Persistent currents and edge states in a magnetic field

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We investigate equilibrium electron currents in an ideal two-dimensional ring (of radii $R_1 < R_2$). The most striking result emerges when the conditions for the existence of edge and bulk states are met, namely $R_2 - R_1 \gg a_H$ where a_H is the magnetic length. If the Fermi energy lies in a gap between two Landau levels, the current (as a function of electron density) displays violent fluctuations (in sign and in absolute value), which is quite unusual for systems without disorder. The fluctuations in sign result from the alternative contributions of inner and outer occupied edge states below the Fermi energy, while those in absolute value originate from the apparent symmetry between the slopes of the energy curves near the two opposite edge states. On the other hand, when the Fermi energy is locked on a Landau level, the current has a plateau as a function of electron density. Its value at a plateau represents the contribution to the current of all the edge states in the lower Landau levels.

I. INTRODUCTION

Persistent current[1] is an equilibrium property of an electronic system with a closed loop, expressing its response to an applied magnetic field. Substantial theoretical efforts have been devoted to the understanding of this phenomena.[2] In two recent experiments on metallic rings, persistent currents have been detected.[3,4]

Numerous theoretical works[2(a)] followed the original idea, and concentrated on a system with a one-dimensional ring geometry and its response to a central flux threading through it. More recently the importance of many channel effects has been addressed, and several authors considered two-dimensional (2D) and 3D cylindrical rings threaded by a flux along the axis.[2(b)] The subtle nature of the averaging procedure has also been considered,[2(c)] as well as the importance of electron-electron interaction.[2(d)] A common feature is that the magnetic field in the sample is not taken into account (while experimentally it is present). In these kinds of geometry, the persistent currents result from the sensitivity of the energy spectrum to the boundary conditions which are determined by the Aharonov-Bohm (AB) flux Φ through the hole. The main efforts were focused on AB periodicity, the role of disorder in the appearance of the half period and thermodynamic aspects of the averaging procedure of many rings. Experimentally, an ideal one-dimensional ring with a flux through its hole is not attainable. In the actual experiments, rings (or even squares) with two radii $R_1 < R_2$ are used such that $W = R_2 - R_1 \ll R_1$. They were put in a weak magnetic field (up to about several flux quanta through the hole) so that the effects of finite width and the magnetic field in the sample are assumed to be limited. Discussion of persistent currents in an annulus of finite width subject to a perpendicular magnetic field was carried out a few years ago.[5(a)] Recently,[5(b)] the inclusion of a weak magnetic field has been suggested within a semiclassical approach based on the Bohr-Zomerfeld quantization condition.

In the present paper we suggest that the study of persistent currents in annular rings in a strong magnetic field is interesting in its own, and contains information about the energy spectrum of edge states in gaps between Landau levels in particular. The existence of these currents was suggested several decades ago. In the study of the quantum Hall effect, Halperin[6] has shown that when a conducting layer with two edges (e.g., a ring or a cylinder) is placed in a strong perpendicular magnetic field, there exist current-carrying electron states which are localized near the boundaries of the sample.

To be more specific, let us denote the energy eigenvalues for an annular ring in a magnetic field and a central flux Φ by $E_n(m;\Phi)$, where n is a radial quantum number which corresponds to the Landau levels and m is the angular momentum. For a fixed value of n and Φ, the set of points $E_n(m;\Phi)$ describes a discrete energy momentum relation. We will study relatively large systems for which several hundreds of m values are relevant for each n. By changing Φ adiabatically between 0 and Φ_0 (here $\Phi_0 = hc/e$ is the magnetic flux quantum) the energies $E_n(m;\Phi)$ move continuously in such a way that $E_n(m;\Phi+\Phi_0) = E_n(m+1;\Phi)$. (This equality is required by gauge invariance since the spectrum must have AB periodicity equal to Φ_0.) It is then useful sometimes to think in terms of continuous lines (which we will call dispersion energy curves). The contribution of an occupied state at energy $E_n(m;\Phi)$ to the current is given by the Byers-Yang[7] relation (see also Ref. 6),

$$I_{nm} = -c\frac{\partial E_n(m;\Phi)}{\partial \Phi} . \tag{1}$$

Our motivation in the present work is based on the interpretation of Eq. (1) as a probe of the energy levels near the edges. Let us assume that the Fermi energy lies in a gap between two Landau levels E_n and E_{n+1}. It then intersects the curve $E_n(m;\Phi)$ at two points. Physically, Eq. (1) tells us that an equilibrium electron current in a two-dimensional conducting film in a strong perpendicular magnetic flux reflects the difference in the slopes of the energy curve at the two edges of the sample. Furthermore, the discrete nature of the spectrum (which is the case for mesoscopic systems) is of crucial importance here since the sign of I_{nm} changes according to whether $E_n(m;\Phi)$ is on the right or left side of the energy curve. Hence, a knowledge of the current as a function of some external parameter (electron density or external magnetic field) is related to the spectroscopy of the pertinent edge states. Motivated by this observation we extensively investigate the characteristics of the persistent (or edge) currents in an ideal ring of radii $R_1<R_2$, for which the Schrödinger equation is solved. We are mainly interested in the situation where edge states as well as bulk (Landau degenerate) states are present, which is the case

$$W=R_2-R_1 \gg a_H \ , \tag{2}$$

where $a_H=(\hbar c/eH)^{1/2}$ is the magnetic length corresponding to the magnetic field H.

This kind of geometry is chosen not only because it is experimentally accessible but also because it is related to the structure of the spectrum. The ring with a finite width geometry is a typical one for which the dispersion curves are not symmetric. Following the physical interpretation of Eq. (1) it is evident that this asymmetry will dominate the behavior of the edge currents. The first thing to notice in this context is that the current will be different from zero even when the additional central flux Φ is absent (in contrast with the cylindrical ring geometry). Thus, the central flux in the present case merely plays the role of an adiabatic parameter through which the energy curves are defined, and not as a trigger of the current.

Our results can be summarized as follows: (a) When the Fermi energy lies in a gap of bulk states, we find that the edge current displays violent fluctuations in sign as a function of electron number N, even when a single electron is added. The fluctuations in absolute value increase with N as the Fermi energy increases between E_n and E_{n+1}. (b) Another result which we find here (which is self-explanatory albeit surprising) is related to the fact that once the Fermi energy reaches a Landau level, it is "locked" on it for a large range of N. This means that the edge current remains constant, and its value represents the contribution from all edge states connected to the lower Landau levels. (c) We also investigate the dependence of the persistent current on the magnetic field and explain our results in the context of Eq. (1).

The subject of equilibrium currents as a response to an applied external magnetic field dates back several decades, but its relevance to mesoscopic systems received much interest recently due mainly to the possible experimental observation of important size and edge effects. Attention is directed toward several mutually overlappig focuses of interest, such as orbital magnetization, de Haas–van Alphen oscillations, persistent currents, etc. In this respect, the present work can be related to any one of these topics for the special case where the geometry of the two-dimensional systems is such that there are two edges and when the magnetic field is very strong. We recall that the experimental setup designed to detect persistent currents[3,4] is based on the ring geometry. Of course, in the actual experiments, the geometrical parameters and the strength of the magnetic field are chosen such as to approach the one-dimensional ring threaded by a flux ideal geometry,

$$R_2 \approx R_1 \ , \quad W \ll a_H \ , \tag{3}$$

which is on the other extreme of condition (2) above. Yet, it is our hope that the present research will intrigue experimental works on conducting rings for which the condition (2) is met. Most likely, it can be achieved in semiconductors with low Fermi energies through the techniques of fabrications of quantum dots (Si-metal-oxide-semiconductor field-effect transistor, heterojunction interfaces, etc.).

II. EDGE CURRENTS IN AN IDEAL RING GEOMETRY

In this section we will solve the Schrödinger equation for two-dimensional noninteracting electrons in a magnetic field within a ring of radii $R_1<R_2$ and determine the energy spectrum. Following Halperin,[6] we will also add a magnetic flux Φ through the hole. Using the one-particle energies, we will compute the persistent current as a function of electron density (for fixed magnetic field) and as a function of the magnetic field (for fixed electron density).

In plane polar coordinates, r,θ, the action of a magnetic field $\mathbf{H}$ and a central flux Φ are conveniently introduced through the symmetric gauge for the vector potential

$$\mathbf{A}=\tfrac{1}{2}\mathbf{H}\times\mathbf{r}+\frac{\mathbf{\Phi}\times\mathbf{r}}{2\pi r^2} \ . \tag{4}$$

The Schrödinger equation for the wave function $\Psi(r,\theta)$ is then

$$\left[\mathbf{p}+\frac{e}{c}\mathbf{A}\right]^2\Psi(r,\theta)=E\Psi(r,\theta) \ , \tag{5}$$

where $\mathbf{p}$ is the momentum operator and E is the energy eigenvalue. We adopt units such that $\hbar=2m^*=1$. Using rotational symmetry (appropriate for the symmetric gauge), the states can be labeled by a radial quantum number n and the angular momentum m. The wave function is conveniently factorized as $\Psi(r,\theta)=e^{im\theta}\psi_{nm}(r)\sqrt{r}$, and the radial equation for a given angular momentum m reads

$$\left[-\frac{d^2}{dr^2}+\left[\frac{m-\phi}{r}-\frac{r}{a_H^2}\right]^2-\frac{1}{4r^2}\right]\psi_{nm}(r)=E_n(m)\psi_{nm}(r)\ , \tag{6}$$

where we have defined $\phi=\Phi/\Phi_0$. The normalization and the boundary conditions are

$$\int_{R_1}^{R_2}|\psi_{nm}(r)|^2 r\,dr=1\ , \tag{7a}$$

$$\psi_{nm}(R_1)=\psi_{nm}(R_2)=0\ . \tag{7b}$$

It is useful at this point to recall the relation between the value of m and the nature of the wave function $\psi_{nm}(r)$ suggested by Halperin:[6] Define the radius r_m through

$$\pi H r_m^2=m\Phi_0-\Phi\ . \tag{8}$$

Then if $R_1<r_m<R_2$ and $|r_m-R_i|\gg a_H$, the states $\psi_{nm}(r)$ are degenerate at the Landau energy,

$$E_n=\frac{2n+1}{a_H^2}\quad(n=0,1,2,\ldots,) \tag{9}$$

and each state $\psi_{nm}(r)$ is peaked at r_m. If, on the other hand r_m-R_1 or R_2-r_m is less than the order of few times a_H (or negative) the state $\psi_{nm}(r)$ is exponentially localized close to R_i $(i=1,2)$ and its energy is higher than E_n. This qualitative feature will be given a quantitative numerical substantiation later on.

The solutions of Eq. (6) can be expressed analytically in terms of confluent hypergeometric functions[8] with parameters $\alpha=-0.5(Ea_H^2-1),\gamma=m-\phi+1$, and argument $z=r^2/2a_H^2$. They are

$$\psi_m(r)=e^{-z/2}z^{m-\phi+1/4}[AM(\alpha,\gamma,z)+BU(\alpha,\gamma,z)]\ . \tag{10}$$

Here $M(\alpha,\gamma,z)$ and $U(\alpha,\gamma,z)$ are two independent solutions of the confluent hypergeometric equation which are, respectively, regular and singular at $r=0$. With $z_i=R_i^2/2a_H^2$, the eigenvalues (as well as the ratio between the constants A and B) are determined by the boundary conditions which lead to

$$M(\alpha,\gamma,z_1)U(\alpha,\gamma,z_2)-M(\alpha,\gamma,z_2)U(\alpha,\gamma,z_1)=0\ . \tag{11}$$

Unfortunately, the implication of the boundary conditions from which the eigenvalues are extracted through Eq. (1) is rather difficult. We therefore found it more useful to diagonalize the Hamiltonian in a standing wave basis of functions,

$$f_k(r)=\left[\frac{2}{W}\right]^{1/2}\sin\left[\frac{k\pi(r-R_1)}{W}\right],\quad k=1,2,\ldots,K\ . \tag{12}$$

The matrix elements $\langle k|r^2|k'\rangle$ and $\langle k|r^{-2}|k'\rangle$ can be stored once for all so that the matrix to be diagonalized is

$$H_{kk'}=\delta_{kk'}\left[\frac{\pi^2k^2}{W^2}-\frac{m-\phi}{a_H^2}\right]+\frac{1}{a_H^4}\langle k|r^2|k'\rangle+[(m-\phi)^2-\tfrac{1}{4}]\langle k|r^{-2}|k'\rangle\ , \tag{13}$$

and the cutoff mode K is chosen so that the spectrum below the Fermi energy is insensitive to its value. For each m, a set of K energies $E_n(m)$ is obtained $(n=0,1,2,\ldots,K-1)$ and when the magnetic field is strong, the dispersion curves $E_n(m;\Phi)$ should be flat (degenerate) far from the edges being equal to the Landau energies (9). This is indeed the case as we can see from Fig. 1(a) and that serves as a test of the numerical procedure. Instead of calculating the current via Eq. (1) [which requires calculations of $E_n(m;\Phi)$ for several values of the flux and then performing numerical derivatives] we found it safer to evaluate the current directly as an integral of the current density along the radial direction. It requires the knowledge of the wave functions which are computed in terms of the eigenvectors of $H_{kk'}$ [Eq. (13)]. For a given wave function $\Psi_{nm}(r,\theta)=\psi_{nm}(r)e^{-im\theta}$, the current density is

$$\mathbf{J}_{nm}=ie(\Psi_{nm}^*\nabla\Psi_{nm}-\Psi_{nm}\nabla\Psi_{nm}^*)-\frac{2e^2}{c}\mathbf{A}\Psi_{nm}\Psi_{nm}^*\ . \tag{14}$$

Apart from the factor e (which we drop henceforth), the component of $\mathbf{J}_{nm}$ along the tangential direction (in the present scheme of units) is

$$J_{nm}(r)=\psi_{nm}(r)e^{-im\theta}\left[\frac{-i}{r}\frac{\partial}{\partial\theta}-\frac{\phi}{r}-\frac{r}{a_H^2}\right]\psi_{nm}(r)e^{im\theta}\ , \tag{15}$$

where the operator within the square brackets is the component of the current operator along the tangential direction. After replacing the derivative operator by im we have the corresponding expression for the current,

$$I_{nm}=\int_{R_1}^{R_2}|\psi_{nm}(r)|^2\left[\frac{m-\phi}{r}-\frac{r}{a_H^2}\right]dr\ . \tag{16}$$

It is somewhat tempting to calculate the current using a discrete version of Eq. (1) with steps of difference $\Delta\Phi=\pm\Phi_0$, where $+$ refers to forward difference and $-$ refers to backward difference. The advantage of this procedure is that it is sufficient to know the spectrum at a single value of the flux in order to calculate the current. The approximate expression is then

$$I_{nm}\approx -c\frac{\Delta E_n(m;\Phi)}{\Delta\Phi}\ . \tag{1$'$}$$

To assess the usefulness of this procedure in the present case let us recall the relation $E_n(m;\Phi\pm\Phi_0)=E_n(m\pm1;\Phi)$ which is dictated by gauge invariance. Equation (1$'$) is then equivalent to

$$I_{nm}\approx -c\frac{E_n(m\pm1;\Phi)-E_n(m;\Phi)}{\pm\Phi_0}\ . \tag{1$''$}$$

Now, let us assume that the Fermi energy E_F crosses the energy curve $E_n(m;\Phi)$ in such a way that the energy of the highest occupied state on the right side is $E_n(m_R;\Phi)$ while the energy of the highest occupied state on the left side is $E_n(m_L;\Phi)$. Thus, $E_n(m_r+1;\Phi)>E_F$, and also $E_n(m_L-1;\Phi)>E_F$. It is now easy to check that if one sums the contributions of the numerators in Eq. (1″) up to the Fermi energy [see Eq. (17) below], the sum will always be positive, being either $E_n(m_R+1;\Phi)-E_n(m_L;\Phi)$ in case one chooses to use a forward difference or $E_n(m_L-1;\Phi)-E_n(m_R;\Phi)$ in the opposite case. The denominator, however, has different sign, being positive in the forward case and negative in the backward case. Thus, beside the inaccuracy of replacing derivative by difference, one arrives at an unacceptable result that the sign of the current depends on the direction of whether one chooses to use forward or backward difference. The unavoidable conclusion is that while Eq. (1′) is successfully applied in other circumstances,[9] it cannot be used in the present calculation scheme.

III. RESULTS AND DISCUSSIONS

Let us first inspect how I_{nm} depends on the quantum numbers (n,m). If $E_n(m;\Phi)=E_n$ (namely, it is degenerate and belongs to the bulk) then clearly $I_{nm}\approx 0$, since $E_n(m;\Phi)$ does not change with Φ. If, on the other hand, $E_n(m;\Phi)$ is on an edge of the energy curve (namely, it is in the gap of bulk states), then I_{nm} will be positive or negative according to whether $E_n(m;\Phi)$ is on the right side (outer edge state) or on the left side (inner edge state).

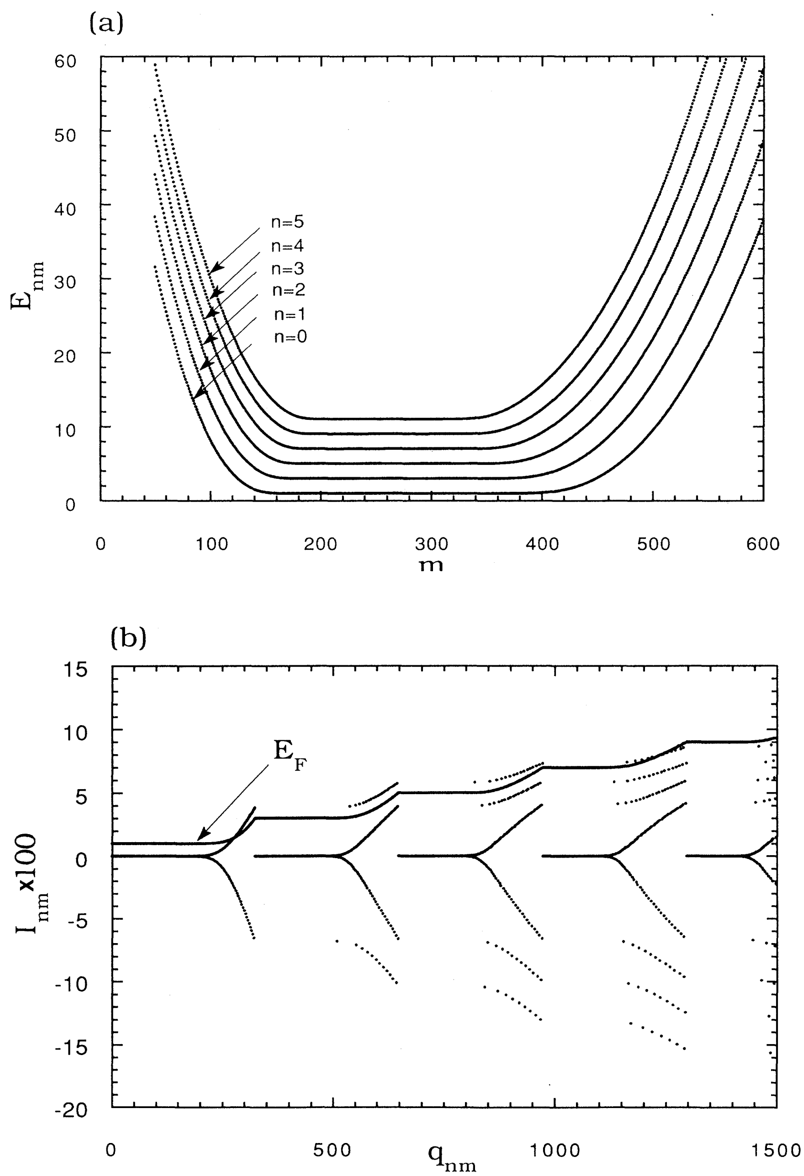
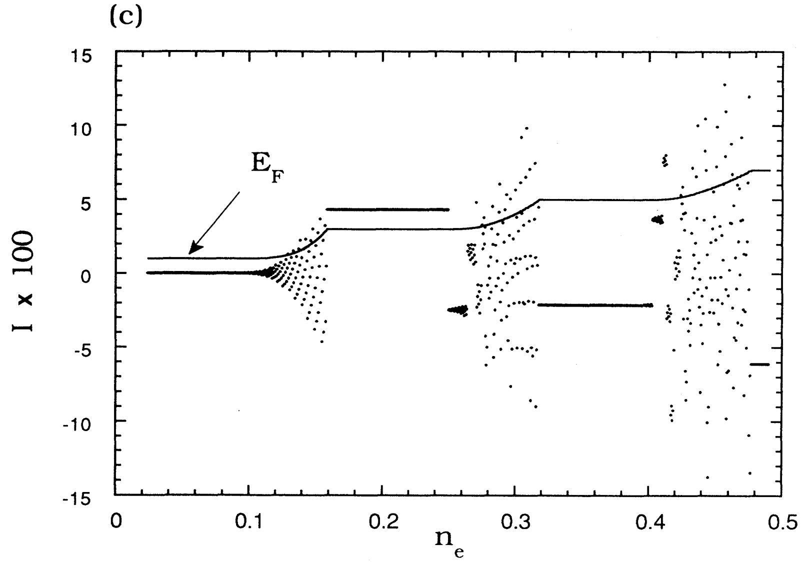


FIG. 1. (a) Energy curves as a function of angular momentum quantum number m for a clean annular ring of radii $R_1=15.86$, $R_2=30.00$, central flux $\phi=0.2$, and magnetic length $a_H=1$. Energies and current are given in units of inverse length square. The flat parts correspond to the values of Landau levels for the infinite system $(2n+1)/a_H^2=1$, 3, 5, 7, and 9. (b) Energy $E_n(m;\Phi)$ (solid line) and current I_{nm} computed from Eq. (16) and multiplied by 100 (dotted line) as a function of (n,m) arranged so that $E_n(m;\Phi)$ is nondecreasing. The radii, field, and flux are the same as in (a). (c) Fermi energy (solid line) and persistent current computed from Eqs. (16) and (17) and multiplied by 100 (dotted line) as a function of electron density $n_e=N/\pi(R_2^2-R_1^2)$ (where N is the number of electrons) corresponding to the geometry and field strength of (a). (d) The result of averaging the current over electron number in an interval of 200. The current is plotted as a function of N where here N is the center of the pertinent interval. The other parameters are as in (a).



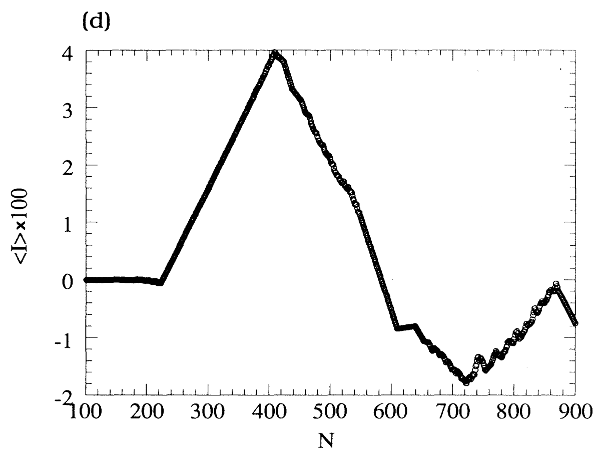


FIG. 1. (*Continued*).

The negative value will be larger in absolute value than the positive one since the slope on the left is larger than that on the right (assuming they both have approximately the same energy, of course). On the other hand, the number of positive contributions within a given energy interval will exceed that of negative contributions since the number of single-particle states per energy interval is inversely proportional to the slope of the energy curve. These considerations are given a quantitative support in Fig. 1(b). We have gathered the pairs (n,m) into a sequence $q(n,m)$ of quantum numbers according to the ascending values of $E_n(m;\Phi)$, and plot $E_n(m;\Phi)$ and I_{nm} as functions of these quantum numbers. We notice the negligible contributions from the bulk states, the larger absolute values of the negative contributions, and the more dense positive contributions. It is also evident that the dispersion curves at the edges are not linear, since if they are linear, the envelopes would have been parallel straight lines.

We can now examine the behavior of the persistent current at zero temperature

$$I=\sum_{nm} I_{nm}\,\theta(E_F-E_n(m;\Phi)) \ , \tag{17}$$

where θ is the step function. The Fermi energy E_F depends on the magnetic field H and on the electron number N (and also on Φ albeit very weakly).

We first consider the dependence of I on N. Experimentally, variation of electron density is achieved relatively easily in terms of an applied gate voltage. In Fig. 1(c) we display the Fermi energy and the persistent current as function of electron density $n_e=N/\pi(R_2^2-R_1^2)$ for the same radii and field strength as in Fig. 1(a), for which the condition (2) holds. In some loose sense Fig. 1(c) is the "integral" of Fig. 1(b). As the Fermi energy moves into the gap of bulk states, the current displays violent oscillations in sign, whose absolute values increase. The sign of the current is reversed almost each time when the new occupied state just below the Fermi energy belongs to a different edge state than its former one. The reason for the apparently irregular pattern of oscillations is the asymmetry of the energy curves.

This is a hallmark of the annulus geometry in contrast with the strip geometry for which the energy curves are symmetric. We stress again that the role of the central flux in the present geometry is not to trigger the current, but just to serve as an adiabatic parameter in terms of which the current can be computed. The specific value of the central flux ($\phi=0.2$ for Fig. 1) is immaterial. Any other value (including $\phi=0$ which correspond to the experimental situation) will give qualitatively the same results.

The magnitude of the current in physical units can be estimated for typical experimental parameters simply by noticing that in our units, the current is about two orders of magnitude smaller than the Fermi energy. Thus, we are tempted to conjecture the estimate $I \approx 10^{-2}(e/h)E_F$, which for E_F in the region of eV corresponds to a current in the region of microamperes.

The plateaus in the persistent current graph are almost self-explanatory. As a function of electron density, the Fermi energy is "locked" on a Landau level, and there is no contribution to the persistent current from the bulk states. The value of the current at each plateau is the sum of the contributions of all the edge states from the lower energy curves below this particular energy. Despite this simple picture, the appearance of these plateaus is quite significant, and represents another hallmark (among many others) in the physics of electrons in strong magnetic fields.

Using arguments related to Fig. 1(b) regarding the nonlinearity of the energy curves at the edges, and the accumulated contributions of the plateaus, it is easy to convince oneself that the current has a good chance to survive an averaging over N. This is substantial in Fig. 1(d), where we plot the average of the current (with respect to

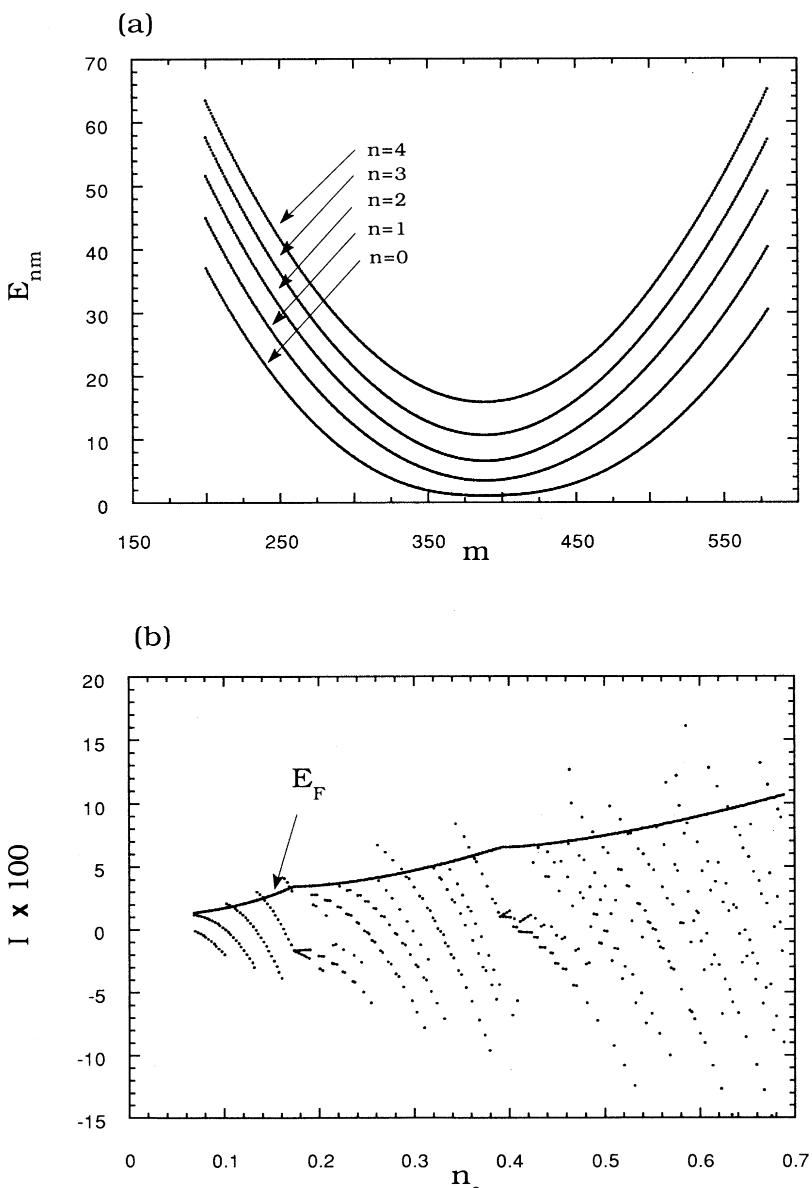


FIG. 2. (a) Same as Fig. 1(a) except that $R_1=25.86$. (b) Same as Fig. 1(c) except that $R_1=25.86$.

electron number) over an interval 2Δ ($=200$ in that case)

$$\langle I \rangle_N \equiv \frac{1}{2\Delta} \sum_{k=N-\Delta}^{k=N+\Delta} I_k \ .$$

The locking effect can be eliminated once the Landau degeneracy is removed. This is possible if the distance W between the edges of the system become smaller, leaving room only for edge states. We test this point in Fig. 2 (a thin ring geometry), for which relation (2) is replaced by

$$R_2, R_1 \gg a_H, \quad W = R_2 - R_1 \approx 4a_H \ . \tag{18}$$

The energy curves plotted in Fig. 2(a) do not have a flat bottom part (except possibly the first one in which the current is zero anyway). Therefore the locking effect should be absent. We also notice that the asymmetry between left and right edges is reduced, and hence, the slopes on the left and on the right become close to each other. In this case the current can be viewed as the sum of terms obtained from a monotonic sequence (higher than linear) with alternating signs. This is exactly what we get as is shown in Fig. 2(b).

To further inspect the effect of asymmetry between the two edge states, we have performed calculations in an angular ring with a small hole (where the asymmetry is pronounced). We notice that in the ring geometry, the relevant parameters are not only W and a_H but also R_1. Indeed, as R_1 becomes smaller, one approaches the origin and the centrifugal term becomes important. Hence, the asymmetry between the two sides of the energy curve is increased. We show, in Fig. 3(a), the energy curves for the geometry of a thick ring, for which

$$R_2 \gg R_1 = 3a_H, \quad W = R_2 - R_1 \gg a_H \ . \tag{19}$$

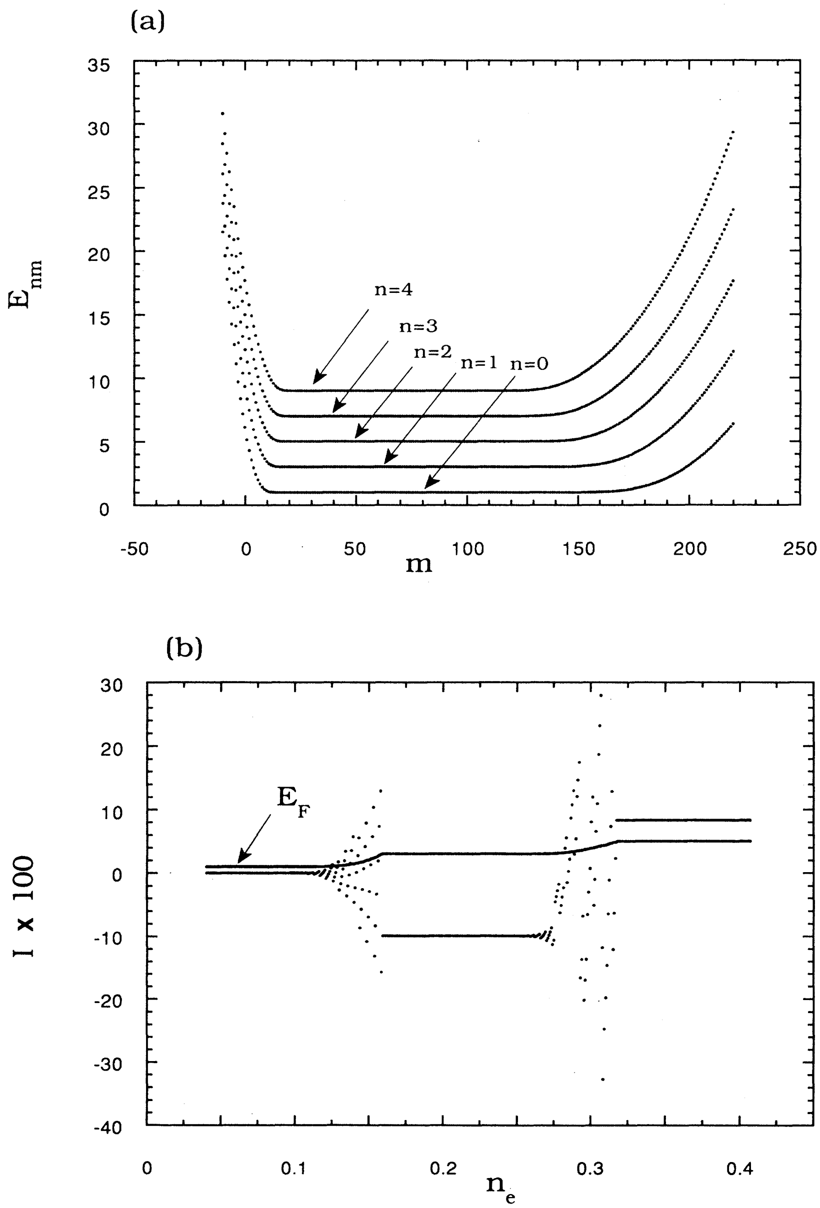
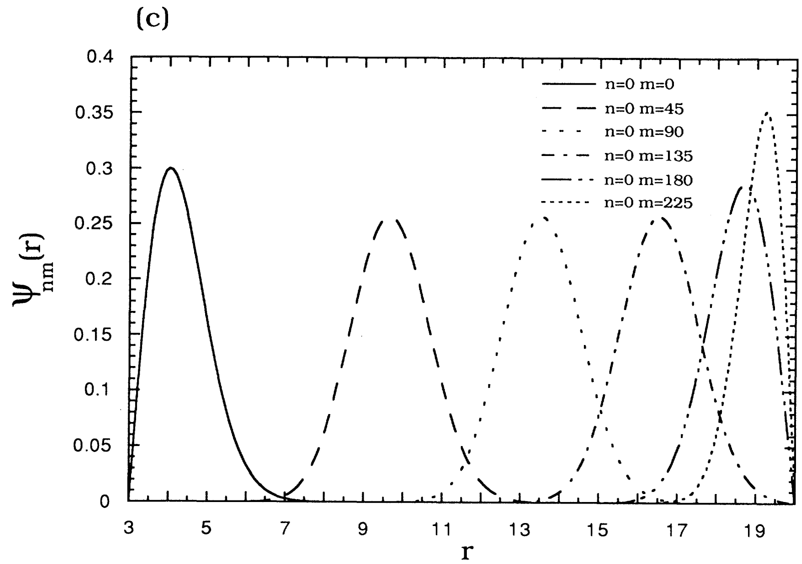


FIG. 3. (a) Same as Fig. 1(a) except that $R_1=3.06$, $R_2=20.00$. (b) Same as Fig. 1(c) except that $R_1=3.06$, $R_2=20.00$. (c) Lowest ($n=0$) radial wave functions for left edge states ($m=0$), bulk states ($m=45$, 90, and 135), and right edge states ($m=180$ and 225). (d) Same as (c) for the second level ($n=1$).



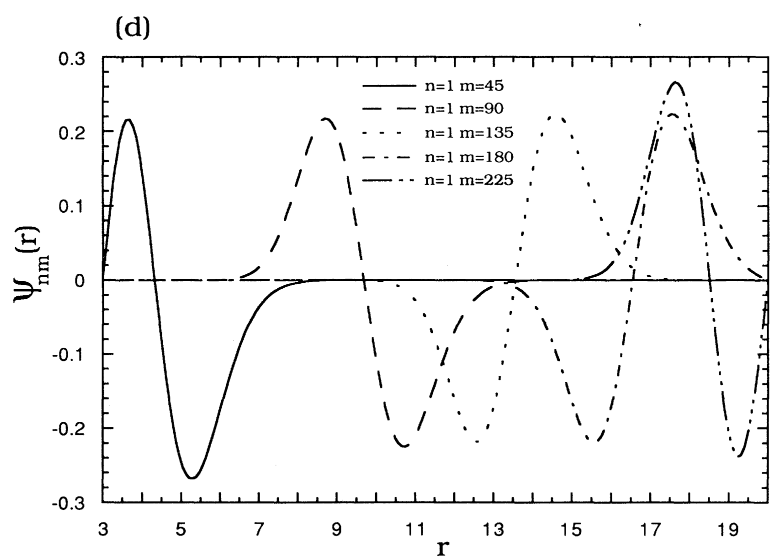


FIG. 3. (*Continued*).

The corresponding Fermi energy and current curves are shown in Fig. 3(b). The first thing to notice is the large value of the absolute value of the current compared with the previous results [e.g., 30 units at density 0.3 compared with 8 units deduced from Fig. 2(b)]. Indeed, steep energy curves on the inner edge results, according to Eq. (1), a large value for the current. At the same time, it is easy to convince oneself that the fact that the curves on the left are very steep implies that each large negative contribution occurs after several positive smaller ones. Therefore, the fluctuations in the present case are much less dense than before. Although the notion of persistent current is related to geometries of surfaces which are not connected simply, a natural question is what will happen in a disk (without a hole in the center). The qualitative behavior is expected to be similar to the one in which the contributions from the inner edge states are neglected. The persistent current will then be a nondecreasing function of the electron number and the locking effect is there also.

We also found it instructive to display some wave functions. In Fig. 3(c) we show some of the lowest level (no radial nodes except at the edges) left and right edge states as well as a few bulk states. As expected, the edge states are highly peaked near their respective edges while the bulk states are more spread. In fact, the bulk states are degenerate, so that an appropriate linear combination of them produces extended states. Some wave functions of the second level (with one radial node) are displayed in Fig. 3(d).

It would have been of importance to analyze the persistent current in a disk geometry, where we expect only one edge states at the disk radius. Unfortunately, we are unable to diagonalize the Hamiltonian with the basis functions (12) due to the singularity at the origin.

Although our main concern in this paper is the current results from the edge states, it is of great interest to study the persistent current in the weak-field limit since this is the domain where experiments have so far been performed (although only with metallic rings). We carried it out for the same ring as in Fig. 1 but with a magnetic field weaker by two orders of magnitude. Thus, we have in this case

$$R_2=30.00>R_1=25.86, \quad a_H=10.00<R_1, \quad a_H\approx 2.5w \ . \tag{20}$$

The energy curves shown in Fig. 4(a) are nearly perfectly symmetric with the center slightly larger than $m=0$. Unlike all previous cases (for which the magnetic field was strong) the dependence of $E_n(m;\Phi)$ on both m and n is not far from quadratic. The parabolic dependence on m implies that the slope of the energy curve is linear with energy, while the almost perfect symmetry implies that the current is computed from an almost perfect sequence with alternating signs [in the sense of the discussion pertaining to Fig. 2(b)]. We therefore expect a regular pattern of fluctuations corresponding globally to several smooth lines, which "bifurcates" each time another level is crossed. Figure 4(b) is a beautiful manifestation of this analysis. We note in passing that in a rectangular geometry of typical length L, irregular fluctuations of the orbital susceptibility have been predicted at an extremely weak magnetic field ($L/a_H \rightarrow 0$).[10]

Dependence of the persistent current on the magnetic field

From an experimental point of view it should be somewhat easier to follow the dependence of equilibrium (persistent) currents in a strong magnetic field on the field strength (than on the electron density). Yet, each case

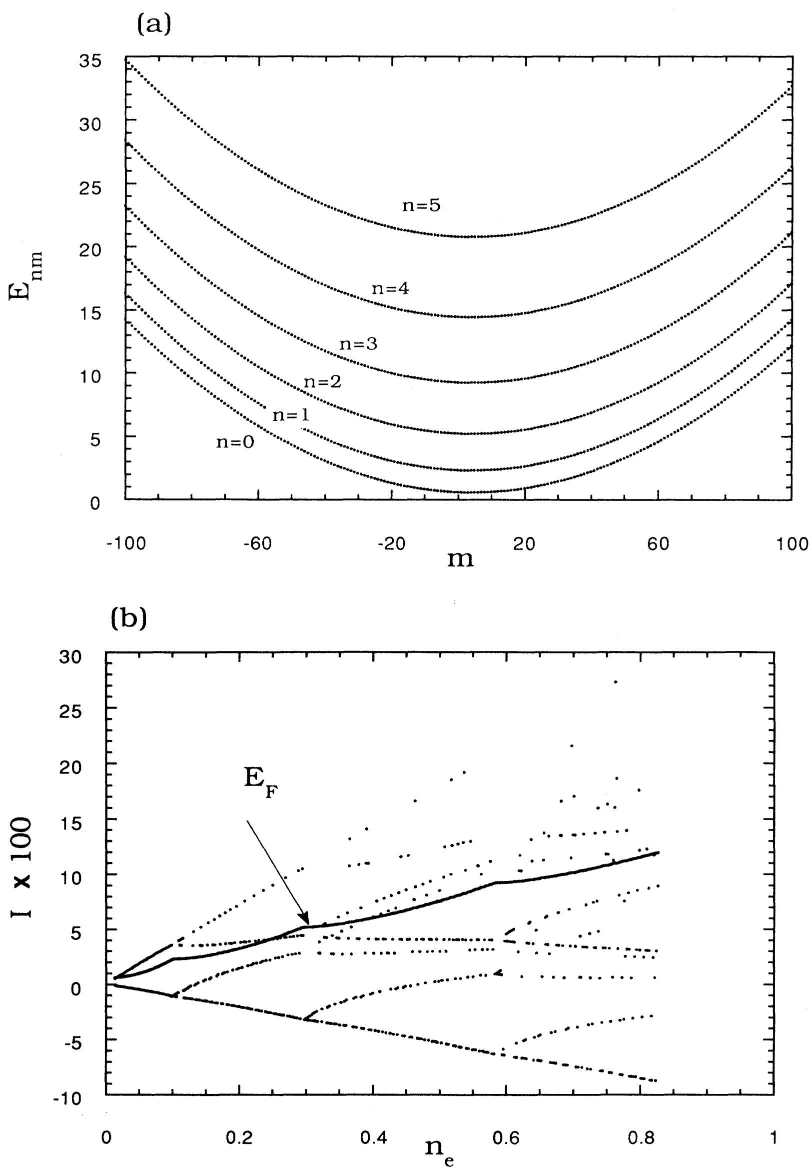


FIG. 4. Results for a system in a relatively weak field. Geometrical parameters are the same as in Fig. 1, with magnetic length $a_H=10.00$. (a) Energy levels. (b) Fermi energy and persistent current.

has its own physical characteristics. Investigation of bulk properties of infinite electronic systems as a function of the magnetic field revealed some of the most beautiful related physical phenomena like Shubnikov–de Haas oscillations and the de Haas–van Alphen effect. They originate from the semiperiodic (sawtooth) behavior of electron population as a function of the inverse of the magnetic field.

It has been pointed out by Sivan and Imry[11] that in the presence of edges, this picture is modified since crossing a Landau level from above puts the Fermi energy at the top of the populated edge states belonging to the lower-energy curves. They analyzed the effect of edge states on the de Haas–van Alphen oscillations within a tight-binding model with diagonal disorder on a relatively small lattice such that the total number N_ϕ of flux quanta through the sample ranged between 7 and 25. It is hence of much interest to study the behavior of the current as a function of the magnetic field in a continuum version and more realistic geometry (albeit pure system). This we have done for the ring geometry of Fig. 1. The number of flux quanta on the area (excluding the hole) ranged between 40 and 1000. (It is more natural in this case to plot the current as a function of $1/N_\phi$.) The results are presented in Figs. 5(a) and 5(b).

First let us inspect the Fermi energy curve. We fix the number of electrons as 200 and start with $N_\phi=1000$, which, in the present geometry corresponds to $a_H=1/\sqrt{2}$. The degeneracy of the Landau level is such that electrons occupy part of the first Landau level only, so that the Fermi energy is $(2n+1)/a_H^2=2$ (for $n=0$). Now we start to decrease the strength of the field and the Fermi energy decreases as well since the energy curve is lowered. At the same time, the degeneracy of the Landau

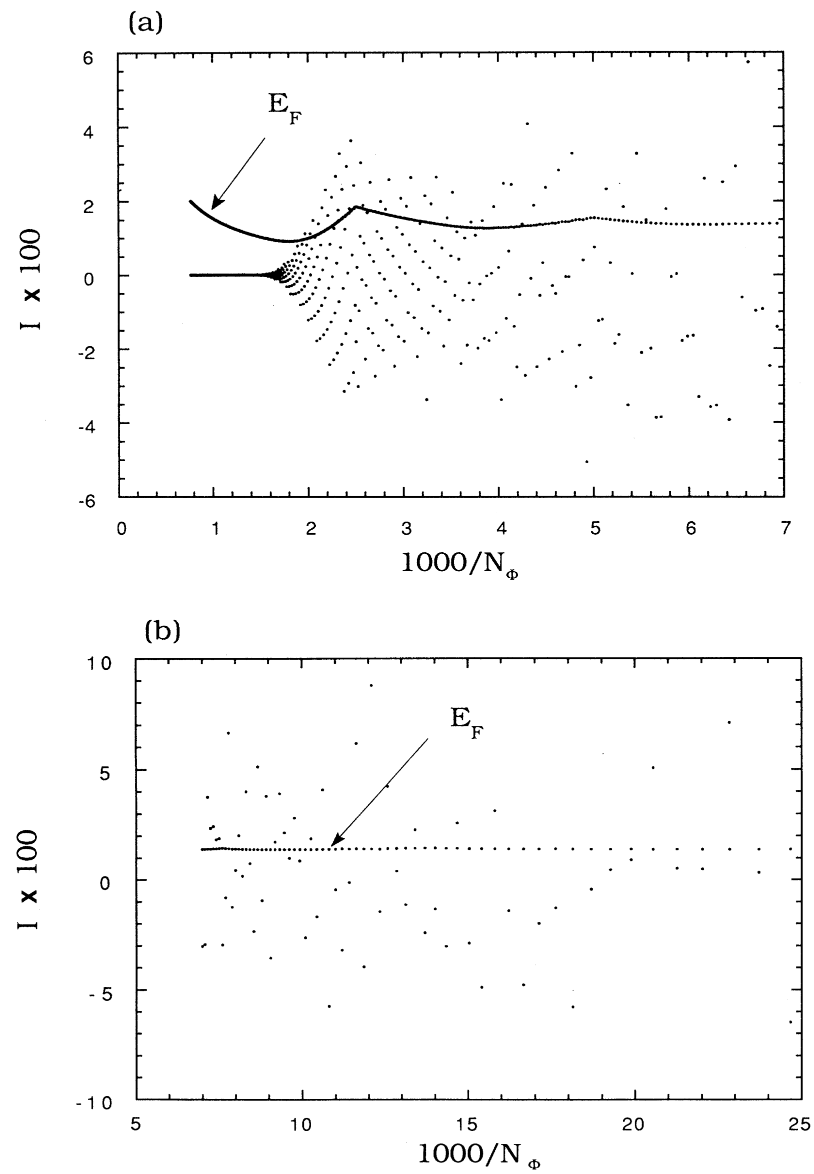


FIG. 5. Fermi energy (solid line) and persistent current (dotted line) as a function of an inverse of the magnetic field for a fixed number of 200 electrons. The abscissa is $1000/N_\phi$ where N_ϕ is the number of flux quanta in the sample between $R_1=25.86$ and $R_2=30.00$. (a) The strong-field domain $1.00<1000/N_\phi<6.50$. (b) The weak-field domain $6.50<1000/N_\phi<25.00$.

level is decreased, so that eventually, the Fermi energy is pushed up, as edge states from the first level are filled. When the number of flux quanta is about 400 [$1000/N_\phi \approx 2.5$ in Fig. 5(a)], electrons start to fill the second Landau level and the Fermi energy stays there for a while and so on. Thus, those parts of Fig. 5(a) in which the energy decreases as a function of the flux correspond to the cases where the Fermi energy is on a Landau level which decreases as $1/H$. On the other hand, when the Fermi energy increases with $1/H$ it means that it is in the gap of bulk states. For an infinite system, the picture is a perfect seesaw. For a system with edges, the pattern is distorted and in the weak-field limit [Fig. 5(b)] almost disappears.

The behavior of the current follows from that of the Fermi energy. When the Fermi energy is attached to the lowest Landau level, we get the same locking effect for the current as we have seen before. Once the Fermi energy is pushed above the Landau level, the fluctuation pattern characteristic of the edge state contribution emerges. When the Fermi energy reaches the second Landau level, we do not observe the locking effect as we have found before, and the fluctuation pattern is almost unaffected. Indeed, any change of the magnetic field simultaneously causes a shift, a dilatation, and a decrease in the degeneracy of the energy curves, and that causes a completely irregular pattern of fluctuations. Therefore, apart from the region where the Fermi energy is attached to the lowest Landau level, there will be no plateau and the currents fluctuate violently. In the weak-field limit [Fig. 5(b)], there are no more degenerate Landau levels and the energy curves are much less steep. This leads to an irregular pattern of fluctuations which is much more dilute. In any case, our results indicate that the modifications due to the finiteness of the system are not smooth but rather irregular. We note in passing that fluctuations of magnetization with the magnetic field in small systems (few tenths of electrons) without edges have been reported recently by Yoshioka and Fukuyama.[12]

Finally, some comments are due concerning the effects of disorder, as well as temperature and inelastic effects. As Halperin has argued, the disorder will localize the bulk states, but will have little effect on the edge states. Therefore, we expect the analysis carried out above to hold also in the presence of weak disorder, except that now, the negligible contribution from the bulk states is due not to the fact that they are independent on the flux but to the fact that they are localized.

At finite temperature, the population is not sharp, and the violent oscillations are expected to be rounded off. This effect has also been noticed in the calculation of magnetization.[12] Of course, once the temperature is higher than the level spacing between two states on the edges, the effects discussed above will be washed out completely.

Inelastic effects like electron-phonon scattering may be important if they can lead to coupling between inner and outer edge states. In a strong magnetic field we expect such a scenario to be highly improbable. Of course, if inelastic scatterings are strong enough so that the length over which phase memory is retained gets smaller than the system size, the concept of persistent currents discussed here itself breaks down.

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