PHYSICAL REVIEW B VOLUME 47, NUMBER 2 1 JANUARY 1993-II

Unusual Doppler effect in He II

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(Received 20 April 1992)

Many years ago Khalatnikov described unusual properties of the Doppler shift for the second sound in He II, first of all the "back-entrainment" effect: at some temperatures (at the beginning of the roton region) the center of the spreading sound moves in the direction opposite to the normal-component velocity v_n [$\Delta u_2=\gamma(T)v_n$, $\gamma(T\sim 0.6K)<0$]. However, the existing theory describes Doppler shift of the first and fourth sounds as a trivial, "kinematic" effect: the center of the spreading sound moves with the velocity of the liquid as a whole [$\Delta u_{1,4}\approx j/p=(1-\rho_n/\rho)v_s$]. We show that the real situation is quite different. We find (1) the coefficient K in the Doppler-shift expression, $\Delta u_{1,4}=(1-K_{1,4}\rho_n/\rho)v_s$, substantially differs from the kinematic value $K=1$:$|K|_{\max}$ reaches some tens. (2) $K_1(T)$ and $K_4(T)$ have different (qualitatively opposite) nontrivial temperature dependences, in particular a high peak (modulo) at the beginning of the roton region. (3) $K_{1,4}(T)$ can be negative: $K_1<0$ in the region of the peak, $K_4<0$ in the phonon region. This implies an "outstripping" effect: the center of the spreading sound moves faster than the flowing superfluid part of the liquid itself.

I. INTRODUCTION

The presence of two types of macroscopic motions and several sound modes in a superfluid provides an unusual manifestation of the Doppler effect. The simplest situation corresponds to a completely locked normal component: the Doppler shift of the fourth sound in He II.[1-3] In Sec. II we consider the theory of this phenomenon. We prove that the existing description of it is inadequate. It claims that the effect is practically "kinematic": The center of the sound sphere moves with the velocity of liquid as a whole, $\mathbf{v}=\mathbf{j}/\rho$, i.e., with its center of mass. Corrections to this result are supposed to be small. We show that the real situation is quite different. The coefficient K_4 ($\equiv K$) in the Doppler-shift expression $\Delta u_4=(1-K\rho_n/\rho)v_s$ substantially differs from the kinematic value $K=1$. The existing theory's formula neglects the thermodynamic derivatives that are proportional to the thermal expansion coefficient $\alpha(T)$. This is not justified in the present case. Moreover, the coefficient K can be expressed in terms of the derivatives of sound velocity (c) and the roton gap (Δ) with respect to the density [$a\equiv(\rho/c)dc/d\rho$, $r\equiv(\rho/\Delta)d\Delta/d\rho$] and the derivatives can be deduced from the experimental values of $\alpha(T)$ (note that the parameters a,r are nearly constant in the most interesting temperature region $T<1$ K).

Our theory predicts some physical effects: (i) At the phonon region of temperatures $T\leq 0.4$ K, we get $K(T)\approx$const<0 ($K\approx -2.7$); i.e., the center of the sound sphere moves faster than the superfluid component itself (an "outstripping" effect). (ii) At the beginning of the roton region of temperatures ($T\approx 0.63$ K), the function $K(T)$ has a sharp peak ($K_{\max}\approx 26.5$).

In Sec. III we investigate the Doppler shift in the case of a "free" normal component. Many years ago, Khalatnikov[4] obtained very striking results for the Doppler shift of second sound: The coefficient $\gamma(T)$ ($\Delta u_2=\gamma v_n$) changes sign at the beginning of the roton region, so that the "entrainment" of the second sound becomes opposite to the direction of the normal-component velocity $\mathbf{v}_n$. Since the difference between $\mathbf{v}_n$ and $\mathbf{v}=\mathbf{j}/\rho$ (at $v_s=0$) is of zeroth order in ρ_n/ρ, it is natural that the difference between Δu_2 and v_n,v is also of zeroth order in ρ_n/ρ. Correspondingly, Δu_2 is described by the calculation to the lowest order in ρ_n/ρ. In particular, one can indeed neglect here the derivatives proportional to $\alpha(T)$. However, for the case of first sound, this approximation is inadequate: The differences between $\Delta u_1,v_s$ and $v=(1-\rho_n/\rho)v_s$ ($v_n=0$) are of first order in ρ_n/ρ. We obtain a nontrivial behavior of the coefficient $K_1(T)$ [$\Delta u_1=(1-K_1\rho_n/\rho)v_s$]: Unlike the case of fourth sound, the function $K_1(T)$ ($\approx$ const) in the phonon region is positive (and large, $K_1\approx 47$), and it becomes negative with large modulus $|K_1|_{\max}\approx 33$) at the beginning of the roton region ($T\approx 0.55$ K) where $|K_1(T)|$ has a high peak. Thus we get here a very large outstripping effect.

Note that although the difference between the kinematic prediction and the real situation in terms of the coefficient $K_{1,4}$ is very large, the maximum distinctions of the total corrections to the Doppler shift correspond to the low temperatures ($T\sim 0.6$ K) where the ratio ρ_n/ρ is small (see Tables I and II). Accordingly, the effect would not be easily observable. in case of K_4 [$\equiv K(T)$], it is more accessible to observe the growth of $K(T)$ at decreasing temperature: if we are interested in corrections of more than, e.g., 0.5%, the corrections are measurable down to $T_{\min}=0.7$ K, whereas the kinematic prediction is $T_{\min}=1$ K. In the case of $K_1(T)$, it becomes accessible to observe directly the outstripping effect, even though the maximal (modulo) value of the total correction, $K_1\rho_n/\rho$, is -0.12% ($T=0.65$ K).

In the conclusion of the paper, we discuss the origin of the nonkinematic character of the Doppler shift in a superfluid.

II. DOPPLER SHIFT OF FOURTH SOUND

(i) The Doppler shift of fourth sound in the He II (hydrodynamic sound with the condition of completely locked normal component) has been investigated by Rudnick *et al.*[1] and is used as a convenient tool to investigate persistent currents and critical velocities in a flowing superfluid.[2,3] In usual hydrodynamics the Doppler shift of a sound mode is an effect with a truly mechanical (kinematic) nature: The center of the spreading sound moves together with the fluid—the medium of the spread. Correspondingly, along the direction of motion, the sound velocity changes according to the formula

$$|u|=|u_0|\pm v_f \ , \tag{1}$$

where u_0 is the sound velocity in an immovable fluid and v_f is the velocity of the fluid. In Ref. 1 the authors suggested for the Doppler shift of fourth sound the simple expression

$$|u|=|u_0|\pm\frac{\rho_s}{\rho}v_s \ . \tag{2}$$

This means that the center of the sound sphere moves together with the liquid (He II) as a whole—the velocity of its center of mass is

$$\mathbf{v}=\frac{\rho_n}{\rho}\mathbf{v}_n+\frac{\rho_s}{\rho}\mathbf{v}_s=\frac{\rho_s}{\rho}\mathbf{v}_s \quad (\mathbf{v}_n=0) \ .$$

Unlike Eq. (1) ($\Delta u = v_f$), there appears a correction which describes the influence of the immovable normal component:

$$\Delta u=\left[1-\frac{\rho_n}{\rho}\right]v_s \ . \tag{3}$$

In Ref. 2 the authors note that formula (2) is not exact and replace it by the expression

$$|u|=|u_0|\pm\left\{\frac{\rho_s}{\rho}+\left[\sigma\frac{\partial(\rho_n/\rho)}{\partial T}\left[\frac{\partial\sigma}{\partial T}\right]^{-1}-\rho\frac{\partial(\rho_n/\rho)}{\partial P}\left[\frac{\partial\rho}{\partial P}\right]^{-1}\right]\right\}v_s \ , \tag{4}$$

where σ, ρ, and T are the specific entropy, pressure, and temperature, respectively. The correction to ρ_s/ρ in the square brackets was assumed to be small. In Ref. 5 it was pointed out that below $T=1.4$ K the correction can be neglected, so that Eq. (2) becomes correct.

We show in this paper that the real situation is quite different: The influence of the normal component substantially differs from the kinematic result [Eqs. (2) and (3)]. The correction in Eq. (4) does not represent the complete contribution to the Doppler shift, because it neglects terms containing thermodynamic derivatives that are proportional to the thermal expansion coefficient,

$$\alpha=\frac{1}{\rho}\left[\frac{\partial\rho}{\partial T}\right]_P=\rho\left[\frac{\partial\sigma}{\partial P}\right]_T ,$$

which are essential here. Moreover, the correction in Eq. (4) is not small: The result (4) also appreciably differs from the kinematic one [Eqs. (2) and (3)].

Using a generalization of Eq. (3),

$$\Delta u=\left[1-K(T)\frac{\rho_n}{\rho}\right]v_s \ , \tag{3'}$$

we obtain the coefficient K as a function of T, which substantially differs from the early supposition, accepted in the literature, $K(T)\approx 1$. At low temperature ($T<0.4$ K) $K(T)$ proves to be a negative and nearly constant quantity: $K\approx -2.7$.[6] The result $K<0$ means that immobilizing the normal component leads to an increase (contrary to natural expectation) of the velocity of center of the sound sphere. With increasing temperature the coefficient K sharply rises to large (positive) value ($K_{max}\sim 26$ at $T\approx 0.6$ K) and then essentially decreases to $K\sim 2$ at $T\sim 2$ K. In conclusion of this section, we consider the corrections to the fourth-sound velocity in an immovable liquid which show that the calculations without the terms proportional to $\alpha(T)$ are here also inadequate.

(ii) When $\mathbf{v}_n=0$, the basic two-fluid hydrodynamic equations are

$$\begin{aligned}&\frac{\partial\rho}{\partial t}+\mathrm{div}\rho_s\mathbf{v}_s=0 \ ,\\ &\frac{\partial(\rho\sigma)}{\partial t}=0 \ ,\\ &\frac{\partial\mathbf{v}_s}{\partial t}+(\mathbf{v}_s\cdot\nabla)\mathbf{v}_s=-\nabla\mu\end{aligned} \tag{5}$$

(see, e.g., Ref. 5). Regarding all the variables in Eq. (5) as functions of T,P,v_s and substituting $T,P,\mathbf{v}_s\neq 0$ with the oscillating corrections $(T',P',\mathbf{v}_s')\propto\exp[i\omega(ux-t)]$, we find the system of linear equations for the amplitudes of T',P',v_s'. The condition for the nonzero solution of the system gives the equation for the fourth-sound velocity u:

$$\begin{vmatrix} -u\frac{\partial\rho}{\partial T}+v_s\frac{\partial\rho_s}{\partial T} & -u\frac{\partial\rho}{\partial P}+v_s\frac{\partial\rho_s}{\partial P} & -u\frac{\partial\rho}{\partial(\frac{1}{2}v_s^2)}v_s+\rho_s \\ \frac{\partial(\rho\sigma)}{\partial T} & \frac{\partial(\rho\sigma)}{\partial P} & \frac{\partial(\rho\sigma)}{\partial(\frac{1}{2}v_s^2)} \\ -\sigma & \frac{1}{\rho} & -u+\frac{\rho_s}{\rho}v_s \end{vmatrix}=0 \ . \tag{6}$$

We support here that $v_s \ll u$.

The partial derivatives are taken here with all other independent variables held constant, i.e.,

$$\frac{\partial}{\partial T}=\left[\frac{\partial}{\partial T}\right]_{P,v_s}, \quad \frac{\partial}{\partial P}=\left[\frac{\partial}{\partial P}\right]_{T,v_s}, \quad \frac{\partial}{\partial(\frac{1}{2}v_s^2)}=\left[\frac{\partial}{\partial(\frac{1}{2}v_s^2)}\right]_{P,T}. \tag{7}$$

From the expression for the chemical potential,

$$d\mu=-\sigma dT+\frac{1}{\rho}dP-\frac{\rho_n}{\rho}v_s dv_s , \tag{8}$$

we find (using the Maxwell relations)

$$\left[\frac{\partial\rho}{\partial v_s^2/2}\right]_{T,P}=\rho^2\left[\frac{\partial(\rho_n/\rho)}{\partial\rho}\right]_{T,v_s},$$
$$\left[\frac{\partial(\rho\sigma)}{\partial v_s^2/2}\right]_{T,P}=\sigma\rho^2\left[\frac{\partial(\rho_n/\rho)}{\partial P}\right]_{T,v_s}+\rho\left[\frac{\partial(\rho_n/\rho)}{\partial T}\right]_{P,v_s}. \tag{9}$$

Using (6) and (9) and the condition $v_s \ll u$, we get

$$u^2-u_0^2+uv_s\left\{-\frac{\rho_s}{\rho}-1+\frac{A+B+C}{D}\right\}=0,$$
$$u=u_0\pm\Delta u , \quad \Delta u=\left[1-\frac{\rho_n}{2\rho}-\frac{A+B+C}{2D}\right]v_s , \tag{10}$$

where

$$u_0^2=\rho_s\frac{B_1}{D}, \tag{11}$$

$$A=\frac{\partial\rho_n}{\partial P}\frac{\partial(\rho\sigma)}{\partial T}-\frac{\partial\rho_n}{\partial T}\frac{\partial(\rho\sigma)}{\partial P},$$
$$B=B_1B_2 , \quad B_1=\left[\frac{1}{\rho}\frac{\partial(\rho\sigma)}{\partial T}+\underline{\sigma\frac{\partial(\rho\sigma)}{\partial P}}\right],$$
$$B_2=\rho^2\frac{\partial(\rho_n/\rho)}{\partial P}, \tag{12}$$
$$C=-\left[\underline{\sigma\rho^2\frac{\partial(\rho_n/\rho)}{\partial P}}+\rho\frac{\partial(\rho_n/\rho)}{\partial T}\right]\left[\frac{1}{\rho}\frac{\partial\rho}{\partial T}+\sigma\frac{\partial\rho}{\partial P}\right],$$
$$D=\frac{\partial\rho}{\partial P}\frac{\partial(\rho\sigma)}{\partial T}-\underline{\frac{\partial\rho}{\partial T}\frac{\partial(\rho\sigma)}{\partial P}}.$$

Underlined terms are of higher order in ρ_n/ρ. They are necessary here only for the corrections to u_0 (11) (in B_1, D).

(iii) As is well known, the thermodynamic derivatives for a Bose liquid can be approximately calculated in terms of ideal gases of excitation—phonons ($\varepsilon=cp$) and rotons [$\varepsilon=\Delta+(p-p_0)^2/2\mu$]. One can see the high precision of such an approach by comparing the results of the calculations, e.g., for the entropy $\sigma(T)$ [Eq. (16)] of He II with the experimental data (see, e.g., Ref. 5).

For the energy F of a Bose liquid, we get[7]

$$F=V\left[f_0(\rho)-\frac{\pi^2}{90}\left[\frac{kT}{\hbar c}\right]^3 kT-kTn_r\right], \tag{13}$$

$$n_r=\frac{2p_0^2(\mu kT)^{1/2}e^{-\Delta/T}}{(2\pi)^{3/2}\hbar^3}, \tag{14}$$

$$\mu=0.16m_{^4\mathrm{He}} , \quad \Delta=8.6\ \mathrm{K} , \quad p_0=\hbar k_0 ,$$
$$k_0=1.92\times10^4\ \mathrm{cm}^{-1} , \quad c=2.4\times10^4\ \mathrm{cm/s} . \tag{15}$$

Thus the entropy $S=mN\sigma=-(\partial F/\partial T)_V$ and the pressure $P=-(\partial F/\partial V)_T$ are

$$\sigma=\frac{2\pi^2k^4T^3}{45c^3\rho\hbar^3}+\frac{kn_r}{\rho}\left[\frac{\Delta}{T}+\frac{3}{2}\right], \tag{16}$$

$$P=f_1(\rho)+f_2(\rho)\frac{\pi^2T^4}{90c^3}\frac{k^4}{\hbar^3}+kTn_r s , \tag{17}$$

where

$$f_1(\rho)=\rho\frac{df_0}{d\rho}-f_0 , \quad f_0=\frac{F(T=0)}{V}$$
$$f_2(\rho)=1+3a , \quad a=\frac{\rho}{c}\frac{dc}{d\rho},$$
$$s=1+\frac{\Delta}{T}r-\delta , \quad r=\frac{\rho}{\Delta}\frac{d\Delta}{d\rho}, \tag{18}$$
$$\delta=\frac{\rho}{2\mu}\frac{d\mu}{d\rho}+\frac{2\rho}{p_0}\frac{dp_0}{d\rho}.$$

We have neglected here the derivatives $(\partial\Delta/\partial T)_\rho$, $(\partial\mu/\partial T)_\rho$, and $(\partial p_0/\partial T)_\rho$ (as, e.g., in Ref. 7). The first two derivatives $[(\partial\Delta/\partial T)_\rho,(\partial\mu/\partial T)_\rho]$ are very small, at least for $T<1.7$ K (see, e.g., Ref. 8); the last one $[(\partial p_0/\partial T)_\rho]$ is negligible at all $T<T_\lambda$: $p_0\propto\rho^{1/3}$. The third term δ in s [see Eqs. (18)] does not contain the large factor Δ/T, unlike the second term. Moreover, the terms in δ [Eqs. (18)] nearly cancel each other:

$$\frac{\rho}{p_0}\frac{dp_0}{d\rho}\approx\frac{1}{3}, \quad \frac{\rho}{\mu}\frac{d\mu}{d\rho}\sim-1.2 \tag{19}$$

(see Ref. 8). Thus we get

$$s\approx1+\frac{\Delta}{T}r ,$$
$$\frac{\partial n_r}{\partial T}\approx\left[\frac{\Delta}{T}+\frac{1}{2}\right]\frac{n_r}{T}, \quad \frac{\partial n_r}{\partial\rho}\approx-\frac{\Delta}{T}r\frac{n_r}{\rho}. \tag{20}$$

Using the expressions for the contributions of phonons and rotons to the normal density,

$$\rho_n=\rho_{\mathrm{ph}}+\rho_r , \quad \rho_{\mathrm{ph}}=\frac{2\pi^2}{45}\left[\frac{kT}{\hbar c}\right]^3\frac{kT}{c^2}, \quad \rho_r=\frac{p_0^2}{3kT}n_r , \tag{21}$$

we get

$$\left[\frac{\partial\rho}{\partial T}\right]_P=-\frac{(\partial\phi/\partial T)_{\rho,P}}{(\partial\phi/\partial\rho)_{T,P}}$$

$$=-\frac{1}{T}\left\{f_2\rho_{\rm ph}+\lambda\rho_r\left[r+\frac{T}{\Delta}\left[1+\frac{r}{2}\right]\right]\right\}, \quad (22)$$

$$\left[\frac{\partial\rho}{\partial P}\right]_T=-\frac{(\partial\phi/\partial P)_{\rho,T}}{(\partial\phi/\partial\rho)_{P,T}}=\frac{1}{c^2}\left[1+\frac{b}{4}\frac{\rho_{\rm ph}}{\rho}+\lambda r^2\frac{\rho_r}{\rho}\right], \quad (23)$$

where

$$\phi\equiv P-f_1-f_2\frac{\pi^2(kT)^4}{90c^3\hbar^3}-kTn_rs\ , \quad (24)$$

$$\frac{df_1}{d\rho}=c^2\ ,\quad b=-c^3\rho\frac{d}{d\rho}\left[\frac{f_2}{c^3}\right],$$

$$\lambda=\frac{3(k\Delta)^2}{(cp_0)^2}\approx 0.179\ . \quad (25)$$

Similarly, we find

$$\left[\frac{\partial\rho_n}{\partial T}\right]_P=\frac{4\rho_{\rm ph}}{T}+\frac{\rho_r}{T}\left[\frac{\Delta}{T}-\frac{1}{2}\right], \quad (26)$$

$$\left[\frac{\partial\rho_n}{\partial P}\right]_T=-\frac{1}{c^2}\left[5a\frac{\rho_{\rm ph}}{\rho}+\left[\frac{\Delta}{T}r-\frac{2}{3}\right]\frac{\rho_r}{\rho}\right], \quad (27)$$

$$\left[\frac{\partial(\rho_n/\rho)}{\partial T}\right]_P=\frac{1}{\rho T}\left[4\rho_{\rm ph}+\frac{\Delta}{T}\left[1-\frac{T}{2\Delta}\right]\rho_r\right], \quad (28)$$

$$\left[\frac{\partial(\rho_n/\rho)}{\partial P}\right]_T=-\frac{1}{\rho^2c^2}\left[(5a+1)\rho_{\rm ph}+\left[\frac{1}{3}+\frac{\Delta}{T}r\right]\rho_r\right], \quad (29)$$

$$\left[\frac{\partial(\rho\sigma)}{\partial T}\right]_P=\frac{c^2}{T^2}\left[3\rho_{\rm ph}+\lambda\left[1+\frac{T}{\Delta}\right]\rho_r\right], \quad (30)$$

$$\left[\frac{\partial(\rho\sigma)}{\partial P}\right]_T=-\frac{1}{\rho T}\left[3a\rho_{\rm ph}+\lambda r\left[1+\frac{T}{2\Delta}\right]\rho_r\right]. \quad (31)$$

In Eqs. (27) and (29), we use Eqs. (19). In Eqs. (26)–(31), we have neglected the terms quadratic in $\rho_{\rm ph},\rho_r$ because they give a very small contribution in the formulas for $\Delta u(10)$, and $u_0(11)$.

Substituting the formulas (21), (22), and (26)–(31) in Eqs. (12), we find A,B_1,B_2,C,D.

For calculation of $\Delta u(10)$ with an accuracy of ρ_n/ρ, we can take in $A,\ldots,D$ only the terms of the lowest order in ρ_n/ρ. In that case we get

$$\frac{A+B+C}{2D}\approx-\frac{\rho_n}{2\rho}+\left\{\frac{\partial P}{\partial\rho}\frac{\partial\rho_n}{\partial P}-\frac{\sigma}{\rho}\frac{\partial\rho_n}{\partial T}\left[\frac{\partial\sigma}{\partial T}\right]^{-1}-\frac{1}{\rho^2}\frac{\partial P}{\partial\rho}\frac{\partial\rho}{\partial T}\frac{\partial\rho_n}{\partial T}\left[\frac{\partial\sigma}{\partial T}\right]^{-1}\right\}, \quad (32)$$

i.e.,

$$\frac{\rho_n}{\rho}K(T)\approx\left\{\frac{\partial P}{\partial\rho}\frac{\partial\rho_n}{\partial P}-\frac{\sigma}{\rho}\frac{\partial\rho_n}{\partial T}\left[\frac{\partial\sigma}{\partial T}\right]^{-1}-\frac{1}{\rho^2}\frac{\partial P}{\partial\rho}\frac{\partial\rho}{\partial T}\frac{\partial\rho_n}{\partial T}\left[\frac{\partial\sigma}{\partial T}\right]^{-1}\right\} \quad (33)$$

[see Eqs. (3′) and (10)]. Accordingly, we find

$$\frac{A+B+C}{2D}=\frac{\rho_n}{\rho}[-\tfrac{1}{2}+K(T)]$$

$$=\frac{\rho_n}{\rho}\left\{-\frac{1}{2}-\frac{5a-g[(\Delta/T)|r|+\frac{2}{3}]}{1+g}+\frac{\{4+g[(\Delta/T)-\frac{1}{2}]\}[a-\frac{1}{3}\lambda g|r|(1+T/2\Delta)]}{(1+g)[1+\frac{1}{3}\lambda g(1+T/\Delta)]}\right\}$$

$$=\frac{\rho_n}{\rho}\,\frac{-(a+\frac{1}{2})+\lambda g^2[(|r|/3)+\frac{1}{18}]+g\{(\Delta/T)(a+|r|)-[(a/2)-\frac{1}{6}]-(\lambda/3)(5a+4|r|+\frac{1}{2})\}}{(1+g)[1+\frac{1}{3}\lambda g(1+T/\Delta)]}, \quad (34)$$

where

$$g(T)=\frac{\rho_r(T)}{\rho_{\rm ph}(T)}=57.6\frac{e^{8.6(1-1/T)}}{T^4\sqrt{T}} \quad (35)$$

(T is measured in kelvin).

We take into account that $r<0$ for He II. [Note that $r<0$ also for the weak-interacting Bose gas (WIBG).[6]] The first term in the numerator on the right-hand side of Eq. (34) corresponds to the phonon's contribution, the second term to the roton's contribution, and the third to the phonon-roton's combined contribution.

In the pure phonon region $g\ll 1$ ($T\leq 0.4$ K; see Table I), we get

$$\frac{A+B+C}{2D}=-\left[a+\frac{1}{2}\right]\frac{\rho_n}{\rho}\ ,$$

$$\Delta u=\left[1+a\frac{\rho_n}{\rho}\right]v_s\ . \quad (36)$$

TABLE I. Fourth-sound Doppler shift.

T	g	K	K^*	ρ_n/ρ	$K\rho_n/\rho$
$T\to 0$	$\to 0$	-2.67	-12	$\to 0$	$\to 0$
0.3	2.5×10^{-5}	-2.66	-11.97	9.5×10^{-7}	-2.5×10^{-6}
0.4	8.9×10^{-3}	-2.04	-11.8	2.9×10^{-6}	-5.9×10^{-6}
0.5	0.24	8.28	-6.55	8.2×10^{-6}	6.8×10^{-5}
0.55	0.75	18.81	-0.47	1.9×10^{-5}	3.6×10^{-4}
0.6	1.86	25.56	2.09	4.3×10^{-5}	1.1×10^{-3}
0.63	2.96	26.56 (max)	6.44	7.3×10^{-5}	1.9×10^{-3}
0.65	3.91	25.72	7.25	1.03×10^{-4}	2.6×10^{-3}
0.67	5.04	25.28	7.78	1.4×10^{-4}	3.5×10^{-3}
0.7	7.21	23.16	8.2	2.3×10^{-4}	5.3×10^{-3}
0.75	12.01	18.93	8.35 (max)	4.8×10^{-4}	9.1×10^{-3}
0.8	18.26	15.06	8.12	8.93×10^{-4}	1.3×10^{-2}
1.0	57.62	6.27	6.85	6.9×10^{-3}	4.3×10^{-2}
1.2	106.6	3.62	5.94	2.6×10^{-2}	9.4×10^{-2}
1.5	164.6	2.4	5.06	0.096	0.23
2	186.8	1.87	4.17	0.35	0.65

Provided $a=(\rho/c)dc/d\rho>0$, Eq. (36) corresponds to an *anomalous* Doppler shift. The coefficient $K(T)$ [cf. Eq. (3′)] is here constant and negative: $K=-a$. The condition $a>0$ is fulfilled for He II [$a\approx 2.67$; see below Eq. (39)]. Let us note that for the general simple model of WIBG we obtain the anomalous Doppler shift too:[6]

$$a_{\rm WIBG}=\tfrac{1}{2}\ ,\quad \Delta u_{\rm WIBG}=\left[1+\frac{\rho_n}{2\rho}\right]v_s\ .$$

Outside the phonon region, one of the roton terms (proportional to g) in the numerator of the fraction on the right-hand side of Eq. (34)—with a large factor $g(\Delta/T)(a+|r|)$—gives the main contribution to K:

$$\frac{A+B+C}{2D}\sim\frac{\rho_n}{\rho}\frac{\Delta}{T}\frac{a+|r|}{1+1/g}\ . \tag{37a}$$

With increasing temperature this term quickly increases owing to the sharp increase of $g(T)$. It becomes comparable (modulo) with the phonon term, exceeds it, and provides an extremely sharp increase of $K(T)$ from negative phonon value $K=-a\approx-2.67$ to the large positive maximum

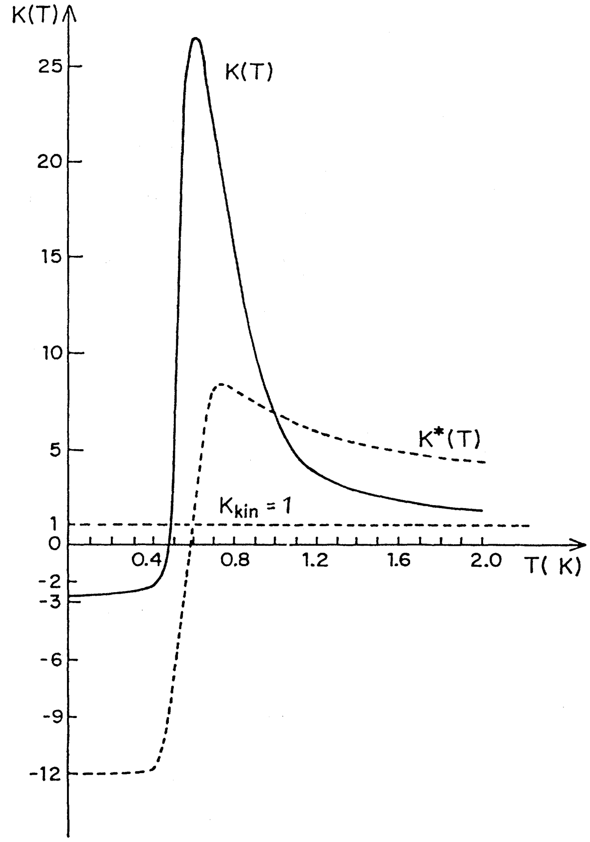

FIG. 1. Coefficient $K(T)$ in the expression for the fourth-sound Doppler shift, $\Delta u=[1-K(T)\rho_n/\rho]v_s$.

$$K_{\max}\sim\frac{\Delta}{T_m}\frac{a+|r|}{1+1/g_m}\sim 33 \tag{37b}$$

[$T_m\approx 0.63$ K, $g_m\approx 2.96$ K; see Eq. (39)]. The correction from other terms somewhat diminishes this result ($33\to 26.56$). A further increase of the temperature leads to diminution of $K(T)$ because of the factor $1/T$ and the increase of g in the term proportional to λg in the denominator [see Eq. (34)]. Eventually, $K(T)$ approaches the kinematic value $K=1$ (but becomes less than 2 only at $T\sim 2$ K; see Table I and Fig. 1).

(iv) From the expression for the thermal-expansion coefficient [see Eq. (22)],

$$\alpha(T)=-\frac{1}{\rho}\left[\frac{\partial\rho}{\partial T}\right]_P\approx\frac{1}{T}\left\{f_2\frac{\rho_{\rm ph}}{\rho}+\lambda\left[\left[1+\frac{T}{2\Delta}\right]r+\frac{T}{\Delta}\right]\frac{\rho_r}{\rho}\right\}$$
$$\approx 10^{-3}\left\{0.117f_2T^3+1.21\frac{e^{8.6(1-1/T)}}{T^{3/2}}\left[\left[1+\frac{T}{17.2}\right]r+\frac{T}{8.6}\right]\right\} \tag{38}$$

[$T<1.2$, where T is measured in kelvin; the experimental data for $\alpha(T)$ are adduced, e.g., in Ref. 5], we find the approximate values of $f_2=(1+3a)$ and r:

$$f_2\approx 9\ (\text{i.e., } a\approx 2.67)\ ,\quad r\approx -0.6\ . \tag{39}$$

Putting Eqs. (34), (35), and (39) in Eq. (10), we obtain

$$\Delta u=\left[1-K(T)\frac{\rho_n}{\rho}\right]v_s\ ,$$

$$K(T)=\frac{1}{2}+\frac{-3.17+0.0458g^2+g(28.09/T-2.14)}{(1+g)[1+0.06g(1+T/8.6)]}\ , \tag{40}$$

$$\frac{\rho_n}{\rho} \approx 1.17\times 10^{-4} T^4 (1+g) \tag{41}$$

(see Table I and Fig. 1).

Thus, with decreasing temperature, the coefficient K grows from the almost kinematic value $K \sim 1$ up to maximum $K \approx 26.56$ at $T \approx 0.63$ K and then sharply drops down to a negative value that means the anomalous Doppler shift.

The fourth line of Table I shows the values of K according to formula (4) [obtained without taking into account terms proportional to $\alpha(T)$]:

$$\Delta u = \left[1 - K^* \frac{\rho_n}{\rho} \right] v_s ,$$

$$\frac{\rho_n}{\rho} K^*(T) \approx \left\{ \frac{\partial P}{\partial \rho}\frac{\partial \rho_n}{\partial P} - \frac{\sigma}{\rho}\frac{\partial \rho_n}{\partial T} \left[\frac{\partial \sigma}{\partial T} \right]^{-1} \right\}$$

$$= \frac{\rho_n}{\rho} \left\{ \frac{-5a + g[(\Delta/T)|r| + \frac{2}{3}]}{1+g} + \frac{\{4+g[(\Delta/T)-\frac{1}{2}]\}(1+\lambda g T/\Delta)}{3(1+g)[1+\frac{1}{3}\lambda g(1+T/\Delta)]} \right\} , \tag{42}$$

$$K^*(T) \approx \frac{-13.3 + g(5.16/T + \frac{2}{3})}{1+g} + \frac{4 + 0.179 g^2 + g(8.6/T - \frac{1}{2})}{3(1+g)(1+0.06g)} .$$

In the case $g \ll 1$ (phonon region), we get

$$K^* = -(5a + \tfrac{4}{3}) , \quad K^*_{\rm WIBG} = \tfrac{23}{6} \quad (a = \tfrac{1}{2}) , \tag{43}$$

i.e.,

$$\Delta u = \left[1 + \left[\frac{4}{3} + 5a \right] \frac{\rho_n}{\rho} \right] ,$$

$$\Delta u_{\rm WIBG} = \left[1 + \frac{23}{6}\frac{\rho_n}{\rho} \right]$$

[cf. correct results, Eqs. (34), (36), and (40)].

We see that the "approximate" formula (4) gives a substantially different result from the exact one especially at $T < 1$ K. Simultaneously, we note that the formula (4), like the correct one (10), does not lead to the result $K \approx 1$, although the peak of $K^*(T)$ at about 0.7 K is not so high and sharp as the peak of $K(T)$. At low temperature ($T < 0.6$ K), it also corresponds to the anomalous Doppler shift; at $T < 0.5$ K K^* becomes much more negative than K.

Finally, in Table I we adduce the total corrections to the Doppler shift—the exact one $[K(T)\rho_n/\rho]$ and the kinematic one (ρ_n/ρ).

Since the difference between $K(T)$ and $K^*(T)$ is proportional to the thermal-expansion coefficient $\alpha(T)$ [see Eqs. (33) and (42)], it is interesting to compare the temperature at which $K = K^*$ with the experimental temperature at which the measured $\alpha(T)$ vanishes. We obtain $T_{K=K^*} \approx 1.0$ (see Table I) and $T_{\alpha=0} \approx 1.1$ (see, e.g., Ref. 5). This reflects the exactness of our numerical calculations of $K(T)$ and $K^*(T)$ on the basis of two main parameters a and r.

(v) Substituting the expressions for B_1 and D in Eq. (11), we can calculate the correction to the fourth-sound velocity $(u_0 - c) \propto \rho_n/\rho$. We get

$$u_0^2 = \rho \left[1 - \frac{\rho_n}{\rho} \right] \frac{B_1}{D} \approx \frac{\partial P}{\partial \rho} \left[1 - \frac{\rho_n}{\rho} + \frac{\partial(\rho\sigma)}{\partial P} \left[\frac{\partial \sigma}{\partial T} \right]^{-1} \left[\sigma + \frac{1}{\rho}\frac{\partial \rho}{\partial T}\frac{\partial P}{\partial \rho} \right] \right]$$

$$\approx c^2 \left[1 - \frac{\rho_n}{\rho} \left[1 + \frac{b/4 + \lambda g r^2}{1+g} - \frac{[3a - \lambda g |r| (1+T/2\Delta)]^2}{3(1+g)[1+\frac{1}{3}\lambda g(1+T/\Delta)]} \right] \right] . \tag{44}$$

At $g \to 0$ we obtain

$$u_0^2 \approx c^2 \left[1 - \frac{\rho_n}{\rho} \left[1 + \frac{b}{4} - 3a^2 \right] \right] ; \tag{45}$$

for the WIBG ($a = \frac{1}{2}$, $b = \frac{15}{4}$),

$$u_0^2 \approx c^3 \left[1 - \frac{19}{16}\frac{\rho_n}{\rho} \right] . \tag{46}$$

Note that the usual approximate formula for u_0 that is obtained by neglecting the derivatives proportional to the thermal-expansion coefficient α [see, e.g., (24.6) in Ref. 7],

$$u_0^2 = \frac{\rho_s}{a}\frac{\partial P}{\partial \rho} + \frac{\rho_s \sigma^2}{\rho \partial\sigma/\partial T} , \tag{47}$$

also does not give the proper correction:

$$u_0^2 \approx c^2 \left[1 - \frac{\rho_n}{\rho} \left\{ 1 + \frac{b/4 + \lambda g r^2}{1+g} - \frac{(1+\lambda g T/\Delta)^2}{3(1+g)[1+\frac{1}{3}\lambda g(1+T/\Delta)]} \right\} \right] ; \tag{48}$$

at $g \to 0$,

$$u_0^2 = c^2 \left[1 - \frac{\rho_n}{\rho} \left[\frac{2}{3} + \frac{b}{4} \right] \right] ;$$

for the WIBG, $u_0^2 = c^2[1 - \frac{77}{48}\rho_n/\rho]$ [cf. Eqs. (45) and (46)].

III. DOPPLER SHIFT OF FIRST SOUND

(i) The consideration of the Doppler shift in the case of a free (unlocked) normal component is based on the complete set of hydrodynamic equations,[4,7]

$$\frac{\partial\rho}{\partial t}+\mathrm{div}\,\mathbf{j}=0\ ,\quad \mathbf{j}=\rho_n\mathbf{v}_n+\rho_s\mathbf{v}_s\ ,$$
$$\frac{\partial j_i}{\partial t}+\nabla_k(p\delta_{ik}+\rho_n v_{ni}v_{nk}+\rho_s v_{si}v_{sk})=0\ ,$$
$$\frac{\partial(\rho\sigma)}{\partial t}+\mathrm{div}(\rho\sigma\mathbf{v}_n)=0\ , \tag{49}$$
$$\frac{\partial\mathbf{v}_s}{\partial t}+(\mathbf{v}_s\cdot\nabla)\mathbf{v}_s=\nabla\mu\ .$$

Let the sound oscillations be propagated in the direction along the x axis, Ox, and let $\mathbf{v}_n\|\mathbf{v}_s\|Ox$. Using the notation

$$\mathbf{v}=\mathbf{j}/\rho\ ,\quad \mathbf{w}=\mathbf{v}_n-\mathbf{v}_s\ ,\quad \mathbf{U}=\mathbf{u}-\mathbf{v}\ , \tag{50}$$

we obtain the following equation for the sound velocity u (owing to the Galilean principle, the quantity $\mathbf{u}$ can appear only in the combination $\mathbf{U}=\mathbf{u}-\mathbf{v}$):

$$\begin{vmatrix} -U\dfrac{\partial\rho}{\partial T} & -U\dfrac{\partial\rho}{\partial P} & -Uw\dfrac{\partial\rho}{\partial w^2/2} & \rho \\ 0 & 1 & \dfrac{2\rho_n\rho_s}{\rho}w & -U\rho \\ -U\rho\dfrac{\partial\sigma}{\partial T}+w\dfrac{\partial(\rho_s\sigma)}{\partial T} & -U\rho\dfrac{\partial\sigma}{\partial P}+w\dfrac{\partial(\rho_s\sigma)}{\partial P} & \rho_s\sigma-Uw\rho\dfrac{\partial\sigma}{\partial w^2/2} & 0 \\ -\sigma+Uw\dfrac{\partial(\rho_n/\rho)}{\partial T} & \dfrac{1}{\rho}+Uw\dfrac{\partial(\rho_n/\rho)}{\partial P} & (U-w)\dfrac{\rho_n}{\rho} & -\left[U+\dfrac{\rho_n}{\rho}w\right] \end{vmatrix}=0\ . \tag{51}$$

Using (9) and the condition $w\ll U$, we get

$$AU^4+BU^2+C+RwU^3+SwU=0\ , \tag{52}$$

$$U_i=u_i^{(0)}+\gamma_i w\ ,$$
$$u_i=u_i^{(0)}+v+\gamma_i w$$
$$=u_i^{(0)}+\left[\gamma_i+\frac{\rho_n}{\rho}\right]v_n+\left[1-\left[\frac{\rho_n}{\rho}+\gamma_i\right]\right]v_s\ , \tag{53}$$

$$(u_{1,2}^{(0)})^2=\frac{-B\pm(B^2-4AC)^{1/2}}{2A}\ , \tag{54}$$

$$\gamma_i=-\frac{R(u_i^{(0)})^2+S}{2[2A(u_i^{(0)})^2-B]}\ , \tag{55}$$

where

$$A=\rho\left[\frac{\partial\rho}{\partial P}\frac{\partial\sigma}{\partial T}-\underline{\frac{\partial\rho}{\partial T}\frac{\partial\sigma}{\partial P}}\right]\ , \tag{56a}$$

$$B=-\rho\left[\frac{\partial\sigma}{\partial T}-\frac{\rho_s\sigma^2}{\rho_n}\frac{\partial\rho}{\partial P}\right]\ , \tag{56b}$$

$$C=\frac{\rho_s\rho\sigma^2}{\rho_n}\ , \tag{56c}$$

$$R=-\frac{\partial\rho}{\partial P}M+R'\ ,\quad S=M+S'\ , \tag{57a}$$

$$M=(2\rho_s+\rho)\frac{\partial\sigma}{\partial T}+\frac{\partial(\rho_s\sigma)}{\partial T}-\frac{\rho\rho_s\sigma}{\rho_n}\frac{\partial(\rho_n/\rho)}{\partial T}-\frac{\rho^2\sigma}{\rho_n}\frac{\partial(\rho_n/\rho)}{\partial T}\ , \tag{57b}$$

$$\underline{R'}=(2\rho_s+\rho)\frac{\partial\rho}{\partial T}\frac{\partial\sigma}{\partial P}+\frac{\partial\rho}{\partial T}\frac{\partial(\rho_s\sigma)}{\partial P}-\frac{\rho\rho_s\sigma}{\rho_n}\frac{\partial\rho}{\partial T}\frac{\partial(\rho_n/\rho)}{\partial P}-\frac{\rho^4\sigma}{\rho_n}\frac{\partial\sigma}{\partial P}\frac{\partial(\rho_n/\rho)}{\partial P}\ , \tag{57c}$$

$$\underline{S'}=\frac{\rho_s\sigma}{\rho}\frac{\partial\rho}{\partial T}+2\rho_s\rho\sigma\frac{\partial\sigma}{\partial P}\ . \tag{57d}$$

The coefficients A,B,C,R,S contain terms of both first and second order in ρ_n/ρ [see Eqs. (22), (23), and (26)–(31)]; the latter are underlined. Note that all the underlined terms are proportional to $\alpha(T)$ or to $\alpha(T)^2$.

In the first approximation, we neglect all the underlined terms; the results coincide (to the same order) with those obtained by Khalatnikov[4] [see Eqs. (53)–(57)]:

$$(u_1^{(0)})^2=\frac{\partial P}{\partial\rho}\ ,\quad (u_2^{(0)})^2=\frac{\rho_s\sigma^2}{\rho_n(\partial\sigma/\partial T)}\ ,$$
$$\gamma_1=0, \tag{58}$$

since

$$R(u_1^{(0)})^2+S=M\left[-\frac{\partial\rho}{\partial P}(u_1^{(0)})^2+1\right]=0\ ,$$

i.e., $U_1=u_1^{(0)}$ or

$$u_1=u_1^{(0)}+v=u_1^{(0)}+\left[1-\frac{\rho_n}{\rho}\right]v_s$$

(if $v_n=0$),

$$\gamma_2=\frac{M}{2\rho\partial\sigma/\partial T}=\left[2-\frac{\sigma}{\rho_n}\frac{\partial\rho_n/\partial T}{\partial\sigma/\partial T}\right]\ , \tag{59}$$

i.e., $U_2 = u_2^{(0)} + \gamma_2 v_n$,

$$u_2 = u_2^{(0)} + \left[\gamma_2 + \frac{\rho_n}{\rho}\right] v_n \approx u_2^{(0)} + \gamma_2 v_n$$

(if $v_s = 0$). Using Eqs. (16), (21), (28), and (30), we find

$$\gamma_2(T) = 2 - \frac{[4+g(\Delta/T-\frac{1}{2})](1+g\lambda T/\Delta)}{[3+g\lambda(1+T/\Delta)](1+g)} . \quad (60)$$

In the phonon region ($g \ll 1$) $\gamma_2(T)$ is positive constant: $\gamma_2(T) \approx \frac{2}{3}$; at the beginning of the roton region ($T_m \approx 0.6$ K), there appears a sharp negative minimum of $\gamma_2(T)$: $\gamma_{\min}(T) \sim -1$ [the term $\Delta/3T(1+1/g)$ plays here a role similar to the role of the term in Eq. (37) for the function $K(T)$; see Eq. (34)]. The result $\gamma < 0$ means an extremely interesting physical effect found by Khalatnikov:[4] the possibility of the entrainment of the second sound in the direction opposite to the normal-component velocity.

The results obtained in the first approximation are adequate for the investigation of the Doppler shift of the second sound, but not for the first sound. In the latter case, we must take into account terms of the next order. The reason for this is the following: The motion of the center of sound "sphere" for the first and second sounds is not connected with the rest frame of either the liquid as a whole or one of its components (superfluid or normal, respectively). Thus we can expect for the Doppler shift velocity something "intermediate" between $\mathbf{v} = \mathbf{j}/\rho$ and $\mathbf{v}_s(\mathbf{v}_n)$. For the second sound, the difference between the corresponding velocities $\mathbf{v}$ and $\mathbf{v}_n$ ($\mathbf{v}_s = 0$) is not small: $(v - v_n)/v_n \propto \rho_s/\rho \approx 1$; therefore, $|\gamma_2 - 1| \sim 1$. However, for the first sound the corresponding difference between v and v_s ($v_n = 0$) is small: $(v - v_s)/v_s \propto \rho_n/\rho$; accordingly, $|\gamma_1 - 1| \propto \rho_n/\rho$. Thus, for the calculation of the "intermediate" Doppler-shift velocity in the case of first sound, we need a higher order of accuracy than in the case of second sound.

If we represent the exact expression for $\gamma_1(55)$ by means of M, R', S' [see Eqs. (57)],

$$\gamma_1 = -\frac{M[(-\partial\rho/\partial P)(u_1^{(0)})^2 + 1] + R'(u_1^{(0)})^2 + S'}{2[2A(u_1^{(0)})^2 - B]} , \quad (61)$$

we can use for all the quantities $M, R', S', A, B, (u_1^{(0)})^2$, except $(u_1^{(0)})^2$ in the numerator, the lowest order in ρ_n/ρ: The terms R', S', and $M(\cdots)$ in the numerator and $A(u_1^{(0)})^2, B$ in the denominator are quantities of the same order (the second and first, respectively). Thus we obtain simpler expressions for M, R', S',

$$M = 2\rho\left[2\frac{\partial\sigma}{\partial T} - \frac{2\sigma}{\rho_n}\frac{\partial\rho_n}{\partial T}\right],$$

$$R' = 3\rho\frac{\partial\sigma}{\partial P}\frac{\partial\rho}{\partial T} + \frac{\partial(\rho\sigma)}{\partial P}\frac{\partial\rho}{\partial T} - \frac{\rho^4\sigma}{\rho_n}\frac{\partial(\rho_n/\rho)}{\partial P}\frac{\partial\sigma}{\partial P} - \frac{\rho^2\sigma}{\rho_n}\frac{\partial\rho}{\partial T}\frac{\partial(\rho_n/\rho)}{\partial P}, \quad (62)$$

$$S' = \sigma\frac{\partial\rho}{\partial T} + 2\rho^2\sigma\frac{\partial\sigma}{\partial P} ;$$

TABLE II. First-sound Doppler shift.

T	g	K_1	ρ_n/ρ	$K_1\rho_n/\rho$
$T \to 0$	$\to 0$	47.5	$\to 0$	$\to 0$
0.3	2.5×10^{-5}	47.47	9.5×10^{-7}	4.5×10^{-5}
0.4	8.9×10^{-3}	43.16	2.9×10^{-6}	1.2×10^{-5}
0.5	0.24	−14.5	8.2×10^{-6}	-1.2×10^{-4}
0.53	0.49	−30.2	1.4×10^{-5}	-4.2×10^{-4}
0.55	0.75	−33.3 (min)	1.9×10^{-5}	-6.3×10^{-4}
0.57	1.1	−31.3	2.6×10^{-5}	-8×10^{-4}
0.6	1.86	−24.0	4.3×10^{-5}	-1.0×10^{-3}
0.65	3.91	−11.4	1.03×10^{-4}	-1.2×10^{-3} (min)
0.7	7.21	−3.5	2.3×10^{-4}	-0.8×10^{-3}
0.8	18.26	−0.2	8.9×10^{-4}	1.7×10^{-4}
1.0	57.26	0.99	6.9×10^{-3}	6.8×10^{-3}
1.2	106.6	0.998	7.6×10^{-2}	2.6×10^{-2}
1.4	148.6	0.95	6.7×10^{-2}	6.3×10^{-2}
1.5	164.2	0.94	0.096	0.09
2	186.8	0.935	0.35	0.33

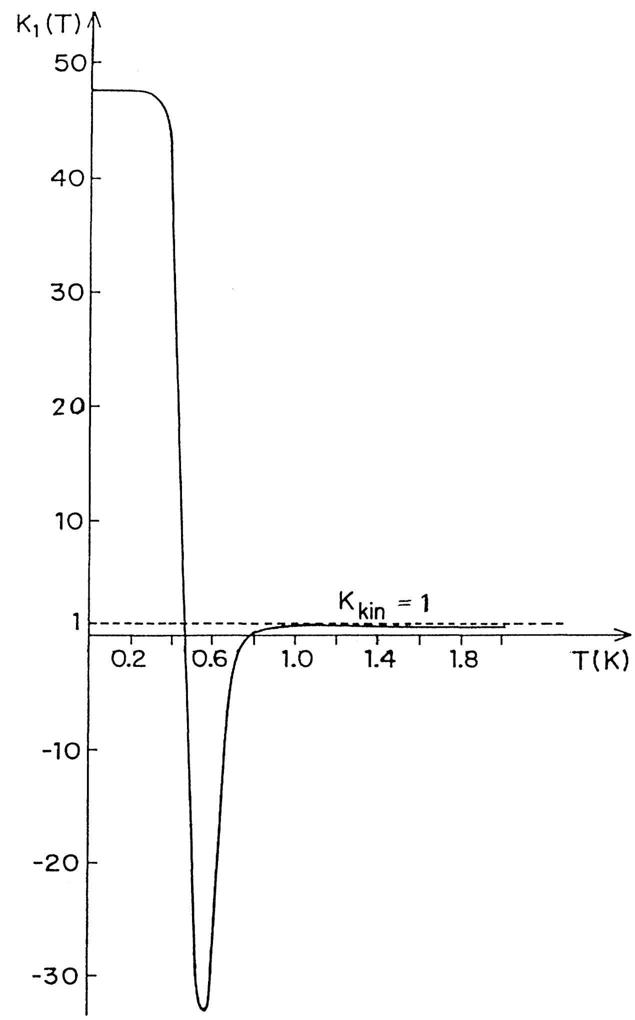

FIG. 2. Coefficient $K_1(T)$ in the expression for the first-sound Doppler shift, $\Delta u_1 = [1 - K_1(T)\rho_n/\rho]v_s$.

the expression for the denominator (the first order in ρ_n/ρ),

$$2A(u_1^{(0)})^2-B=(B^2-4AC)^{1/2}$$

$$=\rho\left[\frac{\partial\sigma}{\partial T}-\frac{\rho\sigma^2}{\rho_n}\frac{\partial\rho}{\partial P}\right]; \tag{63}$$

and $(u_1^{(0)})^2$ for the numerator, with a correction,

$$(u_1^{(0)})^2=\frac{\partial P}{\partial\rho}\left[1+\frac{\partial\rho}{\partial T}\frac{\partial\sigma}{\partial P}\frac{\partial P}{\partial\rho}\Big/\left[\frac{\partial\sigma}{\partial T}-\frac{\rho^2\sigma^2}{\rho_n}\frac{\partial\rho}{\partial p}\right]\right]. \tag{64}$$

We get

$$\gamma_1=M\frac{\partial\rho}{\partial T}\frac{\partial\sigma}{\partial P}\frac{\partial P}{\partial\rho}\Big/2\rho\left[\frac{\partial\sigma}{\partial T}-\frac{\rho\sigma^2}{\rho_n}\frac{\partial\rho}{\partial P}\right]^2-\left[R'\frac{\partial P}{\partial\rho}+S'\right]\Big/2\rho\left[\frac{\partial\sigma}{\partial T}-\frac{\rho\sigma^2}{\rho_n}\frac{\partial\rho}{\partial P}\right]. \tag{65}$$

In the case $v_n=0$ [i.e., $v=(\rho_s/\rho)v_s$, $w=-v_s$] we find

$$U_1=u_1^{(0)}-\gamma_1 v_s\ ,\quad u_1=u_1^{(0)}+\left[1-\left[\frac{\rho_n}{\rho}+\gamma_1\right]\right]v_s \tag{66}$$

[see (53)] or [using the denotation $\gamma_1\equiv(K_1-1)\rho_n/\rho$, which reflects the estimate $\gamma_1\propto\rho_n/\rho$]

$$u_1=u_1^{(0)}+\Delta u_1\ ,\quad \Delta u_1=\left[1-K_1(T)\frac{\rho_n}{\rho}\right]v_s\ , \tag{67}$$

$$\begin{aligned}K_1(T)&=\frac{\partial\rho}{\partial T}\left\{\frac{1}{\rho^2}\frac{\partial\rho}{\partial T}\frac{\partial P}{\partial\rho}\left[2\frac{\partial\sigma}{\partial T}-\frac{\sigma}{\rho_n}\frac{\partial\rho_n}{\partial T}\right]\Big/\left[\frac{\partial\sigma}{\partial T}-\frac{\rho\sigma^2}{\rho_n}\frac{\partial\rho}{\partial P}\right]^2\right.\\&\quad\left.-\left[\frac{\partial P}{\partial\rho}\left[\frac{2}{\rho^2}\frac{\partial\rho}{\partial T}-\frac{\sigma}{\rho_n}\frac{\partial\rho_n}{\partial P}\right]+6\frac{\sigma}{\rho}\right]\Big/\left[\frac{\partial\sigma}{\partial T}-\frac{\rho\sigma^2}{\rho_n}\frac{\partial\rho}{\partial P}\right]\right\}\\&=1+F\left[\frac{F}{G^2}\left\{2\left[3+g\lambda\left[1+\frac{T}{\Delta}\right]\right](1+g)-\left[1+g\lambda\frac{T}{\Delta}\right]\left[4+g\left[\frac{\Delta}{T}-\frac{1}{2}\right]\right]\right\}\right.\\&\quad\left.-\frac{2F(1+g)-(1+g\lambda T/\Delta)\{5a-g[(\Delta/T)|r|+\frac{2}{3}]\}-6(1+g\lambda T/\Delta)(1+g)}{G(1+g)}\right],\end{aligned} \tag{68}$$

$$F\equiv 3a+1-g\lambda\left[|r|\left[1+\frac{T}{2\Delta}\right]-\frac{T}{\Delta}\right],$$

$$G\equiv\left[3+g\lambda\left[1+\frac{T}{\Delta}\right]\right](1+g)-\left[1+g\lambda\frac{T}{\Delta}\right]^2$$

[see Eqs. (22), (23), and (26)–(31)].

Substituting the values of λ [Eq. (25)] and a,r [Eq. (39)], we get

$$\begin{aligned}K_1(T)&=1+F\left[\frac{F}{G^2}\left\{2\left[3+0.179g\left[1+\frac{T}{8.6}\right]\right](1+g)-(1+0.02gT)\left[4+g\left[\frac{8.6}{T}-\frac{1}{2}\right]\right]\right\}\right.\\&\quad\left.-\frac{2F(1+g)-(1+0.02gT)[13.33-g(5.16/T+\frac{2}{3})]-6(1+0.02gT)(1+g)}{G(1+g)}\right],\end{aligned} \tag{69}$$

$$F=\left[9-0.11g\left[1+\frac{T}{17.2}\right]+0.02gT\right],$$

$$G=\left\{\left[3+0.179g\left[1+\frac{T}{8.6}\right]\right](1+g)-(1+0.02gT)^2\right\} \tag{70}$$

(T is measured in kelvin).

In the phonon region ($g\ll1$), $K_1(T)$ is positive constant,

$$K_1(T)=1+(3a+1)(a+\tfrac{5}{2})\approx47.5\ ; \tag{71}$$

its sign corresponds to the normal Doppler effect, but its value is much more than the kinematic one, $K_1=1$. At the beginning of the roton region ($T\sim0.5$ K), the weight of the roton terms—first all of the terms proportional to $g\Delta/T$—sharply increases and provides a deep minimum of the function $K_1(T)$ at $T=T_m\approx 0.55$ K with a large negative value of K_1:

$$K_{1\,\min}\approx -33.3\ . \tag{72}$$

Thus we obtain here a strongly pronounced "anomalous" Doppler shift—the large outstripping effect. At higher temperatures ($T\sim 1$ K), the corrections to the kinematic result $K_1=1$ becomes small (see Table II and Fig. 2).

IV. CONCLUSION

The result that the Doppler shift is greater when part of the fluid is motionless than when the fluid flows as a whole [see (36), (67), and (71)] is highly curious, but it does not contradict any general principles. This is not a mechanical (kinematic) problem, connected with Galilean relativity; it is a question about an inner thermodynamical property of the system. First of all, let us take into account that even fourth sound is not an oscillation of the superfluid component only (as is sometimes claimed in the literature). Such a picture would correspond to the following formula of the Doppler shift:

$$|u|=|u_0|\pm v_s\ . \tag{73}$$

Even the immovable normal component takes part in the spread of fourth sound: There are the oscillations of ρ_n [the normal and superfluid components can convert into one other; cf. Eqs. (5)]. We can therefore expect the Doppler-shift velocity to change: $\Delta u\neq v_s$. (In ^{4}He-^{3}He mixtures, the part of ρ_n connected with ^{3}He does not participate in the oscillations; correspondingly, at small concentrations the influence of ^{3}He on the Doppler-shift fourth sound becomes negligible in comparison with its contribution to ρ_n; this is confirmed by the calculations in Ref. 9), i.e., Eq. (73) becomes correct at $T\to 0$.

Although the normal component takes part in the first- and fourth-sound oscillations, this participation is not on equal grounds, so that formula (2) is not valid. We can expect the correctness of Eq. (2) only for collisionless sound, which does not distinguish between the normal and superfluid components, so that the "rest frame" of its spread is the center-of-mass frame of the Bose system. The anisotropy of the state must contribute here only a small correction, but in the two-fluid hydrodynamic region, the situation is more complicated. The anisotropy connected with the relative motion of the normal and superfluid components leads to an appreciable change in the conditions of the spread of the hydrodynamic mode in different directions (there appears an anisotropy of the local-equilibrium properties of the medium). In the case $\mathbf{v}_n=0$, the contribution of the anisotropy turns out to be so large that the effect exceeds that for the case of the motion of the whole fluid.

Finally, let us note that the temperature which corresponds to a strong change of the Doppler-shift characteristics $K(T)$, $\gamma_2(T)$, and $K_1(T)$ (to the appearance of a peak), nearly the same in all the cases, $T\approx 0.6$ K—the beginning of the "roton" region—is determined by a common cause: the connection of the characteristics with the derivatives $\partial\rho_n/\partial T\propto[4+g(\Delta/T-\frac{1}{2})]$ and

$$\frac{\partial\rho_n}{\partial P}\propto\left[-5a+g\left[\frac{\Delta}{T}|r|+\frac{2}{3}\right]\right]\ ;$$

cf. Eqs. (26), (29), (34), (60), and (68). Note also that this temperature is special for some other characteristics of the superfluid too—see, e.g., the behavior of the second-sound velocity, both in the usual case [Eq. (58)] and in the case of large amplitude.[5]

ACKNOWLEDGMENTS

I would like to thank Professor A. Mann, Professor F. Pobell, and Professor M. Revzen for valuable discussions.

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