epsilon/(k_BTk^2)$, and $\Gamma_{\mathbf{k}}=(k_BT)^2Dk^4/(4\epsilon)$, the noninteracting dynamic relaxation rate. We have also defined a function $m(x)$ by

$$m(x)=x+x\frac{1}{2\sqrt{\pi}}\Gamma\left[-\frac{1}{2},\frac{x^2}{4}\right]. \tag{C20}$$

Expanding in powers of $\hat{\omega}$ we obtain

$$\tilde{S}^0(\mathbf{k},q,\omega)=\frac{\epsilon}{k_BT}\frac{4}{\sqrt{\pi}k^2}\sum_{n=0}^{\infty}i^n b_n(\mathbf{k},q)\hat{\omega}^n \tag{C21}$$

with the coefficients

$$b_n(\mathbf{k},q)=b_n(\hat{q})=2^{n+1}(2n+1)!!\int_0^\infty dx\, e^{-x^2/4}\frac{\cos[(2n+2)\arctan(\hat{q}x/m(x))]}{[(\hat{q}x)^2+m^2(x)]^{n+1}} \tag{C22}$$

with the expansions

$$b_n(q,\mathbf{k})=\begin{cases}\sim\hat{q}^{\,-2n-2} & \text{for } \hat{q}\gg 1\\ b_n-\hat{q}^{\,2}b_n' & \text{for } \hat{q}\ll 1\ ,\end{cases} \qquad \text{(C23)}$$

where the coefficients are given by

$$b_n=2^{n+1}(2n+1)!!\int_0^{\infty}dx\, e^{-x^2/4}m(x)^{-2n-2}\ , \qquad \text{(C24a)}$$

$$b_n'=\frac{(2n+3)!}{n!}\int_0^{\infty}dx\, e^{-x^2/4}x^2m(x)^{-2n-4}\ . \qquad \text{(C24b)}$$

The expression for $S^0(\mathbf{k},q,\omega)$ can now be easily obtained from the above expression for $\tilde{S}^0(\mathbf{k},q,\omega)$ and the FDT, Eq. (C14). In the limit of small frequencies one gets to leading order

$$\tilde{S}^0(\mathbf{k},q,\omega)\approx\frac{\epsilon}{k_BT}\frac{4}{\sqrt{\pi}k^2}[b_0(\hat{q})+ib_1(\hat{q})\hat{\omega}-b_2(\hat{q})\hat{\omega}^2]\ , \qquad \text{(C25a)}$$

$$S^0(\mathbf{k},q,\omega)\approx\frac{2}{\Gamma_{\mathbf{k}}}\frac{\epsilon}{k_BT}\frac{4}{\sqrt{\pi}k^2}[b_1(\hat{q})-b_3(\hat{q})\hat{\omega}^2]$$
$$\approx\frac{2}{\Gamma_{\mathbf{k}}}\frac{\epsilon}{k_BT}\frac{4}{\sqrt{\pi}k^2}\frac{b_1^2(\hat{q})/b_3(\hat{q})}{\hat{\omega}^2+b_1(\hat{q})/b_3(\hat{q})}\ . \qquad \text{(C25b)}$$

In the limit $(\omega\to 0)$ the Fourier transform $S^0(\mathbf{k},q,\omega)$ can also be written in the Lorentzian form

$$S^0(\mathbf{k},q,\omega)\approx A_s(\mathbf{k},q)\frac{2\Gamma_{\text{single}}(\mathbf{k},q)}{\omega^2+\Gamma_{\text{single}}(\mathbf{k},q)^2}\ , \qquad \text{(C26)}$$

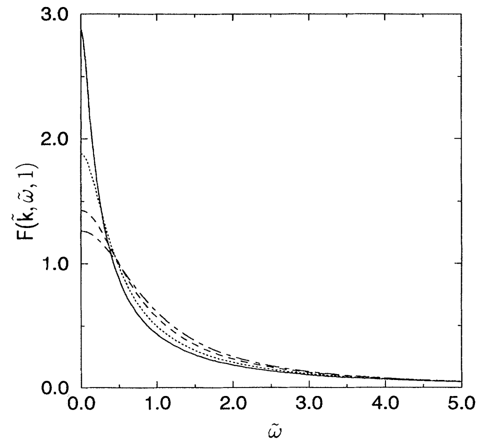

FIG. 6. Scaling functions F for a fixed value of $\kappa=1.0$ and a series for $\tilde{k}^{-1}$ [$=0.0001$ (solid), 0.5 (dotted), 1.0 (dashed), 2.0 (dot-dashed)].

where we have introduced the quantities

$$A_s(\mathbf{k},q)=2\frac{\epsilon}{k_BT}\frac{4}{\sqrt{\pi}k^2}\frac{b_1^{3/2}(\mathbf{k},q)}{b_3^{1/2}(\mathbf{k},q)}\ , \qquad \text{(C27)}$$

and

$$\Gamma_{\text{single}}(\mathbf{k},q)=\Gamma_{\mathbf{k}}\left[\frac{b_1(\mathbf{k},q)}{b_3(\mathbf{k},q)}\right]^{1/2}\ . \qquad \text{(C28)}$$

In the limit $2q\epsilon/(k_BTk^2)\ll 1$, $A_s(\mathbf{k},q)$ reduces to

$$A_s(\mathbf{k},q)=2\frac{\epsilon}{k_BT}\frac{4}{\sqrt{\pi}k^2}\frac{b_1-6(q\epsilon/k_BTk^2)^2b_1'}{b_3-2(q\epsilon/k_BTk^2)^2b_3'}$$
$$\approx 2\frac{k^2k_BT}{\sqrt{\pi}\epsilon}\frac{b_1/b_3}{(k^2k_BT/2\epsilon)^2+q^2[3b_1'/2b_1-b_3'/2b_3]}\ , \qquad \text{(C29)}$$

where b_n, b_n' are constants of order $O(1)$. The formula for $A_s(\mathbf{k},q)$ is quite similar in form to the static structure factor of noninteracting lines

$$S_s^0(\mathbf{k},q)=\frac{k^2k_BT/\epsilon}{q^2+(k^2k_BT/2\epsilon)^2}\ . \qquad \text{(C30)}$$

In the same limit the linewidth reads

$$\Gamma_{\text{single}}(\mathbf{k},q)=\Gamma_{\mathbf{k}}\left[\frac{k^2k_BT}{2\epsilon}\right]^2\times\frac{b_1/b_3}{(k^2k_BT/2\epsilon)^2+q^2[b_1'/2b_1-b_3'/2b_3]}\ . \qquad \text{(C31)}$$

Finally, we discuss the Fourier transform of the single-line correlation function in the limit $q=0$

$$S^0(\mathbf{k},q=0,\omega)=\int_0^L dz\int_{-\infty}^{\infty}dt\, e^{i\omega t}S^0(\mathbf{k},z,t)\ . \qquad \text{(C32)}$$

Upon introducing the scaling variables $\tilde{\omega}=\omega/[\Gamma_{\mathbf{k}}(1+\tilde{k}^{-2})]$, $\tilde{k}^2=k^2Lk_BT/(4\epsilon)$, and $\kappa=\Gamma_{\mathbf{k}}t_{\text{Rouse}}$, where the latter variable gives the ratio of the Rouse time to the decay time of density fluctuations, we obtain

$$S(\mathbf{k},q=0,\omega)=\frac{L}{\Gamma_{\mathbf{k}}(1+\tilde{k}^{-2})}F(\tilde{k},\tilde{\omega},\kappa) \qquad \text{(C33)}$$

with

$$F(\tilde{k},\tilde{\omega},\kappa)=\frac{2}{\sqrt{\kappa}}\int_0^{\infty}d\tau\cos(\tilde{\omega}\tau)\exp\left[-\frac{\tilde{k}^{-2}}{1+\tilde{k}^{-2}}\tau\right]\int_0^{\sqrt{\kappa}}dy\,\exp\left\{-\sqrt{\tau/(1+\tilde{k}^{-2})}\,m[y\sqrt{(1+\tilde{k}^{-2})/\tau}]\right\}\ . \qquad \text{(C34)}$$

The above scaling analysis of the single-line correlation function shows that there are two time scales. First, there is the time scale for the dynamics of the internal modes given by $t_{\text{internal}}=1/\Gamma_{\mathbf{k}}$. Then there is a time scale $t_{\text{Rouse}}=L^2/(D\epsilon)$, which gives the crossover time above which the dynamics starts to be dominated by the center-of-mass mode. Furthermore, we have a length scale $R_G^{2\text{D}}=\sqrt{2k_BTL/\epsilon}$, which describes the projected 2D radius of gyration.

For $\tilde{k}\ll 1$, i.e., for wavelengths much larger than the 2D projected radius of gyration, the dynamics is determined by the center-of-mass motion, and we find for the scaling function

$$F(\tilde{k},\tilde{\omega},\kappa)=\frac{1-e^{-\sqrt{\kappa}}}{\sqrt{\kappa}}\frac{2}{1+\tilde{\omega}^2}\ , \tag{C35}$$

which corresponds to Eq. (C17a), provided κ is small, i.e., the Rouse time is small compared to the typical relaxation time for internal modes.

For $\tilde{k}\gg 1$ the internal modes dominate the dynamic structure factor. In Fig. 6 the scaling function F is shown for a fixed value of $\kappa=1.0$ and a series for $\tilde{k}^{-1}$ $(=0.0001, 0.5, 1.0, 2.0)$. For any value of κ and $\tilde{k}$ the curves are rather well represented by

$$F(\tilde{k},\tilde{\omega},\kappa)\approx\frac{a_0}{1+a_1\tilde{\omega}^{\gamma}}\ , \tag{C36}$$

where the coefficients a_0, a_1 and the exponent γ depend on $\tilde{k}$ and κ. In the limit of small $\tilde{k}$ the scaling function F turns into a Lorentzian with $\gamma=2$ and $a_0=1.264$, $a_1=1.0$, which just corresponds to the Lorentzian shape in the center-of-mass limit [compare Eq. (C17a) and Eq. (C35)]. The exponent γ decreases with increasing $\tilde{k}$. The best nonlinear fit for $\tilde{k}^{-1}=0.0001$ gives $a_0\approx 2.93$, $a_1\approx 5.93$, and $\gamma\approx 1.344$. Hence, the line shape crossover can essentially be characterized in terms of the effective exponent γ. The typical linewidth $\Gamma(k,L)$ of the correlation function is given by the condition $\tilde{\omega}=1$, i.e.,

$$\Gamma(k,L)=\Gamma_{\mathbf{k}}\left[1+\frac{4\epsilon}{k_BT}\frac{1}{k^2L}\right]\ , \tag{C37}$$

which describes the crossover from a dynamics determined by the internal modes to a dynamics governed by the center-of-mass motion.

We note that $S^0(\mathbf{k},z,t)$ and $\tilde{S}^0(\mathbf{k},z,t)$ from Eqs. (C10) and (C11) satisfy the static limit sum rules, as they must,

$$S_s^0(\mathbf{k},q)=S^0(\mathbf{k},q,t=0)$$
$$=\int\frac{d\omega}{2\pi}S^0(\mathbf{k},q,\omega)\ , \tag{C38a}$$

$$\tilde{S}_s^0(\mathbf{k},q)=\tilde{S}^0(\mathbf{k},q,\omega=0)\ . \tag{C38b}$$

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[37]In the regime where $L \gg L_I$, the crossover in the interacting flux-line liquid will occur at $(k_B T/\epsilon)k^2 L_I = 1$.