Localization problem of a two-dimensional lattice in a random magnetic field

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We calculate numerically the conductance of a square lattice of quantum wires (which is equivalent to a tight-binding model) in the presence of a random magnetic field. The magnetic flux per plaquette is uniformly distributed between $-\Phi_0/2$ and $+\Phi_0/2$, where $\Phi_0=hc/e$ is the unit of flux quanta. The existence of localized states is confirmed. Although we cannot reach a definite conclusion, inspection of the conductance distribution $P(g)$ and an analysis of our results within a heuristic finite-size scaling hypothesis are consistent with the existence of extended states, and hence mobility edges.

It is well known since the works of Azbel',[1] Hofstater,[2] and Wannier and co-workers,[3] that the physics of an electron moving in a two-dimensional periodic potential in the presence of a uniform magnetic field is rather rich and complex despite its simple formulation. In recent years it has attracted considerable attention due to its relevance to the quantum Hall effect[4] and to the physics of strongly correlated electron systems. Recently,[5–7] interest has arisen on a somewhat related problem; that of an electron moving in a two-dimensional periodic potential in the presence of a random distribution of magnetic fluxes. While it is difficult to perceive a pertinent experimental setup, it appears to have relevance to other important problems. It was pointed out that flux phases of correlated electron systems are subject to fluctuations which alter some of the mean-field results.[5] It has also been shown that the problem of phase coherence of a fermion in a spin-liquid state of a quantum magnet is reduced to that of a fermion moving in a random magnetic field.[8]

This problem has a pure academic interest of its own. In fact, it was earlier considered[9] to be an example of a different kind of disorder, that of the vector potential, compared with the traditional case in which disorder appears on the site energies or on the amplitudes of the hopping matrix elements in tight-binding models. It is well established[10] that for the latter kind of disorder, all states of a two-dimensional system are localized, although the localization length has an essential singularity at zero disorder.

The situation is different when a two-dimensional disordered system is subject to a uniform magnetic field which breaks time-reversal symmetry. Numerical calculations indicate that, except for a single energy (the center of a Landau band), all states are localized, and that when the energy approaches this value the localization length diverges.[11] It is commonly accepted that there is no band of extended states. In some particular models of disorder extended states are constructed explicitly.[12]

On the other hand, when the system is ordered but the magnetic field is random, the localization problem has not yet been settled. A typical question that can, therefore, be asked (and will be addressed here) is whether there is a genuine metal-insulator transition in such two-dimensional systems, and if the answer is positive, what is the relevant critical exponent? Another interesting question (which we hope to study in the future) is how the localization length in a quasi-one-dimensional system diverges as the disorder goes to zero.

In Ref. 5 the main focus is turned to the density of states and to the wave functions, using diagonalization of a tight-binding Hamiltonian with nearest- and next-nearest-neighbor interactions. Here, we consider as a model a two-dimensional quantum-wire system and evaluate its conductance using the Landauer formula. Quantum-wire systems have proven to be a useful tool in the investigation of quantum (bond) percolation.[13] For regular lattices, quantum-wire networks are equivalent to tight-binding models.[14]

For the sake of self-consistency, we will briefly explain the structure of a quantum-wire system. In its two-dimensional version the model can be considered as a square lattice of narrow wave guides which intersect each other. The sites are labeled (m,n) with $m,n=1,2\ldots,L$. It is assumed that the transverse size of each waveguide tends to zero so that the propagation of waves or the motion of a quantum-mechanical particle (an electron in the present case) between intersections is virtually one dimensional. The basic symmetry and conservation laws of quantum mechanics determine the motion completely once the matching conditions at each intersection are specified. The wave function on a given bond is a linear combination of plane waves $\Psi=(ae^{iks}+be^{-iks})e^{2\pi i\phi s}$ where s is the length measured from the site, k is the wave number, and ϕ is $1/\Phi_0$ times the vector potential integrated along the bond, where $\Phi_0=hc/e$ is the unit of flux quanta. The magnetic flux per plaquette is given by $\Phi_0=\sum_{\text{plaquette}}\phi$.

If the wave functions Ψ_i $(i=1,\ldots,4)$ on four links

meet at a site, the continuity and the current conservation conditions are $\Psi_1=\Psi_2=\Psi_3=\Psi_4$ and $\Psi_1'+\Psi_2'=\Psi_3'+\Psi_4'$ where the prime denotes the covariant derivative. The energy band is given by $E(\text{wires})=k^2$ (we use units such that $\hbar^2/2m^*=1$ where m^* is the effective mass). The mapping on the tight-binding model[10] relates the tight-binding energy E_{TB} and the wave number k through the relation $E_{\text{TB}}=-4t\cos k$. For simplicity, we will set $t=1$ where t is the hopping matrix element of the tight-binding model.

Let an incoming wave e^{ikx} of unit amplitude reach a site $(1,n)$ on the left column of the lattice. Then the wave function at the exit wire attached to site (L,m) on the right is $t_{mn}e^{ikx}$ where t_{mn} is an element of the complex transmission matrix T. Likewise, the wave function at the exit wire attached to site $(1,m)$ on the left is $r_{mn}e^{-ikx}$ where r_{mn} is an element of the complex reflection matrix R. Unitarity (current conservation) requires $\text{Tr}(TT^\dagger)+\text{Tr}(RR^\dagger)=L$. One version of the Landauer formula[15] which we use here relates the dc conductance G to the transmission in a simple way: $G=(e^2/h)\,\text{Tr}(TT^\dagger)\equiv(e^2/h)g$.

The traditional algorithm by which one looks for a mobility edge is attained by calculating the localization length of a very long rectangular sample as a function of the sample width.[16] For the pertinent system, this method has been applied recently by Sugiyama and Nagaosa.[6] We will compare our results with theirs later in this paper. Here, on the other hand, it will be carried out by inspecting the averaged transmission $\langle g(L,E)\rangle$ of a square lattice as a function of its size L at a fixed energy E. This seems to us to be closer in spirit to the scaling theory of localization which concentrates on $g(L,E)$ itself.[17] Besides, it enables us to study the distribution $P(g)$, which, as we believe, can give us additional information on the basic question (the existence of a mobility edge) beyond that supplied by the first moment $\langle g(L,E)\rangle$. In any numerical procedure which pertains to disordered systems it is difficult to get the precise values of the numbers we are interested in. Here, however, the main questions we want to answer are of a qualitative nature, anyway.

The actual technique of calculations is explained elsewhere.[13] The maximum length we could handle was $L=60$. We choose periodic boundary conditions on the transverse (y) direction and calculate the transmission along the longitudinal (x) direction. In order to achieve the smallest localization length, the disorder is taken to be maximal, namely, the normalized flux ϕ is uniformly distributed between $-\frac{1}{2}$ and $+\frac{1}{2}$.

Before calculating the conductance for many samples, we needed to find the band edges. We found, as expected, that the band became narrower. Instead of ranging between $0\le k<\pi$, it shrank to about $0.5\le k\le\pi-0.5$. Henceforth, we will concentrate on the calculations of the conductance for $0.6\le k\le\pi/2$. These calculations concern, as a first step, the evaluation of $\ln(g)$ as a function of $\ln(L)$, since from its slope we can compute the scaling function $\beta(g)$. With our prior knowledge of the band edges, we have chosen five values of k, 0.60, 0.65, 0.70, 0.75, and 1.00, and calculated the conductance of 200 independent samples for each value of k and $L=26,28,\cdots,42$ and 50. For $k=0.60$, 0.75, and 1.00, we have also included calculations for $L=60$. Actually, one should specify exactly what one means by $\ln(g)$. With the tentative assumption that the first three values are located on the localized side while the last two are located on the extended side, we chose to display $\langle\ln(g)\rangle$ and $\ln(\langle g\rangle)$, respectively. The results are shown in Fig. 1. This figure indicates that for $k\le0.65$, $\beta(g)$ is negative (which means an insulator), while for $k\ge0.70$, $\beta(g)\approx0$ (corresponding to the usual Ohmic behavior $g\sim L^{d-2}$ on the metallic side, where d is the dimensionality of the system, and $d=2$ in the present case). However, from this result it is not enough to claim the existence of a mobility edge at, say, $k\approx0.675$, since it may simply mean that the system is localized but that at this energy the localization length exceeds the system size. This sort of uncertainty naturally prevails on any length scale.

We will now present some finer results which may shed more light on this question. Consider first $v(g)$, the variance of $P(g)$. On the insulating side, the conductance is very small and most of the samples are concentrated near $g=0$. The variance is, therefore, also very small. On the metallic side, conductance fluctuations are universal, depending on the pertinent universality class and hence we expect an almost constant (weakly energy dependent) variance. As it turns out, the passage between these two regimes is not monotonic,[18] but is characterized by a relatively large value of the variance in the transition point. In Fig. 2 we plot the variance $v(g)$ for a square of size $L=50$ for 11 values of k between 0.60 and 0.80. It shows a steep increase up to the "transition" point and then it decreases and saturates on the "metallic" side.

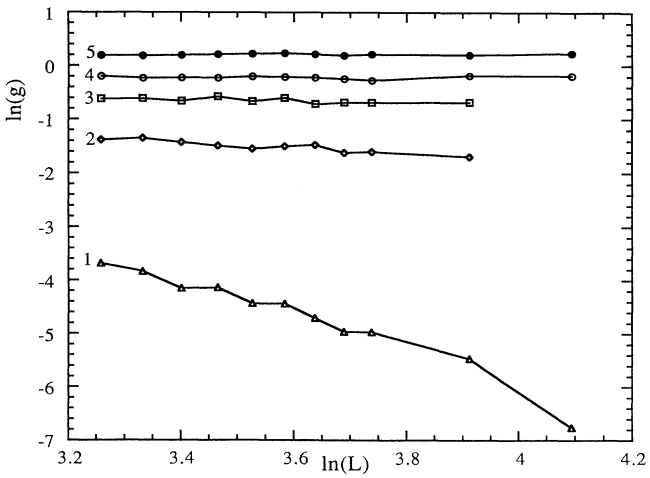

FIG. 1. The function $\ln[g(L,E)]$ vs $\ln(L)$ [whose slope gives the scaling function $\beta(g)$] for a square lattice of quantum wires of size $L=26,28,\ldots,42$ and 50, for wave numbers $k_1=0.60$, $k_2=0.65$, $k_3=0.70$, $k_4=0.75$, and $k_5=1.00$. For k_1, k_4, and k_5 calculations for $L=60$ are also included. For the first three wave numbers the figure displays $\langle\ln[g(L,E)]\rangle$ as a function of $\ln(L)$ while for last two it displays $\ln[\langle g(L,E)\rangle]$ as a function of $\ln(L)$. The statistics are based on 200 samples.

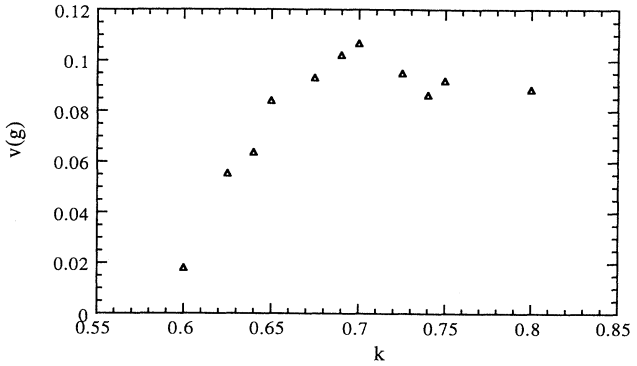

FIG. 2. The variance $v(g)$ of the conductance distribution $P(g)$ as a function of k for $L=50$, based on 200 samples for each value of the wave number.

In the next stage, we carry out two finite-size scaling analyses. The first one assumes the existence of a mobility edge at the onset. The rationale shows that the hypothesis on the existence of a genuine mobility edge (in the thermodynamic limit) is consistent with the results which necessarily pertain to finite-size systems. Analogous to the theory of quantum percolation,[19,12] we assume that at the vicinity of the mobility edge $E \approx E_c$ the averaged conductance of a square of length L has the form

$$\langle g(L,E)\rangle \approx L^{d-2-\gamma} F(\Delta L^{1/\nu}) , \tag{1}$$

where $\Delta = E - E_c$. In Eq. (1) $F(x)$ is an unknown scaling function and the critical exponent ν governs the divergence of the localization length for $E \to E_c$ from below, $\xi \sim (E_c - E)^{-\nu}$. The other critical exponent γ is, in general, unknown but it can be eliminated by noticing that in computing the ratio $\alpha(L,E) \equiv v(g)/\langle g\rangle^2$ between the variance v and the square of the average of the conductance g the prefactors multiplying the scaling function F cancel, and hence, near the mobility edge we have

$$\alpha(L,E) = \frac{v(g)}{\langle g\rangle^2} = \frac{\langle g^2\rangle}{\langle g\rangle^2} - 1 \approx H(\Delta L^{1/\nu}) , \tag{2}$$

where H is a new scaling function. This practical step calls for assuming a reasonable value for E_c (say, $E_c = 0.675^2$ corresponding to the critical value of the wave number drawn from Fig. 1) and then choosing various values of ν and displaying the calculated $\alpha(L,E)$ for as many values of L and $E = k^2$ as possible. If the points fall on a reasonably smooth line, the results are consistent with the hypothesis. In Fig. 3(a) we display the results for $k_c = 0.675$ and $\nu = 1$ while in Fig. 3(b) the results are shown for $k_c = 0.675$ and $\nu = \frac{7}{3}$. It is evident that the latter case represents a better fit. Due to our lack of better statistics we can say only that from the conclusions of Figs. 3(a) and 3(b), one cannot rule out the existence of a mobility edge at $0.65 \le k \le 0.7$ and that in that case, the value of the critical exponent ν is definitely larger than 1. This last figure is consistent with a rigorous bound by Chayes *et al.*[20] that predicts the relation $d\nu > 2$ for random systems.

The second finite-size scaling analysis is based on the suspicion that the transition is nothing but a finite-size effect. If, in the thermodynamic limit, the system is localized, the conductance $g(L,E)$ is a function of the scaling variable $x = L/\xi(E)$, where $\xi(E)$ is the localization length. We expect that the localization length is a monotonic increasing function of energy so long as E is lower than the center of the band. Without prior knowledge we are tempted to conjecture an exponential relation $\xi(E) \sim \exp(\lambda E)$ where λ is an unknown parameter. For energies E close to the band edge, this conjecture has been confirmed numerically (in a tight-binding model) by Sugiyama and Nagaosa[6] who suggest $\lambda = 14.5$. Within this hypothesis, let us examine the behavior of the averaged conductance $\langle g(L,E)\rangle$ as a function of the scaling variable x. If the transition is a finite-size effect and the existence of "extended states" occurs simply because $L < \xi$, we should find an appropriate value of λ such that the results will fall on a smooth line. Small values of x will then correspond to the case $L < \xi$ and the scaled function $\langle g(L,E)\rangle$ as a function of x should then tend to a constant as $x \to 0$. For large x one has $L > \xi$ which means being in the insulating regime. The function should then decay exponentially. If the points cannot be reasonably put on a smooth line it means either that the

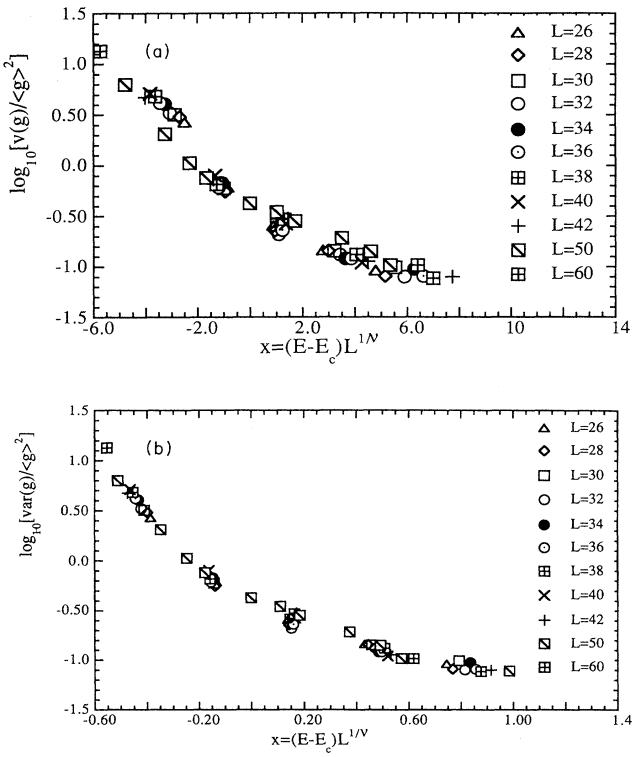

FIG. 3. The quantity $\alpha(L,E)$ defined in Eq. (2) and evaluated at many values of k and L, as a function of the scaling argument $x = (E - E_c)L^{1/\nu}$ for $E_c = k_c^2$ with $k = 0.675$. (a) $\nu = 1$ and (b) $\nu = \frac{7}{3}$. The statistics are based on 200 samples.

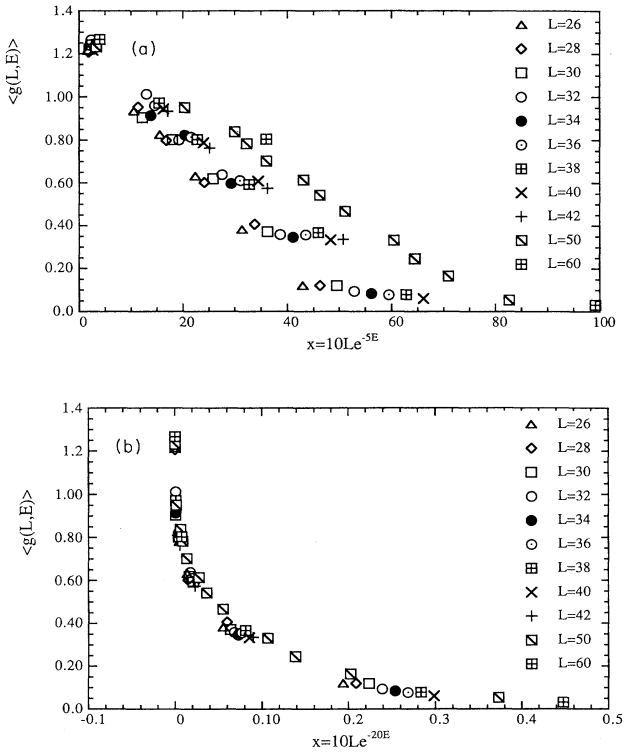

FIG. 4. The averaged conductance $\langle g(L,E)\rangle$ evaluated at many values of k and L, as a function of the scaling argument $x=10L\exp(-\lambda E)$ with $\lambda=5$ (a) and $\lambda=20$ (b).

transition is genuine or that the choice of the functional expression for $\xi(E)$ is not appropriate. An example for this scenario is shown in Fig. 4(a) with the choice $\lambda=5$ and in Fig. 4(b) with the choice $\lambda=20$. So far we were unable to satisfy the three conditions (smooth line, constant limit as $x\to 0$, and exponential decay as $x\to\infty$).

Finally, let us compare our results with previous ones. Pryor and Zee[5] examined (within the tight-binding model) the wave functions for nearest-neighbor hopping and found that most of the states are extended with a tail of localized states near the band edge. Of course, the possibility of having localized states that look like extended ones (since $\xi\geq L$) was noticed, but they believe that the flat part of the density of states contains extended states while the tails of the distribution contain localized states. Sugiyama and Nagaosa[6] used the method of Ref. 16 and performed calculations of the localization length of a very long system ($L\approx 10^4$) as a function of the system width. They did not find evidence for the existence of a mobility edge. Kalmeyer and Zhang[7] arrived at the conclusion that the random magnetic field suppresses localization and that all states are extended. Our results are compatible with those of Ref. 5 and do not contradict those of Ref. 6. Since we have definitely established the existence of localized states we cannot confirm the results of Ref. 7.

In summary, we have investigated the conductance of a square lattice of quantum wires (which is equivalent to a tight-binding model) in the presence of random magnetic flux. The calculations were performed up to lattice dimension $L\times L=60\times 60$. We found a mobility edge at $k\approx 0.675$ which corresponds to the tight-binding energy of -3.123. This transition may be due to finite-size effects. However, a careful inspection of the probability distribution of the conductance, $P(g)$, and finite-size scaling analysis does not rule out the existence of a genuine transition.

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