Phase transitions in Josephson-junction arrays with long-range interaction

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We theoretically investigate ordered and disordered Josephson-junction arrays with long-range interaction. These arrays consist of two orthogonal sets of N parallel superconducting wires that are Josephson coupled to each other at every point of crossing. In this configuration, all wires, regardless of spatial separation, are nearest- or next-nearest neighbors. Using a mean-field approximation we show that the arrays undergo a phase transition to a macroscopically phase-coherent state at a temperature $T_c = NE_J/2k_B$ in the zero-field case. When a magnetic field, corresponding to a strongly commensurate number of flux quanta per unit cell, $f=p/q$, is introduced in an ordered array, we find that $T_c = NE_J/2k_B\sqrt{q}$. For the disordered case, T_c can be defined in four different regions of f. For $f<1/N^2$, $T_c \sim NE_J/2k_B$. For $1/N^2 < f < 1/N$, $T_c = E_J/2k_B\sqrt{f}$, and for $1/N < f < 1$, T_c rises with increasing f, although the exact form is unknown at this time. For $f>1$, T_c asymptotically approaches $\sim 0.85 E_J\sqrt{N}/k_B$. Our Monte Carlo simulations confirm all of our analytical calculations, except that our simulations show that the high-field asymptote approaches $\sim 0.75 E_J\sqrt{N}/k_B$.

I. INTRODUCTION

Two-dimensional arrays of Josephson junctions with short-range interaction are excellent models for exploring two-dimensional phase transitions,[1] flux pinning,[2] and glassy behavior.[3] These arrays consist of superconducting *islands* which are arranged in a geometric lattice (usually triangular or square) and are Josephson coupled to their nearest neighbors.[4] In this paper we present a theoretical and numerical investigation of a different type of Josephson-junction array whose interesting properties in the disordered limit have been theoretically investigated by Vinokur *et al.*[5] The arrays we examine differ from conventional Josephson-junction arrays in that they consist of two orthogonal sets of N parallel superconducting *wires* which are coupled to each other by a Josephson junction at every point of crossing (see Fig. 1). In the system discussed in the accompanying experimental paper,[6] the superconducting wires are lines of Nb film and the Josephson junctions coupling them are made with Al_2O_3 barriers. Because each vertical (horizontal) wire in the array is directly Josephson coupled to every horizontal (vertical) wire, we describe these arrays as having *long-range* interaction.

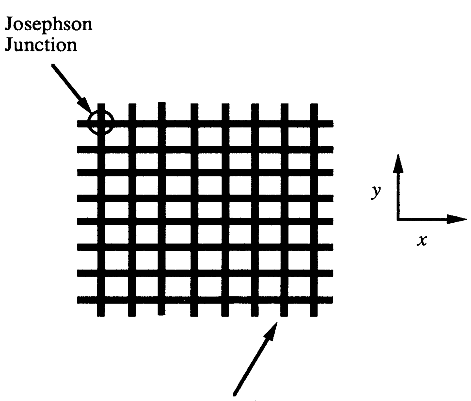

FIG. 1. Schematic drawing of an ordered 8×8 Josephson-junction array with long-range interaction. The straight lines are superconducting wires which are coupled together through Josephson junctions.

For the system we have examined, it is assumed that the Josephson inductance ($\hbar/2ei_c$) of each junction is infinitely greater than both the kinetic and electromagnetic inductance of the wires connecting adjacent junctions. Consequently, for any circulating current flowing through the array of Josephson junctions, one can assume that the phase gradient along any wire in the array arises *only* from the presence of an external magnetic field. (In fact, this condition is hard to satisfy experimentally, except in small arrays, because of the N junctions in parallel coupled to a single wire.[6]) The Hamiltonian of the system is thus given by the sum of individual Josephson-junction energies,

$$H = -\mathrm{Re} E_J \sum_{i=1}^{N} \sum_{j=1}^{N} e^{i(\varphi_i^h - \varphi_j^v - A_{ij})} . \tag{1}$$

Here φ_i^h is the superconducting phase at $x=0$ of the ith horizontal wire, φ_j^v is the phase at $y=0$ of the jth vertical wire, N is the number of wires in each direction or set, E_J is the Josephson-coupling energy, which we take to be constant, and

$$A_{ij} = \frac{2\pi}{\Phi_0} \int_0^{ai} A(x_j)dy ,$$

where $\mathbf{A} = Hx\hat{\mathbf{y}}$, $x_j = aj$, with a being the lattice constant, and Φ_0 is one flux quantum.

Using the Thouless-Anderson-Palmer (TAP) equation,[7] Vinokur *et al.* showed that in the presence of a strong transverse magnetic field disordered arrays with long-range interaction, i.e., those arrays in which the wires of each orthogonal set are randomly displaced, model a *spin glass*,[5] and they investigated the glassy dynamics of such a system. They showed that these arrays undergo a phase transition at a temperature

$$T_c = \frac{E_J N^{1/2}}{2^{3/2} k_B} \left[1 + \frac{2H_0}{H} + \left[1 + \frac{8H_0}{H} \right]^{1/2} \right]^{1/2} , \quad (2)$$

where $H_0 = \Phi_0 / Na^2$ is the field required to generate a flux quantum through the average-sized *strip* between two adjacent wires and H is the applied field.[5] Below this temperature the random phases of the individual wires, φ_i, *freeze* into a macroscopically phase-coherent state such that

$$\langle e^{i\varphi_i} \rangle \neq 0 ,$$

without any long-range periodic order in their values. From (2) we see that, for $H \gg H_0$, T_c approaches, to lowest order in H_0/H, the limiting form

$$T_c = \frac{E_J N^{1/2}}{2k_B} \quad (3)$$

and, for $H_0/N \ll H \ll H_0$,

$$T_c = \frac{E_J N^{1/2}}{2k_B} \left[\frac{H_0}{H} \right]^{1/2} . \quad (4)$$

We have further studied these arrays in both the *ordered* and *disordered* limit by using the more simple mean-field approximation and Monte Carlo (MC) simulations. Despite general agreement between our results for the disordered arrays and those of Vinokur *et al.*,[5] some quantitative discrepancies do exist, the origin of which is unknown at this time.

The body of this paper is organized as follows. In Sec. II we describe our mean-field analysis of ordered and disordered arrays with long-range interaction. In Sec. III we present the results of our MC simulations of these arrays. A discussion of our work follows in Sec. IV.

II. MEAN-FIELD THEORY

A. Ordered arrays

To perform a mean-field analysis of ordered arrays with long-range interaction, we follow closely the mean-field analysis of Shih and Stroud of conventional Josephson-junction arrays.[8] Shih and Stroud obtained the thermodynamic properties of conventional arrays by treating the phases of the individual array islands, φ_i, as classical thermodynamic variables within the canonical ensemble.[8] Because of the long-range interactions in the arrays we now study (in contrast to the arrays previously studied), we are able to find analytical expressions for both the temperature-dependent order parameter of the system and the field-dependent transition temperature.

If we treat the phases of the individual wires in our arrays also as classical thermodynamic variables, we can calculate the order parameter of our system, $\eta_i^h = \langle \exp(i\varphi_i^h) \rangle$, using the expression

$$\eta_i^h \equiv \langle e^{i\varphi_i^h} \rangle \approx \frac{1}{Z_i} \int d\varphi_i^h e^{i\varphi_i^h} e^{-\beta H_i^{\mathrm{MF}}} . \quad (5)$$

Here H_i^{MF} is the contribution of the ith wire to the total mean-field (MF) Hamiltonian and is given by

$$H_i^{\mathrm{MF}} = -E_J \sum_{j=1}^{N} [\cos\varphi_i^h \langle \cos(\varphi_j^v + A_{ij}) \rangle + \sin\varphi_i^h \langle \sin(\varphi_j^v + A_{ij}) \rangle] , \quad (6)$$

where φ_j^v are the phases of the vertical wires which are directly coupled to the horizontal wire of phase φ_i^h. Z_i is the partition function.

$$Z_i = \int d\varphi_i e^{-\beta H_i^{\mathrm{MF}}} , \quad (7)$$

and $\beta = (1/k_B T)$. The expectation values in Eq. (6) are solved self-consistently from equations of the form of Eq. (5).

In zero field, $\eta_i^h = \eta_j^v \equiv \eta$, and consequently Eq. (5) can be rewritten after some reduction as

$$|\eta| = \frac{I_1(NE_J\beta|\eta|)}{I_0(NE_J\beta|\eta|)} , \quad (8)$$

where I_1 and I_0 are Bessel functions of the second kind. Figure 2 shows a numerical solution of $|\eta|$ as a function of $2k_B T/NE_J$ in zero field. Near the array's transition temperature, all the η's are small and, correspondingly, so are the expectation values in Eq. (6). Equation (8) can therefore be expanded for small argument to obtain

$$T_c = \frac{NE_J}{2k_B} . \quad (9)$$

The direct proportionality of the transition temperature to the number of horizontal or vertical wires in the array can be readily understood by recognizing the fact that in order for the system to enter the phase-coherent state, the energy per wire available for fluctuation ($k_B T$) must be on the order of NE_J, the total coupling energy between that wire and the wires orthogonal to it.

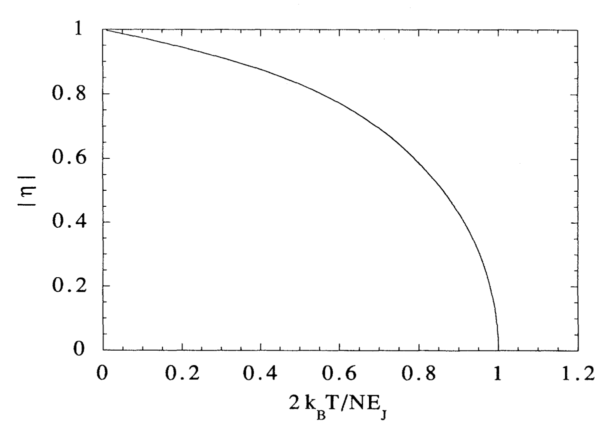

FIG. 2. Numerical solution for the magnitude of the order parameter η as a function of $2k_B T/NE_J$.

The mean energy per wire of the system in zero field can be derived from Eq. (6) by summing it over i. Doing this, we obtain the mean-field Hamiltonian

$$H_{\mathrm{MF}}=-\mathrm{Re}E_J \sum_{i=1}^{N} e^{i\varphi_i^h} \sum_{j=1}^{N} \eta_j^{v*} . \tag{10}$$

The average of H_{MF} is thus

$$\langle H_{\mathrm{MF}} \rangle = -\mathrm{Re}E_J \sum_{i,j=1}^{N} \eta_i^h \eta_j^{v*} . \tag{11}$$

Since, as we have stated above, $\eta_i^h=\eta_j^v\equiv\eta$ in zero field, Eq. (11) becomes

$$\langle H_{\mathrm{MF}} \rangle = -E_J \sum_{i,j=1}^{N} |\eta|^2 = -N^2|\eta|^2 E_J . \tag{12}$$

For $2N$ wires this means that the energy per wire is $-N|\eta|^2E_J/2$. Taking the derivative of Eq. (12) with respect to temperature, we obtain the heat capacity C_v of the system. Figure 3 shows the plot of heat capacity per wire (in units of k_B), $C_v/2Nk_B$, vs temperature. The data show typical mean-field behavior, as well as the classical equipartition value of $\frac{1}{2}k_B$ per degree of freedom (wire) at low temperature.

We draw attention to the important fact that the derived thermodynamic properties of our arrays are all directly *dependent* on N, the array size. As N grows, so does the number of nearest neighbors in the array, and consequently the system becomes more mean-field-like. This behavior is quite unlike that of conventional Josephson-junction arrays in which the number of nearest neighbors remains constant despite the overall array size. The properties of conventional arrays are size *independent*.

In the presence of a magnetic field, corresponding to a commensurate number of flux quanta per unit cell, $f=p/q$ (where p and q are small integers), we expect the ground-state phases and corresponding current configuration to be spatially periodic. Figure 4 shows ground-state configurations for $f=\frac{1}{2}$ and $\frac{1}{3}$. Values at

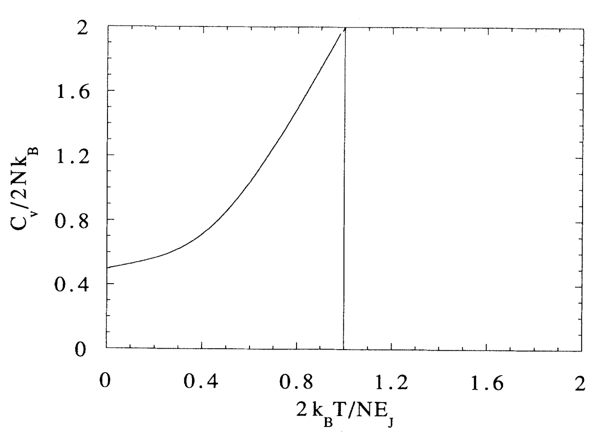

FIG. 3. Heat capacity per wire, $C_v/2Nk_B$, vs normalized temperature $2k_BT/NE_J$ as calculated by our mean-field approximation.

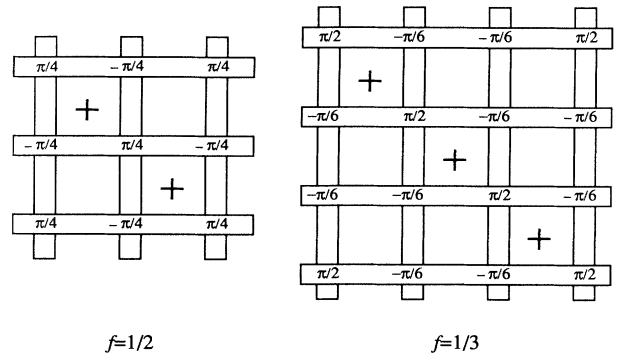

FIG. 4. Ground-state configurations for $f=\frac{1}{2}$ and $\frac{1}{3}$. The values at the intersection of the wires are the gauge-invariant phase difference $\varphi_i^h-\varphi_j^v-(2\pi/\Phi_0)\int_0^{ai}A(x_j)dy$ across the junction at that position. The boldface plus signs denote sites of positive vorticity of the smallest current loop.

the intersection of wires in this figure indicate the gauge-invariant phase difference

$$\varphi_i^h-\varphi_j^v-\frac{2\pi}{\Phi_0}\int_0^{ai}A(x_j)dy$$

across the junction at that position. The sum of these gauge-invariant phase differences (keeping track of the appropriate minus signs) around any closed loop is constrained by

$$\sum \gamma = 2\pi(n-f) , \quad n=\ldots,-2,-1,0,1,2,\ldots ,$$

where n in the ground state is typically either 0 or ± 1. We employ bold plus signs in Fig. 4 to denote the centers of the smallest positive-circulating current loops. If we view the ground states in terms of these positive-circulating currents, we find that the states appear to be very similar to the ground states of conventional four-nearest-neighbor arrays, *despite* the fact that their properties are very different.

To calculate the array's transition temperature in the presence of a magnetic field, we again assume the order parameter η is small. Using the method of Shih and Stroud,[8] the self-consistency equation [Eq. (5)] can be expanded to read

$$\eta_j^h=\tfrac{1}{2}\beta E_J \sum_{k=0}^{N-1} \eta_k^v e^{i(2\pi f jk)} . \tag{13}$$

Equivalently, we can write

$$\eta_k^v=\tfrac{1}{2}\beta E_J \sum_{j=0}^{N-1} \eta_j^h e^{-i(2\pi f jk)} . \tag{14}$$

Substituting Eq. (14) into Eq. (13), we find that

$$\eta_j^h=\tfrac{1}{4}(\beta E_J)^2 \sum_{j'=0}^{N-1} \eta_{j'}^h \sum_{k=0}^{N-1} e^{i(j-j')(2\pi f k)} . \tag{15}$$

If $f=p/q$, where p and q have no common factors and $q<N$, we can sum over k in Eq. (15) to obtain

$$\eta_j^h = \tfrac{1}{4}(\beta E_J)^2 \sum_{j'=0}^{N-1} \eta_{j'}^h N \delta_{j-j',qm} \ , \quad m=0,1,2,\ldots \ . \tag{16}$$

Since the δ function is satisfied N/q times in a field of p/q and $\eta_n = \eta_{n+q}$, we have

$$\eta_j^h = \left[\frac{N\beta E_J}{2}\right]^2 \frac{1}{q}\eta_j^h \ , \tag{17}$$

which directly results in a mean-field transition temperature

$$T_c = \frac{NE_J}{2k_B\sqrt{q}} \tag{18}$$

in a field of $f=p/q$.

Summarizing, we have found the mean-field temperature for a transition to the ordered ground states described above in a field of $f=p/q$ to be $T_c(q)=NE_J/2k_B\sqrt{q}$ for an $N\times N$ array with $q<N$. Since two numerically close values of f can have very different values for their denominator q, $T_c(q)$ is a very discontinuous function in the case of an infinite array (see Fig. 5). These discontinuities are, of course, not present in the case of a *finite* array. By employing a finite-size analysis, we can understand the process by which the discontinuities are smoothed. Combining our previous expressions [Eqs. (13) and (14)] for the linearized self-consistency equations near T_c, we can write

$$\sum_{j'=0}^{N-1}\sum_{k=0}^{N-1}\left[\alpha_{jk}\alpha_{j'k}^{*}\eta_{j'}^h - \frac{4}{(\beta E_J)^2}\eta_j^h\right]=0 \ , \tag{19}$$

where $\alpha_{jk}=\exp(i2\pi fjk)$. This expression is equivalent to the eigenvalue problem derived by Vinokur *et al.* in the disordered case.[5] We have explicitly diagonalized Eq. (19) for a finite 10×10 array, calculating the mean-field transition temperature from the largest eigenvalue. We find that the mean-field transition in this finite array is a continuous function of f (see Fig. 6), as it must be physically, and equal to the infinite-array mean-field value for

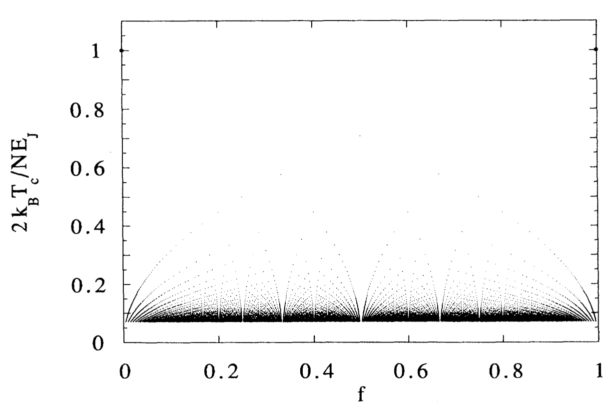

FIG. 5. Normalized transition temperature $2k_BT_c/NE_J$ vs number of flux quanta, f, per unit cell in an infinite-sized ordered array. The discontinuities found in this plot are not found in finite-sized ordered arrays (see Fig. 6).

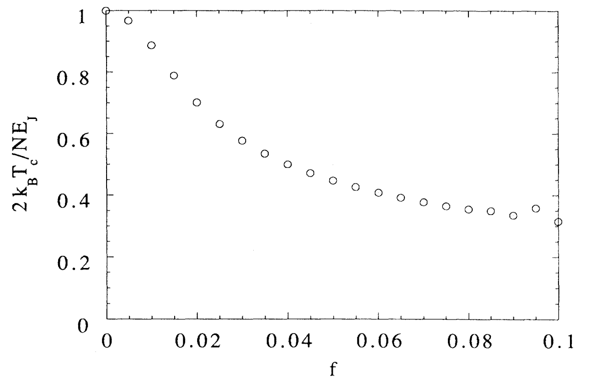

FIG. 6. Normalized transition temperature $2k_BT_c/NE_J$ vs number of flux quanta, f, per unit cell in a 10×10 array. Here $f<1/N$. Unlike in an infinite array (see Fig. 5), T_c is a smooth continuous function of f in a finite-sized array.

commensurate applied fields. We suggest that this continuous change of T_c with applied field is associated with a corresponding continuous change in the ground-state configuration of the finite array. In contrast, the ground states in the infinite array have a $q\times q$ periodicity in a field of $f=p/q$, and hence the current distribution changes discontinuously with f.

Using current conservation in each wire as the equilibrium condition, we have calculated, to first order, states corresponding to f near a commensurate value in the finite array. We find these states to be made up of the commensurate ground states plus additional, small circulating currents analogous to a Meissner shielding state (except that these currents do not significantly shield the applied magnetic field because of the small i_c we have assumed). Near zero field, for example, we expect the ground state to be of this "Meissner" form until $f=1/N$ in an $N\times N$ array. In this regime, T_c drops smoothly from $NE_J/2k_B$ for $f=0$ to its low value of $T_c(f=0)/\sqrt{N}$ for $f=1/N$, the smallest commensurate field.

If one assumes the form $\eta_j=|\eta|e^{i\varphi_j}$ for the order parameter of the jth wire (where φ_j is the ground-state phase of that wire), one can show that after substituting $T/T_c(q)$ for $NE_J\beta/2$ in Eq. (8), the mean-field thermodynamic properties derived in the zero-field case and shown in Figs. 2 and 3 are in fact universal for all fields commensurate with the array.

B. Disordered arrays

We can also use the matrix given in Eq. (19) to analyze the disordered mean-field T_c, as was done by Vinokur *et al.*,[5] by replacing our previous expression for α_{jk} with $\exp[i2\pi f(j+\delta_j)(k+\delta_k)]$. Here δ_j and δ_k represent the deviations of the jth and kth wires from their ordered positions in units of the lattice constant. Figure 7 shows the T_c (identified with the largest eigenvalue) of a 10×10 array averaged over an ensemble of ten systems. Each of these systems has spacings between the parallel wires which are uniformly distributed by up to $\pm0.1\Delta x$, where

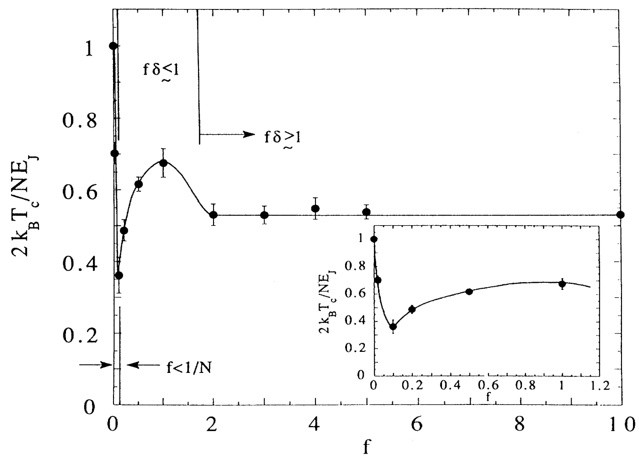

FIG. 7. Normalized mean-field transition temperature $2k_BT_c/NE_J$ vs number of flux quanta, f, per average unit cell in an ensemble-averaged 10×10 disordered array. The solid black line through the data is a guide to the eye. δ refers to the maximum fractional displacement of a wire from its ordered position, i.e., the maximum displacement is $\delta\cdot\Delta x$ and is equal to 0.1 here.

Δx is the mean spacing.

As shown in the figure, the behavior as a function of field can be divided into four general regions. For extremely small fields $f<1/N^2$, T_c approaches the zero-field limit, $T_c=NE_J/2k_B$, since the amount of flux entering the entire array is much smaller than one flux quantum. For small applied field $1/N^2<f<1/N$, T_c decreases as f increases. We note that this region directly corresponds to the one where $H<H_0$ (where, again, H_0 is the field required to generate a flux quantum through the average-sized strip between two adjacent wires) in the analysis of Vinokur *et al.*[5] In the intermediate region $1/N\le f<1$, we observe complicated behavior, which we ascribe to the similarity between the disordered and ordered arrays when f is small. When $f\delta_j<1$, we expect (from our disordered expression for α_{jk}) that the effect of the disorder will be small, and thus the dependence of T_c on f should resemble that of the ordered case. This is seen in Fig. 7 for $1/N\le f<1$, where the T_c, like that in the ordered array, rises (although the exact form is unknown at this time). As f increases to values greater than 1, the effects of disorder become more apparent, and the behavior moves toward the field-independent behavior described by Vinokur *et al.* in Eq. (3).[5] We note, however, that the high-field asymptote of T_c ($\sim 0.85E_J\sqrt{N}/k_B$) predicted by our mean-field analysis is significantly larger than that predicted by Vinokur *et al.* ($E_J\sqrt{N}/2k_B$) for the same-sized array. We will comment on this discrepancy below.

III. MONTE CARLO SIMULATIONS

In addition to our mean-field analysis, we have performed MC simulations of both *ordered* and *disordered* Josephson-junction arrays with long-range interaction. Implementing the well-known Metropolis algorithm,[9] we simulated free-boundary arrays of size $N\times N$, where $N=8$, 20, 30, and 50. Each MC simulation underwent 10 000–20 000 equilibration passes throughout the entire array lattice before executing 20 000–30 000 averaging passes. The results of the disordered arrays we present in this paper are an average of 5 runs (each a different realization of a disordered array) since we found the result of individual simulations to be very noisy.

Using our MC simulations, we looked at the heat capacity per wire, $C_v/2N=[\langle E^2\rangle-\langle E\rangle^2]/2Nk_BT^2$, and phase-coherence modulus (known in other systems as the magnetization modulus[8,10,13]) $M=\langle|\sum_{j=1}^{2N}e^{i\varphi_j}|\rangle$. M is a measure of long-range coherence of the phases, φ_j, in the array.[10] At temperatures well above T_c, $M=0$, indicating that the phases are completely random and not coherent at all. Closer to T_c, M begins to increase and continues to do so until $T=0$. At this temperature, $M=2N$ (in zero field), indicating that the phases have complete long-range coherence. Since it is not gauge invariant, M is not a measurable quantity when a magnetic field is present in the system.

A. Heat capacity

1. Ordered arrays

Figure 8 shows the heat capacity per wire (again, in units of k_B), $C_v/2Nk_B$, of the different-sized arrays we simulated in zero field. As can be seen in the figure, $C_v/2Nk_B$, has a pronounced peak, indicating a phase transition, at a temperature T_c. This peak sharpens and increases in height as we increase the size of the simulated array—a clear indication that the system is mean-field-like as opposed to Kosterlitz-Thouless-like.[11] Comparing the simulated heat-capacity curves with those derived from mean-field theory, we find that the simulated

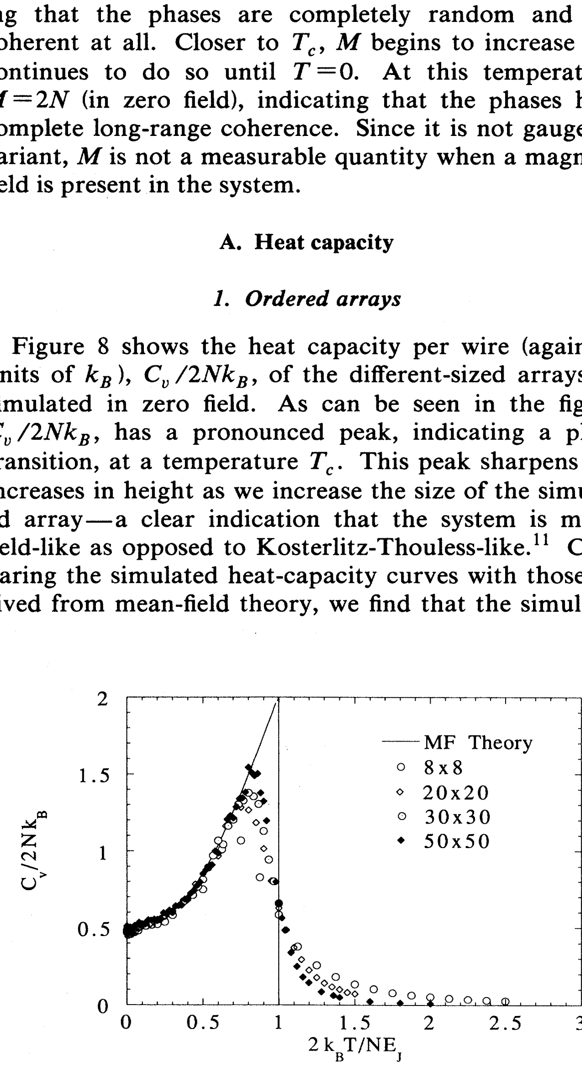

FIG. 8. Heat capacity per wire, $C_v/2Nk_B$, vs normalized temperature $2k_BT/NE_J$ for the ordered array with long-range interaction in zero field as calculated by Monte Carlo simulations and mean-field theory. The size of the simulated array is $N\times N$, where $N=8$, 20, 30, and 50. The transition temperature in the simulations is taken to be the temperature at which the heat-capacity curve has the steepest slope—usually where $C_v/2Nk_B$ is half-maximum.

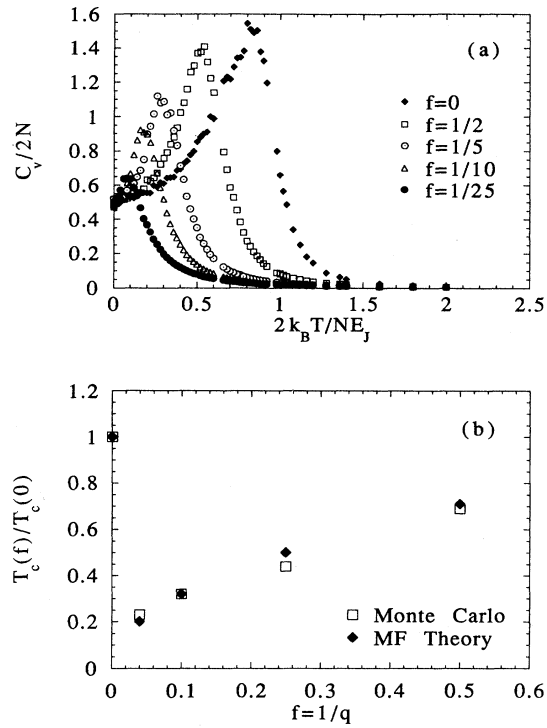

FIG. 9. (a) Heat capacity $C_v/2Nk_B$ vs normalized temperature $2k_BT/NE_J$ for a simulated 50×50 ordered array in the presence of magnetic fields, $f=0$, $\frac{1}{2}$, $\frac{1}{5}$, $\frac{1}{10}$, and $\frac{1}{25}$. (b) Comparison of $T_c(f)/T_c(0)$ vs f between mean-field theory and Monte Carlo simulations.

curves approach the mean-field form as N is increased (see Fig. 8). Since the part of the heat-capacity curve which has the steepest slope (usually where C_v is about half-maximum) corresponds to the discontinuous jump at T_c on the mean-field theory curve, we take the temperature at which this occurs to be T_c. From our MC simulations, we were able to confirm that T_c does indeed scale with N, as was predicted by our MF calculations.[12] When a field $f=p/q$ is present in the ordered-array system, we found that $T_c=NE_J/2k_B\sqrt{q}$ [see Figs. 9(a) and 9(b)], again agreeing with our MF calculations.

2. *Disordered arrays*

To simulate *disordered* arrays, we have introduced disorder into our system by randomly varying the mean distance Δx between the parallel wires of each orthogonal set. Specifically, we have uniformly distributed the spacings by up to $\pm0.1\Delta x$ in the arrays we refer to as having 10% disorder and by up to $\pm0.5\Delta x$ in arrays we refer to as having 50% disorder. Because the disorder introduced into the system is purely geometric[14] and therefore only affects the A_{ij} term in the Hamiltonian [Eq. (1)], disordered and ordered arrays show *identical* behavior in zero field.

Figures 10 and 11 show T_c vs f for the 10% and 50% disordered arrays, respectively. As a basis for comparison, T_c vs f, as derived from the analysis of Vinokur

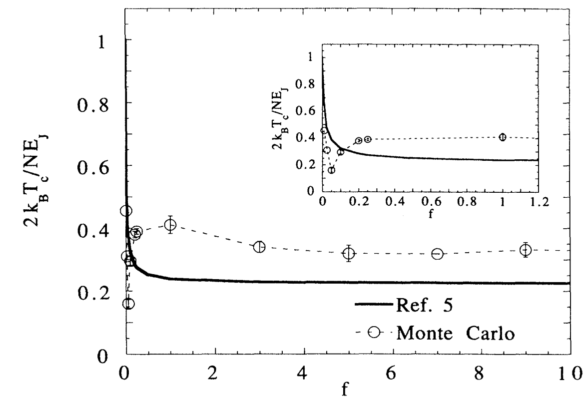

FIG. 10. Normalized transition temperature $2k_BT_c/NE_J$ vs field f for a simulated 10% disordered 20×20 array. All data shown are an average of five MC runs—all of different disordered arrays. For $f\gg1$, T_c asymptotically approaches the value $\sim0.75E_J\sqrt{N}/k_B$. Inset: For $f<1/N^2$, $T_c\approx NE_J/2k_B$. For $1/N^2<f<1/N$, $T_c=E_J/2k_B\sqrt{f}$; for $1/N\le f<1$, T_c rises with f, although the exact form is unknown. The peak at $f\sim1$ can be attributed to the fact that the array is not fully disordered. For comparison, T_c derived from the analysis of Vinokur *et al.* [Eq. (2)] is also shown.

et al. [Eq. (2)], is also shown on the same plots. As before, we can define T_c for four different regions. For $f<1/N^2$, $T_c\approx NE_J/2k_B$. For $1/N^2<f<1/N$, $T_c=E_J/2k_B\sqrt{f}$. This relationship is in agreement with our mean-field-theory analysis and with the first-order approximation of Vinokur *et al.* of T_c for extremely small fields [Eq. (4)] using the relationship $H/H_0=fN$. We note, however, that, as shown in Figs. 10 and 11, T_c slightly differs from the exact form of T_c [Eq. (2)] predicted by Vinokur *et al.*[5] For $1/N\le f\le1$, we see that T_c rises with respect to f, very much as it does in ordered

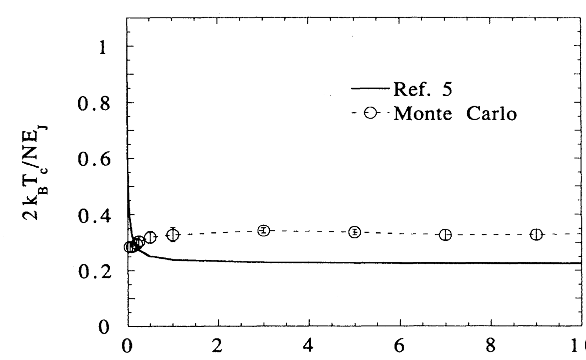

FIG. 11. Normalized transition temperature $2k_BT_c/NE_J$ vs field f for a simulated 50% disordered 20×20 array. All data shown are an average of five MC runs—all of different disordered arrays. T_c scales with N and f in the same way it does in the 10% disordered array. However, no peak in T_c at $f\sim1$ occurs in the 50% disordered array. For comparison, T_c derived from the analysis of Vinokur *et al.* [Eq. (2)] is also shown.

arrays for this region. This behavior was not predicted by Vinokur *et al.*;[5] the exact relationship between T_c and f in this regime is unknown at the present time.

The small peak in T_c at $f \sim 1$ is observable in only the 10%, and not the 50%, disordered array. We attribute this peak, which is also found in our mean-field calculations with the same disorder (see Fig. 7), again to the fact that the array is not fully disordered. When $f \gg 1$, we find that T_c asymptotically approaches $\sim 0.75 E_J \sqrt{N} / k_B$, for both the 10% and 50% disordered arrays. This asymptotic value is lower than that obtained by our mean-field calculations ($T_c \sim 0.85 E_J \sqrt{N} / k_B$). In addition, we note that Vinokur *et al.* predicted that T_c would asymptotically approach $(E_J \sqrt{N})/2k_B$.[5] To see if the discrepancy among the different asymptotes was due to a size effect, we simulated 10% and 100% 50×50 disordered arrays to compare with the results on 20×20 arrays shown in Figs. 10 and 11. The asymptotic value of T_c of these arrays at high fields is still $\sim 0.75 E_J \sqrt{N} / k_B$, suggesting that size is not a crucial factor. The consistency of our results for 10% and 50% randomness suggests that the degree of randomness is not the source of the discrepancy. Thus the discrepancy among the asymptotes derived from our simulations, our mean-field calculations, and the analysis of Vinokur *et al.*[5] is presently of unknown origin.

B. Phase-coherence modulus

Figure 12 shows the phase-coherence modulus M as a function of temperature for array sizes $N = 20$, 30, and 50. As can be seen in the figure, long-range phase coherence sets in gradually in these samples. This particular characteristic—the gradual onset of long-range phase coherence—is very similar to that of ordered conventional arrays.[13] For $T > T_c$, M asymptotically approaches zero as shown in the figure. This approach to zero is slower for smaller arrays than for large ones. In the limit of $N \to \infty$, M would equal zero for $T \geq T_c$, as in the mean-field plot, shown for comparison.

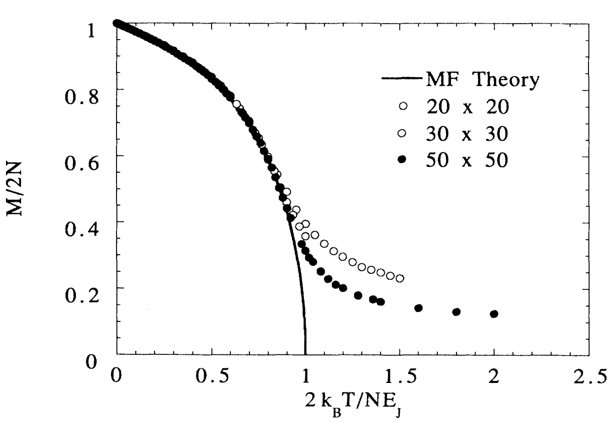

FIG. 12. Normalized phase-coherence modulus $M/2N$ vs normalized temperature $2k_B T/NE_J$ for simulated arrays of size 20×20, 30×30, and 50×50 in zero field. For a comparison, the magnitude of η, originally shown in Fig. 2 and which is equivalent to the normalized phase-coherence modulus for an infinite array in zero field, is also plotted.

IV. DISCUSSION

The size effects we have discussed in arrays with long-range interaction are unlike those found in conventional Josephson-junction arrays. Conventional arrays undergo a Kosterlitz-Thouless (KT) phase transition in which thermally activated vortex-antivortex pairs become bound below a certain temperature T_{KT} which is *independent* of the actual size of the array. T_{KT} is related to the vortex-core energy which does not change with array size for $N \to \infty$ and, hence, neither does T_{KT}.

Finite array size, however, does affect the *nature* of the KT transition in conventional arrays. The resistive behavior of these arrays can be described by the relationship $V \propto I^{a(T)}$. In the ideal case, i.e., an infinite array, the exponent $a(T)$ is predicted to jump from 3 just below T_{KT} to 1 just above.[15] This sharp jump, known as the universal jump in the superfluid density, indicates the unbinding-binding of the vortex-antivortex pairs at T_{KT}. Finite array size leads to deviations from this theoretical prediction. In particular, it causes a smearing of the jump in $a(T)$, in part due to nucleation of free vortices at all temperatures $T > 0$ (Ref. 16) and to the logarithmic spatial interaction of vortices.[17] The overall effect is a less pronounced KT transition in a finite-sized array.

In the case of arrays with long-range interaction, we find that the transition to the superconducting state occurs at a temperature T_c, which is very much *dependent* on the actual size of the array. This dependence on size occurs because the number of nearest neighbors in the array continually grows with increasing array size. Indeed, even in the limit of $N \to \infty$, these novel arrays cannot be considered as models of two-dimensional systems.[18] Because T_c is size dependent and does not depend on any type of vortex-antivortex interaction, we describe arrays with long-range interaction as truly mean-field-like.

We have fabricated both ordered and disordered Josephson-junction arrays with long-rang interaction and have performed both ac-susceptibility and dc-transport measurements on them. Our results are presented in a companion paper.[6]

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