PHYSICAL REVIEW B VOLUME 47, NUMBER 13 1 APRIL 1993-I

Time-dependent Ginzburg-Landau theory for a weak-coupling superconductor

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(Received 2 November 1992)

Using a functional approach to the Keldysh formalism we develop a time-dependent Ginzburg-Landau equation for the long-wavelength and low-frequency dynamics of a weak-coupling superconductor, which is also valid in situations that deviate significantly from thermal equilibrium. The theory takes the interactions between quasiparticles and Cooper pairs into account exactly and is explicitly Galilean-invariant at zero temperature. However, the temporal and spatial dependence of the order parameter is assumed to be such that the effects of Landau damping can be neglected. We also consider the screening properties of the superconductor and show that the inclusion of the polarizability of the electron gas is necessary for a complete description.

I. INTRODUCTION

Even before the advent of the microscopic BCS theory, the phenomenological Ginzburg-Landau[1] theory proved to be a powerful tool in the study of the equilibrium properties of a superconductor in slowly varying electromagnetic fields.[2] Therefore, it came as no surprise that soon after the formulation of the microscopic theory,[3] Gorkov was able to relate both approaches by identifying the energy gap $\Delta(\mathbf{x})$ with the order parameter of the phase transition.[4]

However, for the description of nonequilibrium properties we need an extension of the original Ginzburg-Landau theory, because the time dependence of the order parameter becomes of the utmost importance. Examples are the nucleation and spinodal growth of the superconducting phase,[5] the dynamics of Josephson junctions, and various phase-slip phenomena associated with the motion of vortices. In particular, the decay of the supercurrent in quasi-one-dimensional superconductors due to both temperature and quantum fluctuations seems to be poorly understood at present[6] and may offer the possibility to verify experimentally the predictions of the time-dependent Ginzburg-Landau theory to be derived in this paper.

Theoretically, the above-mentioned phenomena are usually studied at temperatures near the critical temperature T_c, where $\Delta/k_B T$ can be considered as a small quantity and the form of the time-dependent Ginzburg-Landau theory is well established.[7–9] Below T_c the situation is less clear. Abrahams and Tsuneto[8] have made an extensive study of this region and find that a time-dependent Ginzburg-Landau equation can only be derived at zero temperature, since at nonzero temperatures the collective (Bogoliubov-Anderson) mode is Landau damped. Nevertheless, neglecting this process, a time-dependent Ginzburg-Landau theory can be formulated at all temperatures, which is generally used to describe the physics of phase-slip phenomena.[10]

This procedure has, however, an important drawback: since $\Delta/k_B T$ is not small well below T_c, Abrahams and Tsuneto arrive at their results by expanding the equations of motion for $\Delta(\mathbf{x},t)$ around the value of the gap at some point $(\mathbf{x}',t')$, which requires "zeroth-order" Green's functions corresponding to this local gap $\Delta(\mathbf{x}',t')$. Because these are not known one has to resort to a physically motivated approximation. Abrahams and Tsuneto assume that the dynamics of the quasiparticles is fast compared to the dynamics of the Cooper pairs and hence that the quasiparticles are at all times in equilibrium with respect to $\Delta(\mathbf{x}',t')$. Unfortunately, this is not the case in conventional superconductors as pointed out recently by Ao *et al.*[11] They therefore propose a sudden approximation, in which the quasiparticles do not react to a change in the condensate wave function and only provide for a "static" background.

Although the static picture is certainly an improvement over the adiabatic one, there still are some problems associated with it. First, Ao *et al.* include quasiparticle relaxation due to electron-electron, electron-phonon, and electron-impurity scattering in their discussion but neglect the interaction between quasiparticles and condensate, which might induce a more rapid temporal behavior. Second, their time-dependent Ginzburg-Landau theory is not Galilean-invariant at zero temperature, which contradicts BCS theory. From a formal point of view this is somewhat disturbing, although its effect is negligible in the limit of small currents for which the static picture was developed.

In an attempt to overcome these problems we present in Sec. II a functional approach to the Keldysh formalism,[12] which is used in Secs. III and IV to derive an effective long-wavelength and low-frequency action for the energy gap and thus the desired Ginzburg-Landau theory. In Sec. III we first consider the neutral case and in Sec. IV we subsequently show how the electromagnetic screening properties of the charged superconductor can be included in the description. Finally in Sec. V we summarize some conclusions of this work.

II. NONEQUILIBRIUM THEORY

A way in which we can conveniently study nonequilibrium processes and take the interaction between quasiparticles and condensate into account is by means of a functional formulation of the Keldysh theory. Such a formulation was recently developed to discuss the nucleation of Bose-Einstein condensation in a dilute atomic gas[13] and gives the opportunity to follow the time evolution of the system of interest, whenever an initial density matrix $\rho(t_0)$ is specified. In the following we take a $\rho(t_0)$ corresponding to an essentially arbitrary distribution $N(E)$ of quasiparticles with energy E and gap Δ_0, the latter being obtained self-consistently from the appropriate BCS gap equation for the quasiparticle distribution $N(E)$. Subsequently, we determine the equations of motion for $\Delta(\mathbf{x},t)$ at times $t \gg t_0 + O(\hbar/k_B T_c)$ when the transients have died out and the Ginzburg-Landau equation acquires a particular useful form.

Following closely the treatment of the Bose gas, we start the discussion of the Keldysh formalism by writing the generation functional of all Green's functions for a system of fermions with spin states $|\alpha\rangle$ as a functional integral

$$Z[J,J^*] = \int d[\psi^*]d[\psi] \exp\left[\frac{i}{\hbar}S[\psi^*,\psi]\right] \times \exp\left[i\sum_\alpha \int_C dt \int d\mathbf{x}[\psi^*_\alpha(\mathbf{x},t)J_\alpha(\mathbf{x},t) + J^*_\alpha(\mathbf{x},t)\psi_\alpha(\mathbf{x},t)]\right] \tag{1}$$

over the Grassmann variables $\psi_\alpha(\mathbf{x},t)$ and $\psi^*_\alpha(\mathbf{x},t)$ defined on the Keldysh contour C, which consists of a chronological branch from t_0 to infinity and an antichronological branch from infinity to t_0.[14] Here, the sources $J_\alpha(\mathbf{x},t)$ and $J^*_\alpha(\mathbf{x},t)$ are anticommuting c-number fields and the action $S[\psi^*,\psi]$ is given by

$$S[\psi^*,\psi] = \sum_\alpha \int_C dt \int d\mathbf{x}\psi^*_\alpha(\mathbf{x},t)\left[i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2\nabla^2}{2m} + \mu\right]\psi_\alpha(\mathbf{x},t) - \frac{1}{2}\sum_{\alpha,\alpha'}\int_C dt \int d\mathbf{x}\int d\mathbf{x}'\psi^*_\alpha(\mathbf{x},t)\psi^*_{\alpha'}(\mathbf{x}',t)V(\mathbf{x}-\mathbf{x}')\psi_{\alpha'}(\mathbf{x}',t)\psi_\alpha(\mathbf{x},t) \tag{2}$$

for particles with an effective mass m and a spin-independent interaction $V(\mathbf{x}-\mathbf{x}')$. Notice that the chemical potential μ is time dependent in principle, but for a weak-coupling superconductor is well approximated by a constant. Moreover, in the case of singlet pairing the effectively attractive interaction $V(\mathbf{x}-\mathbf{x}')$ can be replaced by the Gorkov potential $-g\delta(\mathbf{x}-\mathbf{x}')$ and the second term on the right-hand side of Eq. (2) becomes

$$g\int_C dt \int d\mathbf{x}\psi^*_\uparrow(\mathbf{x},t)\psi^*_\downarrow(\mathbf{x},t)\psi_\downarrow(\mathbf{x},t)\psi_\uparrow(\mathbf{x},t) ,$$

denoting the spin-up and spin-down states by $|\uparrow\rangle$ and $|\downarrow\rangle$, respectively.

To find the effective action and thus the desired Ginzburg-Landau theory for the order parameter $\langle\Delta(\mathbf{x},t)\rangle = -g\langle\psi_\downarrow(\mathbf{x},t)\psi_\uparrow(\mathbf{x},t)\rangle$ of the phase transition we perform a Hubbard-Stratonovich transformation by multiplying the generating functional $Z[J,J^*]$ by

$$1 \equiv \mathcal{N}\int d[\Delta^*]d[\Delta]\exp\left[-\frac{1}{g}\int_C dt \int d\mathbf{x}|\Delta(\mathbf{x},t) + g\psi_\downarrow(\mathbf{x},t)\psi_\uparrow(\mathbf{x},t)|^2\right] , \tag{3}$$

and evaluating the Gaussian integral over the fermionic fields. The latter is most easily accomplished in Nambu space, which implies the introduction of the vector quantities:

$$\psi(\mathbf{x},t) \equiv \begin{bmatrix}\psi_\downarrow(\mathbf{x},t)\\ \psi^*_\uparrow(\mathbf{x},t)\end{bmatrix}, \quad J(\mathbf{x},t) \equiv \begin{bmatrix}J_\downarrow(\mathbf{x},t)\\ -J^*_\uparrow(\mathbf{x},t)\end{bmatrix} , \tag{4}$$

and their hermitian conjugates. In this manner we find that[9]

$$Z[J,J^*] = \mathcal{N}\int d[\Delta^*]d[\Delta]\exp\left[\frac{i}{\hbar}S[\Delta^*,\Delta]\right]\exp\left[-i\int_C dt \int d\mathbf{x}\int_C dt' \int d\mathbf{x}' J^\dagger(\mathbf{x},t)G(\mathbf{x},t;\ \mathbf{x}',t')J(\mathbf{x}',t')\right] , \tag{5}$$

with the effective action

$$S[\Delta^*,\Delta] = -i\hbar\,\mathrm{Tr}[\ln G^{-1}] - \frac{1}{g}\int_C dt \int d\mathbf{x}|\Delta(\mathbf{x},t)|^2 , \tag{6}$$

expressed in terms of the exact one-particle Green's function

$$iG(\mathbf{x},t;\ \mathbf{x}',t') \equiv \langle T[\psi(\mathbf{x},t)\psi^\dagger(\mathbf{x}',t')]\rangle = \begin{bmatrix}\langle T[\psi_\downarrow(\mathbf{x},t)\psi^\dagger_\downarrow(\mathbf{x}',t')]\rangle & \langle T[\psi_\downarrow(\mathbf{x},t)\psi_\uparrow(\mathbf{x}',t')]\rangle\\ \langle T[\psi^\dagger_\uparrow(\mathbf{x},t)\psi^\dagger_\downarrow(\mathbf{x}',t')]\rangle & \langle T[\psi^\dagger_\uparrow(\mathbf{x},t)\psi_\uparrow(\mathbf{x}',t')]\rangle\end{bmatrix} , \tag{7}$$

where the average is calculated using the initial density matrix $\rho(t_0)$ and T represents the time-ordering operator on the

Keldysh contour. Introducing also a δ function on the contour by means of $\int_C dt'\delta(t,t')=1$, this Green's function obeys the equation of motion,

$$\begin{bmatrix} i\hbar\frac{\partial}{\partial t}+\frac{\hbar^2\nabla^2}{2m}+\mu & \Delta(\mathbf{x},t) \\ \Delta^*(\mathbf{x},t) & -\left(-i\hbar\frac{\partial}{\partial t}+\frac{\hbar^2\nabla^2}{2m}+\mu\right) \end{bmatrix} G(\mathbf{x},t;\mathbf{x}',t')=\hbar\delta(\mathbf{x}-\mathbf{x}')\delta(t,t') , \tag{8}$$

defining also its inverse G^{-1}. The latter plays an important role in the following section where we evaluate the effective action of Eq. (6) and derive a time-dependent Ginzburg-Landau equation for the order-parameter $\langle \Delta(\mathbf{x},t) \rangle$ by means of a gradient expansion.

III. TIME-DEPENDENT GINZBURG-LANDAU THEORY

We now turn to the actual calculation of the action $S[\Delta^*,\Delta]$ and the time-dependent Ginzburg-Landau equation, which follows from the principle of least action, i.e., $\delta S[\Delta^*,\Delta]/\delta\Delta^*(\mathbf{x},t)=0$. As mentioned above we take for $\rho(t_0)$ the density matrix of an ideal gas of quasiparticles with an energy distribution $N(E)$ and the dispersion relation $E(\mathbf{k})=[\epsilon(\mathbf{k})^2+|\Delta_0|^2]^{1/2}$, where $\epsilon(\mathbf{k})=(\hbar^2/2m)\mathbf{k}^2-\mu$ and the energy gap Δ_0 is found from the usual BCS gap equation

$$\int \frac{d\mathbf{k}}{(2\pi)^3}\frac{[1-2N(\mathbf{k})]}{2E(\mathbf{k})}=\frac{1}{g} . \tag{9}$$

We also take the limit $t_0 \to -\infty$, since here we are not interested in the fast [of $O(\hbar/k_B T_c)$] transients, which are especially important for a discussion of the nucleation of the superconducting phase.

We proceed as follows. First, we expand the exact action $S[\Delta^*,\Delta]$ around its minimum $S[\Delta_0^*,\Delta_0]$ by writing $\Delta(\mathbf{x},t)=\Delta_0+\Delta'(\mathbf{x},t)$ and using $\Delta'(\mathbf{x},t)/|\Delta_0|$ as a small parameter. Furthermore, we only consider fluctuations $\Delta'(\mathbf{x},t)$ of long wavelength and small frequency. Explicit calculation at zero temperature shows that this expansior converges quite rapidly and we only have to include quadratic fluctuations to obtain an accurate description of the action in this region. Second, we deduce from this the form of an approximate action $S_{\text{neu}}[\Delta^*,\Delta]$ by requiring that mass is a conserved quantity and that the expansion of this action around $S_{\text{neu}}[\Delta_0^*,\Delta_0]=S[\Delta_0^*,\Delta_0]$ reproduces the previous result.

Performing the first step we introduce the "zeroth-order" normal and anomalous Green's functions associated with the initial density matrix $\rho(t_0)$ by means of the Dyson equation $G^{-1}=G_0^{-1}-\Sigma$ and the self-energy matrix

$$\hbar\Sigma(\mathbf{x},t;\ \mathbf{x}',t')=\begin{bmatrix} 0 & -\Delta'(\mathbf{x},t) \\ -\Delta'^*(\mathbf{x},t) & 0 \end{bmatrix}\delta(\mathbf{x}-\mathbf{x}')\delta(t,t') . \tag{10}$$

Substituting this in Eq. (6) gives

$$S[\Delta^*,\Delta]=-i\hbar\,\mathrm{Tr}[\ln G_0^{-1}]+i\hbar\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{Tr}[(G_0\Sigma)^n] -\frac{1}{g}\int_C dt\int d\mathbf{x}|\Delta_0+\Delta'(\mathbf{x},t)|^2 .$$

The terms linear in $\Delta'^*(\mathbf{x},t)$ and $\Delta'(\mathbf{x},t)$ cancel, because Eq. (9) is equivalent to $\Delta_0=-g\langle T[\psi_\downarrow(\mathbf{x},t)\psi_\uparrow(\mathbf{x},t)]\rangle$, where the field operators evolve in time according to the interaction picture based on G_0^{-1}. Hence, we have

$$S[\Delta^*,\Delta]=S[\Delta_0^*,\Delta_0]+i\hbar\sum_{n=2}^{\infty}\frac{1}{n}\mathrm{Tr}[(G_0\Sigma)^n] -\frac{1}{g}\int_C dt\int d\mathbf{x}|\Delta'(\mathbf{x},t)|^2 . \tag{11}$$

Before we can evaluate the time integrations in this expression we must realize that both $\Delta'(\mathbf{x},t)$ and $G_0(\mathbf{x},t;\ \mathbf{x}',t')$ are functions on the Keldysh contour C. Therefore, the latter can be decomposed into the analytical functions $G_0^>$ and $G_0^<$ by

$$G_0(\mathbf{x},t;\ \mathbf{x}'t')=G_0^>(\mathbf{x},t;\ \mathbf{x}'t')\Theta(t,t') +G_0^<(\mathbf{x},t;\ \mathbf{x}'t')\Theta(t',t) , \tag{12}$$

using the Heaviside function on the contour $\Theta(t,t')$.[14,15] Substituting this decomposition into Eq. (11) and defining the retarded and advanced Green's functions as

$$G_0^{(\pm)}(\mathbf{x},t;\ \mathbf{x}'t')\equiv\pm\Theta(\pm(t-t')) \times[G_0^>(\mathbf{x},t;\ \mathbf{x}'t')-G_0^<(\mathbf{x},t;\ \mathbf{x}'t')] , \tag{13}$$

we arrive at a real time formalism that explicitly reflects the principle of causality, because essentially only retarded quantities are present. In addition, it is now possible to apply a Fourier transformation on both space and time and to find the effective action for the long-wavelength and low-frequency fluctuations by means of a Taylor expansion of the various coefficients.

As an example we take the coefficient of the diagonal $|\Delta'(\mathbf{k},k_0)|^2$ term. The discussion of the nondiagonal terms proportional to $\Delta'^*(\mathbf{k},k_0)\Delta'^*(-\mathbf{k},-k_0)$ and $\Delta'(\mathbf{k},k_0)\Delta'(-\mathbf{k}',-k_0)$ is similar and is not repeated in the following. Physically, three processes are involved in the calculation of the quadratic fluctuations. In terms of the usual coherence factors $u(\mathbf{p})$ and $v(\mathbf{p})$ obeying $|u(\mathbf{p})|^2=\frac{1}{2}[1+\epsilon(\mathbf{p})/E(\mathbf{p})]$, $|v(\mathbf{p})|^2=\frac{1}{2}[1-\epsilon(\mathbf{p})/E(\mathbf{p})]$, and $2u(\mathbf{p})v(\mathbf{p})=-\Delta_0/E(\mathbf{p})$, we find for the contribution due to the production of two quasiparticles:

$$-\frac{1}{2}\int\frac{d\mathbf{k}'}{(2\pi)^3}\frac{1}{\hbar k_0^+-E(\mathbf{k}/2+\mathbf{k}')-E(\mathbf{k}'/2-\mathbf{k}')}\left[1-N\left[\frac{\mathbf{k}}{2}+\mathbf{k}'\right]-N\left[\frac{\mathbf{k}}{2}-\mathbf{k}'\right]\right]\left|u\left[\frac{\mathbf{k}}{2}+\mathbf{k}'\right]\right|^2\left|u\left[\frac{\mathbf{k}}{2}-\mathbf{k}'\right]\right|^2,$$

whereas the contribution due to the creation of two quasiholes becomes

$$\frac{1}{2}\int\frac{d\mathbf{k}'}{(2\pi)^3}\frac{1}{\hbar k_0^++E(\mathbf{k}/2+\mathbf{k}')+E(\mathbf{k}/2-\mathbf{k}')}\left[1-N\left[\frac{\mathbf{k}}{2}+\mathbf{k}'\right]-N\left[\frac{\mathbf{k}}{2}-\mathbf{k}'\right]\right]\left|v\left[\frac{\mathbf{k}}{2}+\mathbf{k}'\right]\right|^2\left|v\left[\frac{\mathbf{k}}{2}-\mathbf{k}'\right]\right|^2.$$

Finally, the possibility of Landau damping associated with the breakup of a Cooper pair into a quasiparticle and a quasihole leads to

$$\int\frac{d\mathbf{k}'}{(2\pi)^3}\frac{1}{\hbar k_0^+-E(\mathbf{k}/2+\mathbf{k}')+E(\mathbf{k}/2-\mathbf{k}')}\left[N\left[\frac{\mathbf{k}}{2}+\mathbf{k}'\right]-N\left[\frac{\mathbf{k}}{2}-\mathbf{k}'\right]\right]\left|u\left[\frac{\mathbf{k}}{2}+\mathbf{k}'\right]\right|^2\left|v\left[\frac{\mathbf{k}}{2}-\mathbf{k}'\right]\right|^2.$$

It is now important to realize that the first two terms have a well-defined Taylor expansion around $(\mathbf{k},k_0)=(\mathbf{0},0)$, because the dispersion relation of the quasiparticles and quasiholes has a gap. Unfortunately, this is not true for the last term, which in principle prevents the formulation of the time-dependent Ginzburg-Landau equation as a simple partial differential equation at nonzero temperatures in agreement with the conclusions of Abrahams and Tsuneto.[8] However, for low frequencies, Landau damping can be neglected and the last term is very accurately approximated by its static value, which is analytic in $\mathbf{k}$ and has a well-behaved long-wavelength expansion at all temperatures. In this manner the evaluation of the effective action for the quadratic fluctuations becomes straightforward although somewhat tedious for the gradient terms.

Making use of particle-hole symmetry and introducing the functions $I_n(|\Delta_0|)$ by means of

$$I_n(|\Delta_0|)\equiv\frac{|\Delta_0|^{2n}}{B(\frac{1}{2},n)}\int_{-\infty}^{\infty}\frac{d\epsilon}{E^{2n+1}}[1-2N(E)]\,, \quad (14)$$

where the Beta function $B(\frac{1}{2},n)$ fixes the normalization such that in the zero-temperature limit $I_n\to 1$, we find for the potential terms in the effective Lagrangian density,

$$-\frac{N(0)}{4}\frac{I_1}{|\Delta_0|^2}[(\Delta_0^*)^2(\Delta')^2+2|\Delta_0|^2|\Delta'|^2+(\Delta_0)^2(\Delta'^*)^2]\,,$$

denoting the density of states for one spin projection at the Fermi surface by $N(0)$. Hence, the most simple form of the Lagrangian density $\mathcal{L}_{\rm pot}(\Delta^*,\Delta)$ that reproduces this result if expanded around Δ_0 and also agrees with the Ginzburg-Landau theory near the critical temperature, is

$$\mathcal{L}_{\rm pot}(\Delta^*,\Delta)=\frac{N(0)}{2}I_1|\Delta|^2\left[1-\frac{|\Delta|^2}{2|\Delta_0|^2}\right]. \quad (15)$$

For the time-derivative terms we find the off-diagonal part

$$\frac{N(0)}{24}I_2\frac{\hbar^2}{|\Delta_0|^2}\left[\frac{(\Delta_0)^2}{|\Delta_0|^2}\Delta'^*\frac{\partial^2\Delta'^*}{\partial t^2}+\Delta'\frac{\partial^2\Delta'}{\partial t^2}\frac{(\Delta_0^*)^2}{|\Delta_0|^2}\right]$$

and the diagonal part

$$-\frac{N(0)}{4}\left[I_1-\frac{I_2}{3}\right]\frac{\hbar^2}{|\Delta_0|^2}\left[\Delta'^*\frac{\partial^2\Delta'}{\partial t^2}\right].$$

Clearly, this result cannot be represented by a Lagrangian density $\mathcal{L}_{\rm time}(\Delta^*,\Delta)$ proportional to $|\partial\Delta/\partial t|^2$ as is usually assumed. We need additional terms of the form $|\Delta\partial\Delta/\partial t|^2$ and $(\Delta^*\partial\Delta/\partial t)^2+(\Delta\partial\Delta^*/\partial t)^2$ that are also of second order in the time derivatives but of fourth order in the order parameter $\Delta(\mathbf{x},t)$. Together with the requirement that we should reproduce the correct theory near the critical temperature we obtain

$$\mathcal{L}_{\rm time}(\Delta^*,\Delta)=\frac{N(0)}{4}\left[I_1+I_2\left[1-\frac{|\Delta|^2}{|\Delta_0|^2}\right]\right]\times\frac{\hbar^2}{|\Delta_0|^2}\left|\frac{\partial\Delta}{\partial t}\right|^2-\frac{N(0)}{24}I_2\frac{\hbar^2}{|\Delta_0|^4}\left[\frac{\partial|\Delta|^2}{\partial t}\right]^2. \quad (16)$$

Finally, we have to determine the gradient part $\mathcal{L}_{\rm grad}(\Delta^*,\Delta)$. In terms of the Yosida function $Y(|\Delta_0|)=\int_{-\infty}^{\infty}d\epsilon(-dN/dE)$, its first derivative

$$Z(|\Delta_0|)\equiv-|\Delta_0|^2\frac{dY}{d|\Delta_0|^2}=\frac{|\Delta_0|^2}{2}\int_{-\infty}^{\infty}d\epsilon\left[\frac{d^2N}{dE^2}\right]\frac{1}{E} \quad (17)$$

and the Fermi velocity v_F we find for the off-diagonal part of the effective Lagrangian,

$$-\frac{N(0)}{24}(1-Y-Z)\frac{\hbar^2v_F^2}{3|\Delta_0|^2}\times\left[\frac{(\Delta_0)^2}{|\Delta_0|^2}\Delta'^*\frac{\partial^2\Delta'^*}{\partial\mathbf{x}^2}+\Delta'\frac{\partial^2\Delta'}{\partial\mathbf{x}^2}\frac{(\Delta_0^*)^2}{|\Delta_0|^2}\right],$$

and for the diagonal part,

$$\frac{N(0)}{6}\left[1-Y-\frac{Z}{2}\right]\frac{\hbar^2v_F^2}{3|\Delta_0|^2}\left[\Delta'^*\frac{\partial^2\Delta'}{\partial\mathbf{x}^2}\right].$$

In a similar way as for the time-derivative terms we deduce from this result that

$$\mathcal{L}_{\rm grad}(\Delta^*,\Delta)=-\frac{1}{2}\frac{\rho_s}{|\Delta|^2}\left|\frac{\hbar}{2m}\frac{\partial\Delta}{\partial\mathbf{x}}\right|^2+\frac{N(0)}{24}(1-Y-Z)\frac{\hbar^2v_F^2}{3|\Delta_0|^4}\left[\frac{\partial|\Delta|^2}{\partial\mathbf{x}}\right]^2, \quad (18)$$

having also terms which are second order in the derivatives but fourth order in the order parameter. The superfluid mass density is given by

$$\rho_S(|\Delta|)=2mn\frac{|\Delta|^2}{|\Delta_0|^2}\left[(1-Y)\left[1-\frac{|\Delta|^2}{2|\Delta_0|^2}\right]-\frac{Z}{2}\left[1-\frac{|\Delta|^2}{|\Delta_0|^2}\right]\right], \tag{19}$$

which reduces to the expected expression $\rho_S=mn(1-Y)$ if $|\Delta|=|\Delta_0|$.[16]

Furthermore, putting $|\Delta_0|=|\Delta|$ we exactly have the result obtained by Werthamer.[17] However, in the spirit of the above calculation we should not identify $|\Delta_0|$ with $|\Delta|$, because this implies local equilibrium of the quasiparticles with the condensate and we are back to the treatment of Abrahams and Tsuneto. Notice that in the situation of a superflow, the momentum dependence of the gap found from Eq. (18) is

$$|\Delta_0(\mathbf{k})|=|\Delta_0|\left[1-\frac{Z}{4}\frac{\hbar^2 v_F^2}{3|\Delta_0|^2}\mathbf{k}^2+O(\mathbf{k}^4)\right], \tag{20}$$

which can also be derived directly from the BCS gap equation. In particular, at zero temperature this implies that below the critical current, $|\Delta_0(\mathbf{k})|$ is independent of momentum, which is a consequence of Galilean invariance.

This concludes the derivation of the time-dependent Ginzburg-Landau theory for a neutral BCS-type superfluid. In summary, the complete nonequilibrium long-wavelength $(\hbar v_F k \ll |\Delta_0|)$ and low-frequency $(\hbar k_0 < \hbar v_F k)$ action of the order parameter is

$$\begin{aligned} S_{\text{neu}}[\Delta^*,\Delta] &= \int dt \int d\mathbf{x}[\mathcal{L}_{\text{time}}(\Delta^*,\Delta)+\mathcal{L}_{\text{grad}}(\Delta^*,\Delta) \\ &\quad +\mathcal{L}_{\text{pot}}(\Delta^*,\Delta)] \\ &\equiv \int dt \int d\mathbf{x}\,\mathcal{L}_{\text{neu}}(\Delta^*,\Delta)\,, \end{aligned} \tag{21}$$

where the various contributions are given in Eqs. (15), (16), (18), and (19). Substituting $\Delta(\mathbf{x},t)=\Delta_0\exp[i\theta(\mathbf{x},t)]$ the Lagrangian density for the phase fluctuations becomes

$$\mathcal{L}_{\text{neu}}(\theta)=\frac{N(0)}{4}\hbar^2 I_1\left[\left[\frac{\partial\theta}{\partial t}\right]^2-v_F^2\frac{1-Y}{3I_1}\left[\frac{\partial\theta}{\partial\mathbf{x}}\right]^2\right], \tag{22}$$

showing explicitly that the Goldstone mode has the velocity $v_F\sqrt{(1-Y)/3I_1}$. Hence, at zero temperature, the velocity is equal to $v_F/\sqrt{3}$ and in agreement with the calculations of Bogoliubov[18] and Anderson.[19] In the next section we discuss how these results are modified in the case of a superconductor where the effect of screening is of great importance for the dynamics of the system.[20]

IV. SCREENING PROPERTIES

We introduce electromagnetism in the functional approach by performing the minimal coupling substitution $i\hbar\partial/\partial t \to i\hbar\partial/\partial t + e\phi$ and $-i\hbar\nabla \to -i\hbar\nabla + e\mathbf{A}$ in $S[\psi^*,\psi]$ and adding the action for the free electromagnetic field, which reduces in the Coulomb gauge to

$$S_{\text{em}}[\phi,\mathbf{A}]=\int_C dt\int d\mathbf{x}\frac{\epsilon_0}{2}\left[-\phi\nabla^2\phi-\mathbf{A}\cdot\frac{\partial^2\mathbf{A}}{\partial t^2}+c^2\mathbf{A}\cdot\nabla^2\mathbf{A}\right]. \tag{23}$$

Applying the same Hubbard-Stratonovich transformation as before and subsequently integrating over the fermionic fields $\psi_\alpha(\mathbf{x},t)$, we find that the order parameter $\Delta(\mathbf{x},t)$ is gauge-invariantly coupled to the electromagnetic field by means of the covariant derivatives $i\hbar\partial/\partial t+2e\phi$ and $-i\hbar\nabla+2e\mathbf{A}$, as is expected of a field with charge $-2e$, and in addition that the photon propagator is dressed due to the polarizability of the electron gas. Hence, the effective action for the charged case acquires the form

$$\begin{aligned} S_{\text{ch}}[\Delta^*,\Delta,\phi,\mathbf{A}] = \int dt \int d\mathbf{x}[&\mathcal{L}_{\text{neu}}(\Delta^*,\Delta)+\mathcal{L}_{\text{em}}(\phi,\mathbf{A}) \\ &+\mathcal{L}_{\text{pol}}(\phi,\mathbf{A}) \\ &+\mathcal{L}_{\text{int}}(\Delta^*,\Delta,\phi,\mathbf{A})]\,. \end{aligned} \tag{24}$$

The interaction Lagrangian density is found from the minimal coupling prescription and is equal to

$$\mathcal{L}_{\text{int}}(\Delta^*,\Delta,\phi,\mathbf{A})=-\delta\rho\phi+\mathbf{J}\cdot\mathbf{A}\,, \tag{25}$$

where the charge-density fluctuation $\delta\rho$ and current density $\mathbf{J}$ are given by

$$\begin{aligned} \delta\rho\equiv\rho+en &= -eN(0)\left[I_1+I_2\left[1-\frac{|\Delta|^2}{|\Delta_0|^2}\right]\right]\frac{1}{|\Delta_0|^2} \\ &\quad\times\left[\frac{i\hbar}{2}\left[\Delta^*\frac{\partial\Delta}{\partial t}-\Delta\frac{\partial\Delta^*}{\partial t}\right]+2e|\Delta|^2\phi\right] \end{aligned} \tag{26}$$

and

$$\mathbf{J}=-\frac{e\rho_S}{2m^2|\Delta|^2}\left[\frac{-i\hbar}{2}\left[\Delta^*\frac{\partial\Delta}{\partial\mathbf{x}}-\Delta\frac{\partial\Delta^*}{\partial\mathbf{x}}\right]+2e|\Delta|^2\mathbf{A}\right], \tag{27}$$

respectively. Because of gauge invariance, the electric charge is locally conserved and the stationary condition $\delta S_{\text{ch}}/\delta\theta(\mathbf{x},t)=0$ can be written as the continuity equation

$$\frac{\partial\,\delta\rho(\mathbf{x},t)}{\delta t}+\nabla\cdot\mathbf{J}(\mathbf{x},t)=0\,, \tag{28}$$

where the density fluctuations are constrained by the Euler-Lagrange equation $\delta S_{\text{ch}}/\delta|\Delta(\mathbf{x},t)|=0$, leading to $\delta\rho(\mathbf{x},t)=0$ in lowest order. However, small deviations from charge neutrality arise in higher orders, which can easily be incorporated if necessary for a particular application of the Ginzburg-Landau equation proposed in this paper.

Finally, to obtain the equations for the electromagnetic potentials also, we have to consider $S_{\rm pol}[\phi, \mathbf{A}]$. It should be noted that this contribution was usually neglected in previous attempts to derive a time-dependent Ginzburg-Landau equation, although it turns out to be essential for a correct description of the screening properties of a superconductor. Introducing the four-vector $A=(\phi/c, \mathbf{A})$ we have

$$S_{\rm pol}[\phi, \mathbf{A}]=\int dt \int d\mathbf{x} \int dt' \int d\mathbf{x}' \frac{1}{2} A^{\mu}(\mathbf{x},t) \times \Pi^{(+)}_{\mu\nu}(\mathbf{x},t;\, \mathbf{x}',t') \times A^{\nu}(\mathbf{x}',t') , \tag{29}$$

with $\Pi^{(+)}_{\mu\nu}$ the retarded polarization tensor of the free-electron gas. Again, due to gauge invariance, $\Pi^{(+)}_{\mu\nu}$ is a transverse tensor and can be written in momentum space as

$$\Pi^{(+)}_{\mu\nu}(k)=\left[g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right]\pi(k) , \tag{30}$$

using $k=(k_0/c,\mathbf{k})$ and the Lorentz metric $g_{\mu\nu}$ with signature $(-1,1,1,1)$. Thus, the evaluation of $S_{\rm pol}[\phi, \mathbf{A}]$ is particularly convenient in the Coulomb gauge. In this gauge and neglecting the small contribution from Landau diamagnetism, we find that

$$S_{\rm pol}[\phi, \mathbf{A}]=\int dt \int d\mathbf{x} \int dt' \int d\mathbf{x}' \frac{1}{2c^2}\phi(\mathbf{x},t) \times \Pi^{(+)}_{00}(\mathbf{x},t;\, \mathbf{x}',t') \times \phi(\mathbf{x}',t') , \tag{31}$$

and $-\Pi^{(+)}_{00}/c^2$ the density-density response function $\chi_{\rho\rho}$.

In the long-wavelength limit we have the well-known result[9]

$$\Pi^{(+)}_{00}(\mathbf{k},k_0) \underset{\frac{kv_F}{k_0}\to 0}{\sim} -\frac{ne^2}{m}\frac{\mathbf{k}^2c^2}{k_0^2} , \tag{32}$$

which formally leads to

$$S_{\rm pol}[\phi, \mathbf{A}]=-\int dt \int d\mathbf{x} \frac{1}{2}\phi(\mathbf{x},t)\frac{ne^2}{m}\frac{\partial^2/\partial\mathbf{x}^2}{\partial^2/\partial t^2}\phi(\mathbf{x},t) . \tag{33}$$

Considering again the Bogoliubov-Anderson mode we find that in the limit of infinite wavelength, a solution to the equations of motion is $\delta\rho=0, \mathbf{J}=\mathbf{A}=0$, $\phi(t)=\phi_0\exp(-i\omega_p t-i\pi/2)$, $\theta(t)=2e\phi_0/\hbar\omega_p\exp(-i\omega_p t)$, and $\omega_p=(ne^2/m\epsilon_0)^{1/2}$ the plasma frequency. Thus, the dispersion of the collective mode is pushed up to the plasma frequency as is expected of a density fluctuation in a system with long-range Coulomb interactions.[19,20]

However, for the purposes of this paper and in agreement with the neglect of Landau damping in the discussion of the neutral superfluid, we are more interested in the region $\hbar k_0<\hbar k v_F$ where it is appropriate to use the static limit of $\Pi^{(+)}_{00}$. In this limit we obtain

$$\Pi^{(+)}_{00}(\mathbf{k},k_0) \underset{\frac{k_0}{kv_F}\to 0}{\sim} \frac{\epsilon_0 c^2}{\lambda^2_{\rm TF}} , \tag{34}$$

with $\lambda_{\rm TF}$ the Thomas-Fermi screening length $(2\epsilon_0\epsilon_F/3ne^2)^{1/2}$. Substituting this into Eq. (31) finally gives

$$S_{\rm pol}[\phi, \mathbf{A}]=\int dt \int d\mathbf{x}\frac{\epsilon_0}{2}\phi(\mathbf{x},t)\frac{1}{\lambda^2_{\rm TF}}\phi(\mathbf{x},t) . \tag{35}$$

The Euler-Lagrange equations for the electromagnetic field are then

$$\left[\nabla^2-\frac{1}{\lambda^2_{\rm TF}}\right]\phi(\mathbf{x},t)=-\frac{\delta\rho(\mathbf{x},t)}{\epsilon_0} , \tag{36a}$$

$$\left[\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\mathbf{A}(\mathbf{x},t)=-\mu_0\mathbf{J}_T(\mathbf{x},t) , \tag{36b}$$

where $\mathbf{J}_T(\mathbf{x},t)$ denotes the transverse part of the current density. Together with Eqs. (26) and (27), we see that the screening of the electric field is due to both the normal as well as the superconducting part of the system, whereas the screening of the magnetic field (the Meissner effect) is only due to the superconducting part. Therefore, the typical length scale associated with the former is $\lambda_{\rm TF}$, while the length scale associated with the latter is equal to the London penetration depth $\lambda_L=(m^2/\mu_0\rho_S e^2)^{1/2}$ for a spatially slowly varying order parameter. Clearly, this intuitively reasonable result is only obtained if the polarizability of the electron gas is taken into account and $S_{\rm pol}[\phi, \mathbf{A}]$ is included in the complete action. Without this contribution, the electric potential is only coupled to the superfluid density fluctuations and only the superconducting part of the system can lead to screening of the electric field.

V. CONCLUSIONS

In summary, we have shown that it is possible to derive a time-dependent Ginzburg-Landau equation for the order parameter of a superconductor in a highly nonequilibrium situation and in the limit that $\hbar k_0<\hbar v_F k \ll |\Delta_0|$. We have also shown how the screening properties of the superconductor can be accounted for. In particular, the different screening lengths for the longitudinal and transverse parts of the electromagnetic field are included in a natural way.

The most important restriction on the applicability of the above theory is caused by the neglect of Landau damping. However, it seems likely that the effects of Landau damping cannot be simply incorporated in the framework of a Ginzburg-Landau theory and requires a more advanced approach, which takes the quasiparticles into account explicitly.[21] Nevertheless, it is of interest to investigate the predictions of the theory presented in this paper, because for a certain application the neglect of Landau damping might turn out to be justified and can in any case be checked in a self-consistent fashion. It will, in particular, be interesting to see if in this way a better agreement with experimental data on the resistance of quasi-one-dimensional superconductors can be obtained.

ACKNOWLEDGMENTS

I am very grateful to Tony Leggett for introducing me to this subject and for numerous illuminating discussions. I also thank Ping Ao, Michel Bijlsma, Ji-Min Duan, and Kieran Mullen for their comments. This work was performed during a visit to the University of Illinois at Urbana-Champaign and was partly supported by the National Science Foundation through Grant No. DMR-8822688.

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