Theory of impure superconductors: Anderson versus Abrikosov and Gor'kov

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(Received 16 October 1992)

The influence of ordinary (nonmagnetic) impurities on the transition temperature of a traditional singlet-pairing superconductor is revisited. The theory of Anderson is juxtaposed with that due to Abrikosov and Gor'kov (AG). Whereas Anderson's treatment leads to no substantial decrease of T_c with impurity substitution, the Abrikosov-Gor'kov theory predicts a large decrease of T_c, linear in the impurity concentration. This latter result is not well known because an unallowed mathematical manipulation canceled a large T_c decrease required by a correct evaluation of the AG theory. The controversy that arises is herein settled in favor of Anderson. The Green's functions employed in the AG method are not appropriate for an impure Bardeen-Cooper-Schrieffer superconductor. It is shown that if the required *projected Green's functions* (not derivable from a Dyson equation) are employed, Anderson's theorem is recovered.

I. INTRODUCTION

The purpose of this study is to show that Anderson's theory[1] of dirty superconductors is in serious conflict with the method developed by Abrikosov and Gor'kov[2] (AG) when the AG analysis is completed without mathematical compromise. The controversy which is then presented can be resolved by observing that the Green's functions employed in the AG treatment are inappropriate. One must use instead *projected Green's functions* which respond only in eigenstate channels having energy within $\hbar\omega_D$ of the Fermi energy E_F. This modified theory agrees with Anderson's theorem: impurity scattering does not significantly alter the superconducting transition temperature T_c. (Our attention here is confined to nonmagnetic solutes.)

The discussion will be facilitated by embracing a number of traditional simplifications: (i) The conduction band has a constant density of states N_0 and has a width $2E_F$ (i.e., electron-hole symmetry). (ii) The superconducting gap parameter Δ of the host superconductor is isotropic in k space (i.e., there will be no decrease in T_c with initial solute additions). (iii) The Fröhlich pairing interaction $V(\mathbf{k}, \mathbf{k}')$ is a constant $(-V)$ independent of $\mathbf{q} \equiv \mathbf{k}' - \mathbf{k}$. (iv) The weak-coupling version of the Bardeen-Cooper-Schrieffer (BCS) theory[3] may be used (i.e, $\lambda \equiv N_0 V < 1$). (v) Solute additions alter neither the band structure, the phonon spectrum, nor the electron-phonon interactions. (vi) Scattering of conduction electrons by solute ions is isotropic. This last simplification is for convenience only; it is not essential. (vii) The Coulomb repulsion between electrons is neglected.

Anderson's theory[1] is based on the observation that the exact eigenstates, with impurities present, occur in time-reversed, degenerate pairs. Accordingly the BCS pairing scheme can be carried out without change. The unitary character of the transformation between scattered states and the momentum representation, together with the first five assumptions listed above, lead to an unchanged BCS gap equation. Accordingly T_c remains equal to T_{c0}, the pure host value. However, in Sec. III we point out that Anderson's theory is valid only to first order in the impurity concentration.

Scattering of the conduction electrons by solute ions does lead, however, to a decrease in the density of states at the Fermi level. This effect is small. Nevertheless we will derive the T_c decrease caused by this term because it will provide a diagnostic tool for interpreting the outcome of the AG method.

We suppose that n is the number of atoms per unit volume and that the scattering potential of each solute ion, on using assumption (vi), is $u\delta(\mathbf{r})$. Accordingly the density of states (per spin) is $N_0 = n/2E_F$, and the scattering rate $1/\tau$ from the "golden rule" is

$$\frac{1}{\tau} = \frac{2\pi}{\hbar} u^2 N_0 nc \, , \tag{1}$$

where c is the atomic fraction of solute ions. The energy shift $\Delta E \equiv E - \epsilon$ of each conduction-band state is given by Brillouin-Wigner perturbation theory

$$\Delta E = \int_{-E_F}^{E_F} \frac{N_0 u^2 nc}{E - \epsilon} d\epsilon \, . \tag{2}$$

(E and ϵ are measured relative to the Fermi level.) The principal value of (2) is

$$\Delta E = -ncu^2 N_0 \ln \left| \frac{E_F - E}{E_F + E} \right| \approx \frac{2N_0 u^2 ncE}{E_F} \, , \tag{3}$$

for $|E| \ll E_F$. Accordingly, the density of states at the Fermi surface is reduced:

$$N = N_0 \left[1 + \frac{2N_0 u^2 nc}{E_F} \right]^{-1} \, . \tag{4}$$

On replacing N_0 by N in the BCS T_c equation, we find

$$T_c \cong T_{c0} - \frac{\hbar T_{c0}}{\pi \lambda E_F \tau} \, . \tag{5}$$

Equation (1) has been used to reexpress the T_c decrease in

terms of the scattering time τ. ($1/\tau$ is, of course, proportional to the solute concentration c.)

The magnitude of the correction term in Eq. (5) can be surmised from the value

$$\frac{\hbar}{E_F \tau} < 10^{-2} , \tag{6}$$

for a 1% concentration of a typical solute. Clearly this small decrease in N and its concomitant reduction in T_c can usually be neglected.

II. EVALUATION OF THE AG THEORY

We will now show that the AG theory leads to a result significantly different from that described above. We set the spin-dependent scattering potential of the AG Hamiltonian to zero and keep only the nonmagnetic potential u. In our notation, the AG T_c equation which follows Eq. (21) of Ref. 2, becomes (with $\tau_s = \infty$)

$$1 = \frac{VT_c}{8\pi^3} \sum_\omega \int \frac{\eta_1}{\omega^2 \eta_1^2 + \epsilon^2} d^3k , \tag{7}$$

where

$$\eta_1 = 1 + \frac{1}{2|\omega|\tau} , \tag{8}$$

τ being the scattering time associated with the solute ions [Eq. (1)] above. The ω's are

$$\omega_n = (2n+1)\pi T_c , \tag{9}$$

for all integers n between $-\infty$ and ∞. ($\hbar$ and k_B have been taken to be 1.) The integral over the electron energy is confined to the narrow strip within ω_D of the Fermi energy. Equation (7) is then, with $\lambda = N_0 V$,

$$1 = \lambda T_c \sum_{n=-\infty}^{\infty} \int_{-\omega_D}^{\omega_D} \frac{\eta_1}{\omega_n^2 \eta_1^2 + \epsilon^2} d\epsilon . \tag{10}$$

AG's next step is to add and subtract $(\omega_n^2 + \epsilon^2)^{-1}$ to (and from) the integrand. The sum involved in the added term

$$\lambda T_c \sum_{n=-\infty}^{\infty} \int_{-\omega_D}^{\omega_D} \frac{d\epsilon}{\omega_n^2 + \epsilon^2} \tag{11}$$

can be evaluated exactly, since[4]

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2 \pi^2 + 4x^2} = \frac{1}{x} \tanh x . \tag{12}$$

Equation (10) can now be written as

$$1 = \lambda \int_{-\omega_D}^{\omega_D} \frac{d\epsilon}{2\epsilon} \tanh \left[\frac{\epsilon}{2T_c} \right]$$

$$+ \lambda T_c \sum_{-\infty}^{\infty} \int_{-\omega_D}^{\omega_D} \left[\frac{\eta_1}{\omega_n^2 \eta_1^2 + \epsilon^2} - \frac{1}{\omega_n^2 + \epsilon^2} \right] d\epsilon . \tag{13}$$

AG evaluate the two integrals of the last term by extending the limits to $\pm\infty$. Then, for each n, the two integrals cancel identically, so that only the first term remains, i.e.,

$$1 = \lambda \int_{-\omega_D}^{\omega_D} \frac{d\epsilon}{2\epsilon} \tanh \left[\frac{\epsilon}{2T_c} \right] , \tag{14}$$

which is the BCS T_c equation for a pure superconductor. Thus follows the (incorrect) deduction from the AG formalism,

$$T_c = T_{c0} , \tag{15}$$

in apparent agreement with Anderson's theorem.

It is not permissible, however, to change the limits from ω_D to ∞ as AG have done. The finite integration limits must be respected in the last two integrals of Eq. (13). Before evaluating these two terms, which we call R, we reexpress the first term of Eq. (13) with the help of an integration by parts (and $x \equiv \epsilon/2T_c$), i.e.,

$$1 = \lambda [\ln(x)\tanh x]_0^{\omega_D/2T_c}$$

$$- \lambda \int_0^{\omega_D/2T_c} \frac{\ln(x)dx}{\cosh^2 x} + \lambda T_c R . \tag{16}$$

Now we can perform the same manipulation with Eq. (14):

$$1 = \lambda [\ln(x)\tanh x]_0^{\omega_D/2T_{c0}} - \lambda \int_0^{\omega_D/2T_{c0}} \frac{\ln(x)dx}{\cosh^2 x} , \tag{17}$$

in which T_{c0} appears instead of T_c. Note that at the upper limits of Eqs. (16) and (17) $\tanh x$ is exponentially close to unity and $\cosh^{-2} x$ is exponentially small. So on subtracting (17) from (16) we obtain

$$0 \cong \ln(T_{c0}/T_c) + T_c R . \tag{18}$$

We return to the evaluation of R, which can be written as

$$R = 4 \sum_{n=0}^{\infty} \int_0^{\omega_D} \left[\frac{\eta_1}{\omega_n^2 \eta_1^2 + \epsilon^2} - \frac{1}{\omega_n^2 + \epsilon^2} \right] d\epsilon . \tag{19}$$

The integrals are elementary:

$$R = 4 \sum_{n=0}^{\infty} \frac{1}{\omega_n} \left[\tan^{-1} \left[\frac{\omega_D}{\omega_n \eta_1} \right] - \tan^{-1} \left[\frac{\omega_D}{\omega_n} \right] \right] . \tag{20}$$

The sums can be carried out to the required precision, to terms linear in $1/\tau$ [Eq. (1)], by using

$$\tan^{-1} x \cong x , \quad x < 1 , \tag{21a}$$

$$\frac{\pi}{2} - \frac{1}{x} , \quad x > 1 . \tag{21b}$$

[It may be noted that $\tan^{-1} x$ equals the mean of (21a) and (21b) at $x = 1$.] Equations (8) and (9), together with[5]

$$\sum_{k=1}^{n} \frac{1}{(2k-1)} \approx \frac{1}{2}[C + \ln(n)] + \ln 2 , \tag{22}$$

where C is Euler's constant, lead to

$$R = \frac{-1}{\pi T_c \omega_D \tau} \left[\ln \left[\frac{2e^C \omega_D}{\pi T_c} \right] + \frac{1}{2} \right] . \tag{23}$$

But the solution to the BCS T_c [Eq. (14)] is

$$T_c = \frac{2e^C \omega_D}{\pi} e^{-1/\lambda} , \tag{24}$$

so the first term in Eq. (23) is just $1/\lambda$. This result, combined with Eq. (18), shows that a significant T_c decrease results from a correct evaluation of the AG theory:

$$T_c = T_{c0}\exp\left[\frac{-1}{\pi\omega_D\tau}\left[\frac{1}{\lambda}+\frac{1}{2}\right]\right] .$$ (25)

[The difference between T_c and T_{c0} has been neglected in Eq. (24). To include the difference would lead only to a higher-order correction.]

The T_c decrease in Eq. (25) is initially linear in $1/\tau$ and therefore is linear in the solute concentration c [cf. Eq. (1)],

$$T_c \cong T_{c0} - \frac{T_{c0}}{\pi\omega_D\tau}\left[\frac{1}{\lambda}+\frac{1}{2}\right] .$$ (26)

This decrease is incorrect theoretically as we elaborate below and is in striking disagreement with experiment. $1/\omega_D\tau$ is not a small quantity. The "physics" of the (incorrect) decrease in Eq. (26) can be surmised by comparing the $1/\lambda$ term with the corresponding term of Eq. (5). The AG formalism has apparently incorporated a reduction in density of states (caused by virtual scattering from solute ions) as if the total bandwidth were $2\hbar\omega_D$ instead of $2E_F$. The remedy, as we will show, is to appreciate that the BCS cutoff must be incorporated in the Green's functions instead of at the end of the calculation.

It is clear from Eqs. (25) and (26) that the important parameter is the scattering time τ of conduction electrons by the (nonmagnetic) impurities. Table I lists residual resistivity data for 16 solute elements in Pb (Ref. 6). The median resistivity per atomic percent of solute is ~ 1 $\mu\Omega$ cm. Since the electron density in Pb is $n_0 = 1.32 \times 10^{23}$ cm^{-3}, and $\sigma = n_0 e^2\tau/m$, one obtains

$$\tau \sim 3 \times 10^{-14} \text{ sec} .$$ (27)

Now the Debye temperature of Pb is ~ 90 K, so

$$\omega_D\tau \sim 0.3$$ (28)

for 1% of a typical solute. Accordingly, the AG theory requires an almost complete suppression of T_c in the 1 to 2% concentration range. However, experimental data for ΔT_c caused by 2% of Ag or Au (Ref. 7) and by even larger concentrations of Cd, Hg, In, and Tl (Ref. 8) show that T_c is reduced (from $T_{c0} = 7.2$ K) by only ~ 0.1 K. Anderson's theorem is well verified (which has been appreciated for three decades).

TABLE I. Residual resistivity, in $\mu\Omega$ cm/at. %, for various impurities in Pb. Data are from Ref. 6.

Ag	3.4	Bi	0.92
Te	3.36	In	0.73
Au	3.22	Tl	0.62
Mg	2.3	Ga	0.55
Cd	1.85	Rh	0.55
Hg	1.36	Pt	0.5
As	1.15	Sn	0.25
Sb	1.18	Pd	0.2

III. LIMITATION OF ANDERSON'S THEOREM

Anderson applied BCS theory to a dirty superconductor[1] by introducing exact scattered states $\psi_{n\sigma}$ for conduction electrons in a metal having nonmagnetic impurities. The relation between the scattered states and the Bloch waves is described by a unitary transformation:

$$\psi_{n\sigma} = \sum_k \phi_{k\sigma}\langle k|n\rangle .$$ (29)

(It is assumed that the Hamiltonian does not depend on spin σ.) The main task is to derive the phonon-mediated interaction H_{ee} in the new (scattered-state) representation. In the momentum representation,[9]

$$H_{ee} = \tfrac{1}{2}\sum_{kk'q\sigma\sigma'} V_{kk'q}c^*_{k-q,\sigma}c^*_{k'+q,\sigma'}c_{k'\sigma'}c_{k,\sigma} .$$ (30)

It is a frequent practice to replace this interaction by a reduced version, which discards all terms which do not play a role in the scattering between $(k\uparrow,-k\downarrow)$ pairs. Thus,

$$H_{\text{red}} = \sum_{kq} V_{kq}c^*_{k-q}c^*_{-k+q}c_{-k}c_k ,$$ (31)

where the states ϕ_k have only spin up and ϕ_{-k} down. H_{red} is merely a notational simplification that recognizes in advance which eigenstates will be paired and so contribute to the BCS condensate. For example, if one were treating a condensate carrying a supercurrent, one would replace (31) by a similar expression with $-k$ replaced by $-k+\kappa$. The two reduced Hamiltonians would have no terms in common, but both are contained in H_{ee} [Eq. (30)].

There are two ways to derive H_{ee} in the scattered-state representation. One method is to carry out the canonical transformation of Fröhlich[9] in the scattered-state basis and the other is to transform H_{ee} [Eq. (30)], using the transformation, Eq. (29). With the latter method, one must replace one-electron energies ϵ_k that appear in the denominators of $V_{kk'q}$ by their corresponding values ϵ_n for the scattered states.[1] We have found that both methods lead to the same H'_{ee}.

The reduced version of this new H'_{ee} anticipates that ψ_n (having spin up) will be paired with its time-reversed counterpart $\psi_{\bar{n}}$ (having spin down). The new reduced Hamiltonian is

$$H'_{\text{red}} = \sum_{nn'} V_{nn'}c^*_{n'}c^*_{\bar{n}'}c_{\bar{n}}c_n ,$$ (32)

where

$$V_{nn'} = \sum_{kk'q} V_{kk'q}\langle k-q|n'\rangle\langle k'+q|\bar{n}'\rangle\langle k'|\bar{n}\rangle^*\langle k|n\rangle^* .$$ (33)

The traditional simplification (iii) mentioned in Sec. I neglects any variations of $V_{kk'q}$ except that if either ϵ_n or $\epsilon_{n'}$ falls outside the $2\hbar\omega_D$ band centered about the Fermi level, $V_{nn'}$ is set equal to zero. Accordingly when both ϵ_n and $\epsilon_{n'}$ fall within the BCS cutoff,

$$V_{nn'} = -V \sum_{kk'q} [\langle -k'|n \rangle \langle k|n \rangle^*]$$

$$\times [\langle k-q|n' \rangle \langle -k'-q|n' \rangle^*] . \quad (34)$$

(We have replaced matrix elements involving $\bar{n}$ and $\bar{n}'$ by their time-reversed equivalents.) Equation (34) cannot be evaluated further as long as the scattered states $\{\psi_n\}$ are general and unspecified. $V_{nn'}$ will not be a constant, and there can be no proof of Anderson's conclusion.

If the sum over k' in Eq. (34) were omitted and instead, k' were set equal to $-k$, then

$$V_{nn'} = -V \left[\sum_k |\langle k|n \rangle|^2 \right] \left[\sum_q |\langle k-q|n' \rangle|^2 \right] . \quad (35)$$

(The isolation of the two factors is possible because the sum over q is independent of k.) The unitary character of the transformation [Eq. (29)] implies that both sums in (35) are unity. Then it would follow that

$$V_{nn'} = -V , \quad (36)$$

and proof of Anderson's theorem would be complete.

However, Eq. (35) is not correct. The only way to obtain it is to transform the *reduced* Hamiltonian for a pure superconductor [Eq. (31)] and then to truncate the result once again for $(n,\bar{n})$ pairing. Such a procedure is not allowed. Truncation of the full Hamiltonian is permitted only once, for the eigenstate pairs that are (finally) involved in the BCS condensate. Since Eq. (35) is equivalent to Eq. (3) of Ref. 1, together with assumption (iii), above, it is apparent that proof of Anderson's theorem in Ref. 1 was incomplete.

We now inquire whether Anderson's theorem can be proved under restricted approximations. The scattered states $\{\psi_n\}$ will be treated by perturbation theory in powers of the solute potential,

$$U(r) = \sum_i u \delta(r - R_i) . \quad (37)$$

$\{R_i\}$ are the solute sites. The scattered states can be evaluated to order u^3 (Ref. 10),

$$\psi_{n(k)} = \phi_k + \sum_\kappa \frac{u}{\epsilon_k - \epsilon_{k+\kappa}} \left[\sum_i e^{-i\kappa \cdot R_i} \right] \phi_{k+\kappa} + \cdots . \quad (38)$$

(The u^2 and u^3 terms are not displayed.) The matrix elements of Eq. (29) are thus specified. The *full* Hamiltonian H_{ee} may then be transformed into this basis and the reduced Hamiltonian [Eqs. (32) and (34)] evaluated to order u^4. The exercise is extremely tedious. We have found that there is considerable compensation for the terms of Eq. (34) which were omitted in Eq. (35). The sum of all u^2 and u^3 terms leads to a correction in Eq. (36) of order $(\hbar\omega_D/E_F)^2$. Since this factor is $\sim 10^{-4}$ (and may be neglected), the theorem is correct to order u^3 and to the linear power in solute concentration c. However, we were unable to recognize a similar compensation for all terms of order u^4 and proportional to c^2.

The foregoing limitation resolves what otherwise might have seemed paradoxical. Reference 1 places little restriction on the complexity of the scattered states. So the imputed result [Eq. (36)] would appear to hold even if the states n and n' were localized in different regions of the sample. The phonon-mediated interaction that connects $(n,\bar{n})$ to $(n',\bar{n}')$ would then have to be zero (since there would then be no spatial overlap of ψ_n with $\psi_{n'}$), thereby contradicting Eq. (36). The cancellation we noted above when using Eq. (38) to the third order is not similarly contradicted since localization does not occur in u^3 order. There have been many studies of the effect of localization on T_c (Ref. 11) but the emphasis has been either on the role of Coulomb interactions or on strong-coupling effects. The limitation we have noted here is related merely to the properties of the unitary transformation [Eq. (29)] and to observing that Eq. (36) is not demonstrable except to order u^3 and to the first power in c. Finally we emphasize that the linear T_c decrease (with c) of the AG theory [Eq. (26)] is proportional to u^2 and so is contradicted by the provable residuum of Anderson's theorem.

IV. PROJECTED GREEN'S-FUNCTION METHOD

In this section we modify the Green's-function method in order to recover a valid theory for an impure superconductor. It will be necessary to utilize at the outset Green's functions which incorporate the BCS cutoff. The usual Green's functions are most easily expressed in terms of the exact eigenstates $\{\psi_n\}$ for the normal (nonsuperconducting) state:

$$G_\omega(s,r) = \sum_n \frac{\psi_n(s)\psi_n^*(r)}{i\omega - \epsilon_n} , \quad (39)$$

where ω is one of the Matsubara values [Eq. (9)]. We define the *projected* Green's functions to be

$$G_\omega^P(s,r) = \sum_n \Theta_n \frac{\psi_n(s)\psi_n^*(r)}{i\omega - \epsilon_n} , \quad (40)$$

where

$$\Theta_n = 1 , \quad \text{if } |\epsilon_n| \leq \hbar\omega_D , \quad (41)$$

$$\Theta_n = 0 , \quad \text{otherwise} .$$

The importance of the cutoff factors $\{\Theta_n\}$ has been emphasized,[12] but it is traditional to suppress them formally. We are unaware of their use to define $G_\omega^P(s,r)$ [Eq. (40)].

The T_c equation employed by AG, Eq. (11) of Ref. 2, involves the Green's functions for the normal state and the pair potential $\Delta^*(r)$. We will argue below that the functions (40) should be used in place of (39). Accordingly,

$$\Delta^*(r) = VT_c \sum_\omega \int \Delta^*(s) G_\omega^P(s,r) G_{-\omega}^P(s,r) d^3s . \quad (42)$$

(We have suppressed spin indices because only nonmagnetic solutes are under consideration.) AG suggest that $\Delta^*(r)$ can be replaced by its average value, $\int \Delta^*(r) d^3r$, so the argument of the ω summation in Eq. (42) becomes

$$\int \int \sum_{n,m} \Theta_n \Theta_m \frac{\psi_n(s)\psi_n^*(r)\psi_m(s)\psi_m^*(r)}{(i\omega - \epsilon_n)(-i\omega - \epsilon_m)} d^3r \, d^3s . \quad (43)$$

Recalling that the $\{\psi_n\}$ here are only the space parts of the one-electron eigenstates, the sum over n can be replaced by the sum of the time-reversed states, $\psi_{\bar{n}} = \psi_n^*$. Accordingly, the integral in d^3r is δ_{nm}, and the subsequent integral in d^3s is just unity (for each remaining term). Thus Eq. (42) becomes

$$\Delta^* = VT_c \sum_\omega \sum_n \Delta^* \frac{\Theta_n}{\omega^2 + \epsilon_n^2} . \tag{44}$$

From Eqs. (9) and (12), this expression reduces to

$$1 = V \sum_n \frac{\Theta_n}{2\epsilon_n} \tanh\left[\frac{\epsilon_n}{2T_c}\right] , \tag{45}$$

which is exactly the BCS T_c equation, e.g., Eq. (14). So Anderson's theorem, $T_c = T_{c0}$, is reproduced.

A formal justification of Eq. (42), with the projected Green's functions, is not available. However, this equation can be interpreted physically in the fashion of other self-consistent strategies. The pair potential $\Delta^*(s)$ "launches" (from the regions near s) time-reversed eigenstate pairs which collaborate to generate a pair potential $\Delta^*(r)$ in the region near r. The pair propagators should be truncated so that only *eigenstates* within $\hbar\omega_D$ of the Fermi energy participate. The BCS cutoff factors $\{\Theta_n\}$ must delineate the *eigenstate* energies $\{\epsilon_n\}$ and not the momentum state energies $\{\epsilon_k\}$ of the basis functions used

to synthesize them. Equation (7) above, taken from the AG theory, employed at the end a cutoff function on $\{\epsilon_k\}$ and thereby imposed a limitation on the momentum components $\{\phi_k\}$, confining them to an energy interval $2\hbar\omega_D$ in the basis space. The scattered states, however, even those with energies $|\epsilon_n| \leq \hbar\omega_D$, require a synthesis using all momentum states within the $(2E_F$ wide) energy band.

Our conclusion is that a correct theory of an inhomogeneous superconductor must incorporate the BCS cutoff at the outset and in the eigenstate representation. Imposing a BCS-like cutoff only at the end of a calculation can lead to invalid results, as shown in Sec. II. Another problem involves use of a Dyson equation or a Dyson expansion, to find Green's functions which include the effect of inhomogeneous perturbations. We have verified that a projected Green's function cannot be obtained by means of a Dyson equation. It is evident that some theories of inhomogeneous superconductivity may need to be reexamined. The problem of magnetic impurities is a particularly important example. The results of such an investigation will be published separately.

ACKNOWLEDGMENTS

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