Wave propagation in isotropic random media with nondiscrete spherical perturbations

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In our studies on the optical properties of artificial anisotropic material, synthesized with methods of nuclear trace and etching technology, we have to investigate wave scattering by nondiscrete cylindrical perturbations. Unfortunately, most previous Twersky-type multiple-scattering theories as well as Keller-type random-medium theories exclusively deal with discrete scattering problems. In this preliminary work we use the theory of stochastic differential equations in order to study the low-frequency limit of scalar and electromagnetic wave scattering from an unbounded isotropic medium into which isotropic nondiscrete (partially overlapping) spherical perturbations are embedded. By taking into proper account the strong singularity of the Green's tensor in the application of the first-order smoothing method it is shown that the effective dielectric tensor is a multiple of the unit dyad and can be calculated approximately via isotropic two-point autocorrelation functions which allow overlap of the spherical scatterers. These random-medium results are compared with those from discrete scattering theory. It is shown that there exists a joint scope of both theories in the limit of small volume fraction and of small size of the perturbations. In the electromagnetic case the degree of agreement between both methods is not as significant as in the scalar case. Finally, we present isotropic correlation functions for overlapping circular cylinders of finite as well as of infinite length.

I. INTRODUCTION

Homogeneous isotropic optical materials can be made anisotropic by methods of nuclear trace technology as one of us has recently suggested (Thielheim[1]). As a result of such procedures as bombarding a plastic sheet with high-energy heavy ions and etching the resulting tracks in the material, an arbitrary large number of partially overlapping parallel cylindrical perturbations is embedded in the originally isotropic optical material bestowing it with properties of anisotropy. In our present work we have undertaken the investigation and theoretical description of these optical properties by methods of random-medium theory. The fundamental questions are whether such a material can be described approximately by means of an effective dielectric tensor and how coherent modes of plane electromagnetic waves can propagate within an optical material of this type in which nondiscrete scatterers are embedded.

The problem of wave scattering by random media has been studied extensively (Tatarskii,[3] Frisch,[4] Ishimaru,[5,6] Twersky,[7–13] Keller,[14–16] Karal and Keller,[17] Keller and Vizetti,[18] Tsang and Kong,[19–21] Tsang, Kong, and Newton,[22] and Sobczyk[23]) and has found applications, e.g., in the propagation of waves through turbulent atmospheres (Tatarskii[3,24] and Ishimaru[6]) and in microwave remote sensing (Stogryn,[25] Tsang and Kong,[19,20] Fischer,[26] Tan and Fung[27] and Zuniga and Kong[28]).

However, according to the authors' knowledge, the bulk of the Twersky-type multiple scattering theories as well as the Keller-type random-medium theories exclusively deal with scattering problems in stochastic media with discrete spherical or cylindrical perturbation geometries.

In this preliminary work, which is a first approach to a closed theory of artificial anisotropic media of Thielheim-type, we will thus discuss the theory of stochastic differential equations applied to the propagation of scalar or electromagnetic waves in unbounded isotropic random media into which partially overlapping spherically shaped scatterers are embedded. To determine the macroscopic properties of such media a rigorous study of the basic mathematical methods for calculating the effective permittivity $\epsilon_{\rm eff}$ and the effective dielectric tensor $\bar{\epsilon}_{\rm eff}$ of an isotropic randomly inhomogeneous medium is imperative (cf. Sec. II). We here make use of the first-order smoothing approximation (Primas,[2] Keller and Karal,[29] Karal and Keller,[17] Frisch,[4] Tatarskii,[30] Ryzhov and Tamoikin[31]) which is equivalent to the Twersky integral equations (Ishimaru[5]). For scalar wave propagation, in the limit of low frequency, this renormalization method (Tsang and Kong[32]) is valid for both weak and strong fluctuations of refractive index (Tatarskii and Gertsenshtein[33]) and by familiar integral equation techniques yields the desired approximation of $\epsilon_{\rm eff}$ in the decomposed form of a constant and a wavelength-dependent part.

In the electromagnetic case additional secular terms are generated by the strong singularity of the Green's tensor at the origin (van Bladel,[34] Tai,[35] Ryzhov and Tamoikin,[31] and Stogryn[36]) which are absent in the scalar case. The removal of these perturbation terms constitutes the main difficulty in the computation of $\bar{\epsilon}_{\rm eff}$, and the final result is that $\bar{\epsilon}_{\rm eff}$ can be represented as a multiple of the unit dyad. Here we base our considerations on the results of Ryzhov *et al.*,[37] Ryzhov and Tamoikin,[31] Tamoikin,[38] Karal and Keller,[17] Tsang and Kong,[32] and Tsang, Kong, and Newton[22] in deriving weak and strong fluctuation theories for a random medium with a spherically symmetric correlation function and randomly distributed discrete scatterers.

The applicability of this general mathematical theory of effective permittivities to our specific model with isotropic nondiscrete stochastic spherical perturbations (cf. Sec. III) mainly depends on the determination of the two-point autocorrelation functions which allow overlap of the embedded spherical or cylindrical scatterers. As far as we know such correlation functions have not yet been studied explicitly in the literature and will be established here for the first time (cf. Sec. IV). Consequently, we succeed in the approximative computation of ϵ_{eff} and $\overline{\epsilon}_{\mathrm{eff}}$ in the validity range of the smoothing approximation. Again, random-medium results of this type for nondiscrete scattering problems do not seem to be available in the vast literature and a comparison with corresponding results from discrete scattering theory yields the remarkable fact that there exists a joint scope of both theories in the limit of small volume fraction and of small size of the spherical perturbations. However, in the electromagnetic case the degree of agreement between the two methods is not so significant as in the scalar case. For discrete scatterer problems similar investigations have been made, e.g., by Tsang and Kong[32] and Tsang, Kong, and Newton.[22]

Finally, note the interesting fact that the formal structure of our autocorrelation functions for overlapping circular cylinders (cf. Sec. IV) is essentially identical with that of the MTF (modular transfer function) of an imaging system with a circular aperture of constant diameter (Ishimaru[6]).

II. WAVE PROPAGATION AND EFFECTIVE PERMITTIVITIES

We here discuss wave propagation within a layer of an unbounded inhomogeneous random medium with a spherically symmetric correlation function. In the limit of low frequency we want to compute approximately the effective permittivities for the scalar and electromagnetic cases, respectively.

(i) In the scalar case, wave propagation is governed by the random Helmholtz wave equation for the random monochromatic scalar wave function $\Psi(\mathbf{r})$

$$\Delta\Psi(\mathbf{r})+k_0^2\epsilon(\mathbf{r})\Psi(\mathbf{r})=0 \ , \qquad (1)$$

where k_0 is the vacuum wave number. The medium is characterized by the random dielectric constant.

$$\epsilon(\mathbf{r})=\langle \epsilon \rangle+\omega(\mathbf{r}) \ , \qquad (2)$$

$\langle \epsilon \rangle$ is the ensemble average (independent of the spatial variable $\mathbf{r}$) and the perturbation function ω has zero mean, i.e.,

$$\langle \omega \rangle=0 \ . \qquad (3)$$

The statistical properties of $\epsilon(\mathbf{r})$ are described by the autocorrelation function

$$A(\Delta\mathbf{r})=\langle \epsilon(\mathbf{r})\epsilon(\mathbf{r}+\Delta\mathbf{r})\rangle-\langle \epsilon \rangle^2 \ . \qquad (4)$$

Hence our aim is to compute the effective permittivity ϵ_{eff} incorporated in the averaged wave equation (1) for the mean coherent wave amplitude $\langle \Psi(\mathbf{r})\rangle$

$$\Delta\langle \Psi(\mathbf{r})\rangle+k_0^2\epsilon_{\mathrm{eff}}\langle \Psi(\mathbf{r})\rangle=0 \ . \qquad (5)$$

In view of Eqs. (1)–(3) we obtain by familiar methods the exact stochastic integro-differential equation

$$L\langle \Psi(\mathbf{r})\rangle=k_0^4\int G_0(\mathbf{r},\mathbf{r}')\langle \omega(\mathbf{r})\omega(\mathbf{r}')\Psi(\mathbf{r}')\rangle d^3r' \ , \qquad (6)$$

where L is the Helmholtz operator $L\equiv\Delta+k_0^2\langle \epsilon \rangle$ and the free-space Green's function G_0 is defined by

$$G_0(\mathbf{r},\mathbf{r}')=-\frac{\exp(ik_0\sqrt{\langle \epsilon \rangle}|\mathbf{r}-\mathbf{r}'|)}{4\pi|\mathbf{r}-\mathbf{r}'|} \qquad (7)$$

satisfying

$$\Delta G_0(\mathbf{r},\mathbf{r}')+k_0^2\langle \epsilon \rangle G_0(\mathbf{r},\mathbf{r}')=\delta^3(\mathbf{r}-\mathbf{r}') \ . \qquad (8)$$

In order to obtain ϵ_{eff} from Eqs. (5) and (6) we make use of the first-order smoothing approximation (Primas[2]; Tatarskii and Gertsenshtein[33]; Frisch[4]) for the stochastic integral (6) which is applicable as long as the correlation length l is small compared with the wavelength, i.e.,

$$\sqrt{\langle \omega^2 \rangle}k_0^2l^2 \ll 1 \ . \qquad (9)$$

Under the condition (9) we obtain the approximation

$$L\langle \Psi(\mathbf{r})\rangle\approx k_0^4\int G_0(\mathbf{r},\mathbf{r}')A(\mathbf{r}-\mathbf{r}')\langle \Psi(\mathbf{r}')\rangle d^3r' \ . \qquad (10)$$

For low frequency, most of the contribution to the integral (10) comes from the neighborhood of $\mathbf{r}=\mathbf{r}'$ on the wavelength scale. Hence we can approximate Eq. (10) by

$$\mathbf{L}\langle \Psi(\mathbf{r})\rangle\approx k_0^4\langle \Psi(\mathbf{r})\rangle\int G_0(|\mathbf{r}'|)A(\mathbf{r}')d^3r' \ . \qquad (11)$$

Since A is isotropic, i.e., $A(\mathbf{r})=A(|\mathbf{r}|)$ the comparison of Eq. (5) with Eq. (11) yields, via Eq. (7), the effective dielectric constant

$$\epsilon_{\mathrm{eff}}\approx\langle \epsilon \rangle+k_0^2\int_0^\infty\exp(i\sqrt{\langle \epsilon \rangle}k_0r)A(r)r\,dr \ , \qquad (12)$$

valid under the smoothing condition (9).

(ii) We now turn to the electromagnetic case. Wave propagation here is governed by the random Maxwell wave equation for the random electric field $\mathbf{E}(\mathbf{r})$

$$-\nabla\times[\nabla\times\mathbf{E}(\mathbf{r})]+k_0^2\epsilon(\mathbf{r})\mathbf{E}(\mathbf{r})=0 \ , \qquad (13)$$

where the permeability is dropped.

In contrast to the scalar case we use the decomposition

$$\epsilon(\mathbf{r})=\epsilon_c+\omega(\mathbf{r}) \ , \qquad (14)$$

where the value of $\epsilon_c\neq\langle \epsilon \rangle$ is a deterministic constant to be determined by a criterion of elimination of secular terms which are absent in the scalar case.

Hence our aim is to compute the effective dielectric tensor $\overline{\epsilon}_{\mathrm{eff}}$ incorporated in the averaged wave equation (13) for the mean field $\langle \mathbf{E}(\mathbf{r})\rangle$:

$$-\nabla\times[\nabla\times\langle \mathbf{E}(\mathbf{r})\rangle]+k_0^2\overline{\epsilon}_{\mathrm{eff}}\langle \mathbf{E}(\mathbf{r})\rangle=0 \ . \qquad (15)$$

Here we review, modify, and extend results of Ryzhov *et al.*,[37] Ryzhov and Tamoikin,[31] Tsang and Kong,[32] and

Tsang, Kong, and Newton.[22] By Eqs. (13) and (14) we obtain

$$-\nabla\times[\nabla\times\mathbf{E}(\mathbf{r})]+k_0^2\epsilon_c\mathbf{E}(\mathbf{r})=-k_0^2\omega(\mathbf{r})\mathbf{E}(\mathbf{r}) \ . \tag{16}$$

In order to solve Eq. (16) we need the Green's tensor $\overline{G}_0(\mathbf{r})$ satisfying the random vector wave equation

$$-\nabla\times[\nabla\times\overline{G}_0(\mathbf{r})]+k_0^2\epsilon_c\overline{G}_0(\mathbf{r})=\overline{I}\delta^3(\mathbf{r}) \ , \tag{17}$$

where $\overline{I}$ is the unit tensor.

Applying the distributional Fourier transform method to Eq. (17) yields, through familiar techniques,

$$\overline{G}_0(\mathbf{r})=\left[\overline{I}+\frac{\Delta}{k_0^2\epsilon_c}\right]G_0(\mathbf{r}) \ , \tag{18}$$

with $\Delta=\nabla^2$ and

$$G_0(\mathbf{r})=-\frac{\exp(ik_0\sqrt{\epsilon_c}|\mathbf{r}|)}{4\pi|\mathbf{r}|} \ . \tag{19}$$

Hence Eq. (16) is equivalent to the integral equation

$$\mathbf{E}(\mathbf{r})=\mathbf{E}_{\rm inc}(\mathbf{r})-k_0^2\int\overline{G}_0(\mathbf{r}-\mathbf{r}')\mathbf{E}(\mathbf{r}')\omega(\mathbf{r}')d^3r' \ , \tag{20}$$

where $\mathbf{E}_{\rm inc}$ is a solution of the homogeneous wave equation and $\overline{G}_0(\mathbf{r},\mathbf{r}')\equiv\overline{G}_0(\mathbf{r}-\mathbf{r}')$ in view of the translation invariance of the medium.

The strong singularity of $\overline{G}_0$ at the origin ($\mathbf{r}=\mathbf{r}'$) requires the decomposition (Ryzkov *et al.*[37] and Tamoikin[38])

$$\overline{G}_0(\mathbf{r},\mathbf{r}')=PV_s\overline{G}_0(\mathbf{r},\mathbf{r}')+(3k_0^2\epsilon_c)^{-1}\overline{I}\delta^3(\mathbf{r}-\mathbf{r}') \ , \tag{21}$$

where PV_s stands for the principal value of $\overline{G}_0$ with respect to a spherical infinitesimal volume excluded about the singularity. Substituting Eq. (21) in Eq. (20) yields the integral equation with a nonsingular kernel:

$$\left[1+\frac{\omega(\mathbf{r})}{3\epsilon_c}\right]\mathbf{E}(\mathbf{r})=\mathbf{E}_{\rm inc}(\mathbf{r}) -k_0^2\int\overline{G}{}_0^{(s)}(\mathbf{r}-\mathbf{r}')\mathbf{E}(\mathbf{r}')\omega(\mathbf{r}')d^3r' \ , \tag{22}$$

where

$$\overline{G}{}_0^{(s)}(\mathbf{r}-\mathbf{r}')\equiv PV_s\overline{G}_0(\mathbf{r}-\mathbf{r}') \ . \tag{23}$$

Hence we obtain

$$\mathbf{F}(\mathbf{r})=\mathbf{E}_{\rm inc}(\mathbf{r})-k_0^2\int\overline{G}{}_0^{(s)}(\mathbf{r}-\mathbf{r}')\mathbf{F}(\mathbf{r}')\xi(\mathbf{r}')d^3r' \ , \tag{24}$$

where the random field $\mathbf{F}$ is given by

$$\mathbf{F}(\mathbf{r})\equiv\frac{\epsilon(\mathbf{r})+2\epsilon_c}{3\epsilon_c}\mathbf{E}(\mathbf{r}) \ , \tag{25}$$

and the perturbation function ξ is defined by

$$\xi(\mathbf{r})\equiv3\epsilon_c\frac{\epsilon(\mathbf{r})-\epsilon_c}{\epsilon(\mathbf{r})+2\epsilon_c} \ . \tag{26}$$

By iteration we obtain from Eq. (24)

$$\mathbf{F}(\mathbf{r})=\mathbf{E}_{\rm inc}(\mathbf{r})-k_0^2\int\overline{G}{}_0^{(s)}(\mathbf{r}-\mathbf{r}')\xi(\mathbf{r}')\mathbf{E}_{\rm inc}(\mathbf{r}')d^3r' +k_0^4\int\int\overline{G}{}_0^{(s)}(\mathbf{r}-\mathbf{r}')\overline{G}{}_0^{(s)}(\mathbf{r}'-\mathbf{r}'') \times\xi(\mathbf{r}')\xi(\mathbf{r}'')\mathbf{F}(\mathbf{r}'')d^3r'd^3r'' \ . \tag{27}$$

For the removal of secular terms we have the following condition:

$$\langle\xi(\mathbf{r})\rangle=0 \tag{28}$$

or equivalently

$$\left\langle\frac{\epsilon(\mathbf{r})-\epsilon_c}{\epsilon(\mathbf{r})+2\epsilon_c}\right\rangle=0 \ . \tag{29}$$

Hence ϵ_c is determined by Eq. (29).

In view of Eqs. (27) and (28) we obtain for the mean field

$$\langle\mathbf{F}(\mathbf{r})\rangle=\mathbf{E}_{\rm inc}(\mathbf{r})+k_0^4\int\int\overline{G}{}_0^{(s)}(\mathbf{r}-\mathbf{r}')\overline{G}{}_0^{(s)}(\mathbf{r}'-\mathbf{r}'') \times\langle\xi(\mathbf{r}')\xi(\mathbf{r}'')F(\mathbf{r}'')\rangle \times d^3r'd^3r'' \ . \tag{30}$$

We now make use of the first-order "smoothing" approximation for the stochastic integral (30), which is applicable as long as the correlation length l is small compared to the wavelength, i.e.,

$$\sqrt{\langle\xi^2\rangle}k_0^2l^2\ll1 \ . \tag{31}$$

Under the condition (31) one has

$$\langle\xi(\mathbf{r}')\xi(\mathbf{r}'')F(\mathbf{r}'')\rangle\approx A_\xi(\mathbf{r}',\mathbf{r}'')\langle\mathbf{F}(\mathbf{r}'')\rangle \ , \tag{32}$$

where A_ξ denotes the autocorrelation function of the function ξ

$$A_\xi(\mathbf{r}',\mathbf{r}'')\equiv A_\xi(\mathbf{r}'-\mathbf{r}')\equiv\langle\xi(\mathbf{r}')\xi(\mathbf{r}'')\rangle \ , \tag{33}$$

which is assumed to depend only on $\eta=\mathbf{r}'-\mathbf{r}''$. Hence the mean field can be approximated by

$$\langle\mathbf{F}(\mathbf{r})\rangle\approx\mathbf{E}_{\rm inc}(\mathbf{r})+k_0^4\int\int\overline{G}{}_0^{(s)}(\mathbf{r}-\mathbf{r}')\overline{G}{}_0^{(s)}(\mathbf{r}'-\mathbf{r}'') \times\langle\mathbf{F}(\mathbf{r}'')\rangle A_\xi(\mathbf{r}',r'') \times d^3r'\,d^3r'' \ . \tag{34}$$

For low frequency we obtain the approximation

$$\int\overline{G}{}_0^{(s)}(\mathbf{r}'-\mathbf{r}'')\langle\mathbf{F}(\mathbf{r}'')\rangle A_\xi(\mathbf{r},\mathbf{r}'')d^3r'' \approx\langle\mathbf{F}(\mathbf{r}')\rangle\int\overline{G}{}_0^{(s)}(\mathbf{r}'')A_\xi(\mathbf{r}'')d^3r'' \ . \tag{35}$$

Combining Eqs. (34) and (35) finally yields

$$\langle\mathbf{F}(\mathbf{r})\rangle\approx\mathbf{E}_{\rm inc}(\mathbf{r})-k_0^2\int\overline{G}{}_0^{(s)}(\mathbf{r}-\mathbf{r}')\overline{\xi}\langle\mathbf{F}(\mathbf{r}')\rangle d^3r' \ , \tag{36}$$

where the tensor $\overline{\xi}$ is defined by

$$\overline{\xi}\equiv-k_0^2\int\overline{G}{}_0^{(s)}(\mathbf{r})A_\xi(\mathbf{r})d^3r \ . \tag{37}$$

We now are able to approximately calculate $\overline{\epsilon}_{\rm eff}$ via the tensor $\overline{\xi}$. We assume that the correlation function A_ξ is isotropic, i.e.,

$$A_\xi(\mathbf{r})=A_\xi(r) \ , \tag{38}$$

and that the components $\xi_{\mu\nu}$ of $\bar{\xi}$ satisfy $\xi_{\mu\nu} \ll 1$. By use of a well-known principle of variational computation we obtain by comparison of the averaged versions of Eqs. (24) and (36):

$$\langle \xi(\mathbf{r})\mathbf{F}(\mathbf{r})\rangle = \bar{\xi}\langle \mathbf{F}(\mathbf{r})\rangle \ . \tag{39}$$

A similar argument applied to Eqs. (13) and (15) yields

$$\langle \epsilon(\mathbf{r})\mathbf{E}(\mathbf{r})\rangle = \bar{\epsilon}_{\text{eff}}\langle \mathbf{E}(\mathbf{r})\rangle \ . \tag{40}$$

Note that Eqs. (39) and (40) only hold approximately.

Through Eq. (25), (26), and (39) we get

$$\langle \epsilon(\mathbf{r})\mathbf{E}(\mathbf{r})\rangle = \epsilon_c\langle \mathbf{E}(\mathbf{r})\rangle + \bar{\xi}\langle \mathbf{F}(\mathbf{r})\rangle \ . \tag{41}$$

Inserting the averaged form of Eq. (25) into Eq. (41) leads to

$$\langle \epsilon(\mathbf{r})\mathbf{E}(\mathbf{r})\rangle = \epsilon_c\langle \mathbf{E}(\mathbf{r})\rangle + \tfrac{2}{3}\bar{\xi}\langle \mathbf{E}(\mathbf{r})\rangle + \frac{1}{3\epsilon_c}\bar{\xi}\langle \epsilon(\mathbf{r})\mathbf{E}(\mathbf{r})\rangle \tag{42}$$

and the comparison with Eq. (40) shows that

$$\bar{\epsilon}_{\text{eff}} = \bar{I}\epsilon_c + \left[\bar{I} - \frac{1}{3\epsilon_c}\bar{\xi}\right]^{-1}\bar{\xi} \ . \tag{43}$$

Since $\xi_{\mu\nu} \ll 1$ we can neglect higher than linear terms in $\bar{\xi}$. Hence

$$\bar{\epsilon}_{\text{eff}} \approx \bar{I}\epsilon_c + \bar{\xi} \tag{44}$$

and through Eq. (37) we obtain

$$\bar{\epsilon}_{\text{eff}} \approx \bar{I}\epsilon_c - k_0^2 \int \bar{G}_0^{(s)}(\mathbf{r}) A_{\xi}(\mathbf{r}) d^3 r \ . \tag{45}$$

In order to show that $\bar{\epsilon}_{\text{eff}}$ is a multiple of the unit tensor we make use of the decomposition (Tartarski and Gertsenshtein[33])

$$\bar{G}_0^{(s)}(\mathbf{r}) = PV_s G_1(r)\bar{I} + PV_s G_2(r)\mathbf{e}_r\mathbf{e}_r \ , \tag{46}$$

where $\mathbf{e}_r$ is the radial unit vector with $\int \mathbf{e}_r\mathbf{e}_r d^2\omega = \frac{1}{3}\bar{I}$ and

$$G_1(r) \equiv (1 - ik_0\sqrt{\epsilon_c}r - k_0^2\epsilon_c r^2)\frac{\exp(ik_0\sqrt{\epsilon_c}r)}{4\pi k_0^2\epsilon_c r^3} \ , \tag{47}$$

$$G_2(r) \equiv (-3 + 3ik_0\sqrt{\epsilon_c}r + k_0^2\epsilon_c r^2)\frac{\exp(ik_0\sqrt{\epsilon_c}r)}{4\pi k_0^2\epsilon_c r^3} \ . \tag{48}$$

Inserting these expressions into Eq. (45) finally yields the effective dielectric tensor in the case of an isotropic correlation function

$$\bar{\epsilon}_{\text{eff}} \approx \bar{I}\left[\epsilon_c + \frac{2}{3}k_0^2\int_0^\infty r A_{\xi}(r)\exp(ik_0\sqrt{\epsilon_c}r)dr\right] \ , \tag{49}$$

valid under the smoothing condition (31).

III. WAVE SCATTERING BY STOCHASTIC OVERLAPPING SPHERES

In this section the random-medium theory of the previous section will be used to discuss the propagation of scalar and vector electromagnetic waves scattered by random medium with stochastic overlapping spherical perturbations. In order to calculate the effective permittivities we have to introduce new two-point autocorrelation functions. Furthermore, we will compare the random medium results with those from the Mie-Foldy-Lax multiple scattering theory. The aim is to determine a joint scope of both theories. For this purpose the mathematical formalism of the previous section will now be applied to the specific model of an unbounded homogeneous isotropic background material with dielectric constant ϵ_0 in which randomly distributed isotropic spherical scatterers are embedded which, in this case, are partially overlapping spheres of dielectric constant ϵ_1 and radius ρ.

The random process is non-Gaussian, as the random dielectric function $\epsilon(\mathbf{r})$ can take on either of the two values ϵ_0 or ϵ_1. Without loss of generality we discuss the case of two stochastic spheres with radii ρ and centers at the origin and the point Q, respectively, so that $\text{dist}(0,Q) = |\mathbf{r}| = r$. In what follows $v \in [0,1]$ is the relative volume fraction of spheres. For small v (practically no overlapping of spheres) we have

$$v = \tfrac{4}{3}\pi N\rho^3 \ , \tag{50}$$

where N is the number density of the spheres. We obviously have

$$P[\epsilon(0) = \epsilon_0] = 1 - v \quad \text{and} \quad P[\epsilon(0) = \epsilon_1] = v \ , \tag{51}$$

where P stands for probability.

The corresponding two-point autocorrelation function

$$A^{(s)}(\mathbf{r}) = \langle \epsilon(\mathbf{r})\epsilon(\mathbf{r}')\rangle - \langle \epsilon\rangle^2 \tag{52}$$

is here given by

$$A^{(s)}(r) = \begin{cases} v(1-v)(\epsilon_1-\epsilon_0)^2(1-r/2\rho)^2(1+r/4\rho) \\ \quad r \le 2\rho \\ 0 \quad r > 2\rho \end{cases} \ , \tag{53}$$

and has not yet been mentioned in the literature of random-medium theory.

Details of the calculation of Eq. (53) will be given in the next chapter, where we also will introduce two more correlation functions for circular cylinders of infinite or finite length, which—as far as we know—have not yet been discussed in literature.

(i) We first turn to the calculation of the effective dielectric constant in the scalar case. In view of Eqs. (12) and (53) a straightforward calculation yields the new approximation.

$$\epsilon_{\text{eff}} \approx \langle \epsilon\rangle + v(1-v)(\epsilon_1-\epsilon_0)^2 \times \left[-\langle \epsilon\rangle^{-1} + \frac{3i\rho k_0}{\sqrt{\langle \epsilon\rangle}} j_1(\sqrt{\langle \epsilon\rangle}\rho k_0) \times h_1(\sqrt{\langle \epsilon\rangle}\rho k_0)\right] \ , \tag{54}$$

where $j_1(z)$ and $h_1(z)$ are the spherical Bessel and Hankel functions of order 1, respectively, and

$$\langle \epsilon \rangle = \epsilon_0 + v(\epsilon_1 - \epsilon_0) .$$

The approximation (54) is valid for $\rho k_0 \ll 1$ and $v \in [0,1]$.

Hence we get

$$\epsilon_{\text{eff}} \underset{\rho k_0 \ll 1}{\longrightarrow} \langle \epsilon \rangle + v(1-v)(\epsilon_1 - \epsilon_0)^2 \times \left[\frac{2}{5}(\rho k_0)^2 + \frac{i}{3}\sqrt{\langle \epsilon \rangle}(\rho k_0)^3 \right] \tag{55}$$

and the double limit becomes

$$\epsilon_{\text{eff}} \underset{\substack{\rho k_0 \ll 1 \\ v \ll 1}}{\longrightarrow} \langle \epsilon \rangle + v(\epsilon_1 - \epsilon_0)^2 \left[\frac{2}{5}(\rho k_0)^2 + \frac{i}{3}\sqrt{\epsilon_0}(\rho k_0)^3 \right] . \tag{56}$$

It is well known that the theory of multiple scattering also holds for $\rho k_0 \gg 1$ though only if $v \ll 1$. Thus we are led to the hypothesis that the range where both theories are applicable is given by the range $\rho k_0 \ll 1$ and $v \ll 1$.

To verify this suggestion we use the Mie-Foldy-Lax theory in order to solve the scattering problem of a scalar wave by a single sphere with radius ρ and dielectric constant ϵ_1 in a medium with dielectric constant ϵ_0.

For this purpose we consider the scalar wave equation in spherical polar coordinates. If the incident wave is given by

$$\Psi_{\text{inc}}(\mathbf{r}) = \exp(i\sqrt{\epsilon_0}k_0 z)$$

then the scattering wave has the following series expansion

$$\Psi_S(\mathbf{r}) = \sum_{n \geq 0} i^n (2n+1) a_n h_n(\sqrt{\epsilon_0}k_0 r) P_n[\cos(\vartheta)] \quad (r = |\mathbf{r}| > \rho) , \tag{57}$$

where P_n is the Legendre polynomial and ϑ denotes the polar angle relative to the z axis. The scattering coefficients a_n originally due to Mie are given by

$$a_n = -\frac{j_n(\sqrt{\epsilon_1}\rho k_0) j_n'(\sqrt{\epsilon_0}\rho k_0) - \sqrt{(\epsilon_1/\epsilon_0)} j_n(\sqrt{\epsilon_0}\rho k_0) j_n'(\sqrt{\epsilon_1}\rho k_0)}{j_n(\sqrt{\epsilon_1}\rho k_0) h_n'(\sqrt{\epsilon_1}\rho k_0) - \sqrt{\epsilon_1/\epsilon_0} h_n(\sqrt{\epsilon_0}\rho k_0) j_n'(\sqrt{\epsilon_1}\rho k_0)} \tag{58}$$

and the forward-scattering amplitude $F_s(0)$ in view of Eq. (57) becomes

$$F_s(0) = -\frac{i}{\sqrt{\epsilon_0}k_0} \sum_{n \geq 0} (2n+1) a_n . \tag{59}$$

Under these premises the theory of multiple scattering yields the following effective dielectric constant

$$\epsilon^* \approx \epsilon_0 + 4\pi N k_0^{-2} F_s(0) , \tag{60}$$

where again N is the number density of the spheres.

If $\rho k_0 \ll 1$ then the expansion of a_0 and a_1 via Eq. (58) yields

$$a_0 \underset{\rho k_0 \ll 1}{\longrightarrow} \frac{i}{3}\sqrt{\epsilon_0}(\rho k_0)^3(\epsilon_1 - \epsilon_0) \times \left[1 + \frac{1}{5}(\rho k_0)^2(2\epsilon_1 - 3\epsilon_0) + \frac{i}{3}\sqrt{\epsilon_0}(\rho k_0)^3(\epsilon_1 - \epsilon_0) \right] \tag{61}$$

and

$$a_1 \underset{\rho k_0 \ll 1}{\longrightarrow} \frac{i}{45}\epsilon_0^{3/2}(\rho k_0)^5(\epsilon_1 - \epsilon_0) .$$

Inserting these expressions into Eqs. (59) and (60) results in the approximation

$$\epsilon^* \underset{\rho k_0 \ll 1}{\longrightarrow} \epsilon_0 + \frac{4\pi}{3} N \rho^3 (\epsilon_1 - \epsilon_0) \times \left[1 + (\epsilon_1 - \epsilon_0) \left[\frac{2}{5}(\rho k_0)^2 + \frac{i}{3}\sqrt{\epsilon_0}(\rho k_0)^3 \right] \right] , \tag{62}$$

which, in view of Eq. (50), coincides with Eq. (56). Hence in the scalar case we have the remarkable result that both theories yield identical results up to the third order in ρk_0.

(ii) We now turn to the electromagnetic case. In order to compute the effective dielectric tensor $\bar{\epsilon}_{\text{eff}}$ we need the value of ϵ_c as well as the correlation function $A_\xi^{(s)}$ of the function $\xi(\mathbf{r})$. From Eqs. (29) and (51) we obtain the well-known Bruggemann formula (Bruggemann,[40] Tsang and Kong,[32] Tsang, Kong, and Newton[22])

$$v\frac{\epsilon_1 - \epsilon_c}{\epsilon_1 + 2\epsilon_c} + (1-v)\frac{\epsilon_0 - \epsilon_c}{\epsilon_0 + 2\epsilon_c} = 0 . \tag{63}$$

This quadratic equation for ϵ_c yields

$$\epsilon_c = \eta_1^{(s)} + \eta_2^{(s)} \tag{64}$$

with

$$\eta_1^{(s)}=\tfrac{1}{4}[2\epsilon_0-\epsilon_1+3v(\epsilon_1-\epsilon_0)]\ ,$$
$$\eta_2^{(s)}=\tfrac{1}{4}\{(2\epsilon_0+\epsilon_1)^2+3v(\epsilon_1-\epsilon_0)[2(2\epsilon_0-\epsilon_1)+3v(\epsilon_1-\epsilon_0)]\}^{1/2}\ . \tag{65}$$

Since $\langle\xi\rangle=0$ we have $A_\xi^{(s)}(\mathbf{r})=\langle\xi(0)\xi(\mathbf{r})\rangle$. Hence $A_\xi^{(s)}(0)=\langle\xi^2\rangle^{(s)}$ and Eq. (53) leads to

$$A_\xi^{(s)}(r)=\begin{cases}\langle\xi^2\rangle^{(s)}\left[1-\dfrac{r}{2\rho}\right]^2\left[1+\dfrac{r}{4\rho}\right]\ , & r\le 2\rho\\ 0\ , & r>2\rho\ .\end{cases} \tag{66}$$

The variance is easily computed by means of Eqs. (26) and (51):

$$\langle\xi^2\rangle^{(s)}=9\epsilon_c^2\left[v\left[\frac{\epsilon_1-\epsilon_c}{\epsilon_1+2\epsilon_c}\right]^2+(1-v)\left[\frac{\epsilon_0-\epsilon_c}{\epsilon_0+2\epsilon_c}\right]^2\right]\ . \tag{67}$$

By simple manipulations we obtain

$$\langle\xi^2\rangle^{(s)}=\frac{9v(1-v)(\epsilon_1-\epsilon_0)^2\epsilon_c^2}{[v\epsilon_0+(1-v)\epsilon_1+2\epsilon_c]^2}\ . \tag{68}$$

Inserting these expressions into Eq. (49) yields

$$\overline{\epsilon}_{\text{eff}}=\overline{I}\left[\epsilon_c+\frac{2}{3}k_0^2\langle\xi^2\rangle^{(s)}\int_0^{2\rho}r\left[1-\frac{r}{2\rho}\right]^2\left[1+\frac{r}{4\rho}\right]\times\exp(ik_0\sqrt{\epsilon_c}\,r)dr\right]\ . \tag{69}$$

Hence we finally get the new approximation

$$\overline{\epsilon}_{\text{eff}}=\overline{I}\left[\epsilon_c+\frac{2}{3}\langle\xi^2\rangle^{(s)}\left[-\frac{1}{\epsilon_c}+\frac{3i\rho k_0}{\sqrt{\epsilon_c}}j_1(\sqrt{\epsilon_c}\rho k_0)\times h_1(\sqrt{\epsilon_c}\rho k_0)\right]\right]\ . \tag{70}$$

Here ϵ_c and $\langle\xi^2\rangle^{(s)}$ are defined by Eqs. (64), (65), and (68) and j_1,h_1 denote the spherical Bessel and Hankel functions of order 1, respectively. Thus we have shown by methods of the theory of stochastic differential equations that $\overline{\epsilon}_{\text{eff}}$ exists under the special "smoothing" condition $\rho k_0\ll 1$.

In the limit of small volume v we obtain the following limiting results from Eqs. (64), (65), and (68) in the first order in v (Tsang, Kong, and Newton[22]):

$$\epsilon_c\underset{v\ll 1}{\longrightarrow}\epsilon_0\left[1+3v\frac{\epsilon_1-\epsilon_0}{\epsilon_1+2\epsilon_0}\right]\ , \tag{71}$$

$$\langle\xi^2\rangle^{(s)}\underset{v\ll 1}{\longrightarrow}v\left[\frac{3\epsilon_0(\epsilon_1-\epsilon_0)}{\epsilon_1+2\epsilon_0}\right]^2\ . \tag{72}$$

In the limit of small radius ρ of the spherical scatterers we obtain from Eq. (70)

$$\overline{\epsilon}_{\text{eff}}\underset{\rho k_0\ll 1}{\longrightarrow}\overline{I}\left[\epsilon_c+\frac{2}{3}\langle\xi^2\rangle^{(s)}\left[\frac{2}{5}(\rho k_0)^2+\frac{i}{3}\sqrt{\epsilon_c}(\rho k_0)^3\right]\right]\ . \tag{73}$$

Through Eqs. (71) and (72) we thus finally get in the ranges $\rho k_0\ll 1$ and $v\ll 1$:

$$\overline{\epsilon}_{\text{eff}}\underset{\substack{\rho k_0\ll 1\\ v\ll 1}}{\longrightarrow}\overline{I}\epsilon_0\left[1+3v\kappa+6v\epsilon_0\kappa^2\left[\frac{2}{5}(\rho k_0)^2+\frac{i}{3}\sqrt{\epsilon_0}(\rho k_0)^3\right]\right] \tag{74}$$

with

$$\kappa=\frac{\epsilon_1-\epsilon_0}{\epsilon_1+2\epsilon_0}\ . \tag{75}$$

We will now compare these results with those from the theory of multiple scattering. As in the scalar case the medium under consideration can be characterized by an effective dielectric constant of the form

$$\epsilon^*\approx\epsilon_0+4\pi N k_0^{-2}F_s(0)\ . \tag{76}$$

Here ϵ^* is a scalar since $F_s(0)$ is the copolarized forward-scattering amplitude which, in view of the radial symmetry of the scattering problem, does not depend on the direction of propagation or of polarization of the wave. The behavior of ϵ^* in the range $\rho k_0\ll 1$, $v\ll 1$ can be studied by means of the results of the Mie-Foldy-Lax theory.

It is well known that

$$F_s(0)=\frac{i}{2k_0\sqrt{\epsilon_0}}\sum_{n\ge 1}(2n+1)(a_n+b_n)\ . \tag{77}$$

The scattering coefficients are given by

$$a_n=\frac{\alpha\Psi_n(k_0\rho\sqrt{\epsilon_1})\Psi_n'(k_0\rho\sqrt{\epsilon_0})-\Psi_n(k_0\rho\sqrt{\epsilon_0})\Psi_n'(k_0\rho\sqrt{\epsilon_1})}{\alpha\Psi_n(k_0\rho\sqrt{\epsilon_1})\Xi_n'(k_0\rho\sqrt{\epsilon_0})-\Xi_n(k_0\rho\sqrt{\epsilon_0})\Psi_n'(k_0\rho\sqrt{\epsilon_1})}\ , \tag{78}$$

$$b_n=\frac{\Psi_n(k_0\rho\sqrt{\epsilon_1})\Psi_n'(k_0\rho\sqrt{\epsilon_0})-\alpha\Psi_n(k_0\rho\sqrt{\epsilon_0})\Psi_n'(k_0\rho\sqrt{\epsilon_1})}{\Psi_n(k_0\rho\sqrt{\epsilon_1})\Xi_n'(k_0\rho\sqrt{\epsilon_0})-\alpha\Xi_n(k_0\rho\sqrt{\epsilon_0})\Psi_n'(k_0\rho\sqrt{\epsilon_1})}\ , \tag{79}$$

with $\alpha=\sqrt{\epsilon_1/\epsilon_0}$ and

$$\Psi_n(z)=zj_n(z)\ ,\quad \Xi_n(z)=zh_n(z)\ , \tag{80}$$

where $j_n(z), h_n(z)$ are the spherical Bessel and Hankel functions of order n, respectively.

In the expansion of $F_s(0)$ in terms of small ρk_0 up to the sixth-order substantial contributions only result from the coefficients a_1, a_2, and b_1. Through Eqs. (78) and (79) we thus obtain

$$\begin{aligned} a_1 \underset{\rho k_0 \ll 1}{\longrightarrow} & -\frac{2i}{3}\epsilon_0^{3/2}(\rho k_0)^3\frac{\epsilon_1-\epsilon_0}{\epsilon_1+2\epsilon_0} \\ & -\frac{2i}{5}\epsilon_0^{5/2}(\rho k_0)^5\frac{(\epsilon_1-2\epsilon_0)(\epsilon_1-\epsilon_0)}{(\epsilon_1+2\epsilon_0)^2} \\ & +\frac{4}{9}\epsilon_0^3(\rho k_0)^6\left[\frac{\epsilon_1-\epsilon_0}{\epsilon_1+2\epsilon_0}\right]^2 , \\ a_2 \underset{\rho k_0 \ll 1}{\longrightarrow} & -\frac{i}{15}\epsilon_0^{5/2}(\rho k_0)^5\frac{\epsilon_1-\epsilon_0}{2\epsilon_1+3\epsilon_0}\ , \\ b_1 \underset{\rho k_0 \ll 1}{\longrightarrow} & -\frac{i}{45}\epsilon_0^{3/2}(\rho k_0)^5(\epsilon_1-\epsilon_0)\ . \end{aligned} \tag{81}$$

Inserting these expressions into Eqs. (77) and (76) results, through Eq. (50), in the approximation

$$\epsilon^* \underset{\rho k_0 \ll 1}{\longrightarrow} \epsilon_0\left[1+3v\kappa+v\kappa^2\left[\frac{\epsilon_1^2+27\epsilon_0\epsilon_1+38\epsilon_0^2}{2\epsilon_1+3\epsilon_0}\frac{1}{5}(\rho k_0)^2 + 2i\epsilon^{3/2}(\rho k_0)^3\right]\right] \tag{82}$$

with

$$\kappa=\frac{\epsilon_1-\epsilon_0}{\epsilon_1+2\epsilon_0}\ .$$

Here the comparison with Eq. (74) shows that there is a discrepancy in the quadratic term of ρk_0 which is only negligible in the case of weak fluctuation of the refractive index. Hence the degree of agreement is not as significant as in the scalar case.

IV. AUTOCORRELATION FUNCTIONS FOR SPHERES AND FOR CIRCULAR CYLINDERS

We here shall calculate the two-point autocorrelation function for overlapping spheres already suggested in Sec. III. In addition to that we shall also introduce the correlation functions for circular cylinders of finite or infinite length.

To deduce Eq. (53) we make use of the fact that the material is invariant under translations. Hence without loss of generality we may consider two stochastic spheres S_1 and S_2 with radii ρ and centers at the origin and at the point Q, respectively, so that the distance between 0 and Q is $|\mathbf{r}|=r$. Then we have $S_1\cap S_2\neq 0$ $(r\le 2\rho)$ and $S_1\cap S_2=0$ $(r>2\rho)$. The volumes V_i of S_i are $V_1=V_2=\frac{4}{3}\pi\rho^3$.

If $D_i\equiv S_i/(S_1\cap S_2)$ $(i=1,2)$ then the volumes V_{D_i} of D_i become

$$V_{D_1}=V_{D_2}=\begin{cases}\pi r(\rho^2-r^2/12)\ , & r\le 2\rho\\ V_i\ , & r>2\rho\ .\end{cases} \tag{83}$$

The following considerations are based upon the simple equivalence assertion: S_1 contains no (at least one) center of a perturbation sphere if and only if $\epsilon(0)=\epsilon_0(=\epsilon_1)$.

We thus have to calculate

$$A^{(s)}(r)=\langle\epsilon(0)\epsilon(\mathbf{r})\rangle-\langle\epsilon\rangle^2\ .$$

The second-order moment can be written as

$$\langle\epsilon(0)\epsilon(\mathbf{r})\rangle=\sum_{i=0}^{1}\epsilon_i P(\epsilon_i)\langle\epsilon(\mathbf{r})|\epsilon(0)=\epsilon_i\rangle\ , \tag{84}$$

where $P(\epsilon_i)$ is the probability measure of the event $\epsilon(0)=\epsilon_i$ and $\langle\epsilon(\mathbf{r})|\epsilon(0)=\epsilon_i\rangle$ is the conditional average. Obviously $P(\epsilon_0)=1-v$ and $P(\epsilon_1)=v$. Hence the evaluation of Eq. (84) reduces to the calculation of the conditional averages.

It should be noted that the value of $\langle\epsilon(\mathbf{r})\rangle$ at the point Q essentially depends on the number density of the centers of perturbation spheres contained in S_2. Hence by definition of the mean the condition $\epsilon(0)=\epsilon_0$ implies

$$\langle\epsilon(\mathbf{r})|\epsilon(0)=\epsilon_0\rangle=\left[1-\frac{V_{D_2}}{V_2}\right]\epsilon_0+\frac{V_{D_2}}{V_2}\langle\epsilon\rangle \tag{85}$$

and in the alternative case $\epsilon(0)=\epsilon_1$

$$\langle\epsilon(\mathbf{r})|\epsilon(0)=\epsilon_1\rangle=\left[1-\frac{V_{D_1}}{V_1}\right]\epsilon_1+\frac{V_{D_1}}{V_1}\langle\epsilon\rangle \tag{86}$$

where

$$\langle\epsilon\rangle=\epsilon_0+v(\epsilon_1-\epsilon_0)\ . \tag{87}$$

Now Eqs. (83) and (87) furnish

$$\langle\epsilon(\mathbf{r})|\epsilon(0)=\epsilon_0\rangle =\begin{cases}\epsilon_0+v(\epsilon_1-\epsilon_0)\dfrac{r}{4\rho^3}\left[3\rho^2-\dfrac{r^2}{4}\right], & r\le 2\rho\\ \langle\epsilon\rangle\ , & r>2\rho\end{cases} \tag{88}$$

and

$$\langle\epsilon(\mathbf{r})|\epsilon(0)=\epsilon_1\rangle =\begin{cases}\epsilon_1-(1-v)(\epsilon_1-\epsilon_0)\dfrac{r}{4\rho^3}\left[3\rho^2-\dfrac{r^2}{4}\right], & r\le 2\rho\\ \langle\epsilon\rangle\ , & r>2\rho\ .\end{cases} \tag{89}$$

Inserting Eqs. (88) and (89) into Eq. (84) we obtain by straight-forward computation

$$\langle \epsilon(\mathbf{r})\epsilon(0)\rangle = \begin{cases} (1-v)\epsilon_0^2+v\epsilon_1^2-v(1-v)(\epsilon_1-\epsilon_0)^2\dfrac{r}{4\rho^3}\left[3\rho^2+\dfrac{r^2}{4}\right], & r\le 2\rho \\ \langle\epsilon\rangle[\epsilon_0(1-v)+\epsilon_1 v], & r>2\rho \end{cases} \tag{90}$$

which yields the desired correlation function (53).

If the embedded perturbations are overlapping circular cylinders of infinite length parallel to the z axis with radius ρ then the autocorrelation function has the form

$$A^{(c)}(R) = \begin{cases} v_c(1-v_c)(\epsilon_1-\epsilon_0)^2\dfrac{2}{\pi}[\arccos(R/2\rho)-(R/2\rho)\sqrt{1-(R/2\rho)^2}], & R\le 2\rho \\ 0, & R>2\rho \end{cases} \tag{91}$$

where v_z is the relative volume fraction of the cylinders and $R\equiv\sqrt{x^2+y^2}$.

For overlapping circular cylinders of finite length $2l$ the correlation function is given by

$$A^{(c)}(R,|z_Q|) = \begin{cases} v_c(1-v_c)(\epsilon_1-\epsilon_0)^2[\arccos(R/2\rho)-(R/2\rho)\sqrt{1-(R/2\rho)^2}]\left[\dfrac{2}{\pi}-\dfrac{|z_Q|}{l\pi}\right], & R\le 2\rho,\ \ |z_Q|\le 2l \\ 0, \quad R>2\rho \text{ or } |z_Q|>2l, \end{cases} \tag{92}$$

where z_Q is the z coordinate of the center Q of the second cylinder and, without loss of generality, the center of the first one is taken at the origin.

Note that the formal structure of the correlation functions (91) and (92) is essentially identical with that of the MTF of an imaging system with a circular aperature of constant diameter (Ishimaru[6]).

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