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Fluxon dynamics in one-dimensional Josephson-junction arrays

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Current-voltage characteristics of an annular array of small Josephson junctions with different values of single-loop inductances are calculated numerically and compared with that for a continuum annular Josephson junction. The resonances between the moving fluxon and the linear waves caused by the array discreteness induce steps on the current-voltage characteristics which do not exist in the continuum case. The voltage position of the steps is found to be in good agreement with the kinematic approach based on the Frenkel-Kontorova model. Our results show that in Josephson devices the discreteness may lead to strong superradiant emission of electromagnetic waves by moving fluxons.

Discrete arrays of *overdamped* Josephson junctions have gained interest after the first flux-flow device was built on a basis of a high-T_c superconducting film.[1] This device consists of a row of weakened bridges and its design follows a suggestion made by Likharev.[2] Experimental studies of flux flow in a discrete parallel array of weak links based on low-T_c superconductors have also been reported in the literature.[3] Recently the flux-flow dynamics in underdamped one-dimensional (1D) arrays has been studied numerically by Filatrella, Matarazzo, and Pagano.[4]

A limited amount of work has been devoted to the study of 1D arrays in the *underdamped* limit. A discretized Josephson transmission line was at the basis of the idea of the so-called phase-mode logic suggested by Nakajima *et al.*[5] One of the advantages of the discrete array with respect to the long Josephson junction is the fact that the maximum velocity of electromagnetic wave propagation (so-called, Swihart velocity) is much higher than in the continuum case. This property increases the operation frequency of the discrete array with respect to the continuum long Josephson junction. In spite of this encouraging perspective, no systematic comparison between the discrete and continuum cases has been done so far. The study of fluxon dynamics and the radiation produced by the moving fluxon in 1D arrays may also provide an insight into more complex properties of underdamped 2D arrays of Josephson junctions.[6]

In this paper we study numerically and analytically the fluxon dynamics in 1D arrays of Josephson junctions for different degrees of the discreteness and compare the results with the continuum case. The discreteness leads to the radiation of small-amplitude linear waves by a moving fluxon. In order to avoid the influence of the array boundaries on the effects induced by discreteness we consider a Josephson transmission line with periodic boundary conditions. A similar continuous system has been already realized in experiments using annular Josephson junctions.[7]

A 1D parallel array of Josephson junctions is described by the discretized version of the perturbed sine-Gordon equation

$$\frac{d^2\varphi_n}{dt^2}+\alpha\frac{d\varphi_n}{dt}+\sin\varphi_n+\gamma-\frac{1}{a^2}(\varphi_{n-1}-2\varphi_n+\varphi_{n+1})=0\,, \tag{1}$$

where $1\leq n\leq N$, N is the number of junctions which are assumed to be identical, and φ_n is the superconducting phase difference on the nth junction. In order to simplify the comparison with the continuum case all the parameters in Eqs. (1) are written in a standard notation used for long Josephson junctions:[8] the spatial coordinate x is normalized to the effective Josephson penetration depth $\lambda_J=[\Phi_0/2\pi\mu_0\Lambda J_c]^{1/2}$, $\Phi_0=2.07\times10^{-15}$ Wb is the magnetic flux quantum, Λ is the averaged magnetic field penetration depth, the time t is normalized to the inverse plasma frequency $\omega_0^{-1}=[\Phi_0 C/J_c]^{1/2}$, C is the averaged capacitance per unit area of the array, α is the dissipation coefficient, and γ is the bias current normalized to the critical current density J_c calculated as an average for the total area of the array. The discreteness parameter $a=\beta_L^{1/2}=(2\pi L_0 I_c/\Phi_0)^{1/2}$ is also measured in units of the effective λ_J, where L_0 is the inductance of a single cell of the array and I_c is the critical current of each junction. In the limit of $a\to0$ model (1) corresponds to the continuum case.

Currie *et al.*[9] showed by numerical simulations that in model (1) the discreteness leads to the radiation of small-amplitude waves ("phonons") by a moving kink. Peyrard and Kruskal[10] investigated the energy losses by a moving kink due to this radiation for a nondissipative ($\alpha=0$) highly discrete limit. They studied the motion of the kink in a very long system and found quasisteady states when the kink velocity remains unchanged during long time intervals. In such a state the kink almost does not excite any radiation. In this paper we investigate the conditions under which the resonances between fluxons moving in the discrete system with dissipation and the linear waves radiated by them are generated. We calculate the effect of these resonances on a simply measurable experimental parameter, i.e., the current-voltage characteristics of the array.

Let us discuss first a simple kinematic approach which leads to an estimate of the frequency of the radiation as a function of the fluxon velocity and system parameters. With $\alpha = \gamma = 0$ Eq. (1) correspond to the well-known Frenkel-Kontorova model. The dispersion law for linearized waves $\varphi_n = \varphi^{(0)} \exp(i\omega t - ikan)$ is

$$\omega^2 = 1 + \frac{4}{a^2} \sin^2 \left(\frac{ka}{2} \right) . \tag{2}$$

The wave number k takes its values inside the Brillouin band

$$0 \le k \le \frac{2\pi}{a} . \tag{3}$$

The *phase velocity* of the linear waves is

$$v(k) \equiv \frac{\omega}{k} = \frac{1}{k} \sqrt{1 + \frac{4}{a^2} \sin^2 \left(\frac{ka}{2} \right)} . \tag{4}$$

In the interval (3), the phase velocity takes the values $v_{\min} \le v(k) \le \infty$, where

$$v_{\min} \equiv v_{k=2\pi/a} = \frac{a}{2\pi} . \tag{5}$$

Kink (fluxon) solutions do not exist in model (1), since they exist only in exactly integrable models (e.g., in the continuum sine-Gordon model). A quasicontinuum (QC) kink exists "approximately" in model (1) if the size of the kink ℓ is much larger than the lattice spacing a. In the quasicontinuum limit the kinks may exist stably if their velocity v_{fl} is less than the Swihart velocity [which is equal to unity in the normalized units of Eq. (1)]. The phase velocity (4) of the waves excited by the moving fluxon coincides with the fluxon velocity, i.e., $v_{\mathrm{fl}} = v(k)$. This is possible provided that in Eq. (5) $v_{\min} < 1$, i.e., $a < 2\pi$.

Let the *chain of moving fluxons* have a spatial period L. This periodicity can be provided by the external magnetic field in case of a flux-flow regime in a linear array, or it could be just the length of the system in case of an annular array with one trapped fluxon. During the time period T between sequential fluxons passing through a certain point of the array, the local lattice oscillations change their phase by the amount $\Delta\phi = \omega(k)T = \omega(k)L/v_{\mathrm{fl}}$. The *superradiant* excitation (phase locking) of the linear waves (2) by the chain of fluxons takes place if $\Delta\phi$ is a multiple of 2π. If the fluxon frequency is much lower than the linear wave frequency (for a rarified fluxon chain) the resonance condition is

$$kL = 2\pi m , \quad m = 1, 2, 3, \ldots . \tag{6}$$

Taking into account Eq. (2) and the equality of phase velocities of the fluxon and its radiation we obtain

$$v_m = \frac{L}{2\pi m} \sqrt{1 + \frac{4}{a^2} \sin^2 \left(\frac{\pi m a}{L} \right)} . \tag{7}$$

Equation (7) gives the values of the velocity v_{fl} for which, with a given spatial period L, the fluxon chain (or just one fluxon passing through the same point of the periodic array with the length L) generates the resonant superradiant emission. The integer m takes finite values in the interval

$$m_{\min} < m < \frac{L}{a} , \tag{8}$$

where $m_{\min}$ corresponds to the highest velocity (7) below unity. In contrast, low velocities v_m are attained for high resonance numbers (8).

In real experiment the system that we discuss here can be realized as a parallel array of Josephson junctions [Superconducting quantum interference device (SQUID) array] placed in the external magnetic field which defines the fluxon chain period L. In order to study only the interaction of a moving fluxon with its radiation we performed numerical simulations for periodic boundary conditions which eliminate boundaries effects. In this case in Eqs. (1) the spatial points $n = 0$ and $n = N + 1$ are assumed to be equivalent to $n = N$ and $n = 1$ correspondingly, and $\varphi_{N+1} = \varphi_1 + 2\pi M_{\mathrm{fl}}$, where M_{fl} is the number of fluxons trapped in the array. In the simulations presented here we considered the simplest case with one fluxon ($M_{\mathrm{fl}} = 1$), so that $L = \ell + a$. The integration was performed using the fourth-order Runge-Kutta scheme with the time step equal to 0.01 or 0.025.

Figure 1 shows the calculated current-voltage (IV) characteristics of the annular array with the parameters $N = 10$, $a = 1.0$, and $\alpha = 0.1$, which could be achieved in a typical experiment. In order to see clearly the hysteresis between the neighboring steps, the current γ was swept several times up and down. In each sequential point of the IV curve the initial conditions were taken from the stationary state achieved in the previous point. The dashed line shows the IV curve for the continuum case calculated for the system with the same normalized

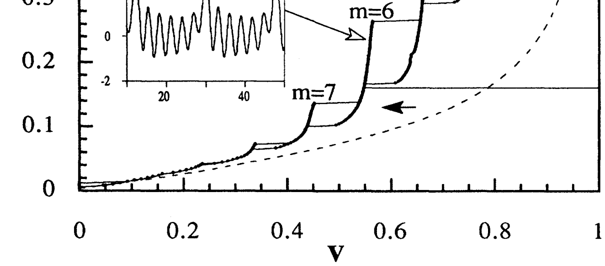

FIG. 1. IV characteristics of the 1D annular Josephson-junction array with the parameters $N = 10$, $a = 1.0$, and $\alpha = 0.1$. Solid arrows indicate the direction of switching in the hysteresis loop. The dashed line shows the IV curve for the continuum case calculated for the system with the same normalized length. The insets show the voltage evolution for two different points of the IV curve measured in the middle point of the array.

length $L = (N+1) \times a = 11.0$ and $a = 0.05$. In both cases the voltage $\langle \varphi_t \rangle$ is normalized to $2\pi/L$ which is the maximum voltage of the first zero field step of the continuous long junction having the length L. In these units the voltage is equal to the average fluxon velocity v in the array.

The difference between the discrete and the continuum cases is clearly seen from Fig. 1. The two curves tend to be close to each other for the low values of the bias current. The IV curve for the discrete array consists of a series of equally spaced current singularities. At high bias the shift of the discrete array curve towards the lower velocities is very clear. One can expect this type of behavior from the simple intuitive argument that in the QC approximation the Lorenz contracted size of a fluxon cannot become smaller than the discreteness parameter a. Thus, even for the large driving force the fluxon in the discrete case cannot move faster than a certain maximum velocity $\tilde{v}(a) < 1$.

The discreteness leads to pinning of a static fluxon by the lattice. In order to start its motion the fluxon has to overcome the Peierls-Nabarro barrier E_{PN}, well known in the dislocation theory. This barrier can be evaluated as the difference of energy between two fluxon positions in the array: with the center of mass placed at one of the junctions and with the center of mass placed between the junctions.[10,11] The evidence of the barrier E_{PN} manifests itself by the existence a small critical current $\gamma_{\text{cr}} = 0.014$ in Fig. 1 corresponding to the point where the curve begins at zero voltage. For $a = 1$ the height of the barrier is of the order of 10^{-2} of the fluxon energy, but E_{PN} increases rapidly for $a > 1$: for $a = 2.0$ we found the critical current $\gamma_{\text{cr}} = 0.31$.

The voltage position of the resonances in the IV curve was found to be independent of the dissipation parameter α in the range between 0.05 and 0.3 (by the voltage position of each step we assumed the maximum voltage attained at the step). As expected, for the larger α the steps become less vertical and the hysteresis between them disappears.

The inset of Fig. 1 shows the voltage evolution in the fifth junction of the array for two different bias points of the calculated IV curve. One can see that in both cases each fluxon-induced voltage pulse is followed by the smaller amplitude oscillation. A more detailed analysis of the time evolution of the local magnetic field $\frac{d\varphi}{dx} \sim (\varphi_{n+1} - \varphi_{n-1})/(2a)$ shows that the small amplitude waves propagate through the array with the same velocity v as the fluxon. The frequency of these oscillations is in a reasonable agreement with formulas (6) and (2) with $m = 5$ and $m = 6$ for the two different points correspondingly. Naturally, the integer m is the number of oscillation periods of the smaller waves per each fluxon pulse.

The negative differential resistance which is clearly seen at the highest step at $v \approx 0.77$ was found only for small damping; it disappears for $\alpha > 0.15$. A possible reason for this negative slope can be a strong nonlinear coupling between the fluxon mode and the waves described by Eq. (2) which appears if the amplitude of these waves becomes sufficiently large. For the continuous system with spatial modulation somewhat similar behavior has been found in numerical simulations[12] and also detected experimentally.[13]

The comparison of the numerically calculated voltage positions of the steps with Eq. (7) is shown in Fig. 2(a) for a particular value of the discreteness parameter $a = 0.625$. In the simulations of the IV curve we found seven steps corresponding to the resonance numbers m from 4 to 10. The agreement with theory (solid line) is fairly good. However, the numerically evaluated step positions are always slightly below the predicted values. This discrepancy is increasing for the high fluxon velocities (close to unity), as expected due to the discreteness-limited Lorenz contraction mentioned above.

In order to make a detailed comparison with theory we calculated a series of IV curves for different values of the discreteness parameter a. The product $N \times a$ was fixed and equal to 10. In agreement with Eq. (7), when a is decreasing the voltage spacing between neighboring steps is also decreasing, each step corresponding to a particular value of m moves down in voltage, and the overall shape of the IV curve becomes closer to the continuum case. The height of steps for small a becomes lower and the hysteresis loops disappear. In order to determine correctly the voltage positions of steps the simulations for

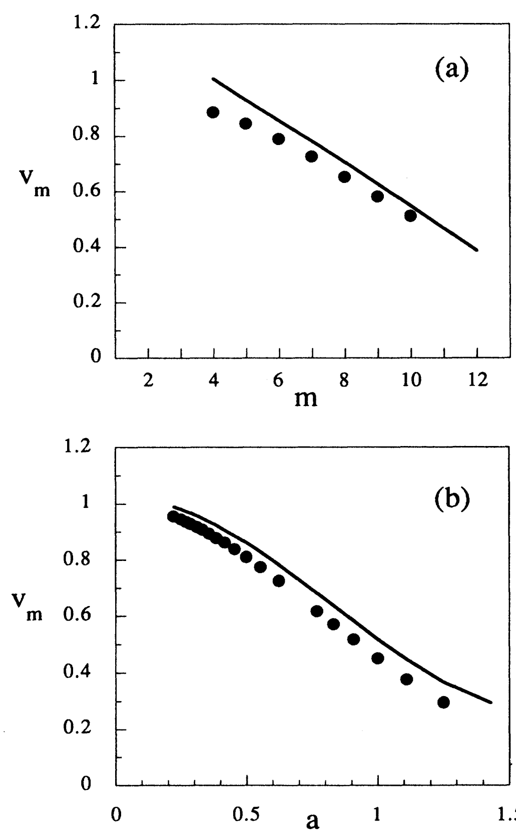

FIG. 2. The voltage position of the discreteness-induced resonances v_m as a function of the resonance number m for $a = 0.625$ (a) and of the array spacing a for $m = 7$ (b). Squares correspond to the numerical results; the solid line is given by Eq. (7).

$a < 0.4$ have been performed with $\alpha = 0.05$. The summary of the results obtained from the simulations of IV curves with different a is shown in Fig. 2(b) for $m = 7$. The agreement with Eq. (7) is rather good. As expected, the theoretical approach presented above works better in the QC approximation, i.e., for small a.

Fine resonant structures similar to that shown in Fig. 1 have been found in numerical simulations with open boundary conditions,[14] however, in the latter case the explanation of the results is more difficult due to fluxon reflections at the boundaries.

The system that we discuss here can be considered as an array of SQUID's. A different way of analyzing the physical phenomena described above is to consider the current singularities of Fig. 1 as due to the resonances of single interferometers[15] excited by the fluxon motion. When the fluxon passes through a single SQUID it excites the oscillations with the frequency $f_{\mathrm{SQUID}} \sim 1/a$ determined by the SQUID inductance and the capacitance of the junctions. The profile of the propagating waves (2) is given by the phase shift between the SQUID resonant oscillations in neighboring junctions which depends on the fluxon velocity. Thus, if the fluxon has the velocity v_m, it arrives again to the same SQUID cell after m periods of SQUID oscillations have passed since the previous passage of the fluxon. A decreasing of SQUID inductances $(a \to 0)$ yields the rise of the SQUID resonant frequency and the higher m for the same fluxon velocities.

We would like to point out the difference between the fluxon radiation in 1D arrays of Josephson junctions and the behavior found in 2D underdamped arrays of junctions investigated so far.[16,17] Numerical simulations[17] showed that the large energy loss by a moving vortex observed in experiment[16] is caused by the excitation of plasmalike oscillations in the junctions. The main difference is that in the 2D case the phase oscillations (analogous to spin waves) are induced by a moving vortex in the every junction it passes and they do not propagate along the array, whereas in the 1D case considered here the oscillations are traveling waves which propagate in the system with the same phase velocity as the vortex. The dynamical models for 2D arrays discussed until now have neglected the self-inductance of each cell in the array, thus the effective Swihart velocity does not exist for this model ($\lambda_J = \infty$). The question of whether the true ballistic motion of vortices is possible in 2D arrays with moderate self-inductance is in our opinion very interesting and deserves further study.

We have investigated soliton dynamics in a strongly discretized underdamped sine-Gordon system focusing attention on the effects that can be observed in experiment on fluxon motion in 1D parallel Josephson junction arrays (SQUID arrays). We find that the resonances between the moving fluxon and the linear waves caused by the array discreteness generate a steplike structure in the IV characteristics which does not exist in the continuum case. The numerically calculated voltage positions of the steps are in reasonable agreement with the kinematic analysis based on the Frenkel-Kontorova model. Our results suggest that in the Josephson devices the discreteness may lead to strong superradiant emission of electromagnetic waves by moving fluxons.

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