ac and dc properties of Josephson-junction arrays with long-range interaction

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We present an experimental investigation of arrays with long-range interaction, together with a model to explain observed deviations from the predictions of earlier theoretical work. These arrays consist of N horizontal and N vertical superconducting filaments arranged in two parallel planes separated by an oxide layer, so that every wire is Josephson coupled to every other wire in the array as nearest- or next-nearest neighbors. We have performed ac-susceptibility and dc-transport measurements on both ordered and disordered arrays. Our ac measurements show a strong feature at a temperature T_c, which we interpret as a transition to the macroscopically phase-coherent state. We find that this feature is field dependent in ordered arrays, but not in disordered arrays. dc-transport measurements reveal that these arrays have unexpectedly low critical currents and show voltage steps in their I-V curves; moreover, they are *hysteretic* despite the fact that they consist of *nonhysteretic* junctions. Our analysis of these results shows that the finite ratio of the wire inductance to the Josephson inductance cannot be ignored since it limits the effective number of Josephson junctions along a wire to $N_{\text{eff}} \ll N$.

I. INTRODUCTION

In a recent paper,[1] we presented our theoretical and numerical investigation of a type of Josephson-junction array whose interesting properties in the disordered limit had already been theoretically studied by Vinokur *et al.*[2] These arrays consist of two orthogonal sets of N parallel superconducting *wires* which are coupled by a Josephson junction at every point of crossing (see Fig. 1). As a result of this unique geometry, all the wires in the array are Josephson coupled to each other as nearest- or next-nearest neighbors. This is in contrast to conventional Josephson-junction arrays in which the superconducting elements are *islands*—each one of which is coupled to only four or six nearest-neighbor islands.[3] Whereas conventional arrays have *short-range* interaction, the special arrays we will discuss have *long-range* interaction.

Through Monte Carlo simulations and a mean-field analysis,[1] we have shown that, in zero magnetic field, arrays with long-range interaction undergo a phase transition to a macroscopically phase-coherent state at the temperature $T_c = NE_J/2k_B$, where $E_J = \hbar i_c/2e$ and i_c is the critical current of a single junction in the array. When a field, corresponding to a commensurate number of flux quanta per unit cell, $f = p/q$ (where p and q are small integers), is applied to the ordered arrays, T_c is reduced to $NE_J/2k_B\sqrt{q}$, provided $q < N$. For positionally disordered arrays, i.e., arrays in which the distance between the parallel wires of each orthogonal set is randomly varied, T_c is defined for different regions. When $f < 1/N^2$, the field is ineffective and $T_c \approx NE_J/2k_B$. When $1/N^2 < f < 1/N$, $T_c \approx E_J/2k_B\sqrt{f}$, varying from $\sim NE_J/2k_B$ down to $\sim \sqrt{N}E_J/2k_B$ as f increases. The field at which $f = 1/N$ is the same as the field H_0 defined by Vinokur *et al.*,[2] which is sufficient to generate a flux quantum through the average-sized strip between two adjacent wires, thus causing sufficient phase variation so that the currents add *incoherently*, yielding the $\sqrt{N}$ dependence. After a transition regime when $1/N \le f < 1$, which depends on the strength of the disorder, T_c approaches an asymptotic value for $f \gg 1$, which is $\sim 0.75\sqrt{N}E_J/k_B$ according to our simulations and $\sim 0.5\sqrt{N}E_J/k_B$ according to Vinokur *et al.* The source of this minor quantitative discrepancy is not known at this time.

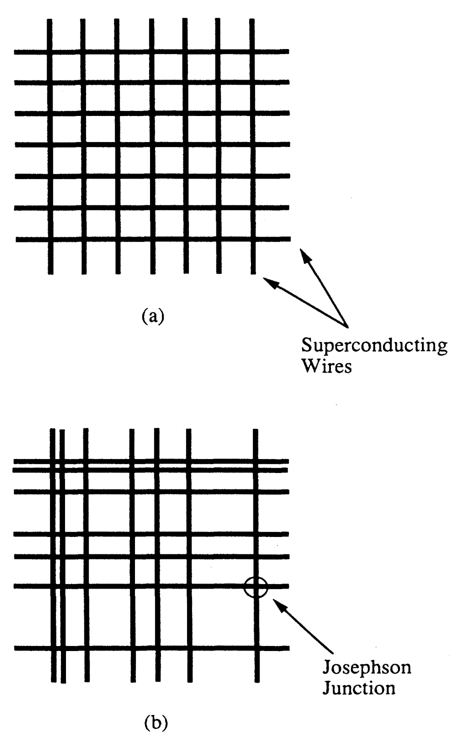

FIG. 1. Sections of an (a) ordered and (b) disordered array. The straight lines are the superconducting wires. Each wire is coupled to every orthogonal wire by a Josephson junction.

In this paper we present an *experimental* investigation of these arrays. Our arrays consist of niobium (Nb) superconducting wires separated by an aluminum-oxide (Al_2O_3) barrier; the wires therefore are coupled to each other via tunnel junctions. We have performed both dc-transport and ac-susceptibility measurements on these arrays: the former to make quantitative measurements on such nonlinear properties as the critical current of the system and the latter to detect the onset of long-range phase coherence. A description of our sample fabrication and measurement techniques can be found in Sec. II of this paper. Our results, presented in Sec. III, show that real arrays with long-range interaction are far *less* simple than theoretical work[1,2] on a simplified model would suggest. Finally, in Sec. IV, we present our analysis of the experimental results and our preliminary conclusions about their interpretation.

II. SAMPLE FABRICATION AND MEASUREMENT TECHNIQUES

As we have noted, in zero field the array's transition to the phase-coherent state is predicted to occur at a temperature $T_c = NE_J/2k_B$. In order that T_c lie well below T_{c0}, the superconducting transition temperature of the individual wires, the product NE_J must be less than $\sim k_B T_{c0}$. For our fabricated arrays, N is chosen to be large (600 or 1000) so that (i) the macroscopic size of the array would be large enough for us to conduct ac-susceptibility measurements and (ii) localized defects and inhomogeneities would have less effect on the basic properties of the arrays. Consequently, for $T_c < T_{c0}$, i_c needs to be extremely small, $i_c < 0.6$ nA. One way of attempting to achieve this experimentally is to employ *tunnel* junctions, and in fact we use Nb-Al_2O_3-Nb junctions. By increasing the thickness of the oxide layer of our junctions, we can reduce the single-junction critical current i_c in our arrays.

Because of our unique array geometry, we are unable to fabricate Nb-Al_2O_3-Nb junctions in an ideal manner, i.e., by fabricating a Nb-Al_2O_3-Nb trilayer *in situ* and reactive-ion etching the Nb to create the junctions.[4] Instead, we find that we must fabricate the junctions in several steps which are *not in situ:* This lowers the achievable quality of the junctions. First, we sputter 2000 Å of Nb onto a previously patterned oxidized-silicon substrate. Following a lift-off and repatterning process, we clean the surface of the Nb using a rf argon plasma. This cleaning is crucial since it removes the native oxides NbO_x that formed when the sample had been exposed to air.[5] We then immediately sputter a thin layer (25 Å) of aluminum onto the sample and oxidize it in pure oxygen for 10 min at 500 mT. Once the chamber is reevacuated, we sputter an additional layer (25 Å) of Al directly onto the oxide layer and subsequently oxidize it for an additional 30 min at 500 mT. This two step aluminum-oxidation process is used to ensure a thick oxide barrier and hence junctions with small critical currents. Following the oxidation, we sputter, *in situ,* an additional layer (2000 Å) of Nb onto the sample to form our counter electrodes. A simple lift-off is then performed and our sample is diced.

The ordered arrays we studied consist of either 1000×1000 or 600×600 Nb wires with a lattice constant of either 5.5 or 10 μm, respectively. The disordered arrays consist of 600×600 Nb wires, which are arranged such that the distance between adjacent wires, Δx, is normally distributed with a mean distance $\Delta \bar{x}$ of 10 μm and a standard deviation $\sigma_{\Delta x}$ of 2 μm. All wires, whether in the ordered or disordered arrays, are 2.5 μm wide, making the junctions in the arrays 2.5 $\mu\text{m} \times 2.5$ μm. Two single junctions with the same geometry as those in the arrays are made concurrently on the same substrate on either side of the array. The critical currents of these junctions typically agreed within 20–30%, allowing us to estimate i_c of the junctions in the array and also giving some indication of its homogeneity.

Because of the array configuration, "busbar" electrodes on two opposing edges of the array to feed in a uniform-bias current are not included in the overall design of the array, as they usually are in conventional arrays. The busbars are omitted because they would be shorted together by the superconducting wires of one set, and consequently the N^2 junctions would not play any role in the dynamics of the system. Likewise, busbars on two perpendicular array edges are not included since each busbar would short the wires of its set, effectively transforming the entire array into one very large Josephson junction.

We perform two-probe measurements (in the sense that separate voltage and current leads go down to the same pad) on our fabricated arrays.[6] (Four-probe measurements, however, are performed on the single junctions.) Two different current-feed orientations are tried on each of the arrays. In orientation A [Fig. 2(a)], the current is injected into one wire and extracted from another of the *same* set. In orientation B [Fig. 2(b)], the current is injected into one wire of one set and extracted from a wire of the *other* set. As will be shown in the following section, these two different current-feed orientations result in very different I-V curves.

In addition to dc-transport measurements, we also perform contactless ac measurements on our arrays. To accomplish this, we use a two-coil mutual-inductance technique similar to that outlined by Jeanneret *et al.*[7] Specifically, we use a drive coil, which consists of approximately 23 equally spaced turns, and a pair of astatically wound receive coils, each of which consists of two layers of approximately 15 turns.[8] The receive coils are coaxially mounted within the drive coil and adjusted so that they are as close to the sample as possible while still being balanced with respect to the drive coil. The in- and out-of-phase components of the voltage δV at the receive coil (due to the screening currents flowing through the sample in response to an ac current with frequency ω) are detected by conventional lock-in techniques.

All dc and ac measurements are performed in a closed screened room. For the dc measurements, 1-kΩ resistors are attached to the leads coming from the sample. These resistors, nominally at the same temperature as the sample, help to filter out room-temperature noise. In addition, a dc-voltage source is used to power the heater that we use to control the temperature for our dc measure-

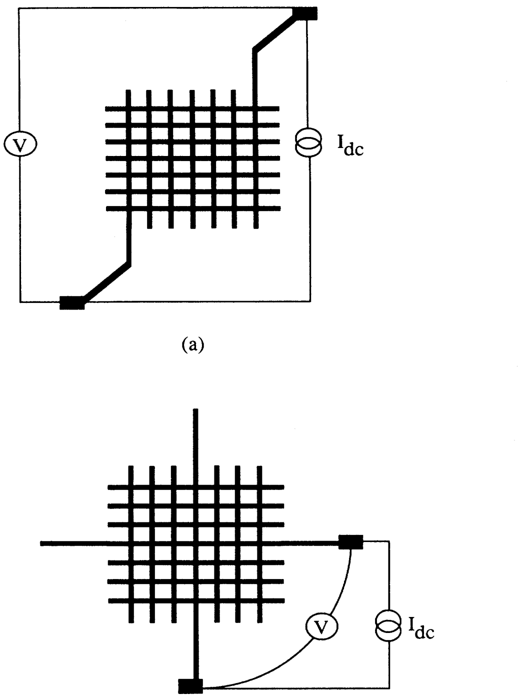

FIG. 2. Two different current-feed orientations we tried. (a) Current-feed orientation A: the current is fed into one wire of one set and taken out of another of the *same* set. (b) Current-feed orientation B: the current is fed into one wire of one set and taken out of a wire of the *other* set.

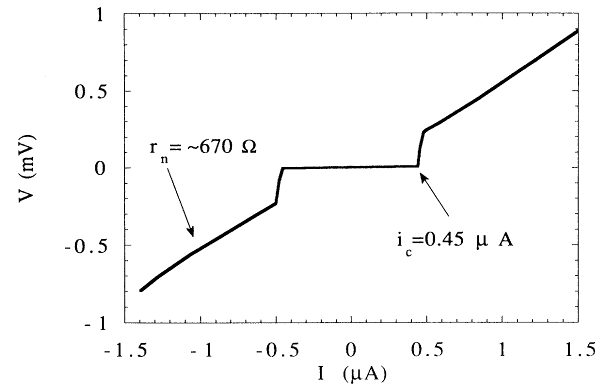

FIG. 3. Representative I-V curve of one of our fabricated single junctions. The junction is nonhysteretic, has a critical current i_c of 0.45 μA, and a normal-state resistance r_n of 670 Ω.

ments since we found that our digital temperature controller can send noise down to our sample and "smear" the I-V curves.

III. RESULTS

A. dc measurements

All of the single junctions we measured have normal-state resistances r_n of approximately 670 Ω and critical currents i_c ranging from 0.45 to 1.5 μA at $T=1.7$ K. (The $i_c r_n$ products of these junctions, therefore, range from 0.3 to 1 mV, compared to the ideal value of $i_{c0} r_n = \pi\Delta/2e = 2.1$ mV). All the junctions appear to be resistively shunted, indicating nonideal tunnel barriers. Figure 3 shows a representative I-V curve of one of the single junctions we measured. As can be seen, the junction is clearly nonhysteretic and has a critical current i_c of 0.45 μA and a normal-state resistance r_n of 670 Ω. These rather disappointing results reflect our less-than-ideal means of fabricating Nb-Al_2O_3-Nb junctions. We would like to emphasize, however, that the rather poor quality of the junctions should not drastically affect the arrays' properties since they are expected to be determined primarily by the *magnitude* of i_c. It is clear, however, that the i_c values obtained (even at $T=4$ K, where we have measured i_c to be as low as 0.08 μA) are $\sim$100–1000 times larger than the nA range required to conform to our theoretical target.

Upon measuring the various arrays in current-feed orientation A, we found that the array critical currents I_c range from 11.3 to 33.0 μA at $T=1.7$ K. In current-feed orientation B, we found I_c ranging from 2 to 5 μA. The I_c values we obtained for either current-feed orientation are far less than the $I_c = N i_c \sim 1$ mA values that might be anticipated from having $N(\approx 1000)$ junctions, each with critical current $i_c(\approx 1\mu$ A), in parallel. Also striking is the fact that both the ordered and disordered arrays, in either current-feed orientation, show *hysteretic* behavior [Figs. 4(a) and 4(b)], even though each consists of *nonhysteretic* junctions. Sweeping the dc current up and down, we observe that the arrays produce a range of critical (I_c) and retrapping (I_r) currents [see inset to Fig. 4(b)]. One might attribute this premature switching to thermal-activation processes occurring within the arrays, but this seems unlikely since none of the single junctions show any signs of such switching. Alternatively, this switching may be a consequence of the multiply connected topology of the array.

The observation of voltage "steps" in both the critical- and retrapping-current portions of the array I-V curves is surprising. As shown in Fig. 4(b), the steps along the upsweep portion of the curves are much more visible than those along the downsweep portion. Premature switching from one step to another is again observed [see inset to Fig. 4(b)]. Comparing the I-V curves resulting from the two different current-feed orientations [Figs. 4(a) and 4(b)], we observe many more steps in orientation B, i.e., where the current is injected in one wire of one set and extracted from a wire of the other set. However, curves derived from both current-feed orientations show a decrease in the number of steps as the temperature is increased. The resistance R_1 of the array after the first step was measured to be 35 Ω when the current was injected in and extracted from wires of the same set [Fig. 4(a)] and to be $R_1 = 71\Omega$ when the current is injected in a wire of one set and extracted from a wire of the other set [Fig. 4(b), inset].

If we zero-field-cool (ZFC) the *ordered* arrays in

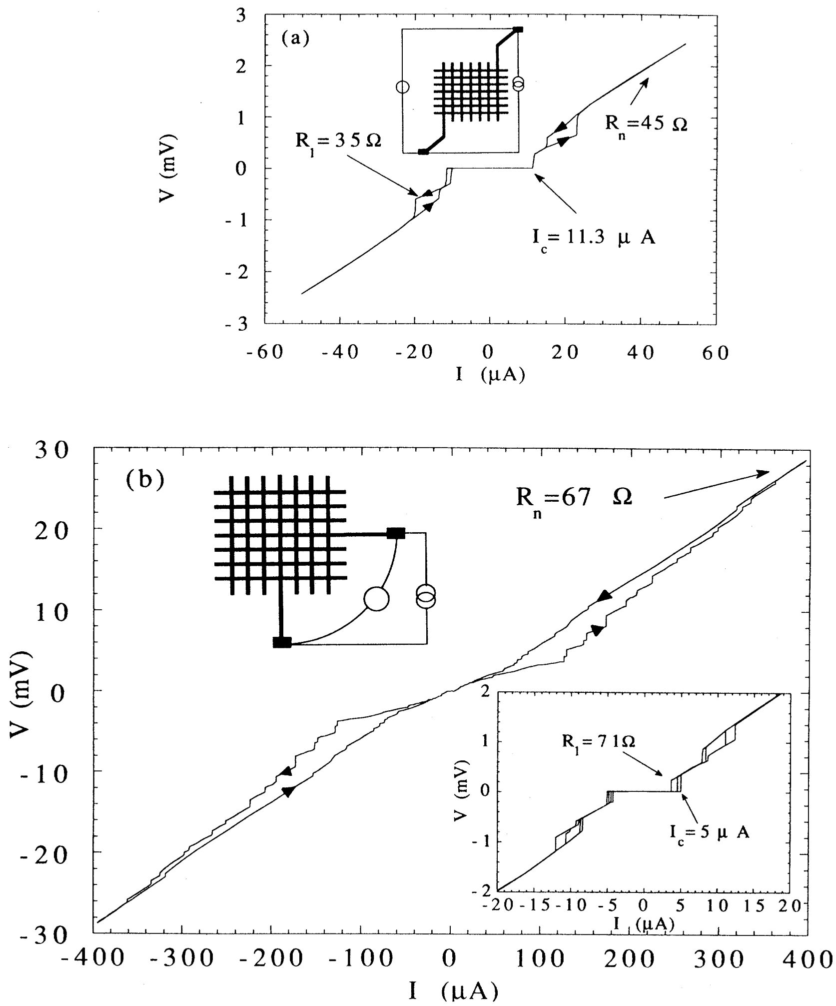

FIG. 4. I-V curves obtained at $T=1.7$ K from the two different current orientations shown in Fig. 2. (a) I-V curve for current-feed orientation A. $I_c=11.3\ \mu$A and $R_1=35\ \Omega$. (b) I-V curve for current-feed orientation B. Inset: magnification of region near origin. $I_c=5\mu$A and $R_1=71\Omega$. All curves show hysteresis and steplike structure.

current-feed orientation B, we find that I_c is dramatically affected when a field is subsequently applied. For each ZFC run, we changed the field strength with the overall effect that I_c periodically oscillates with respect to field strength (see Fig. 5). The period of oscillation corresponds to an area of $\sim 400\mu\text{m}^2$; the possible significance of this area will be discussed briefly in Sec. IV. We note that I_c is symmetric about $f=0$ and remains zero for $|f|\geq 0.27$. Unlike those of the ordered arrays, the critical currents of *disordered* arrays do not show any visible oscillation with varying field strengths; rather, they show a constant nonzero value with respect to the different field strengths we tried (from $f=0$ up to 5.75).

B. ac measurements

All of the ac-susceptibility results we present in this paper were obtained using a frequency range of 80–100 kHz, since we achieved maximum signal output in this range. The ac drive-current amplitude $I_{D\omega}$ was on the order of microamperes and produced fields of 10–100 μG at the sample. Such fields are small enough for the sample to give a linear response, in the sense that doubling the drive $I_{D\omega}$ doubles the sample's response $\delta V(\omega)$. (We have confirmed this experimentally.)

In neither the ordered nor the disordered arrays do we observe the superconducting transition of Nb (which, for our films, occurs at ~ 8.8 K). The individual Nb wires give a negligible diamagnetic response to the ac field because of their very small dimensions. We do observe, however, a strong signal, corresponding to the array's broadened phase transition to a macroscopically phase-coherent state, at temperatures which range from 3 to 5 K, depending on the sample [see Figs. 6(a) and 6(b)]. A magnetic field ($f=1/q$, where $q=2,3,4,5,6$) applied to the ordered arrays leads to a marked reduction in the transition temperature T_c. As shown in Fig. 7, small fields (i.e., large q) suppress T_c more than do large fields

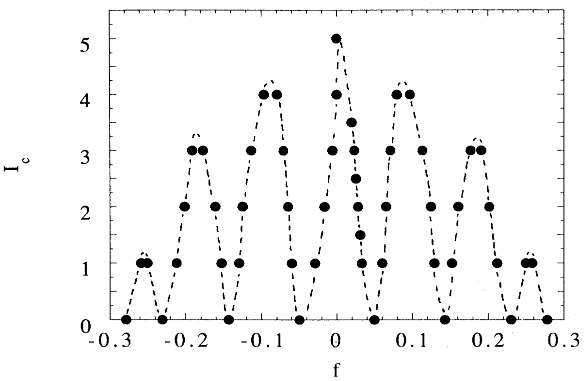

FIG. 5. Critical current I_c vs field of an ordered 1000×1000 array with a lattice constant of 5.5 μm and in current-feed orientation B. The critical current shows a periodic oscillation with decreasing amplitude when the magnetic-field strength is varied. The period of oscillation corresponds to an area of $\sim 400\ \mu m^2$. I_c is zero for $|f| \geq 0.27$.

(i.e., small q). This qualitative behavior corresponds well with that predicted by our mean-field theory,[1] which is also plotted in Fig. 7, but the observed depression of T_c is only about a quarter as large as the ideal prediction. Although we have not measured T_c for deliberately incommensurate fields, we expect that $T_c(f)$ is a continuous function with much substructure.

The T_c of *disordered* arrays, on the other hand, is not observed to be appreciably affected by a magnetic field between $H=0$ and $H=H_0$ or even much larger fields and whether zero-field cooled or field cooled (FC). This surprising observation is contrary to that theoretically predicted by Vinokur *et al.*[2] and by us.[1] Our initial thought was that, even in nominally zero field, an inhomogeneous stray field might be present that was large enough to cause the array to be in the high-field, constant-T_c regime of a disordered array. This regime, as defined by theory,[1,2] corresponds to $H > H_0$, where H_0 is the field value giving one flux quantum through an average-sized strip between two adjacent wires, which corresponds to $f=1/N$. For our disordered arrays, this H_0 is only 0.3 mG. Thus, if an inhomogeneous stray field were present that could not be nulled everywhere to <0.3 mG, the array would be in the high-field, constant-T_c regime. Although this interpretation now seems implausible in view of the evidence presented below that our arrays show an effective size $N_{\text{eff}} \ll N$, this consideration remains a challenge to future experiments on samples which better fulfill the conditions of the theory.

If we sweep the transverse magnetic field at temperatures near the transition, we find that the ac response $\delta V(f)$ of the ordered arrays, like that of conventional arrays,[9] shows a complex oscillatory behavior. Figures 8(a)–8(c) show plots of the ac response vs field. Here we see that strong peaks in both the in- and out-of-phase components of the signal develop as we decrease the temperature from just above T_c to just below.[10] These peaks correspond to commensurate field strengths $f=0,\frac{1}{2},\frac{1}{3},\ldots,\frac{1}{8}$. As shown in Fig. 9, only a single peak

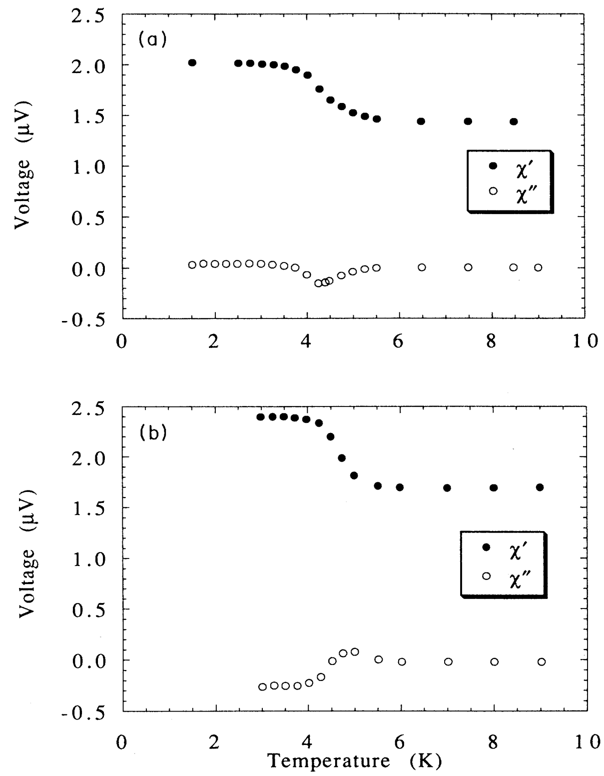

FIG. 6. ac response vs temperature for the (a) 1000×1000 ordered and (b) 600×600 disordered array. χ' and χ'' are the components of the measured signal which are in and out of phase relative to the reference phase (giving a pure in-phase signal in the normal state). The ac amplitude used to make the measurement was 0.5 μA for the ordered array and 0.25 μA for the disordered array; the frequency used was 100 kHz. The strong signal at $T\sim4$ K in both arrays indicates the arrays' transitions to the macroscopically phase-coherent state.

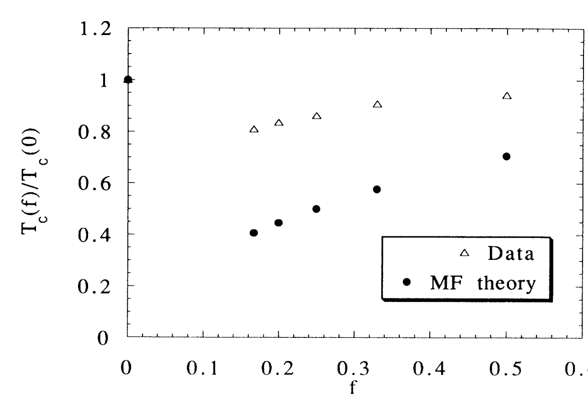

FIG. 7. Normalized transition temperature $T_c(f)/T_c(0)$ vs number of flux quanta, f, per unit cell for our experimental data obtained from a 1000×1000 ordered array and our results from mean-field theory. The data correspond to fields $f=1/q$, where $q=2,3,4,5,6$.

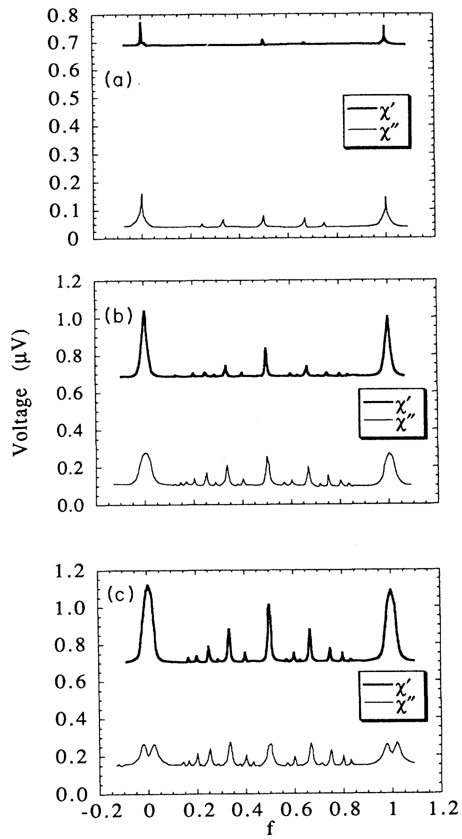

FIG. 8. Real and imaginary components of the complex ac response of a 1000×1000 ordered array to a varying magnetic field. The data were taken at (a) $T=4.25$ K, (b) $T=3.75$ K, and (c) $T=3.5$ K at a frequency of 100 kHz. The excitation current used was 2.1 μA. As temperature is decreased from just above T_c (which for this array is ~ 4 K) to well below T_c, the number of peaks increases. These peaks correspond to array commensurability with field strengths $f=0, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{8}$.

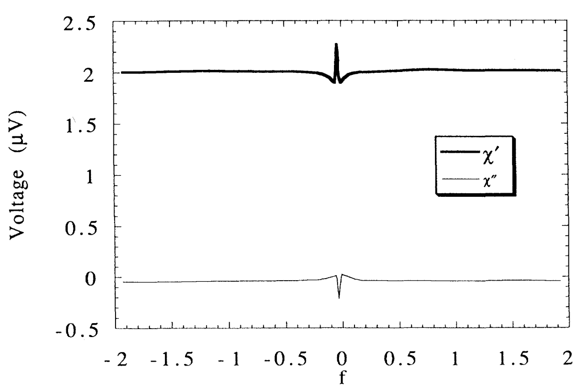

FIG. 9. Complex ac response of a 600×600 disordered array to a varying magnetic field. Only one peak, corresponding to $f=0$ is observed. The T_c of the array was 5.0 K and the data shown were taken at $T=4.75$ K and a frequency of 87.0 kHz. The excitation current used was 0.5 μA.

corresponding to $f=0$ appears in our disordered-array data. The fact that the disordered arrays do *not* show any commensurability effects is qualitatively consistent with the fact that the cell size in the lattice varies too much for any nonzero field to be commensurate with it.

IV. DISCUSSION

Given our experimental results, it is evident that the properties of real arrays with long-range interaction are not as straightforward as previous theoretical work[1,2] would suggest. As stated in the previous section, the zero-field critical current I_c of the arrays is much less than the value Ni_c expected naively from the fact that there are N junctions in parallel which take current from the wire into which it is originally fed. In addition, we observed our arrays undergoing a phase transition at $T_c \sim 4$ K, despite the fact that i_c is ~ 100 times greater than that required to obtain that value of T_c from the theoretical model.[1,2] Obviously, the theoretical model should be refined to account for these major discrepancies.

We suggest that the major oversimplification of the standard theoretical model is that it assumes that both the kinetic and electromagnetic inductance of the wires connecting adjacent junctions are negligible compared to the Josephson inductance ($\hbar/2ei_c$) of the junctions in the array. Consequently, the phase gradient along any wire in the array was assumed to arise *only* from the presence of an external magnetic field and not from the small currents circulating through the weak Josephson junctions. The Hamiltonian describing this system is given by the sum of individual Josephson-junction energies:

$$H = -\operatorname{Re} E_J \sum_{i=1}^{N} \sum_{j=1}^{N} \exp[i(\varphi_i^h - \varphi_j^v - A_{ij})] \ . \tag{1}$$

Here φ_i^h is the superconducting phase at $x=0$ of the ith horizontal wire, φ_j^v is the phase at $y=0$ of the jth vertical wire,

$$A_{ij} = \frac{2\pi}{\Phi_0} \int_i^j \mathbf{A} \cdot d\mathbf{l} \ ,$$

where $\mathbf{A} = Hx\hat{\mathbf{y}}$ (with H being the externally applied field), and Φ_0 is the flux quantum. As will be shown below, this assumption—that the fields and phase gradients produced by circulating currents are negligible—is incorrect when dealing with our experimental system.

A. dc properties

We start by considering the case of dc bias, in which we are feeding current in one wire and out another, in zero applied magnetic field. The current-induced action is dominated by the spreading of the current from the feed wire into the $N \sim 1000$ cross wires and the symmetrical ingathering of the current into the exit wire. It is convenient to take advantage of this symmetry and assign equal and opposite phases to the two drive wires. Since the current spreads so widely in the cross wires, thus causing much smaller phase gradients in them, we can to a good approximation treat only the phase along the current-fed wire, taking all the cross wires as having the same (approximately zero) phase because of the symmetric boundary conditions at the two leads. The phase difference between the positions of cross wires n and $n+1$ along the current-fed wire in an array is $(2\pi I_n L_w/\Phi_0)$, where I_n is the current flowing through the wire at the nth junction site and L_w is the inductance (kinetic plus electromagnetic) of the wire segment in a cell defined by the cross wires, the current-fed wire, and the wire parallel to it (see Fig. 10). In the simple case of a small number of extremely weak Josephson junctions in an array, I_n is small, because not much current is needed to reach the critical current of all the junctions. Consequently, the phase gradient along the current-fed wire is small and we expect that $I_c \approx N i_c$. However, in the case of a large number of stronger Josephson junctions, I_n can be quite large, because a huge current is needed to reach the critical current of all the junctions in parallel. As a result, the phase gradient along the current-fed wire can no longer be ignored. It is this current-induced phase shift along the wire which limits the effective number of junctions in parallel to some $N_{\rm eff} \ll N$ such that $I_c = N_{\rm eff} i_c$; this explains why I_c is far less than might have been expected.

Based on the considerations outlined in the previous paragraph, we can proceed to estimate $N_{\rm eff}$ as follows: The difference in the phases at successive junctions along the current-feed wire is given by

$$\frac{d\varphi_n}{dn} = \frac{2\pi I_n L_w}{\Phi_0} . \tag{2}$$

Taking the derivative of Eq. (2) and noting that dI_n/dn stems from the current $I_c \sin\varphi_n$ transferred to the nth cross wire (all of which are assumed to have phase of approximately zero as noted above), we obtain the sine-Gordon equation

$$\frac{d^2\varphi_n}{dn^2} = \frac{1}{\lambda_n^2}\sin\varphi_n , \tag{3a}$$

where

$$\lambda_n = \left[\frac{\Phi_0}{2\pi L_w i_c}\right]^{1/2} . \tag{3b}$$

Note that Eq. (3a) has the same form as the familiar pendulum equation.

For small phase differences, i.e., $I \ll I_c$ and $\varphi_n \ll 1$, we can linearize Eq. (3a) to get

$$\frac{d^2\varphi_n}{dn^2} = \frac{\varphi_n}{\lambda_n^2} . \tag{4}$$

This has the solution $\varphi_n \sim e^{\pm n/\lambda_n}$. If $\lambda_n \ll N$, this implies that the current will leak off the current-fed wire before getting very far from the input. Then, if φ_0 is the applied phase, we have $\varphi_n = \varphi_0 e^{-n/\lambda_n}$ and total current

$$I = i_c \sum \sin\varphi_n \approx i_c \sum \varphi_n = i_c \varphi_0 \lambda_n . \tag{5}$$

The total phase difference, including the symmetric part of equal magnitude φ_0 from the exit current wire, is $2\varphi_0 = \varphi_{\rm tot}$. Thus, in the linear regime,

$$\frac{dI}{d\varphi_{\rm tot}} = \frac{1}{2}\lambda_n i_c , \tag{6}$$

so that the array acts like $(N_{\rm eff})_{\rm linear} = \lambda_n/2$ junctions in parallel.

To find I_c of the array, we need to return to the nonlinear equation (3a), since we must consider the case in which φ_n is no longer small. Fortunately, we can draw on the analogy to a pendulum described by (3a), in which φ_n is the angle of the pendulum from the *top* of its circular arc at time t_n, with both φ and t being treated as continuous variables and with $\lambda_n^{-2} = g/L$, where L is the length of the pendulum. The nonlinear generalization of the exponentially decaying solution to Eq. (4) discussed above corresponds to the motion of the pendulum launched from an angle φ_0 at $t=0$ with a velocity $d\varphi/dt|_0$ such that it just comes to rest (exponentially) at the top of the circle. Conservation of energy can be used to find the relation between φ_0 and $d\varphi/dt|_0$ such that this particular motion ensues. Converted back to the problem at hand, this relation is

$$\left[\frac{d\varphi_n}{dn}\right]^2\Bigg|_0 = \frac{2}{\lambda_n^2}(1-\cos\varphi_0) . \tag{7}$$

Recalling Eq. (2), we see that this equation relates the square of the current injected at $n=0$ to the corresponding value of the phase φ_0. The maximum possible current then clearly occurs for $\varphi_0 = \pi$, in which case Eq. (7) yields $d\varphi/dn|_0 = 2/\lambda_n$. Substituting in from Eq. (2), we obtain

$$I_{\rm max} = 2\lambda_n i_c = \left[\frac{2\Phi_0 i_c}{\pi L_w}\right]^{1/2} . \tag{8}$$

Note that, by this nonlinear criterion, $(N_{\rm eff})_{I_c} = 2\lambda_n$, as opposed to the $\lambda_n/2$ obtained above from the linear-response criterion. Thus, in general, we expect an $N_{\rm eff} \sim \lambda_n$, with the numerical coefficient depending on which physical property is being considered.

The pendulum analogy can also be used to gain insight into the behavior at current inputs exceeding $I_{\rm max}$. This case corresponds to the pendulum going round and

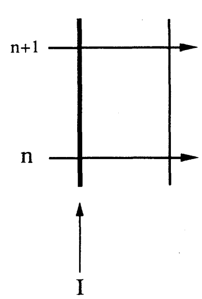

FIG. 10. Schematic of a section of an array in which current is being fed in one wire.

round, accelerating and decelerating under gravity, but continuing round in the same direction, whatever the initial value of phase φ_0. In the array case, this corresponds to current feeding out into the cross wires where the pendulum is rising and current feeding back in from the cross wires where it is accelerating down again. Since the net current outflow cancels out over a full cycle, the maximum net supercurrent is that found in Eq. (8), which corresponds to a single half cycle from $\varphi=\pi$ to 0. Thus any current in excess of $I_{\max}$ must feed out to the cross wires as a normal current, causing a resistive voltage to develop. Insofar as the superconductivity in the wire does not break down and each wire remains an equipotential along its length, one might expect the resistance above $I_{\max}$ to correspond to the full N junctions in parallel, instead of $N_{\text{eff}} \ll N$. However, that is not what is found experimentally. Instead, the resistive slope R_1 above $I_{\max}$ corresponds quite closely to only N_{eff} junctions in parallel, so that the $I_c R_1$ product turns out to be very similar to that of a single junction. Although we have not yet been able to extend our solution from the static regime below I_c to this much more complex dynamic regime, we conclude from the empirical data that the current flow is restricted to only some $N_{\text{eff}} \approx \lambda_n$ junctions at each lead wire in the resistive as well as the zero-voltage regime.

To examine the quantitative comparison of our model with our experimental data, we first calculate the total inductance $L_w = L_{\text{em}} + L_{\text{kin}}$ of an array wire. We estimate the electromagnetic inductance L_{em} using the formula[11]

$$L_{\text{em}} \approx 2l \left[\ln\frac{4l}{p} + \frac{1}{2} + 0.1118\frac{p}{l} \right] \times 10^{-7}\text{H} ,$$

where l is the length (in one cell) and p is the perimeter ($2d+2w$, where d is the thickness and w is the width) of the given wire. For our specific dimensions, $L_{\text{em}} \approx 2.2\times 10^{-12}$ H. We estimate the kinetic inductance L_{kin} using the definition

$$L_{\text{kin}} = \mu_0 \lambda^2 \frac{l}{\sigma} ,$$

where λ is the penetration depth of the Nb wire and σ is the cross-sectional area of the wire.[12] If we estimate $\lambda = 1000$ Å for our nonideal Nb wires, we find $L_{\text{kin}} \approx 10^{-13}$ H. Since this is much less than L_{em}, $L_w \approx L_{\text{em}}$. When we insert this estimated L_w and the different measured i_c's into Eq. (8), we find that the predicted I_c is a slight overestimate of our experimentally observed I_c. This overestimate may well simply reflect the crudeness of the model. Alternatively, it might be explained by a noting that in the parallel problem of H_{c1} in a long Josephson junction the solution corresponding to that above actually describes a metastable state, which is thermodynamically stable only up to a field $(2/\pi)$ times smaller.[13] If we apply that factor of $2/\pi$ here, we might expect the apparent critical current to be

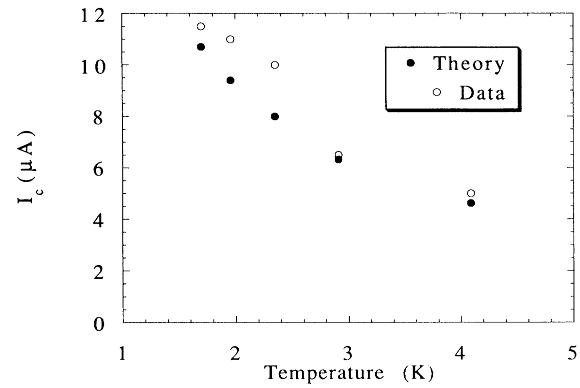

FIG. 11. Comparison between our experimental data and our model calculation of the critical current I_c of an array.

$$I_c = \frac{4}{\pi}\lambda_n i_c = \frac{2}{\pi}\left[\frac{2\Phi_0 i_c}{\pi L_w}\right]^{1/2} . \tag{9}$$

As shown in Fig. 11, the values for I_c obtained from Eq. (9) are in surprisingly good agreement with those observed from our arrays for the current-feed configuration A (both feed wires in same set), which gives us some confidence in the general model.

As stated earlier, the theoretical results obtained by Vinokur *et al.*[2] and by us[1] are for wires whose electromagnetic inductances are so small that, in zero field, there is no phase gradient along them. What we have just shown is that the electromagnetic inductance ($\sim 10^{-12}$ H per cell) is *not* completely negligible compared to the Josephson inductance ($\sim 10^{-9}$ H) of the junctions in the array. Consequently, to be realistic, a finite ratio of these two inductances *must* be included as an additional parameter for the variation of the phase along a wire. Such an inclusion leads to a sine-Gordon equation [Eq. (3a)] that has a characteristic dimensionless length scale λ_n given by the *square root* of the inductance ratio. Thus, for an $N\times N$ array, the inductance ratio would have to be $\geq N^2$ ($\sim 10^{-6}$ in our case) in order that $\lambda_n \geq N$, so that the simple theory would be valid. For our arrays, $\lambda_n \sim 46$ at $T_c = 4$ K (where $i_c \sim 0.08\mu$A) and ~ 20 at $T=1.7$ K (where $i_c \sim 0.45\mu$A). Consequently, the effective number of junctions, N_{eff}, coupled to the current-fed wire, is not 1000; rather, it is [using Eq. (9)] $I_c/i_c = 4\lambda_n/\pi \sim 59$ at T_c and ~ 25 at $T=1.7$ K. As noted above, the resistance of the first step, R_1, in Fig. 4(a) measured at $T=1.7$ K also corresponds to approximately N_{eff} junctions—each with $r_n \sim 670\Omega$—in parallel.

The behavior of the array in the dc-bias case in which current is injected into one wire of one set and extracted from a wire of the other set (current-feed configuration B) is much more complicated at high current levels than the case we have just discussed. Suffice it to say, that near the critical-current level the array again acts as if some effective number $N'_{\text{eff}} \ll N$ of junctions in parallel determine the critical current and initial resistance of the array. As indicated in the inset of Fig. 4(b), $I_c \sim 5\mu$A and $R_1 \sim 71\Omega$. These values correspond to approximately 11 junctions in parallel. This number for $N'_{\rm eff}$ can be related to the data we obtained by measuring I_c vs field (see Fig. 5). As we have already noted, one period of oscillation in this figure corresponds to an extended "Josephson junction" with an area of 400 μm^2 or a 13×1 unit "block" of cells. Thus the measurements of I_c, R_l, and the magnetoresistance all give a reasonably consistent estimate of the effective number of junctions through which the current spreads from the feed wires.

The discussion above has focused on the initial breakdown in superconductivity at I_c. However, the experimental *I-V* curves show two successive breakdowns at current values differing roughly by a factor of 2. We interpret these as reflecting the independent breakdown of superconductivity separately at the entrance and exit wires for the current, giving rise to additive voltages since they are in series. If the array were ideal, these two breakdowns would be superimposed at the same current value. Given the expected nonuniformity across these large and nonideal arrays, however, it is plausible that the local values of i_c could vary by a factor of 4 between the two sides of the array, causing the observed factor of 2 difference in $I_c \sim \lambda_n i_c$. Such inhomogeneities could also account for the numerically different results found for I_c in the current-feed configurations *A* and *B*. However, for current-feed orientation *B*, the current flows in a more complicated manner, since there are many more inequivalent paths, corresponding to many different sets of parallel junctions, through which the current can flow. Therefore it is not surprising that the resulting *I-V* curves at *high* currents, when breakdowns are occurring at other junctions than those directly involving the feedwires, have many more steps than those of current-feed orientation *A*.

Contrast the step behavior in our arrays to that found in conventional arrays. Both van der Zant *et al.*[14] and Tighe, Johnson, and Tinkham[15] report that the *I-V* curves of underdamped conventional arrays display steps in the *I-V* curves which correspond to the simultaneous switching of an entire single row of junctions across the width of the array into the resistive state. In their case there are as many steps as there are rows in the array and they are all equivalent in size. In our case the step structure is more complicated and has a less direct interpretation. As the temperature is lowered, the number of voltage steps increases. This may be a consequence of there being a greater number of independent units of size λ_n, since this length gets smaller as i_c increases with decreasing temperature. The nonuniform step widths are most likely due to nonuniformities in the array.

We attribute the curiously hysteretic behavior of the arrays to collective phenomena occurring in the array, which we can only qualitatively explain. The initial condition of the array largely determines the paths along which the current flows in the array. When we begin to sweep the current up, the array is in a *static* state. When we begin to sweep the current down from the above I_c, the array is in a *dynamic* state. Since the two initial conditions are very different, we expect the current to flow in different patterns even at the same total current, thus producing hysteresis.

B. ac measurements

The results of our ac measurements of T_c are the least well understood. We have measured transition temperatures in the rather narrow range 3–5 K in both our ordered and disordered arrays. As stated earlier, the measured single-junction critical current at the T_c of the array, $i_c(T_c)=0.08\mu A$, so that $E_J(T_c)/k_B=1.9$ K. This is ~ 250 times larger than that required to fit the T_c data if one uses $N\sim 1000$ in the formula $T_c=NE_J/2k_B$ given by the analysis of Vinokur *et al.*[2] or our mean-field approximation.[1] If the phase transition we measured in our arrays is indeed of the sort which is predicted, it would suggest that each wire in the array is effectively coupled to only $N''(\sim 4) \ll N(\sim 1000)$ wires. Our calculation of an $N_{\rm eff}\sim 10-50$ for interpreting our dc critical-current measurements provides a qualitative basis for such a reduced effective number, but this N'' need not *equal* the $N_{\rm eff}$ which gives the value of I_c. Since T_c is determined by the onset of *weak* phase ordering, it might be reasonable to expect N'' to be more like $(N_{\rm eff})_{\rm linear}$ (which we found to be $\lambda_n/2$) rather than the nonlinear value $(N_{\rm eff})_{I_c}(\sim 2\lambda_n)$. If so, this would suggest values of N'' which were ~ 4 times smaller than $(N_{\rm eff})_{I_c}$, that is, $N''\sim 3-12$. This would at least account for the rough order of magnitude of what is found. These numbers for N'' are so small that a relatively large external field $f\sim 1/N''$ would be required in the disordered array case to substantially reduce T_c by replacing N'' by $\sqrt{N''}$. This is certainly a subject for future investigation.

The ac response of the ordered arrays measured under a swept magnetic field, shown in Fig. 8, shows that they, like conventional arrays, display commensurability with certain fields $f=p/q$. Here the intensity of the feature falls monotonically with q and vanishes for $q\geq 8$. This means that the array wires have phase gradients, caused by the magnetic fields, which lead to small circulating current loops. We have already shown in a previous paper that these circulating loops resemble those found in conventional arrays under the same field strength.[1] In that work we also showed that finite-size $N\times N$ arrays display commensurate structure only for $q\leq N$. In the present context, one might expect that result to become $q\leq N''$, and indeed the features do fade out for q in this range. The same line of reasoning may account qualitatively for the fact that the depression of T_c by commensurate fields shown in Fig. 7 is less than predicted for an infinite array. The disordered arrays do not show any commensurability with any fields because, as stated previously, the cell size in the array varies too much for any field to be commensurate with the array lattice.

Thus we see that the unique geometry of arrays with long-range interaction leads to the arrays having properties which are quite different from those of conventional arrays. As we have shown in this paper, the wires in the arrays must be treated as "imperfect", i.e., there is a current-induced variation of the phase along any given wire in the array. This phase gradient is due to the fact that the electromagnetic inductance is *not* negligible compared to the Josephson inductance of the junctions. The ratio of these two inductances leads to a characteristic dimensionless length scale λ_n which limits the effective size of the array, at least in the dc-biased case.

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