Specific heat for a strong-coupling superconductor with logarithmic electronic density of states

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We find that a Van Hove singularity can affect the temperature (T) variation of the electronic-specific-heat difference between the superconducting and normal states $\Delta C(T)$. It shifts the zero in this quantity away, rather than towards T_c as is indicated in some experiments in the copper oxides. To carry out the calculations we have generalized the Bardeen-Stephan formula for the free-energy difference between the normal and superconducting states, and the Eliashberg equations on the imaginary-frequency axis to include explicitly a logarithmic electronic density of states (a Van Hove singularity) with normal- and paramagnetic-impurity scattering included.

I. INTRODUCTION

There have been several recent attempts to explain high-T_c superconductivity in the copper oxides in terms of a Van Hove singularity in the electronic density of states (EDOS).[1-6] In some of these approaches an underlying electron-phonon model is assumed. This assumption leads naturally to large variation in the isotope-effect coefficient[5-11] as observed. More recent work, however, reveals difficulties with this model when a quantitative understanding of the measured isotope coefficient is attempted.[12,13] A joint phonon plus electronic mechanism is more consistent with the data, but this is no way invalidates by itself the Van Hove scenario.[6] Recently Tseui *et al.*[14] reported success in explaining the observed variation of the specific-heat jump to T_c ratio $[\Delta C(T_c)/T_c]$ in oxygen-deficient $YBa_2Cu_3O_{7-\delta}$ polycrystals[15-17] with a Van Hove singularity in the density of states.

In ordinary Eliashberg theory the energy dependence in the electronic density of states is assumed not to be important and a constant value is taken in the energy region of importance for superconductivity. In this limit it has been shown that the normalized ratio of the specific-heat slope at T_c to the jump $R=\{T_c[d\Delta C(T)/dT]\}_{T_c}/\Delta C(T_c)$ is bounded above to a value of approximately 5.0 whatever the shape, origin, and size of the electron-boson spectral density that is responsible for the pairing.[18-20] On the other hand, initial experiments in $YBa_2Cu_3O_7$ by Junod *et al.*[21] and by Loram and Mirza[22] have given values of R slightly beyond the theoretically predicted range. An even larger value has been obtained by Schilling, Ott, and Hulliger[23] in $Bi_{1.6}Pb_{0.4}Sr_2Ca_2Cu_3O_{10}$. Such large values cannot be understood at present within the framework of conventional Eliashberg theory for any electron-boson exchange mechanism with an S-wave order parameter.

Eliashberg theory has been extended to include variation with energy (ϵ) in the electronic density of state $[N(\epsilon)]$ at the Fermi surface by many authors.[24-30] This generalized formalism has been applied to many properties including the specific heat.[26,30] The existing results employ a Lorentzian form superimposed on a constant background to account for energy dependence in $N(\epsilon)$ and have been limited to the jump at T_c.

In this paper we wish to generalize previous work in two directions. First we wish to use a Van Hove singularity form for the electronic density of states $N(\epsilon)$. Second, we want to consider the specific-heat difference between normal and superconducting states $[\Delta C(T)]$ at finite temperatures below T_c. We will particularly be interested in the shift in position of the node observed in $\Delta C(T)$ near T_c when the Van Hove singularity is introduced. The larger the value of the slope to jump, the closer to $T=T_c$ is $\Delta C(T)=0$ expected to occur. It can be argued that fluctuations due to the two-dimensional nature of the CuO plane could strongly affect the value of R at T_c but it is much harder to see how they could importantly affect the position of the node in $\Delta C(T)$ which occurs at a temperature much below T_c in a region where fluctuations are not likely to be very important and certainly not dominant.

This paper is structured as follows. In Sec. II, we give the generalized Eliashberg formalism needed to include a Van Hove singularity in $N(\epsilon)$. Some details of the derivations are found in the Appendix. Results are presented in Sec. III and a conclusion is found in Sec. IV.

II. ELIASHBERG EQUATIONS AND FREE-ENERGY FORMULA

The Eliashberg equations on the imaginary axis in the superconducting state for general EDOS are given[1] by

$$Z(i\omega_n)\omega_n=\omega_n+\pi T\sum_{m=-\infty}^{\infty}\lambda(n-m)\left[\frac{1}{\pi}\int_{-\infty}^{\infty}d\epsilon\frac{N(\epsilon)}{N_0}\frac{Z(i\omega_m)\omega_m}{[-\det\hat{G}^{-1}(\epsilon,i\omega_m)]}\right], \tag{1a}$$

$$\chi(i\omega_n) = -\pi T \sum_{m=-\infty}^{\infty} \lambda(n-m) \left[\frac{1}{\pi} \int_{-\infty}^{\infty} d\epsilon \frac{N(\epsilon)}{N_0} \frac{\epsilon + \chi(i\omega_m)}{[-\det\hat{G}^{-1}(\epsilon, i\omega_m)]} \right], \tag{1b}$$

$$\phi(i\omega_n) = \pi T \sum_{m=-\infty}^{\infty} [\lambda(n-m) - \mu^* \theta(\omega_c - |\omega_m|)] \left[\frac{1}{\pi} \int_{-\infty}^{\infty} d\epsilon \frac{N(\epsilon)}{N_0} \frac{\phi(i\omega_m)}{[-\det\hat{G}^{-1}(\epsilon, i\omega_m)]} \right], \tag{1c}$$

$$\det\hat{G}^{-1}(\epsilon, i\omega_n) = [i\omega_n Z(i\omega_n)]^2 - [\epsilon + \chi(i\omega_n)]^2 - \phi^2(i\omega_n), \tag{1d}$$

and

$$\lambda(n-m) = \int_0^{\infty} d\Omega \alpha^2(\Omega) F(\Omega) \frac{2\Omega}{\Omega^2 + (\omega_n - \omega_m)^2}, \tag{1e}$$

where $Z(i\omega_n)$ is the renormalization function; $\omega_n = (2n+1)\pi T$, $n = 0, \pm 1, \pm 2, \ldots$, are the Matsubara frequencies; T is the temperature; $\alpha^2(\Omega)F(\Omega)$ is the electron-boson spectral density at a frequency Ω; $N(\epsilon)$ is the electronic density of states; $\hat{G}$ is the renormalized electron propagator matrix; $\chi(\omega)$ is the shift in the chemical potential due to the electron-boson interaction; and $\phi(\omega)$ is the renormalized pairing energy in the superconducting state. The density of states $N(\epsilon)$ in this particular calculation is of the form

$$N(\epsilon) = N_0 \left[r - \frac{s}{(\epsilon-\delta)^2 + D^2} \ln \left(\left| \frac{\epsilon - \delta}{E_f} \right| \right) \right]. \tag{2}$$

The Lorentzian factor $1/[(\epsilon-\delta)^2 + D^2]$ is used to damp out the logarithmic part at large energy values in order to make the density of states physical. The filling factor δ is the position of the singularity with respect to the Fermi surface which is chosen at $\epsilon = 0$ by convention, at $T = 0$. The Fermi energy is E_f and D is an adjustable damping parameter. The strength of the Van Hove singularity is given by s, and the parameter r is a constant background. Using Eqs. (2), (A1), (A2), and (A3) in performing the ϵ integral, Eq. (1b) becomes

$$\chi_n = \frac{s\pi T}{2D} \sum_{m=-\infty}^{\infty} \lambda(n-m) \left\{ \frac{\delta'_m \ln(D^2 \sqrt{\delta'^2_m + \tilde{\omega}^2_m + \phi^2_m}/E_f^3) - (\sqrt{\tilde{\omega}^2_m + \phi^2_m} + D)\tan^{-1}(\delta'_m/\sqrt{\omega^2_m + \phi^2_m})}{\delta'^2_m + (D + \sqrt{\tilde{\omega}^2_m + \phi^2_m})^2} + \frac{\delta'_m \ln(D^2/E_f \sqrt{\delta'^2_m + \tilde{\omega}^2_m + \phi^2_m}) + (\sqrt{\tilde{\omega}^2_m + \phi^2_m} - D)\tan^{-1}(\delta'_m/\sqrt{\tilde{\omega}^2_m + \phi^2_m})}{\delta'^2_m + (D - \sqrt{\tilde{\omega}^2_m + \phi^2_m})^2} \right\}, \tag{3a}$$

where

$$\chi_n = \chi(i\omega_n), \tag{3b}$$

$$\delta'_n = \delta + \chi_n, \tag{3c}$$

$$\tilde{\omega}_n = Z(i\omega_n)\omega_n. \tag{3d}$$

$\tilde{\omega}_n$ is called the renormalized Matsubara frequency.
Also, Eqs. (1a) and (1c) now become

$$\tilde{\omega}_n = \omega_n + \pi T \sum_{m=-\infty}^{\infty} \lambda(n-m) \frac{\tilde{\omega}_m \hat{N}(\tilde{\omega}_m)}{\sqrt{\tilde{\omega}^2_m + \phi^2_m}} + \pi(t^+ + t^-) \frac{\tilde{\omega}_n \hat{N}(\tilde{\omega}_n)}{\sqrt{\tilde{\omega}^2_n + \phi^2_n}}, \tag{4}$$

$$\phi_n = \pi T \sum_{m=-\infty}^{\infty} [\lambda(n-m) - \mu^* \theta(\omega_c - |\omega_m|)] \frac{\phi m \hat{N}(\tilde{\omega}_m)}{\sqrt{\tilde{\omega}^2_m + \phi^2_m}} + \pi(t^+ - t^-) \frac{\phi_n \hat{N}(\tilde{\omega}_n)}{\sqrt{\tilde{\omega}^2_n + \phi^2_n}}, \tag{5}$$

$$\hat{N}(\tilde{\omega}_n) = r - \frac{s}{2D} \left\{ \frac{(\sqrt{\tilde{\omega}^2_n + \phi^2_n} + D)\ln(D\sqrt{\delta'^2_n + \tilde{\omega}^2_n + \phi^2_n}/E_f^2) + \delta'_n \tan^{-1}(\delta'_n/\sqrt{\tilde{\omega}^2_n + \phi^2_n})}{\delta'^2_n + (D + \sqrt{\tilde{\omega}^2_n + \phi^2_n})^2} + \frac{(\sqrt{\tilde{\omega}^2_n + \phi^2_n} - D)\ln(D/\sqrt{\delta'^2_n + \tilde{\omega}^2_n + \phi^2_n}) - \delta'_n \tan^{-1}(\delta'_n/\sqrt{\tilde{\omega}^2_n + \phi^2_n})}{\delta'^2_n + (D - \sqrt{\tilde{\omega}^2_n + \phi^2_n})^2} \right\}. \tag{6}$$

The Eliashberg equations in the normal state are obtained by setting $\phi_n=0$ and

$$\chi_{0n}=\frac{s\pi T}{2D}\sum_{m=-\infty}^{\infty}\lambda(n-m)\left\{\frac{\delta'_{0m}\ln(D^2\sqrt{\delta'^2_m+\tilde{\omega}^2_{0m}}/E_f^3)-(|\tilde{\omega}_{0m}|+D)\tan^{-1}(\delta'_{0m}/|\tilde{\omega}_{0m}|)}{\delta'^2_{0m}+(D+|\tilde{\omega}_{0m}|)^2}\right.$$
$$\left.+\frac{\delta'_{0m}\ln(D^2/E_f\sqrt{\delta'^2_{0m}+\tilde{\omega}^2_{0m}})+(|\tilde{\omega}_{0m}|-D)\tan^{-1}(\delta'_{0m}/|\tilde{\omega}_{0m}|)}{\delta'^2_{0m}+(D-|\tilde{\omega}_{0m}|)^2}\right\}, \tag{7a}$$

$$\delta'_{0n}=\delta+\chi_{0n}, \tag{7b}$$

$$\tilde{\omega}_{0n}=\omega_n+\pi T\sum_{m=-\infty}^{\infty}\lambda(n-m)\operatorname{sgn}(\tilde{\omega}_{0m})\hat{N}_0(\tilde{\omega}_{0m})+\pi(t^++t^-)\operatorname{sgn}(\tilde{\omega}_{0n})\hat{N}_0(\tilde{\omega}_{0n}), \tag{8}$$

$$\hat{N}_0(\tilde{\omega}_{0n})=r-\frac{s}{2D}\left\{\frac{(|\tilde{\omega}_{0m}|+D)\ln(D\sqrt{\delta'^2_{0n}+\tilde{\omega}^2_{0n}}/E_f^2)+\delta'_{0n}\tan^{-1}(\delta'_{0n}/|\tilde{\omega}_{0m}|)}{\delta'^2_{0n}+(D+|\tilde{\omega}_{0m}|)^2}\right.$$
$$\left.+\frac{(|\tilde{\omega}_{0m}|-D)\ln(D/\sqrt{\delta'^2_{0n}+\tilde{\omega}^2_{0n}})-\delta'_{0n}\tan^{-1}(\delta'_{0n}/|\tilde{\omega}_{0m}|)}{\delta'^2_{0n}+(D-|\tilde{\omega}_{0m}|)^2}\right\}. \tag{9}$$

For the free-energy difference between the superconducting and normal states we use the *Bardeen-Stephan formula:*[26]

$$\Omega_s-\Omega_n=\frac{-1}{\beta}\sum_p\left[\ln\left[\frac{\Phi_s(p)}{\Phi_n(p)}\right]+[\sigma_{1s}(p)-\sigma_{1n}(p)][G_s(p)+G_n(p)]-\sigma_2(p)F(p)\right], \tag{10}$$

$p=(\mathbf{p},i\omega_n)$, $1/\beta=T$, and

$$\frac{-1}{\beta}\sum_p=\frac{-1}{\beta}\sum_{n=-\infty}^{n=\infty}\int\frac{d^3p}{(2\pi)^3}=\frac{-1}{\beta}\sum_{n=-\infty}^{n=\infty}\int_{-\infty}^{\infty}N(\epsilon)d\epsilon \tag{11}$$

and

$$\Phi_s(p)=-[\tilde{\omega}_n^2+\phi_n^2+(\epsilon_p+\chi_n)^2], \tag{12a}$$

$$\sigma_{1s}(p)=i\omega_n-i\tilde{\omega}_n, \tag{12b}$$

$$G_s(p)=\frac{i\tilde{\omega}_n+\epsilon+\chi_n}{\Phi_s(p)}, \tag{12c}$$

$$\sigma_2(p)=\phi_n, \tag{12d}$$

$$F(p)=\frac{-\sigma_2(p)}{\Phi_s(p)}. \tag{12e}$$

Normal-state variables are obtained by replacing the subscript s with n and by setting $\phi_n=0$, $\tilde{\omega}_n=\tilde{\omega}_{0n}$, and $\chi_n=\chi_{0n}$:

$$\sigma_{1s}(p)-\sigma_{1n}(p)=i\tilde{\omega}_{0n}-i\tilde{\omega}_n. \tag{13}$$

The final expression for the free energy is

$$\Omega_s-\Omega_n=\frac{-2}{\beta}N_0\sum_{n=1}^{n=\infty}\left\{\pi r\left[2\left(\sqrt{\tilde\omega_n^2+\phi_n^2}-\tilde\omega_{0n}\right)-\frac{\tilde\omega_n^2+\phi_n^2-\tilde\omega_n\tilde\omega_{0n}}{\sqrt{\tilde\omega_n^2+\phi_n^2}}+(\tilde\omega_{0n}-\tilde\omega_n)\right]\right.$$

$$+\frac{s\pi}{2D}\frac{\tilde\omega_n^2+\phi_n^2-\tilde\omega_n\tilde\omega_{0n}}{\sqrt{\tilde\omega_n^2+\phi_n^2}}$$

$$\times\left[\frac{\left(\sqrt{\tilde\omega_n^2+\phi_n^2}+D\right)\ln\left(D\sqrt{\delta_n'^2+\tilde\omega_n^2+\phi_n^2}/E_f^2\right)+\delta_n'\tan^{-1}\left(\delta_n'/\sqrt{\tilde\omega_n^2+\phi_n^2}\right)}{\delta_n'^2+\left(D+\sqrt{\tilde\omega_n^2+\phi_n^2}\right)^2}\right.$$

$$\left.+\frac{\left(\sqrt{\tilde\omega_n^2+\phi_n^2}-D\right)\ln\left(D/\sqrt{\delta_n'^2+\tilde\omega_n^2+\phi_n^2}\right)-\delta_n'\tan^{-1}\left(\delta_n'/\sqrt{\tilde\omega_n^2+\phi_n^2}\right)}{\delta_n'^2+\left(D-\sqrt{\tilde\omega_n^2+\phi_n^2}\right)^2}\right]$$

$$-\frac{s\pi}{2D}(\tilde\omega_{0n}-\tilde\omega_n)\left[\frac{(\tilde\omega_{0n}+D)\ln\left(D\sqrt{\delta_{0n}'^2+\tilde\omega_{0n}^2}/E_f^2\right)+\delta_{0n}'\tan^{-1}(\delta_{0n}'/\tilde\omega_{0n})}{\delta_{0n}'^2+(D+\tilde\omega_{0n})^2}\right.$$

$$\left.+\frac{(\tilde\omega_{0n}-D)\ln\left(D/\sqrt{\delta_{0n}'^2+\tilde\omega_{0n}^2}\right)-\delta_{0n}'\tan^{-1}(\delta_{0n}'/\tilde\omega_{0n})}{\delta_{0n}'^2+(D-\tilde\omega_{0n})^2}\right]$$

$$\left.-s\int_{-\infty}^{\infty}\frac{\ln[|(\epsilon-\delta)/E_f|]}{(\epsilon-\delta)^2+D^2}\ln\left[\frac{(\epsilon+\chi_n)^2+\tilde\omega_n^2+\phi_n^2}{(\epsilon+\chi_{0n})^2+\tilde\omega_{0n}^2}\right]d\epsilon\right\}, \tag{14}$$

which can be written in the following form:

$$\Omega_s-\Omega_n=\frac{-2}{\beta}N_0\sum_{n=1}^{n=\infty}\left\{\pi 2r\left[\sqrt{\tilde\omega_n^2+\phi_n^2}-\tilde\omega_{0n}\right]-\pi\hat N(\tilde\omega_n)\frac{\tilde\omega_n^2+\phi_n^2-\tilde\omega_n\tilde\omega_{0n}}{\sqrt{\tilde\omega_n^2+\phi_n^2}}+\pi\hat N_0(\tilde\omega_{0n})(\tilde\omega_{0n}-\tilde\omega_n)\right.$$

$$\left.-s\int_{-\infty}^{\infty}\frac{\ln[|(\epsilon-\delta)/E_f|]}{(\epsilon-\delta)^2+D^2}\ln\left[\frac{(\epsilon+\chi_n)^2+\tilde\omega_n^2+\phi_n^2}{(\epsilon+\chi_{0n})^2+\tilde\omega_{0n}^2}\right]d\epsilon\right\}. \tag{15}$$

III. NUMERICAL RESULTS AND DISCUSSION

In all our numerical results we will take $t^+=t^-=0$ for simplicity. It is not at all difficult to include normal- and paramagnetic-impurity scatterings. Also, to start we consider the critical temperature which follows Eqs. (4) and (5) linearized in the gap value ϕ_m. Specification of the electron-boson spectral density $\alpha^2F(\Omega)$ corresponds to a specification of mechanism. We find, however, that T_c is mainly dependent on the characteristic boson energy used $\omega_{\ln}$ (Refs. 31–33) defined by

$$\omega_{\ln}=\exp\left[\frac{2}{\lambda}\int_0^\infty\frac{\alpha^2F(\omega)}{\omega}\ln(\omega)d\omega\right]$$

and on the value of the mass renormalization λ:

$$\lambda=2\int_0^\infty\frac{\alpha^2F(\omega)}{\omega}d\omega\ .$$

Thus, we take for $\alpha^2F(\omega)$ a δ function at energy $\omega_E=20.0$ meV and a value of $\lambda=2.0$ [N_0 in Eq. (2) has been incorporated into this value of λ]. This in no way implies that we are dealing exclusively with phonons. We could have taken λ much smaller and ω_E larger which might be more appropriate for an exciton mechanism. The results that we now present would not change qualitatively. To specify the density of states we fix the background in Eq. (2) at $r=1.0$, the Fermi energy $E_f=500$ meV, and the damping factor in the Van Hove form $D=40$ meV. The dotted curve of Fig. 1 gives T_c starting at 80 K as a function of filling factor δ for a case with the strength of the Van Hove singularity s set equal to 6000 (meV).[2] We see that T_c drops rapidly towards 40 K as we move away from the logarithmic singularity in $N(\epsilon)$ as is expected. Thus, a Van Hove singularity can certainly greatly increase T_c over its value when it is not present. Note in this comparison that $T_c=40$ K is the value of the critical temperature that comes from the constant background only.

The solid curve of Fig. 1 applies to the same parameters as the dotted curve but now the filling factor is fixed

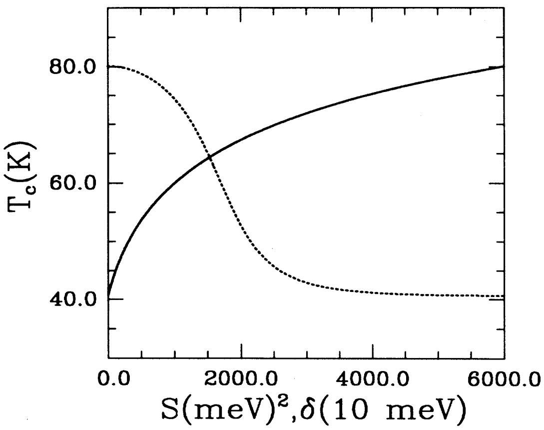

FIG. 1. Transition temperature for Eliashberg superconductor with $\omega_E=20$ meV, $\lambda=2.0$, $r=1.0$, $\mu^*=0.1$, $E_f=500$ meV, $D=40$ meV, and (i) $s=6000$ (meV)^2 (dotted curve) vs the filling factor δ, (ii) $\delta=0$ (solid curve) vs the logarithmic strength s.

at $\delta=0$ and the strength of the Van Hove singularity s is increased. The curve starts at 40.0 K for $s=0$ and increases to 80 K when $s=6000$ (meV).^{2} The rise of T_c from 40 to 80 K is much more gradual when s is changed than is the drop in T_c through the same range that we found on increasing δ at fixed value of s.

Results for the specific-heat difference $[\Delta C(T)]$ between superconducting and normal states follows from our free-energy difference formula, Eq. (14), for $\Delta F \equiv \Omega_s - \Omega_n$ through the thermodynamic relationship

$$\Delta C(T) = -T\frac{d^2\Delta F}{dT^2} ,$$

which follows from a specific choice of spectral density $\alpha^2F(\Omega)$ and of $N(\epsilon)$. In Fig. 2, we compare results of a

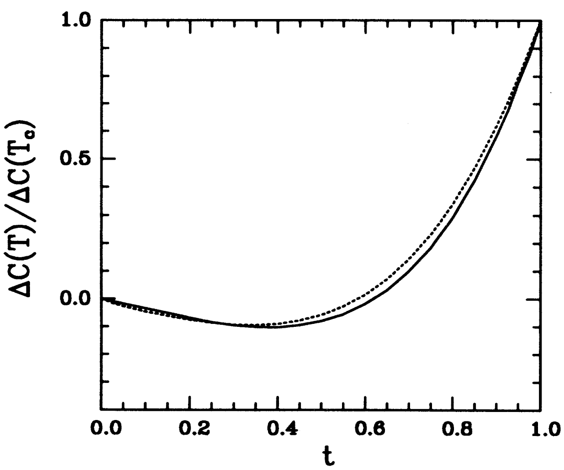

FIG. 2. Specific-heat difference $[\Delta C(T)]$ between superconducting and normal states with $T_c=101$ K, $\omega_E=70$ meV, $\lambda=1.4$, $\mu^*=0.1$, $E_f=500$ meV, $D=40$ meV, and (i) $r=1.0$, and $s=0$ (meV)^2 (solid curve) vs temperature T, (ii) $r=0$, and $s=740$ (meV)^2 (dotted curve) vs T_c.

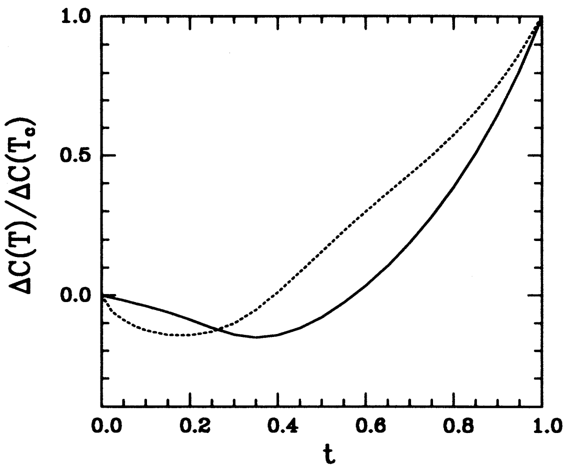

FIG. 3. Specific-heat difference $[\Delta C(T)]$ between superconducting and normal states with $T_c=89.4$ K, $\omega_E=20$ meV, $\lambda=6.5$, $\mu^*=0.1$, $E_f=500$ meV, $D=40$ meV, and (i) $r=1.0$, and $s=0$ (meV)^2 (solid curve) vs temperature T, (ii) $r=0$, and $s=2140$ (meV)^2 (dotted curve) vs T_c.

constant density-of-state calculation (solid line) with $T_c=101$ K with similar results (dotted curve) for the case $\mu^*=0.1$, $\lambda=1.4$ with a δ function at 70 meV for $\alpha^2F(\Omega)$, background $r=0$, strength of Van Hove singularity $s=740$ (meV),^{2} width of damping factor $D=40$ meV, Fermi energy equal to 500 meV, and filling factor equal to zero ($\delta=0$). In both curves the critical temperature is the same. With energy dependence the slope at T_c is reduced over the constant case and the zero in $\Delta C(T)$ is pushed toward a lower temperature although the effect is not numerically large, at lower temperatures solid and dotted curves cross and end up going to zero at $T=0$ of course. In Fig. 3, large differences between constant and nonconstant density-of-state results are shown for the case $T_c=89.4$ K with $\mu^*=0.1$ with a δ spectral density at $\omega_E=20$ meV (more in the phonon region) with $\lambda=6.5$. In this case the zero for the dotted curve is pushed to even lower values of temperature than is the corresponding curve of Fig. 2. Strong-coupling effects seem to push the node in $[\Delta C(T)]$ to lower temperatures. We can safely conclude from our work that the ratio R of the slope to jump at T_c of $[\Delta C(T)]$ is reduced rather than increased over the constant density-of-state curve and that the node in $[\Delta C(T)]$ is pushed to lower rather than higher temperatures. It is clear that the introduction of a Van Hove singularity for the electronic density of states into an Eliashberg formalism does not help explain recent experimental data in the oxides.[21–23] It is found that $[\Delta C(T)]$ crosses the axis at a reduced value of temperature t much closer to 1 than is predicted in BCS theory and that R is also very much larger.

IV. CONCLUSIONS

We have derived generalized expressions for the free-energy difference between superconducting and normal states which apply to an Eliashberg superconductor with a Van Hove singularity in its electronic density of state $N(\epsilon)$. Eliashberg equations appropriate to the same model are also given. The formulas are evaluated numerically to obtain the specific-heat difference between superconducting and normal states $\Delta C(T)$ at any temperature T below T_c. It is found that the zero in this quantity is pushed towards lower energy with the introduction of a Van Hove singularity in $N(\epsilon)$. Experiments in the copper oxides indicate instead that the node in $\Delta C(T)$ is much closer to $T=T_c$ than in a BCS superconductor. Such results can therefore not be explained, at least in our simple treatment of the Van Hove singularity in $N(\epsilon)$.

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APPENDIX

$$\int_{-\infty}^{\infty} \frac{1}{(\epsilon-\delta)^2+D^2}\frac{d\epsilon}{\epsilon^2+b^2}=\frac{\pi}{Db}\frac{(b+D)}{\delta^2+(b+D)^2}, \tag{A1}$$

$$\int_{-\infty}^{\infty} \frac{\ln(|\epsilon-\delta|)}{(\epsilon-\delta)^2+D^2}\frac{d\epsilon}{\epsilon^2+b^2}=\frac{\pi}{2Db}\left\{\frac{(b+D)\ln(D\sqrt{b^2+\delta^2})+\delta\tan^{-1}(\delta/b)}{\delta^2+(b+D)^2}+\frac{(b-D)\ln(D/\sqrt{b^2+\delta^2})-\delta\tan^{-1}(\delta/b)}{\delta^2+(b-D)^2}\right\}, \tag{A2}$$

$$\int_{-\infty}^{\infty} \frac{\ln(|\epsilon-\delta|/E_f)}{(\epsilon-\delta)^2+D^2}\frac{\epsilon d\epsilon}{\epsilon^2+b^2}=\frac{\pi}{2D}\left\{\frac{2\delta\ln(D/Ef)+\delta\ln[\sqrt{(b^2+\delta^2)/Ef^2}]-(b+D)\tan^{-1}(\delta/b)}{\delta^2+(b+D)^2}\right.$$
$$\left.+\frac{2\delta\ln(D/Ef)-\delta\ln[\sqrt{(b^2+\delta^2)/Ef^2}]+(b-D)\tan^{-1}(\delta/b)}{\delta^2+(b-D)^2}\right\}. \tag{A3}$$

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