Theory of the half-filled Landau level

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A two-dimensional electron system in an external magnetic field, with Landau-level filling factor $\nu=\frac{1}{2}$, can be transformed to a mathematically equivalent system of fermions interacting with a Chern-Simons gauge field such that the average effective magnetic field acting on the fermions is zero. If one ignores fluctuations in the gauge field, this implies that for a system with no impurity scattering, there should be a *well-defined Fermi surface* for the fermions. When gauge fluctuations are taken into account, we find that there can be infrared divergent corrections to the quasiparticle propagator, which we interpret as a divergence in the effective mass m^*, whose form depends on the nature of the assumed electron-electron interaction $v(\mathbf{r})$. For long-range interactions that fall off slower than $1/r$ at large separation r, we find no infrared divergences; for short-range repulsive interactions, we find power-law divergences; while for Coulomb interactions, we find logarithmic corrections to m^*. Nevertheless, we argue that many features of the Fermi surface are likely to exist in all these cases. In the presence of a weak impurity-scattering potential, we predict a finite resistivity ρ_{xx} at low temperatures, whose value we can estimate. We compute an anomaly in surface acoustic wave propagation that agrees qualitatively with recent experiments. We also make predictions for the size of the energy gap in the fractional quantized Hall state at $\nu=p/(2p+1)$, where p is an integer. Finally, we discuss the implications of our picture for the electronic specific heat and various other physical properties at $\nu=\frac{1}{2}$, we discuss the generalization to other filling fractions with even denominators, and we discuss the overall phase diagram that results from combining our picture with previous theories that apply to the regime where impurity scattering is dominant.

I. INTRODUCTION

When the fractional quantized Hall effect was discovered in 1982 by Tsui, Stormer, and Gossard,[1] we were given only a first glimpse of the remarkably complex behavior of two-dimensional electron systems with a partially filled Landau level. In the intervening decade, as samples with higher and higher mobility have been prepared, and measurements have been extended to lower temperature and stronger magnetic fields, there have been observed an increasing number of Hall plateaus corresponding to various filling fractions ν with odd denominators.[2,3] A few fractions with even denominators have also been seen in higher Landau levels,[4,5] but not, so far, in a single-layer system, in the lowest Landau level.

From a theoretical point of view, the occurrence of Hall plateaus at filling fractions with odd denominators can be understood, to a great extent, through the original theoretical analysis of Laughlin,[6] and various subsequent extensions, such as the hierarchical construction of quantized Hall states.[2,7–13] The essential correctness of these explanations is well established in the case of the strongest fractional Hall plateaus, such as those at $\nu=\frac{1}{3}$ and $\frac{2}{3}$, where the energy gap is large, and the Laughlin trial wave function is the exact ground state for a system with short-range repulsive interactions.[8] Moreover, numerical solutions of the ground state for small finite systems with Coulomb interactions show a very high degree of overlap with the Laughlin trial functions. As one considers filling fractions with larger and larger denominators, however, the evidence for the correctness of our general theoretical picture becomes more and more tenuous, as our ability to calculate quantitative properties of the quantized Hall state becomes progressively weaker, the energy differences between competing states of possible interest become progressively smaller, and the ability to obtain meaningful results from finite-size system calculations becomes progressively more questionable. Nevertheless, it is at least plausible to believe that the most essential features of the observed plateaus with odd denominators such as the quasiparticle charge and statistics, and the quantum numbers of the ground state and low-lying excited states, are correctly described by the conventional hierarchical constructions.[2,13]

By contrast, the behavior of a two-dimensional electron system in the vicinity of a filling fraction with even

denominator is very poorly understood. As an example, from the point of view of the hierarchical construction, the state at filling fraction $\nu=\frac{1}{2}$ is only described as the limit of an infinite sequence of odd fractions with larger and larger denominators, such as the sequence $\nu=\frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \ldots$. Each state of this sequence is a daughter state of the state which proceeds it, but as the density of quasiparticles added at each stage is relatively large on the scale set by the effective magnetic length or the size of the quasiparticle, and the form of the quasiparticle interaction is not known on these length scales, the extrapolation to $\nu=\frac{1}{2}$ cannot be made with any degree of confidence. Moreover, it is not even clear that the state which is constructed by taking the limit of the sequence $\nu=\frac{2}{3}, \frac{3}{5}, \frac{4}{7}\ldots$, which approaches $\nu=\frac{1}{2}$ from above, is necessarily the same as the state which is obtained by the approach from below.[9]

An alternative approach to understanding the behavior at $\nu=\frac{1}{2}$ is suggested by the ideas of transmutability of statistics for particles in two-dimensional systems.[9,14-31] In particular, it is possible to introduce a Chern-Simons gauge field that interacts with the electrons, which is equivalent to attaching to each electron a "magnetic flux tube." The quantum-mechanical model thus obtained will have identical properties to those of the original electron system without the fictitious gauge field, provided that the statistics of the particles in the new model are appropriately modified. Specifically, let $\tilde{\phi}$ denote the strength of the flux tube, in units of the flux quantum $2\pi\hbar$. (The fictitious charge of each particle which interacts with the fictitious gauge field has been chosen to have unit strength.) Then we must impose the condition that if two particles are interchanged in a clockwise manner around a curve which encloses no other particles, then the wave function must change by a phase factor $e^{i\theta}$, where[14]

$$\theta=\pi(\tilde{\phi}+1) \pmod{2\pi} . \tag{1.1}$$

If we choose $\tilde{\phi}$ to be an even integer, then the wave function still obeys the conditions for fermions, and indeed the quantum-mechanical problem is really unchanged. However, there are approximations which seem natural in the transformed representation but which might not be so evident in the original formulation. For example, if we make the choice $\tilde{\phi}=2$, and if the external magnetic field B is chosen such that there are precisely two flux quanta per electron (i.e., $\nu=\frac{1}{2}$), then the average of the fictitious magnetic field arising from the flux tubes precisely cancels the external magnetic field. Thus if we ignore fluctuations in the gauge field, we are led to a greatly simplified model, *a system of spinless fermions in zero magnetic field.*

It is the purpose of the present paper to follow this line of argument to its logical conclusion. We assume that the ground state at $\nu=\frac{1}{2}$ (and at certain other fractions with even denominator) in fact contains a Fermi surface for some type of quasiparticle, which can be obtained adiabatically from the mean-field state of transformed fermions with $\tilde{\phi}=2n$ flux quanta attached. We obtain various predictions based on this and we find that a number of these predictions are in agreement with known experimental observations. In particular, we are able to provide an explanation for the anomalies in surface acoustic wave propagation which have been recently reported near $\nu=\frac{1}{2}$.[32,33]

We also probe the self-consistency of our underlying assumption by trying to calculate some of the effects of the fluctuations of the fictitious magnetic field about its average value. Although we encounter some divergences in the course of these calculations, which depend in a crucial manner on the nature of the interaction between the electrons, we conclude that, at least for the physically important case of Coulomb interactions, the essential features of a Fermi-liquid theory are likely to be correct. The necessary modifications of Fermi-liquid theory, which are particularly large in the case of short-range interactions, will be also explored in Sec. VI below.

The idea of attaching an even number $2n$ of quanta of fictitious flux to each electron, at $\nu=(2n)^{-1}$, was used in the work of Moore and Read[17] to construct an order parameter for the spin-singlet quantized Hall state of Haldane and Rezayi,[34] analogous to the order parameters previously used for quantized Hall states with odd denominators,[18-20] and to construct a new class of spin-polarized states involving a Pfaffian. In both the Haldane-Rezayi state and the Pfaffian state, the "mean-field theory," which describes the fermions in the zero *average* magnetic field, contains pairs, in the manner of the BCS theory of superconductivity. In the Haldane-Rezayi state, where both spin states are occupied, the pairs are singlets, so that the pairing function has even parity (a d wave in the simplest case). In the Pfaffian state for spinless or spin-polarized electrons, the pairing has odd parity (a p wave in the simplest case). The Pfaffian state was further discussed in a similar manner together with other pairing states of odd parity by Greiter, Wen, and Wilczek.[21,22] The latter authors also emphasized the possibility of attaching the flux to the electrons in an adiabatic manner, passing continuously from a p-wave BCS state in zero field, to a fractional quantized Hall state at $\nu=\frac{1}{2}$, with an energy gap that hopefully remains open at all intermediate stages, for a suitably chosen potential. In all the above works, the goal was to construct a state which is an incompressible fluid, and this has been demonstrated for suitable Hamiltonians. Thus these states may describe the even-denominator quantized Hall plateaus which have been actually observed under some special conditions—e.g., in the second Landau level, at $\nu=\frac{5}{2}$, in a single-layer system,[4,5] and at $\nu=\frac{1}{2}$, in certain double-layer or thick-layer systems.[35,36] In the present paper, however, we shall be concerned with the more usual behavior at $\nu=\frac{1}{2}$, where no quantized Hall plateau is observed.[2,4,32,37]

The idea of attaching an even number of flux quanta to each electron, in order to convert a state at one value of the filling factor ν into a state with an average magnetic field corresponding to another simpler filling factor, is also central to the work of Jain,[23] who used it to construct trial wave functions for fractions of the form

$$\nu=\frac{p}{2np+1} , \tag{1.2}$$

where p is a positive or negative integer. In this case, the transformed fermions are in an integer quantized Hall state, with p Landau levels occupied. The sequence of states obtained from (1.2), with $n=1$, is in fact the principal sequence of odd-denominator quantized Hall states,[38] tending toward $\nu=\frac{1}{2}$ from above or below as $p\to+\infty$ or $p\to-\infty$, which includes the most prominent fractional Hall plateaus that are observed experimentally in the lowest Landau level.

The existence of prominent Hall plateaus with associated vanishing of the resistivity ρ_{xx} at the sequence of fractions $\nu=p/(2p+1)$, and the success of Jain's trial wave functions, were in fact among the principal motivations for our attempt to develop a theory in which there is a Fermi surface at $\nu=\frac{1}{2}$. If the density of electrons n_e is held fixed, then the magnetic field B corresponding to $\nu=p/(2p+1)$ satisfies

$$\Delta B\equiv(B-B_{1/2})=\left[\frac{2\pi\hbar c}{e}\right]\frac{n_e}{p}\ , \tag{1.3}$$

where $B_{1/2}=4\pi\hbar c n_e/e$ is the magnetic field corresponding to $\nu=\frac{1}{2}$ at this density. This equation has the same form as the relation which determines the positions of the integer quantized Hall plateaus in the vicinity of $B=0$, except that the deviation ΔB appears here on the left-hand side instead of the total magnetic field B. But the integer Hall plateaus are properly regarded as an extreme form of the de Haas–Shubnikov effect, which is a direct consequence of the Fermi surface at $B=0$.

The underlying method of the present paper, which is the idea of treating a many-body system of fermions with attached flux tubes by starting from a mean-field theory in which the fermions see only the average value of the fictitious magnetic field, was used with dramatic effect by Laughlin[31] in his argument for the occurrence of superconductivity in a collection of semions—particles with half-Fermi statistics, which can be obtained from fermions or bosons by attaching a flux tube with $\tilde{\phi}=\pm\frac{1}{2}$. In subsequent papers, Laughlin and various co-workers, as well as other authors, have developed a systematic perturbation theory for taking into account the fluctuations of the fictitious magnetic field about its average value, and have explored the consequence of a variety of approximations, including the Hartree-Fock approximation and the random-phase approximation (RPA), for anyon models of superconductivity.[39–43] A related study of the fractional quantized Hall states at $\nu=p/(2p+1)$, which employs a transformation to fermions with two flux quanta attached, and which includes a discussion of the RPA in this case, was presented recently by Lopez and Fradkin.[25]

Overall, there is substantial evidence supporting the validity of the underlying method for the case of anyon superconductivity and for fractional quantized Hall states with a large energy gap.[14,25,40,44] In the present case, however, we wish to use the method in a gapless system. Thus the self-consistency of the approach rests on much more delicate arguments, dependent on the vanishing of phase space for scattering events close to the Fermi surface, and requiring that the corresponding matrix elements not diverge too rapidly as the Fermi surface is approached. Hence the validity of the method cannot be taken for granted, and it will be of great interest to see whether further theoretical and experimental work supports the results we have obtained so far.

The situation considered in the present paper is the ideal case of a very high mobility sample, where the random potential due to impurities is either neglected or treated as a weak perturbation. This should be contrasted with recent work by several other authors who have considered the case of a Landau level near a filling fraction with even denominator under circumstances where the disorder is relatively large.[27,28,45] In that case, it appears that there is no Fermi surface, and no metallic state over a range of densities, but rather a direct impurity-driven transition between one quantized Hall state and another. Authors who studied the dirty case have also made use of a Chern-Simons gauge field, but in contrast to the approach we use here they chose to transform the electrons into *bosons,* with an *odd* number of flux quanta attached to each boson.[27,28,46]

The outline of our paper is as follows. In Sec. II, we define the mathematical transformation we use, and obtain the key results of mean-field theory and the RPA response functions for the system at $\nu=\frac{1}{2}$, in the absence of impurity scattering. The mean-field theory results are extended to the fractional quantized Hall states at $\nu=p/(2p+1)$ in Sec. III. In Sec. IV, we discuss the value of the effective mass m^* which one obtains in the case of a Coulomb interaction between electrons, if one fits the mean-field formula for the energy gaps at $\nu=p/(2p+1)$ to numerical results for the energy gap in small systems of electrons on a sphere. We also discuss some implications for the interaction parameters of a Fermi-liquid theory. Effects of impurities are included, within the RPA theory, in Sec. V, where we also estimate the electrical resistivity ρ_{xx} using a simple model for the impurities, at $\nu=\frac{1}{2}$.

The effects of gauge fluctuations, not taken into account in the mean-field theory or the simple RPA response functions, are examined in Sec. VI. We discuss the contribution of collective-mode fluctuations to the specific heat at low temperatures, and we discuss the singular corrections to the one-fermion self-energy that arise from scattering by dynamical gauge fluctuations, in a clean sample, for various forms of electron-electron interaction. In Sec. VI D, we speculate on the modifications to Fermi-liquid theory which are likely to be implied by these singular self-energy corrections. In VI F, we discuss implications for the size of the energy gap at $\nu=p/(2p+1)$, while in VI G we examine the effects of fluctuations on the electrical resistance at $\nu=\frac{1}{2}$, in the dirty case where impurity scattering is moderately large.

Various experimental consequences of our theory are discussed in Sec. VII, including the predictions of an anomaly in the propagation of surface acoustic waves near $\nu=\frac{1}{2}$, referred to above. In Sec. VIII, we extend our analysis at $\nu=\frac{1}{2}$ to various other filling fractions with even denominator, and in Sec. IX we discuss the overall

phase diagram, at at low temperatures, as a function of the magnetic field B and the strength of the impurity scattering potential, which one obtains by combining our analysis with the earlier work on impurity-driven phase transitions between different quantized Hall states. A brief summary of our conclusions is given in Sec. X.

Two appendixes contain a derivation of the RPA using the Lagrangian formalism, and a discussion of the wave-vector-dependent resistivity tensor in the presence of impurities in the vicinity of $\nu=\frac{1}{2}$.

II. MEAN-FIELD THEORY AND THE RPA AT $\nu=\frac{1}{2}$

We consider a two-dimensional system of spinless electrons, governed by a Hamiltonian of the form

$$H=K+V \ , \tag{2.1}$$

where K is the kinetic-energy operator,

$$K\equiv\frac{1}{2m_b}\int d^2r\,\psi_e^+(\mathbf{r})[-i\nabla+e\,\mathbf{A}(\mathbf{r})]^2\psi_e(\mathbf{r}) \ , \tag{2.2}$$

and V is a potential-energy operator which depends only on the positions of the electrons. The operator $\psi_e^+(\mathbf{r})$ is the creation operator for an electron at point $\mathbf{r}$, the vector potential $\mathbf{A}(\mathbf{r})$ is due to a uniform external magnetic field B which points in the z direction normal to the plane, and m_b is the band mass of the electrons. We employ units where $\hbar=c=1$, and the electron charge is $-e$.

We next introduce a "quasiparticle" creation operator $\psi^+(\mathbf{r})$ which is related to ψ_e^+ by

$$\psi^+(\mathbf{r})=\psi_e^+(\mathbf{r})\exp\left[-i\tilde{\phi}\int d^2r'\arg(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r}')\right] \ , \tag{2.3}$$

where $\arg(\mathbf{r}-\mathbf{r}')$ is the angle the vector $(\mathbf{r}-\mathbf{r}')$ forms with the x axis, and

$$\rho(\mathbf{r})\equiv\psi_e^+(\mathbf{r})\psi_e(\mathbf{r})=\psi^+(\mathbf{r})\psi(\mathbf{r}) \tag{2.4}$$

is the density of particles at point $\mathbf{r}$. The operators $\psi^+(r)$ and $\psi(r)$ will obey the usual fermion commutation relations provided that $\tilde{\phi}$ is an even integer. Here we shall primarily consider the case $\tilde{\phi}=2$; however, we shall keep track of $\tilde{\phi}$ in our equations for later generality. In terms of the transformed operators $\psi^+(\mathbf{r})$, we may write the kinetic operator in the following form:[16]

$$K=\frac{1}{2m_b}\int d^2r\,\psi^+(\mathbf{r})[-i\nabla+e\,\mathbf{A}(\mathbf{r})-\mathbf{a}(\mathbf{r})]^2\psi(\mathbf{r}) \ , \tag{2.5}$$

where

$$\mathbf{a}(\mathbf{r})\equiv\tilde{\phi}\int d^2r'\mathbf{g}(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r}') \ , \tag{2.6}$$

$$\mathbf{g}(\mathbf{r})\equiv(\hat{z}\times\mathbf{r})/r^2 \ . \tag{2.7}$$

For the potential-energy operator, we assume a two-body interaction of the form

$$V=\tfrac{1}{2}\int d^2r\,d^2r'v(\mathbf{r}-\mathbf{r}'):\rho(\mathbf{r})\rho(\mathbf{r}'): \ , \tag{2.8}$$

where the colons represent normal ordering with respect to creation and annihilation operators. Our primary interest is in the case of Coulomb interaction

$$v(\mathbf{r}-\mathbf{r}')=\frac{e^2}{\varepsilon|\mathbf{r}-\mathbf{r}'|} \ , \tag{2.9}$$

where ε is the background dielectric constant. However, we shall also consider finite-range repulsive interactions, and potentials which fall off slower than $|\mathbf{r}-\mathbf{r}'|^{-1}$. In the case of the Coulomb interaction, or other slow power laws, we must also assume an interaction with a uniform neutralizing background, which cancels out the infinite $k=0$ Fourier component of the two-body potential v.

We shall study the Hamiltonian H using a perturbation approach based on its formulation in terms of the quasiparticle operators ψ and ψ^+. Because the relation between $\psi_e^+(r)$ and $\psi^+(r)$ is highly nonlocal, quantities such as the one-electron Green function $G_e(r,t)=\langle iT[\psi_e^+(r,t)\psi_e(0,0)]\rangle$ cannot be readily obtained with this approach. However, quantities such as the response function for the density, which has a local expression in terms of ψ and ψ^+, are natural objects for study.

As a starting point, we define a mean-field Hamiltonian[31]

$$H_0=\frac{1}{2m^*}\int\psi^+[-i\nabla+e\Delta\,\mathbf{A}(\mathbf{r})]^2\psi d^2r \ , \tag{2.10}$$

where m^* is an effective mass and $\Delta\,\mathbf{A}$ is a mean-field vector potential which satisfies

$$\nabla\times(\Delta\,\mathbf{A})=\Delta B\equiv B-2\pi\tilde{\phi}n_e/e \ , \tag{2.11}$$

where n_e is the mean value of the electron density $\rho(\mathbf{r})$. Within the Hartree approximation or the RPA, the effective mass m^* is equal to the band mass m_b; we introduce a different symbol, however, for later generality. In this section, we shall restrict ourselves to the case of a half-filled Landau level ($\nu\equiv2\pi n_e/Be=\frac{1}{2}$) so that ΔB is precisely zero, and H_0 is just the Hamiltonian of a set of fermions in zero magnetic field. The ground state of H_0 is then a filled Fermi sea, with Fermi wave vector

$$k_F=(4\pi n_e)^{1/2}=1/l_0 \ , \tag{2.12}$$

where l_0 is the magnetic length, defined by

$$l_0\equiv(eB)^{-1/2} \ . \tag{2.13}$$

The linear-response function for the density and current can be readily calculated in the RPA, or time-dependent Hartree approximation. We write the response function in the form

$$j_\mu=eK_{\mu\nu}(\mathbf{q},\omega)A_\nu^{\text{ext}} \ , \tag{2.14}$$

where μ and ν take on the values $(0,x,y)$; A_ν^{ext} is an external perturbing electromagnetic scalar or vector potential at wave vector $\mathbf{q}$ and frequency ω, and j_μ is the induced change in the particle density and current. We shall work in the Coulomb gauge, so that the longitudinal part of $\mathbf{A}$ is equal to zero. Moreover, the longitudinal part of $\mathbf{j}$ is simply equal to $(\omega/q)j_0$. Therefore, it is possible to treat $K_{\mu\nu}$ as a 2×2 matrix, in which the indices taken on the values 0 and 1, denoting, respectively, the time component and transverse space component of a vector. (Our sign convention is such that, for $\mathbf{q}\|\hat{x}$, the

direction $\mu=1$ refers to the positive y direction.)

The RPA equations for $K_{\mu\nu}$ may be written as

$$K=K^0[1+UK^0]^{-1}$$
$$=K^0-K^0[K^0+U^{-1}]^{-1}K^0 , \tag{2.15}$$

where

$$U\equiv\begin{bmatrix} v(\mathbf{q}) & \frac{2\pi i\tilde{\phi}}{q} \\ \frac{-2\pi i\tilde{\phi}}{q} & 0 \end{bmatrix} \tag{2.16}$$

and $K^0_{\mu\nu}$ is the response function of the noninteracting Fermi system, governed by the Hamiltonian H_0. The off-diagonal matrix elements of the matrix U arise from the interaction with the Chern-Simons gauge field while $v(\mathbf{q})$ is the Fourier transform of the two-body interaction $v(\mathbf{r}-\mathbf{r}')$. For the Coulomb potential (2.9), this is given by

$$v(q)=\frac{2\pi e^2}{\varepsilon q} . \tag{2.17}$$

The unperturbed response function $K^0_{\mu\nu}$ has only diagonal elements, which may be written as

$$K^0_{00}=\int\frac{f(\omega_{k+q})-f(\omega_k)}{\omega-\omega_{\mathbf{k}+\mathbf{q}}+\omega_{\mathbf{k}}+i\eta}\frac{d^2k}{(2\pi)^2} , \tag{2.18}$$

$$K^0_{11}=\frac{-n_e}{m^*}+\int\left[\frac{k_y}{m^*}\right]^2\frac{f(\omega_{k+q})-f(\omega_k)}{\omega-\omega_{\mathbf{k}+\mathbf{q}}+\omega_{\mathbf{k}}+i\eta}\frac{d^2k}{(2\pi)^2} . \tag{2.19}$$

(See Fig. 1.)

The RPA equations may be derived using the Hamiltonian formalism by following in a straightforward manner the methods developed for the anyon superconductor by Hanna, Fetter, and Laughlin and others.[39,40,42] An alternative derivation, using the Lagrangian approach,[41] is reviewed in Appendix A below.

We first consider the static response function $K_{\mu\nu}(q,0)$ for wave vectors $q\ll k_F$. In the limit of small q, we find

$$K^0_{00}=\frac{m^*}{2\pi}+O(q^2) , \tag{2.20}$$

$$K^0_{11}=-\frac{q^2}{12\pi m^*}+O(q^4) . \tag{2.21}$$

Carrying out the matrix inversions and multiplications specified in Eq. (2.15), we find, for the RPA density-response function (compressibility),

$$K_{00}(q,0)\approx\frac{m^*}{m^*v(q)+2\pi(1+\tilde{\phi}^2/6)} . \tag{2.22}$$

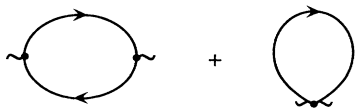

FIG. 1. Diagrams which contribute to the bare density-current response function $K^0_{\mu\nu}$. The second diagram gives the diamagnetic contribution $-n_e/m^*$ to K^0_{11}.

For an interaction potential, such as the Coulomb potential, which diverges for $q\to 0$, the static density response vanishes as $1/v(q)$ for $q\to 0$, just as for the Fermi system in zero magnetic field. On the other hand, if $v(q)$ is finite for $q\to 0$, the RPA compressibility is also finite and differs from the zero-field RPA compressibility only through the term $2\pi\tilde{\phi}^2/6$ in the denominator of Eq. (2.22).

Next we consider the behavior at finite frequency, for $q\ll k_F$. In the limit where $\omega\gg qk_F/m^*$, we have

$$K^0_{00}\approx\frac{-n_e}{m^*}\frac{q^2}{\omega^2} , \tag{2.23}$$

$$K^0_{11}\approx\frac{-n_e}{m^*} , \tag{2.24}$$

so that

$$(K^0+U^{-1})\approx\begin{bmatrix} \frac{-n_eq^2}{m^*\omega^2} & \frac{-iq}{2\pi\tilde{\phi}} \\ \frac{iq}{2\pi\tilde{\phi}} & \frac{-n_e}{m^*} \end{bmatrix} . \tag{2.25}$$

We see that $\det(K^0+U^{-1})$ vanishes, and we obtain a pole in $K_{\mu\nu}$, when

$$\pm\omega=\frac{2\pi n_e\tilde{\phi}}{m^*} . \tag{2.26}$$

This frequency is just the cyclotron frequency in the external magnetic field B for a system of particles of effective mass m^*.

We also find structure in $K_{\mu\nu}$ for $\omega\ll qv_F$. Here we have

$$\mathrm{Im}K^0_{00}\approx\frac{m^*}{2\pi}\frac{\omega}{qv_F} , \tag{2.27}$$

$$\mathrm{Im}K^0_{11}\approx\frac{2n_e\omega}{k_Fq} . \tag{2.28}$$

The imaginary part of K^0_{00} is small compared to the real part, for $\omega\ll qv_F$, and thus can be neglected. On the other hand, the imaginary part of K^0_{11} may be large relative to the real part and must be kept. Thus we find, in this regime,

$$\det(K^0+U^{-1})\approx\frac{in_e}{\pi}\frac{\omega}{qv_F}-\frac{q^2}{(2\pi\tilde{\phi})^2}\left[1+\frac{\tilde{\phi}^2}{6}+\frac{m^*v(q)}{2\pi}\right] . \tag{2.29}$$

If we continue this result analytically into the lower half-plane for ω, we find a zero of the determinant, and hence a pole in $K_{\mu\nu}$, at

$$\omega\approx-i\frac{q^3k_F}{4\pi n_e\tilde{\phi}^2m^*}\left[1+\frac{\tilde{\phi}^2}{6}+\frac{m^*v(q)}{2\pi}\right] . \tag{2.30}$$

Thus we find a relaxation rate for long-wavelength density fluctuations that varies as q^3 in the case of a finite-

range interaction potential, but varies as q^2 in the case of a Coulomb interaction.

We recall that within the RPA, $m^*=m_b$, so that Eq. (2.26) is in agreement with Kohn's theorem, which states that the pole in K_{00} should be at the bare cyclotron frequency $\omega_c=eB/m_b$. At a later point, we shall treat m^* as a parameter in the theory and a Fermi-liquid interaction parameter will then have to be introduced to satisfy Kohn's theorem. This will be discussed in Sec. IV. It is also worth noting that the RPA expressions give correctly the expected result that the f-sum rule is exhausted by the cyclotron mode in the limit $q\rightarrow 0$ and that the overdamped mode given by Eq. (2.30) contributes a spectral weight which vanishes as q^4.

As will be discussed further in Sec. VI below, the matrix $D\equiv[K^0+U^{-1}]^{-1}$ may be interpreted as the dynamically screened interaction, or as the propagator for combined gauge and potential fluctuations. The correlation function for fluctuations in the transverse vector potential $\mathbf{a}(\mathbf{q})$ is proportional to $\mathrm{Im}D_{11}(q,\omega)$, where

$$D_{11}(q,\omega)=K_{00}^0/\det[K^0+U^{-1}] = \left[\frac{2\pi\tilde{\phi}}{q}\right]^2 K_{00}(q,\omega) . \tag{2.31}$$

For $\omega\ll v_F q$, with $q\ll k_F$, we may replace K_{00}^0 by its limiting value $m^*/2\pi$, and use Eq. (2.29) for the value of the determinant.

For a general value of $\mathbf{q}$, the RPA density-response function $K_{00}(\mathbf{q},\omega)$ will have a nonzero imaginary part anywhere that $\mathrm{Im}K_{00}^0(q,\omega)\neq 0$. For $0<q<2k_F$, this occurs at all frequencies such that $2m^*|\omega|<q^2+2k_Fq$. For $q>2k_F$, the region where $\mathrm{Im}K_{00}(q,\omega)\neq 0$ is given by

$$q(q-2k_F)<2m^*\omega<q(q+2k_F) . \tag{2.32}$$

This leads to the familiar Kohn anomaly at $q=2k_F$ in the static response function $K_{00}(q,0)$, which for a two-dimensional Fermi surface has the form

$$K_{00}(q,0)\approx K_{00}(2k_F,0)-\mathrm{const}(q-2k_F)^{1/2} \tag{2.33}$$

for $q>2k_F$.

In principle, the Kohn anomaly in K_{00} should be reflected in Friedel oscillations in the screening charge density surrounding an impurity in the electron layer; i.e., we expect that, for $r\rightarrow\infty$,

$$\langle\rho(r)\rangle-n_e\propto\frac{\cos(2k_Fr+\delta)}{r^2} , \tag{2.34a}$$

where $\langle\rho(r)\rangle$ is the charge density at a distance r from the impurity, and δ is a phase shift which depends on the potential near the impurity. Similarly, near an edge of the sample or other straight-line perturbation, we expect to find

$$\langle\rho(x)\rangle-n_e\propto\frac{\cos(2k_Fx+\delta)}{x^{3/2}} . \tag{2.34b}$$

Unfortunately, the wave vector $2k_F$ corresponds to a wavelength $\lambda=\pi l_0$ which is not very convenient for experimental measurements. Moreover, the amplitude of the Kohn anomaly or of the associated Friedel oscillations is likely to be quite small. The amplitude of the singularity in the RPA response K_{00} will be generally reduced relative to that in K_{00}^0 when there is a strong repulsive interaction between electrons. More importantly, in the case where the electron-electron interaction is large compared to the cyclotron energy so that electrons are confined to the lowest Landau level, the matrix elements of the density operator are reduced by a factor of $e^{-q^2l_0^2/4}$, which comes from overlap between two states in the lowest Landau level separated by a wave vector q in the Landau gauge. Since the density response K_{00}^0 is proportional to the square of the density-matrix element, we guess that the form factor leads to a reduction in the size of the $2k_F$ anomaly by a factor

$$e^{-2k_F^2l_0^2}=e^{-2} . \tag{2.35}$$

III. FRACTIONAL QUANTIZED HALL STATES AT $\nu=p/(2p+1)$

Perhaps the most direct demonstration of the existence of a sharp Fermi surface in an ordinary metal is the observation of oscillations in the magnetization and transport properties as a function of the inverse applied magnetic field, in high-quality samples at low temperatures. In a two-dimensional system, these oscillations are magnified in importance, leading to energy gaps at the Fermi energy and vanishing of the longitudinal resistance, associated with the integer quantized Hall effect, at magnetic fields given by the condition $eB=2\pi n_e/p$, where p is a positive or negative integer. In an ideal system of noninteracting spinless electrons, the energy gap for an integer quantized Hall state is just the cyclotron energy, $\omega_c=|eB|/m$. If electron-electron interactions are taken into account, the energy gap E_g is found to be shifted somewhat from ω_c due to the difference in exchange and correlation energies between the lowest empty level and the highest filled Landau level. (The energy gap is defined as the smallest energy necessary to create a positive and a negative charged carrier, infinitely far apart. In a sample with very low impurity concentration, the thermal activation energy for electrical resistance should be just $E_g/2$.)

If there exists a Fermi surface for a half-filled Landau level, then the behavior of the system when the magnetic field deviates slightly from the field $B_{1/2}=4\pi n_e$, at which the Landau level is precisely half full, should resemble the behavior of a noninteracting electron system near $B=0$. In particular, we would expect to find an energy gap in the excitation spectrum when the external magnetic field satisfies

$$e\Delta B=\frac{2\pi n_e}{p} , \tag{3.1}$$

where

$$e\Delta B\equiv eB-4\pi n_e . \tag{3.2}$$

As was noted in the Introduction, this condition corresponds precisely to the condition $\nu=p/(2p+1)$, which characterizes the most prominent series of fractional quantized Hall plateaus converging to $\nu=\frac{1}{2}$ from above

or below.[23,25] Moreover, in the spirit of mean-field theory, we are led to define an effective cyclotron energy

$$\Delta\omega_c^* \equiv e|\Delta B|/m^* \tag{3.3}$$

and to use this quantity as an estimate of the energy gap $E_g^{(\nu)}$ at $\nu=p/(2p+1)$, i.e.,

$$E_g^{(\nu)} \cong \frac{2\pi n_e}{m^*|p|} . \tag{3.4}$$

The reader may recall that the energy gap for a quantized Hall state is defined as the lowest energy necessary to create a fractionally charged quasiparticle and quasihole, separated by a large distance, and that the elementary excitations in the quantized Hall state with $\nu=p/(2p+1)$ have charge $\pm qe$, with $q=1/(2p+1)$.[6-9] On the other hand, the effective cyclotron energy $\Delta\omega_c^*$ has been obtained as a mean-field energy gap for the energy spectrum associated with the creation operator $\psi^+(\mathbf{r})$, which adds a quasiparticle of charge $-e$ to the system. The apparent discrepancy between these definitions may be understood in terms of the effect of the flux quanta attached to the electron in the definition of ψ^+. For a quantized Hall state with $\nu=p/(2p+1)$, the addition of two flux quanta creates a local charge deficit of $2p/(2p+1)$ electrons, with the missing charge being pushed away to the boundary of the system. Thus the operator $\psi^+(\mathbf{r})$ applied to the quantized Hall state leads to a net charge of $-e/(2p+1)$ in the vicinity of $\mathbf{r}$ and a net charge of $-2pe/(2p+1)$ on the boundary of the system. If $\mathbf{r}$ and $\mathbf{r}'$ are two points far apart, then the product of operators $\psi^+(\mathbf{r})\psi(\mathbf{r}')$ creates only a pair of fractional charges at $\mathbf{r}$ and $\mathbf{r}'$, as the boundary charges cancel each other in this case.

Since an extra electron added to or removed from a fractional quantized Hall state will break up into $(2p+1)$ quasiparticle or quasiholes with charge $\pm e/(2p+1)$, the jump in the chemical potential for electrons $\Delta\mu$ which occurs as one crosses from one side of the fractional Hall state to another is related to $E_g^{(\nu)}$ by[6,7,38]

$$\Delta\mu = |2p+1|E_g^{(\nu)} . \tag{3.5}$$

Thus if $E_g^{(\nu)}$ is proportional to $1/p$, the jump in chemical potential is independent of p, for $p\to\infty$. This result, which seems surprising at first glance, is actually consistent with the following thermodynamic argument. If we reason by analogy with the quantum oscillations of a noninteracting electron system in weak magnetic field, we would expect that the ground-state energy per electron near $\nu=\frac{1}{2}$ can be written in the form[47]

$$E(n_e,\nu) \approx E_0(n_e,\nu) + \frac{(\Delta\omega_c^*)^2}{E_F} P\left[\frac{2\pi n_e}{e\Delta B}\right] , \tag{3.6}$$

where E_0 is an analytic function of its arguments, while $P(s)$ is a periodic function, with period unity, and with a discontinuity in slope $\Delta P'>0$ which occurs each time the argument s crosses an integer value. The chemical potential μ is defined as the derivative of the total energy $NE(n_e,\nu)$ with respect to the total particle number N, at constant magnetic field and volume. From (3.1–(3.3) and (3.6), we therefore find

$$\Delta\mu = \frac{\Delta P'}{E_F}(\Delta\omega_c^*)^2(2p^2+p) , \tag{3.7}$$

which is indeed consistent with (3.4) and (3.5).

The orbital moment M per electron may be obtained from $E(n_e,\nu)$ from

$$M = -\left.\frac{\partial E}{\partial B}\right|_{n_e} . \tag{3.8}$$

Equation (3.6) then implies that there is a discontinuity in the magnetic moment at the fractional quantized Hall state which is given by

$$\Delta M = \frac{e\nu}{2\pi}\Delta\mu , \tag{3.9}$$

which is also independent of p, for $p\to\infty$.

The current situation may be contrasted with the de Haas–van Alphen effect near $B=0$, where ΔM is independent of p but $\Delta\mu\propto 1/p$ for $p\to\infty$. This origin of the difference becomes clear if one replaces ΔB by B in Eq. (3.6) and repeats the derivation. Of course, Eq. (3.9) also applies to the integer quantized Hall effect, with $\nu=p$ in that case.

If magnetic field B is held fixed, and the electron density is varied from a value with $\nu<\frac{1}{2}$ to a value with $\nu>\frac{1}{2}$, then we pass through an infinite number of quantized Hall states of the form $\nu=p/(2p+1)$, and, according to Eq. (3.7), the sum of the jumps $\Delta\mu$ is positively infinite. Since the *net* change in μ must be finite, it follows that there must be regions of decreasing μ between the quantized Hall states of large $|p|$, enough to cancel the infinite increase at the Hall states themselves. For a system with short-range interactions, this would violate the thermodynamic requirement that $d\mu/dn_e$ must be ≥ 0 in any equilibrium state. It follows that there must be an infinite number of phase transitions, with discontinuities in density, which eliminate some or all of the filling fractions between the principal quantized Hall states. Effectively, this means that fractions with even denominators of the form $\nu=(2p+1)/(4p+4)$, which lie roughly midway between the values $p/(2p+1)$, must be unstable for large p in the case of short-range interactions.[38]

For the case of long-range Coulomb interactions, there is no thermodynamic requirement that $d\mu/dn_e$ be ≥ 0, as the diverging Coulomb energy will prevent phase separation on the macroscopic length scale, if the positive background is held fixed. Nevertheless it seems likely that a system of electrons with a filling fraction midway between two quantized Hall states of the form $\nu=p/(2p+1)$, for large p, will tend to break up locally into domains of the two adjoining stable densities. Presumably this would lead to a periodic array of domain walls, and hence a lowering of the translational symmetry of the system.

The conclusion that, for a system with short-range forces, there should be an infinite number of first-order phase transitions in the neighborhood of $\nu=\frac{1}{2}$ does not imply that the point $\nu=\frac{1}{2}$ is itself necessarily unstable. If Eq. (3.6) is correct, then the first derivative $\partial E/\partial n_e|_B$ is well defined at $\nu=\frac{1}{2}$, so that the chemical potential $\mu_{1/2}$ is also unique and well defined. Although the second

derivative $\partial^2 E/\partial n_e^2|_B$ is not well defined, the stability condition at $\nu=\frac{1}{2}$ only requires an inequality, viz.,

$$E(n_e^0+\delta n_e)\geq E(n_e^0)+\mu_{1/2}\delta n_e \ , \tag{3.10}$$

where $E(n_e)$ is here the energy per particle at a fixed value of B, and n_e^0 is the density at $\nu=\frac{1}{2}$. If $E_0(n_e)$ is the contribution to $E(n_e)$ that comes from the analytic term on the right-hand side of (3.6), and P_0 is the minimum value of the periodic function $P(s)$, which presumably occurs when s is an integer, then the inequality (3.10) will be satisfied for small values of δn_e provided that

$$E_0''(n_e^0)+\frac{16\pi}{n_e m^*}P_0>0 \ . \tag{3.11}$$

In a case where $\partial^2 E/\partial n_e^2$ is ill defined, one may ask what is the relation between the static density-response function $K_{00}(\mathbf{q},0)$, discussed in Sec. II, and the energy curve $E(n_e)$, in the limit where $\mathbf{q}\to 0$. It seems most likely that the response to an infinitesimal density perturbation at finite $\mathbf{q}$ should be insensitive to the precise value of the magnetic field, and should be the same at small finite temperature as it is at $T=0$. This means that we should effectively replace the periodic function $P(s)$ by $\bar{P}$, its average over the period of its argument. For a short-range potential, then, we would expect that, precisely at $\nu=\frac{1}{2}$,

$$\lim_{q\to 0} K_{00}(q,0)=\left[E_0''(n_e^0)+\frac{16\pi}{n_e m^*}\bar{P}\right]^{-1} \ . \tag{3.12}$$

For a Coulomb potential, where the $k=0$ part of the electron-electron interaction is canceled by the positive background and is thus excluded from the definition of E, the relation corresponding to (3.12), at $\nu=\frac{1}{2}$, is

$$\lim_{q\to 0}\left[\frac{1}{K_{00}(q,0)}-\frac{2\pi e^2}{\varepsilon q}\right]=E_0''(n_e^0)+\frac{16\pi}{n_e m^*}\bar{P} \ . \tag{3.13}$$

Slightly away from $\nu=\frac{1}{2}$, the value of $K_{00}(q,0)$ will depend sensitively on the order of limits as $q\to 0$ and $\nu\to\frac{1}{2}$. For fixed finite q, as $\nu\to\frac{1}{2}$, we would expect the value of $K_{00}(q,0)$ to approach the same value as it has precisely at $\nu=\frac{1}{2}$. On the other hand, if the limit $q\to 0$ is taken at fixed ν, the value approaches the uniform compressibility, which is a rapidly oscillating function of $\nu-\frac{1}{2}$. For the stable fractional Hall states with $\nu=p/(2p+1)$, we expect that $K_{00}(q,0)\to 0$, proportional to q^2, for $q\to 0$.

The density-current response function $K_{\mu\nu}(q,\omega)$ can be calculated in the RPA, at $\nu=p/(2p+1)$ using formulas that have been developed previously for models of anyon superconductivity which are mathematically quite similar to the fractional quantum Hall states. Formulas for the RPA response functions in the fractional quantized Hall case were in fact presented by Lopez and Fradkin,[25] but were not evaluated explicitly except in the limit of $q=0$. Our present formulation is somewhat simpler than that of Lopez and Fradkin in that we work with 2×2 matrices, while they use 3×3 matrices.

In order to calculate $K_{\mu\nu}$ at $\nu=p/(2p+1)$ we employ the basic RPA equation, (2.15), except that now we must use for $K^0_{\mu\nu}$ the response function for noninteracting electrons in the effective magnetic field ΔB, defined by Eq. (3.2). The matrix K^0 may be written in the general form

$$K^0_{\mu\nu}(q,\omega)=\frac{-n_e}{m^*}\delta_{\mu 1}\delta_{\nu 1}+\sum_{s=1}^{\infty}\frac{M^{(s)}_{\mu\nu}(q)e^{-q^2 l_p^2/2}}{(\omega+i\eta)^2-s^2(\Delta\omega_c^*)^2} \ , \tag{3.14}$$

where $M^{(s)}_{\mu\nu}$ has a polynomial dependence on q, $\Delta\omega_c^*$ is the effective cyclotron frequency defined by (3.3), and l_p is the magnetic length in the effective magnetic field ΔB. This is related to the magnetic length l_0 in the true magnetic field B by

$$l_p^2=|2p+1|l_0^2 \ . \tag{3.15}$$

Expressions for $M^{(s)}_{\mu\nu}$ in terms of the associated Laguerre polynomials may be found, e.g., in Chen *et al.*[41] In general, for larger values of s, the polynomials begin and end with the higher powers of q.

It can be seen without much difficulty that, for any fixed value of q, the matrix $K_{\mu\nu}(q,\omega)$ has a discrete series of poles at frequencies $\omega_n(q)$, which are in fact restricted to lie on the real axis. (This follows from the fact that there is no mechanism for dissipation in the problem.)

For any fixed s, the residue in $K^0_{\mu\nu}$ decreases exponentially for sufficiently large q, and hence there will be a corresponding pole (or pair of poles) in $K_{\mu\nu}$ which will be very little shifted from the value $s\Delta\omega_c^*$. Also, in the limit $q\to 0$, the residues in $K^0_{\mu\nu}$ vanish rapidly for all s other than $s=1$. This again leads to poles in $K_{\mu\nu}$, at frequencies $\approx s\Delta\omega_c^*$, for $s\geq 2$, whose weights vanish for $q\to 0$. The entire weight of the response function $K_{\mu\nu}$ appears, for $q\to 0$, in a single pole at the cyclotron frequency ω_c, which is equal in the RPA to $|2p+1|\Delta\omega_c^*$. (Recall that, within the RPA, $m^*=m_b$.) This result has previously been found by Lopez and Fradkin.[25] The pole at ω_c arises from the $s=1$ term and the diamagnetic first term in the expression (3.14) for $K^0_{\mu\nu}$, so there is no pole in $K_{\mu\nu}$ at $\omega=\Delta\omega_c^*$.

We note that the lowest branch of the excitation spectrum $\omega_1(q)$ may be identified with the quasiexciton mode, which for large values of q is described as a quasiparticle and quasihole, of charge $\pm e/(2p+1)$, separated by a distance $\hat{z}\times\mathbf{q}l_p^2$.[48,49] (Note that this gives a dipole moment eql_0^2.) As we have seen in the RPA, the frequency of the lowest excitation mode is equal to $\Delta\omega_c^*$ (which is the energy gap $E_g^{(\nu)}$) for large values of q, but $\omega_1(q)\to 2\Delta\omega_c^*$ for $q\to 0$.

If one averages over a frequency interval that is large compared to $\Delta\omega_c^*$, then the diagonal matrix elements $K^0_{00}(q,\omega)$ and $K^0_{11}(q,\omega)$ will be well approximated by the corresponding response functions in zero magnetic field, provided that ql_p is large compared to unity. The necessary conditions to approximate the RPA response function $K(q,\omega)$ at $\nu=p/(2p+1)$ by its value at $\nu=\frac{1}{2}$ are somewhat more complicated, since the weight of the $K(q,\omega)$, outside of the pole at ω_c is shifted to lower frequencies than in the case of $K^0(q,\omega)$. Thus in the case of short-range interactions, where the characteristic relaxation rate at $\nu=\frac{1}{2}$ is of the form $\omega_q\propto q^3$, we would expect to find $(ql_0)^3>\Delta\omega_c^*/\omega_c$ as a necessary condition to ap-

proximate $K(q,\omega)$ by its value at $\nu=\frac{1}{2}$. For much of the interesting range of frequencies and wave vectors, the RPA response function may be well approximated by using the semiclassical formulas presented in Appendix B, below.

IV. VALUE OF THE EFFECTIVE MASS

Within the simple Hartree theory which we have considered so far, the effective mass m^* of the quasiparticles is simply the single-electron band mass m_b for a single electron in the two-dimensional system. (For GaAs wells, m_b is approximately $0.07m_e$, where m_e is the free-electron mass.) In fact, however, the situation is more complicated. In the limit where the electron-electron interaction vanishes compared to the cyclotron energy ω_c, states in a given Landau level all become degenerate, and thus we must have $m^*\to\infty$. More generally, if $(e^2/\varepsilon l_0)$ is small compared to $\hbar\omega_c$, we can neglect mixing between Landau levels, and then all energies are proportional to $(e^2/\varepsilon l_0)$. Thus we find, using dimensional analysis, that if a finite effective mass exists it should have the form

$$\frac{\hbar^2}{m^*}=\frac{C}{(4\pi n_e)^{1/2}}\frac{e^2}{\varepsilon}\ , \tag{4.1}$$

where C is a dimensionless constant and $(4\pi n_e)^{1/2}$ is equal to l_0^{-1} at $\nu=\frac{1}{2}$. For a given material, if ν is held fixed, the value of $\hbar\omega_c$ increases with increasing B faster than $(e^2/\varepsilon l_0)$ and so Eq. (4.1) should be valid in the limit of large magnetic fields. A crude estimate, given below, suggests that the constant C has a value

$$C\approx 0.3\ . \tag{4.2}$$

Using the dielectric constant $\varepsilon=12.6$ appropriate for GaAs, we obtain from Eq. (4.1) the result

$$m^*\approx 0.27m_e\approx 4m_b \tag{4.3}$$

for magnetic field $B=10$ T. The value of m^* will of course increase $\propto B^{1/2}$ for larger values of B.

The way we have estimated the value of C is by fitting to the known energy gaps for the fractional quantized Hall states at $\nu=\frac{1}{3}$ and $\frac{2}{3}$ to formula (3.4) for the values of the energy gaps for the quantized Hall states with $\nu=p/(2p+1)$. In the strong magnetic-field limit, $\hbar\omega_0\gg e^2/\varepsilon l_0$, the energy gap in a fractional quantized Hall state of filling factor ν can be written in the form

$$E_g^{(\nu)}=g^{(\nu)}\frac{e^2}{\varepsilon l_0}\ , \tag{4.4}$$

where $g^{(\nu)}$ is a dimensionless constant and l_0 is the magnetic length which is related to the electron density at filling factor ν by

$$l_0=(\nu/2\pi n_e)^{1/2}\ . \tag{4.5}$$

Under conditions where only one spin state is occupied, there is an electron hole symmetry which requires that

$$g^{(\nu)}=g^{(1-\nu)}\ . \tag{4.6}$$

d'Ambrumenil and Morf[49] have given estimates of the energy-gap coefficients $g^{(\nu)}$, for the first three fractional Hall states in the series $\nu=p/(2p+1)$, based on extrapolation of exact diagonalizations of small systems on a sphere. Their results are

$$g^{(1/3)}=0.102\pm 0.003\ , \tag{4.7}$$

$$g^{(2/5)}=0.063\pm 0.004\ , \tag{4.8a}$$

$$g^{(3/7)}=0.049\pm 0.018\ . \tag{4.8b}$$

In trying to compare these results with the predictions of Eq. (3.4) for $\nu=p/(2p+1)$, we immediately encounter an ambiguity. Although Eq. (3.4) is well defined in the limit $p\to\infty$, when $\nu\to\frac{1}{2}$, if we wish to use the formula for small values of p, the answer depends upon whether one assumes that m^* is a function only of the electron density n_e, a function of the magnetic length l_0, or of some combination of the two. We have chosen to resolve this ambiguity by *defining* m^* to be the effective mass at $\nu=\frac{1}{2}$ for the given value of n_e, so that m^* is determined by n_e, and to incorporate the fact that (3.4) is only strictly correct asymptotically for $p\to\infty$, by writing

$$E_g=\frac{2\pi n_e}{|p|m^*}(2\nu)^{1/2}h(\nu)\ , \tag{4.9}$$

where $h(\nu)$ is an analytic function of ν in the vicinity of $\nu=\frac{1}{2}$, with the value $h=1$ at $\nu=\frac{1}{2}$. The factor $(2\nu)^{1/2}=[2p/(2p+1)]^{1/2}$ is also analytic in the vicinity of $\nu=\frac{1}{2}$, and has been inserted explicitly to facilitate the incorporation of electron-hole symmetry. If Eq. (4.1) is used for the effective mass, then (4.9) leads to the result

$$g^{(\nu)}=\frac{C}{|2p+1|}h(\nu)\ . \tag{4.10}$$

If ν is a fraction of the form $p/(2p+1)$, then $(1-\nu)$ is a fraction of the same form, with p replaced by $-(p+1)$. Equation (4.10) is therefore compatible with the symmetry requirement (4.6) if and only if $h(\nu)$ is symmetric about $\nu=\frac{1}{2}$.

The simplest guess for $h(\nu)$, which is consistent with the above requirements, is that $h(\nu)=1$, independent of ν. This then leads to the final result

$$E_g^{(\nu)}\approx\frac{C}{|2p+1|}\frac{e^2}{\varepsilon l_0}\ . \tag{4.11}$$

Note that $|2p+1|$ is just the denominator of the fraction ν. We see that (4.11) is in excellent agreement with the numerical results for $\nu=\frac{1}{3}$ and $\frac{2}{5}$, provided that we choose $C\approx 0.31$. The resulting estimate $g^{(3/7)}=0.044$ also agrees with the numerical value (4.8b), within the relatively large quoted uncertainties.

The fact that the effective mass m^* is quite different from the band mass m_b suggests that if a Fermi-liquid theory is correct near $\nu=\frac{1}{2}$, the Landau interaction parameters must be large, and must be taken into account in the computation of quantities of physical interest. In particular, the interaction parameters should be taken into account in the computation of the density-current response function for small q and ω.

As an example, let us consider the compressibility

$(dn/d\mu)$, defined by (3.12) for the case of short-range interactions, or by the inverse of Eq. (3.13) in the case of Coulomb interactions. In Fermi-liquid theory, one finds that

$$\left[\frac{dn}{d\mu}\right]^{-1}=\frac{2\pi}{m^*}\left[1+\frac{\tilde{\phi}^2}{6}\right]+u_0\ , \tag{4.12}$$

where u_0 is the appropriately normalized Landau interaction parameter for uniform dilations of the Fermi surface. The RPA result is obtained from (4.12) by setting $u_0=v(0)$ for a short-range interaction, and $u_0=0$ in the Coulomb case.

The requirements of Galilean invariance dictate that the cyclotron resonance pole in $K(q,\omega)$ is determined by the bare band mass rather than by m^*; viz., $\hbar\omega_c=eB/m_b$. Moreover, the form of $K(q,\omega)$ in the limit $q\to 0$ for any fixed $\omega\neq 0$ is also determined by the bare mass. For example, the components K_{00} and K_{10} in this limit are given by

$$K_{00}(q,\omega)\sim\frac{n_e q^2}{m_b(\omega_c^2-\omega^2)}\ , \tag{4.13}$$

$$K_{10}(q,\omega)\sim -iq\frac{Be}{n_e}\left[1-\frac{\omega^2}{\omega_c^2}\right]^{-1}\ . \tag{4.14}$$

It can be shown that in order for these relations to be satisfied in a Landau Fermi-liquid theory, it is necessary that the p-wave Landau parameter u_1, with an appropriate normalization, obeys the relation

$$u_1=\frac{1}{m_b}-\frac{1}{m^*}\ . \tag{4.15}$$

When interactions beyond the RPA are properly included, the density-current response functions at high frequencies are not directly determined by the effective mass m^* and the Landau interaction parameters, which strictly describe only low-energy long-wavelength fluctuations. Although (4.15) is *necessary* for (4.13) and (4.14) to be valid at low frequencies, the validity of the latter equations at high frequencies, including the correct locations of the cyclotron pole, implies additional constraints relating the one-fermion propagator and the interaction vertices at all frequencies.

In the limit where ΔB is small, so that the effective cyclotron frequency $\Delta\omega_c^*$ is small compared to the Fermi energy, it should be correct to use Fermi-liquid theory to calculate $\Delta\omega_c^*$, which we identify with the energy gap $E_g^{(\nu)}$ for the quantized Hall states at $\nu=p/(2p+1)$. On the basis of a preliminary analysis, we believe that the Landau interaction parameters do not affect the value of the effective cyclotron frequency $\Delta\omega_c^*$ or of the energy gap $E_g^{(\nu)}$ for small values of ΔB, and therefore (4.9) should be correct for large p, if Fermi-liquid theory applies. However, we find that there are potentially divergent corrections to m^*, as will be discussed in Sec. VI.

We expect that the lowest pole $\omega_1(q)$ of the density-response function should coincide with $\Delta\omega_c^*$ at large values of q, as we found in the RPA approximation which we discussed in Sec. IV; however, there is no simple relative between $\Delta\omega_c^*$ and $\omega_1(q)$ for $q\to 0$.

V. EFFECTS OF IMPURITIES

In order to make any comparisons between theoretical predictions and the results of actual experiments, it is necessary to take into account the effects of scattering by impurities or other types of disorder. A peculiar feature of the model with a Chern-Simons gauge field is that one of the principal effects of an impurity potential is to produce a static fluctuation in the fictitious magnetic field, consequent to the induced modulation in the density of particles.

We begin by considering the effect of a positively charged impurity, located at a distance $d_s\gg l_0$ away from the electron layer, above the origin in the x,y plane.

The impurity will give rise to a bare potential $v_0(r)=(e^2/\varepsilon)(r^2+d_s^2)^{-1/2}$ whose Fourier transform is

$$v_0(\mathbf{q})=\frac{2\pi e}{\varepsilon q}e^{-qd_s}\ . \tag{5.1}$$

This will induce a fluctuation in the electron density in the plane which is given by

$$\rho(q)=ev_0(q)K_{00}(q,\omega=0)\ , \tag{5.2}$$

where K_{00} is the density-response function defined in Sec. II. For $q\ll 2k_F$, we expect that $(K_{00})^{-1}$ is dominated by the Coulomb interaction term, so that $K_{00}\approx\varepsilon q/(2\pi e^2)$. Hence, we find that the induced electron density is

$$\rho(\mathbf{q})\approx e^{-qd_s}\ . \tag{5.3}$$

(This result is simply the charge density induced in a perfect metallic plane by a unit point charge a distance d_s away from it.)

The charge density (5.3) leads to a vector potential $\mathbf{a}(\mathbf{q})$ which satisfies $\mathbf{q}\cdot\mathbf{a}(q)=0$ and

$$i\mathbf{q}\times\mathbf{a}(q)=2\pi\tilde{\phi}\rho(\mathbf{q})\ . \tag{5.4}$$

The vector potential gives rise to a matrix element for scattering a fermion from a wave vector $\mathbf{k}$ to wave vector $\mathbf{k}+\mathbf{q}$, which is given by

$$M_{\mathbf{k},\mathbf{k}+\mathbf{q}}=\frac{\mathbf{a}(\mathbf{q})\cdot\mathbf{k}}{m^*}\approx\frac{2\pi\tilde{\phi}e^{-qd_s}k_F}{m^*iq}\ . \tag{5.5}$$

Note that the requirement $q\ll 2k_F$ implies that if $\mathbf{k}$ and $\mathbf{k}+\mathbf{q}$ have the same energy, then $\mathbf{q}$ must be approximately perpendicular to $\mathbf{k}$. In addition to the vector potential term there will be a scattering potential due to the scalar Chern-Simons potential $a_0(\mathbf{q})$ and to the screened impurity Coulomb potential, which in the RPA we may write as

$$v_{\text{eff}}(\mathbf{q})\approx\frac{2\pi}{m^*}\rho(\mathbf{q})\ . \tag{5.6}$$

However, this term will clearly be less important than the vector potential term for $q\ll 2k_F$.

If we assume that there is a density n_{imp} of charged impurities in a doping layer that is located a distance d_s

from the electron layer, and if we assume the positions of the impurities within the layer are completely random, then we may use Eq. (5.5) to calculate the transport scattering rate for the quasiparticle in the lowest Born approximation. Writing $q=2k_F\sin\alpha$, we find

$$\frac{1}{\tau_{\rm tr}}=\frac{4m^*n_{\rm imp}}{\pi}\int_0^{\pi/2}d\alpha|M_{\mathbf{k},\mathbf{k}+\mathbf{q}}|^2\alpha^2$$
$$=\frac{n_{\rm imp}}{m^*}\frac{\pi\tilde{\phi}^2}{k_Fd_s}\ . \tag{5.7}$$

Note that the total scattering rate, where the factor α^2 is missing from the integrand in (5.7), has a divergence due to small-angle scattering, unless some kind of infrared cutoff is incorporated in the calculation.

Now we would like to use this result to calculate the dissipative part of the resistivity ρ_{xx}. In a Hall experiment, with electrical current $-ej$ in the x direction, the rate at which energy is put into the electron system is given by

$$-ejE_x=e^2j^2\rho_{xx}\ . \tag{5.8}$$

If we identify the drift velocity of the quasiparticle with the drift velocity $\mathbf{v}_d=\mathbf{j}/n_e$ of the electrons, then the rate of momentum transfer from the impurities to the quasiparticle system is given by $dP/dt=-P/\tau_{\rm tr}$, where $P=n_em^*v_d$ is the momentum per unit area. (We do not distinguish here between the effective mass m^* and the bare mass m_b. A preliminary analysis suggests that our final results should not, in fact, be affected by the renormalization of the effective mass.) The rate of energy dissipation in the quasiparticle system is then given by

$$-v_d\frac{dP}{dt}=\frac{j^2m^*}{n_e\tau_{\rm tr}}\ . \tag{5.9}$$

Comparing (5.7)–(5.9), we obtain

$$\rho_{xx}\approx\frac{n_{\rm imp}}{n_e}\frac{\pi\tilde{\phi}^2}{k_Fd_se^2}\ . \tag{5.10}$$

A more formal derivation of the relationship between ρ_{xx} and $\tau_{\rm tr}$ may be found in Appendix B, below.

For an ideal modulation-doped sample, the number of charged impurities in the doping layer is just equal to the number of electrons, so that $n_{\rm imp}=n_e$. Setting $\tilde{\phi}=2$ for the case of interest, and restoring the factors of $\hbar$, we thus find

$$\rho_{xx}=\frac{1}{(k_Fd_s)}\frac{4\pi\hbar}{e^2}\ . \tag{5.11}$$

Recall that $4\pi\hbar/e^2$ is the Hall resistance ρ_{yx} at $\nu=\frac{1}{2}$, and $k_F=1/l_0$, where l_0 is the magnetic length.

Actually the assumption that there are no correlations in the positions of charged impurities within the doping layer may tend to overestimate the scattering of the charged impurities. Most frequently, only a fraction of the donors in a doping layer are ionized, so that redistribution of bound electrons within the donor layer may give rise to a significant screening and hence a reduction of the amplitude of the random impurity potential at long wavelengths.

For a typical value of $k_Fd_s\approx15$, Eq. (5.11) leads to a value of ρ_{xx}, at $\nu=\frac{1}{2}$, which is of order 3000 Ω/square. Experimental values found by Willett *et al.*[33] are of the order of 1000 Ω/square, however, which is a factor of 3 smaller than our estimate. Similarly, measurements reported by Stormer *et al.*[50] gave values for ρ_{xx} at $\nu=\frac{1}{2}$ of 1500 and 2000 Ω/□ on two high mobility samples with $k_Fd_s=9.4$ and 16.5, respectively. It is not clear whether the reduction in potential fluctuations due to screening within the doping layer could account for the discrepancy between theory and experiment or for the differences among experimental results.

The model of uncorrelated charged impurities in a remote doping layer can also be used to calculate the resistivity of the electron layer in zero magnetic field. In this case, of course, we have no Chern-Simons field so that term (5.5) is not present in the scattering matrix element. Thus the scattering is due to the potential term, which is still given by (5.6), for the case of spinless electrons. The resulting resistivity for spinless electrons in zero field is then

$$\rho_{xx}^0=\frac{n_{\rm imp}}{n_e}\frac{1}{4(k_Fd_s)^3}\left[\frac{2\pi\hbar}{e^2}\right]\ , \tag{5.12}$$

which is smaller than (5.11) by a factor of $(8k_F^2d_s^2)^{-1}$. If one takes into account the fact that actually two spin states should be occupied when $B=0$, one finds that (5.6) is reduced by a factor of 2, and hence one finds

$$\rho_{xx}^0=\frac{n_{\rm imp}}{n_e}\frac{1}{16(k_Fd_s)^3}\left[\frac{2\pi\hbar}{e^2}\right]\ . \tag{5.13}$$

However, the value of k_F is now equal to $(2\pi n_e)^{1/2}$ rather than $(4\pi n_e)^{1/2}$, the value for the spin-polarized state at $\nu=\frac{1}{2}$, so that the double-spin occupancy leads only to a further reduction of ρ_{xx}^0 by a factor $1/\sqrt{2}$.

The value of the zero-field resistivity obtained from (5.13) for $(k_Fd_s)\approx10$ is actually smaller, by an order of magnitude, than the zero-field resistivity in the samples employed by Willett *et al.* This suggests that the zero-field mobility is actually determined by a small number of residual charged impurities that are much closer to the electron layer than the impurities in the doping layer. These close impurities should be much less important for the resistivity at $\nu=\frac{1}{2}$, however, because the value of ρ_{xx} is in any case much larger at $\nu=\frac{1}{2}$ than it is at $B=0$, and because the distance dependence of the impurity scattering in Eq. (5.11) is much weaker than in the zero-field case (5.13).

The prediction of a finite value of ρ_{xx} would be modified, at very low temperatures, by the effects of "weak localization."[51] The quasiparticle scattering mechanism arising from the Chern-Simons field is not invariant under time reversal, so that we might expect that weak-localization effects would be similar to those for noninteracting electrons in a system with spin scattering, or random magnetic fields, but zero average field. Then if the value of ρ_{xx} obtained from (5.11) is sufficiently small

(e.g., if $k_F d_s$ is very large), we would expect that at $T=0$ there is a length-scale-dependent resistivity $\rho_{xx}(L)$ which satisfies

$$[\rho_{xx}(L)]^2 \approx [\rho_{xx}(l_0)]^2 + \text{const}\frac{h^2}{e^4}\ln(L/l_0) . \quad (5.14)$$

We shall argue in Sec. VI G, however, that logarithmic corrections due to Coulomb interactions are more important than this weak-localization effect.

VI. EFFECTS OF LONG-WAVELENGTH FLUCTUATIONS IN THE GAUGE FIELD

A. Propagator for gauge fluctuations

As shown in Sec. II and Appendix A, there is an interaction term which couples charge-density and current fluctuations, which may be represented by the exchange of a gauge field.[52] In the Coulomb gauge, $\nabla\cdot\mathbf{a}=0$, so that in two dimensions there are two independent components of the gauge field: a_0 and the transverse component a_1. We define the gauge-field propagator at wave vector $\mathbf{q}$ by

$$\mathcal{D}_{\mu\nu}(\mathbf{q},\tau) = -\langle T_\tau[a_\mu(\mathbf{q},\tau)a_\nu^+(q,0)]\rangle , \quad (6.1)$$

where τ is an imaginary time, T_τ is the time-ordering operator, and $\mu,\nu=0$ or 1. Its Fourier component $\mathcal{D}_{\mu\nu}(\mathbf{q},\omega_n)$ at the Matsubara frequency $\omega_n=2\pi nT$, is given, in the RPA, by $\mathcal{D}_{\mu\nu}(\mathbf{q},\omega_n)=D_{\mu\nu}(q,i|\omega_n|)$, with

$$D=(K^0+U^{-1})^{-1} . \quad (6.2)$$

where U is the matrix defined in Eq. (2.16), and K^0 is the density-current response function for the unperturbed system, described by the Hamiltonian (2.10), with $\Delta\mathbf{A}=0$ in the case where impurities are absent. The general formula for K^0 in this case was stated in (2.18) and (2.19) for frequencies ω on the real axis. Here, however, we shall also consider the case with impurities. We note that in our notation, the scalar potential $a_0(\mathbf{q})$ includes fluctuations arising from the electron-electron interaction $v(q)$ as well as from the Chern-Simons field.

The most important physics is dominated by the low-lying mode of the gauge propagator and its behavior depends on the presence or absence of Coulomb interactions and on disorder which gives rise to a mean free path $l=v_F\tau_{\rm tr}$. Thus there are four separate cases to consider. It is useful to display the following compact form which is valid for $\omega<v_F q$:

$$D^{-1}(q,\omega)=\begin{bmatrix} \mathcal{K}^0_{00}(q,\omega) & -i\dfrac{q}{2\pi\tilde{\phi}} \\ \dfrac{iq}{2\pi\tilde{\phi}} & i\omega\gamma_q - q^2\tilde{\chi}(q) \end{bmatrix} , \quad (6.3)$$

where

$$\gamma_q=\frac{2n_e}{m^* q v_F} \quad \text{clean:}\ \ ql>1 , \quad (6.4a)$$

$$\frac{n_e}{m^*}\frac{l}{v_F}\equiv\frac{1}{e^2\rho_0} \quad \text{disordered:}\ \ ql<1 , \quad (6.4b)$$

and

$$\tilde{\chi}(q)=\chi_0+\frac{v(q)}{(2\pi\tilde{\phi})^2} , \quad (6.5)$$

where

$$\chi_0=\frac{1}{12\pi m^*} . \quad (6.6)$$

Note that χ_0 is the Landau diamagnetic susceptibility of a two-dimensional (2D) Fermi gas. For long-range Coulomb interactions, $v(q)\sim q^{-1}$, which then gives the dominant contribution to Eq. (6.5).

The scattering rate $1/\tau_{\rm tr}$ should be identified with the quasiparticle scattering rate calculated in Sec. V, and as previously it is primarily determined by the static gauge field fluctuations produced by the impurity potential. The quantity ρ_0 is just the value of ρ_{xx} at $\nu=\frac{1}{2}$, which was obtained, using this scattering rate, in Eq. (5.10).

The density-density function $K^0_{00}(q,\omega)$ takes the form

$$K^0_{00}(q,\omega)=\begin{cases} \dfrac{m^*}{2\pi} & \text{clean:}\ \ ql>1 \quad (6.7a) \\ \dfrac{m^*}{2\pi}\dfrac{Dq^2}{Dq^2-i\omega} & \text{disordered:}\ \ ql,\omega\tau_{\rm tr}<1 , \quad (6.7b) \end{cases}$$

where $D=\frac{1}{2}v_F l$ is the diffusion constant. The diffusion pole in Eq. (6.7b) is responsible for the anomalous thermodynamic and transport properties of the disordered interacting Fermi gas.[53] The four regions of long- or short-range interaction and clean or disordered limit correspond to different limits of Eqs. (6.4), (6.5), and (6.7).

B. Gauge fluctuation contribution to the specific heat

As a first application, let us consider the contributions to the free energy and specific heat due to the low-lying gauge fluctuations. As discussed in Appendix A, the effective Lagrangian takes the form $\sum_{ij}a_i D_{ij}^{-1}a_j$. Therefore, integrating out the a_i field leads to

$$\Delta F' = \frac{T}{2}\sum_{q,n}\ln\det\mathcal{D}^{-1}(q,\omega_n) . \quad (6.8)$$

Standard contour integration leads to the form

$$\Delta F' = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int\frac{d^2q}{(2\pi)^2}\frac{1}{e^{\beta\omega}-1} \times\tan^{-1}\left[\frac{\text{Im}\det D^{-1}(q,\omega)}{\text{Re}\det D^{-1}(q,\omega)}\right] , \quad (6.9a)$$

$$\det D^{-1}(q,\omega)=K^0_{00}(q,\omega)[i\omega\gamma-\tilde{\chi}(q)q^2]-\frac{q^2}{(2\pi\tilde{\phi})^2} . \quad (6.9b)$$

Let us first consider the clean case, where $\det D^{-1}$ is given by Eq. (2.29) and

$$\frac{\mathrm{Im}\det D^{-1}}{\mathrm{Re}\det D^{-1}} = -\frac{n_e(2\pi\tilde{\phi})^2}{\pi v_F}\frac{\omega}{q^3}\frac{1}{\left[1+\frac{\tilde{\phi}^2}{6}+\frac{m^*}{2\pi}v(q)\right]} . \tag{6.10}$$

For short-range interactions, we have

$$\Delta S = \frac{-d(\Delta F')}{dT} = \frac{2}{T^2}\int_0^\infty \frac{d\omega}{2\pi}\int \frac{d^2q}{(2\pi)^2}\frac{\omega}{(e^{\beta\omega/2}-e^{-\beta\omega/2})^2} \times \tan^{-1}\left[C\frac{\omega}{q^3}\right] , \tag{6.11}$$

where $C=(n_e/\pi v_F)(2\pi\tilde{\phi})^2/[1+\tilde{\phi}^2/6+m^*v(0)/2\pi]$. The q integration can be done by scaling, giving $\sim(C\omega)^{2/3}$. The ω integration can be estimated by replacing $\omega^2(e^{\beta\omega/2}-e^{-\beta\omega/2})^{-2}$ by T^2 for $\omega<3T$ and zero for $\omega>3T$. Thus

$$\Delta S \approx \int_0^T d\omega\,\omega^{-1}(C\omega)^{2/3} \approx C^{2/3}T^{2/3} . \tag{6.12}$$

For Coulomb interactions, $v(q)=2\pi e^2/|q|$, the argument of $\tan^{-1}$ in Eq. (6.11) is now $C'\omega/q^2$, where $C'=(n_e/\pi v_F)(2\pi\tilde{\phi})^2(m^*e^2)^{-1}$. The q integration is now ultraviolet divergent. After introducing an upper cutoff $\approx 2k_F$, we obtain $1/4\pi C'\omega\ln[(2k_F)^2/C'\omega]$. The ω integration can be now evaluated, yielding

$$\Delta S \approx \frac{C'}{12}T\ln[(2k_F)^2/C'T] . \tag{6.13}$$

Thus we see that the overdamped collective mode identified in Sec. II still gives a contribution to the low-temperature entropy which dominates over the quasiparticle contributions of a Landau Fermi liquid. The damping rate goes from q^3 to q^2 in the presence of long-range Coulomb interaction, and the contribution to the entropy is reduced from $T^{2/3}$ to $T\ln T$.

In the presence of disorder, the transverse conductivity given by Eq. (6.4a) is replaced by the q-independent conductivity of (6.4b). From D_{11}^{-1} in Eq. (6.3), we see that the damping of the transverse gauge field alone is now q^2 and q with short-range and Coulomb interactions, respectively. To calculate ΔS, it is essential to include the diffusion structure of K_{00}^0 given by (6.7b). In the case of Coulomb interactions, it is easy to verify that the ratio

$$\frac{\mathrm{Im}\det D^{-1}(q,\omega)}{\mathrm{Re}\det D^{-1}(q,\omega)} \approx \frac{-\omega}{Dq^2} . \tag{6.14}$$

This leads to

$$\Delta S = \frac{T}{D}\ln\left[\frac{Dk_F^2}{T}\right] . \tag{6.15}$$

This is the same result as has been obtained previously[53] for a disordered electron gas with no magnetic field, and is a consequence of the diffusion pole in Eq. (6.7b). In the present case, the damping rate of the transverse gauge field is proportional to q, and its contribution to ΔS is unimportant compared with (6.15).

In the case of disordered short-range interactions, both diffusion and transverse modes contribute and we again obtain $\Delta S \sim T\ln T$ with a more complicated prefactor.

C. Effect on the fermion propagator

We now turn to the important question of the effect of interactions via the gauge fluctuations on the propagator for the fermions. The lowest-order contribution to the fermion self-energy is given by the Feynman diagram in Fig. 2, where the wiggly double line is the RPA gauge propagator $D_{\mu\nu}$. We limit ourselves, in this subsection, to the clean case where impurity scattering is neglected.

Let us first consider the contribution from the transverse gauge fluctuations, given by D_{11}. From Eq. (6.3),

$$D_{11}(q,\omega) = \frac{1}{i\omega\gamma_q - q^2\tilde{\chi}'} , \tag{6.16}$$

where

$$\tilde{\chi}' = \chi_0' + v(q)/(2\pi\tilde{\phi})^2 , \qquad \chi_0' = \frac{1}{2\pi m^*}\left[\frac{1}{6}+\frac{1}{\tilde{\phi}^2}\right] . \tag{6.17}$$

The propagators D_{01} and D_{00} involve factors of q and ω/q, respectively, in the numerator. Since the important region in ω is $\sim q^3$, we expect that these will be less important in the small q, ω limit.

We note that the problem of a Fermi gas coupled to a transverse gauge field without the Chern-Simons term has been considered before, in a different context, by various authors.[54–57] In that case, the propagator D_{11} for the transverse gauge field has precisely the same form as Eq. (6.16) except for the replacement of $\tilde{\chi}'$ by χ_0. Thus for the case of short-range interactions we can immediately adopt the results already known in the literature. There are also some treatments of the effect of an external magnetic field,[58,59] in which case D has exactly the form of (6.3), with the off-diagonal term $iq/(2\pi\tilde{\phi})^2$ replaced by $iq\sigma_{xy}$.

We begin by examining the Green function for the "quasiparticle" ψ introduced in Eq. (2.3). Note that this is not to be confused with the Green function of the physical electron ψ_e. The (retarded) self-energy is easily calculated to lowest order in the coupling to a transverse gauge field, and is given by

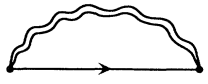

FIG. 2. The leading contribution to the fermion self-energy. The wiggly double line represents the propagator $D_{\mu\nu}$ for the combined Chern-Simons gauge field and the electron-electron interaction $v(q)$, including the effects of screening by the fermion bubbles. The solid line represents a fermion propagator.

$$\Sigma(\mathbf{k},\varepsilon)=\int\frac{d\mathbf{q}}{(2\pi)^2}\int_0^\infty\frac{d\omega}{\pi}\left|\frac{\mathbf{k}\times\hat{q}}{m^*}\right|^2\mathrm{Im}D_{11}(q,\omega)\times\left\{\frac{1+n_B(\omega)-n_f(\xi_{\mathbf{k}+\mathbf{q}})}{\varepsilon+i\eta-\xi_{\mathbf{k}+\mathbf{q}}-\omega}+\frac{n_B(\omega)+n_F(\xi_{\mathbf{k}+\mathbf{q}})}{\varepsilon+i\eta-\xi_{\mathbf{k}+\mathbf{q}}+\omega}\right\},\quad(6.18)$$

where $\xi_k=k^2/2m^*-E_F$ and n_B, n_F are the Bose and Fermi factors. The important point is that

$$\mathrm{Im}D_{11}(q,\omega)=-\frac{\omega\gamma_q^{-1}}{\omega^2+(\tilde{\chi}'\gamma_q^{-1}q^2)^2}\quad(6.19)$$

diverges for small q and ω, leading to a singularity in $\Sigma(k,\varepsilon)$. When evaluated on the mass shell, we find

$$\Sigma''(k,\xi_k)\approx\frac{k_F}{2\pi m^*}\int_0^{\xi_k}\frac{d\omega}{\pi}\int_0^\infty dq\,\mathrm{Im}D_{11}(q,\omega)[n_B(\omega)+1]\,,\quad(6.20)$$

where $q=2k_F\sin(\theta/2)$ is the momentum transfer. For short-range interactions, $\tilde{\chi}'$ is a constant, $\gamma_q^{-1}\sim q/k_0$ where $k_0=2n_e/m^*v_F$, and the q integration can be done by scaling, yielding $\omega^{-1/3}(\tilde{\chi}')^{-2/3}k_0^{-1/3}$. At $T=0$, the ω integration gives the result

$$-\Sigma''(k,\xi_k)\approx\frac{C}{m^*}\xi_k^{2/3}\,,\quad(6.21)$$

where $C=k_F\tilde{\chi}'^{-2/3}k_0^{-1/3}$. This calculation can also be done for Σ' and Σ'' for general k and ε. We obtain

$$-\Sigma''(k,\varepsilon)\approx\begin{cases}\dfrac{C}{m^*}|\varepsilon|^{2/3}, & |\varepsilon|>\dfrac{\chi'}{k_0}\left|\dfrac{\xi_k}{v_F}\right|^3 \quad(6.22a)\\ \dfrac{C'}{m^*}\left[\dfrac{k_0}{\chi'}\right]^{4/3}\dfrac{\varepsilon^2}{\xi_k^4}, & |\varepsilon|<\dfrac{\chi'}{k_0}\left|\dfrac{\xi_k}{v_F}\right|^3 \quad(6.22b)\end{cases}$$

$$\Sigma'(k,\varepsilon)\approx\begin{cases}-\dfrac{C''}{m^*}|\varepsilon|^{2/3}\mathrm{sgn}\varepsilon, & |\varepsilon|>\dfrac{\chi'}{k_0}\left|\dfrac{\xi_k}{v_F}\right|^3 \quad(6.23a)\\ -\dfrac{C''}{m^*}\left[\dfrac{k_0}{\chi'}\right]^{1/3}\dfrac{\varepsilon}{|\xi_k|}, & |\varepsilon|<\dfrac{\chi'}{k_0}\left|\dfrac{\xi_k}{v_F}\right|^3 \quad(6.23b)\end{cases}$$

where C', C'', and C''' differ from C by factors of order unity. Similar results have been obtained by Blok and Monien.[60] If we attempt to analyze the Green function in terms of the conventional Fermi-liquid quasiparticle pole, $G(k,\varepsilon)=z/(\varepsilon-\xi_k^*)$, where $z=(1-\partial\Sigma/\partial\varepsilon)^{-1}$, we find that the spectral weight vanishes since $\partial\Sigma/\partial\varepsilon$ diverges. Furthermore the Fermi velocity v_F^* defined by $\xi_k^*=v_F^*(k-k_F)$ is given by $v_F^*=v_F(1+\partial\Sigma/\partial\xi_k)/(1-\partial\Sigma/\partial\varepsilon)$. Near the mass shell $\varepsilon=\xi_k^*$, we see from Eq. (6.23a) that Σ' has no singular dependence on ξ_k while $\partial\Sigma/\partial\varepsilon$ diverges. Hence we conclude that v_F^* vanishes, while the effective mass $m^*\equiv k_F/v_F^*$ is found to diverge at the Fermi surface. This suggests that at least some modifications to Fermi-liquid theory are necessary in this case.

The above calculation can easily be repeated for the case of long-range interactions, where $v(q)\to\infty$ for $q\to0$, using the appropriate form for $D_{11}(q,\omega)$ in that case. For the case of a Coulomb interaction with $v(q)\propto1/q$, we find a contribution to $\Sigma'(k,\varepsilon)$ at $\varepsilon=\xi_k$, of form $\varepsilon|\ln\varepsilon|$, which suggests that there is a logarithmically diverging correction to the effective mass in this case. On the other hand, for ultra-long-range interactions such that $v(q)$ diverges as $1/q^x$, for $q\to0$ with $1<x\le2$, we find $|\Sigma''(q,\xi_k)|\ll|\xi_k|$ for $|\xi_k|\to0$, while Σ' is linear in ξ_k. Thus there is no obvious violation of the conditions for a normal Fermi-liquid theory in this case.

There are other pathologies occurring in the one-fermion Green function, however, which appear even in the case of an ultra-long-range interaction. An indication of this may be found in a singular contribution to the self-energy which comes from long wavelengths and *high* frequencies in the contribution of D_{00} to the diagram in Fig. 2. Specifically, in this limit we can make the approximation

$$D_{00}(q,\omega)\approx\frac{(2\pi\tilde{\phi})^2}{q^2}K_{11}(q,\omega)=\frac{(2\pi\tilde{\phi})^2}{q^2}\frac{n_e}{m_b}\frac{\omega^2}{\omega_c^2-(\omega+i\eta)^2}\,,\quad(6.24)$$

where ω_c is the cyclotron frequency $\omega_c=eB/m_b$, and we use the band mass m_b rather than m^* here to emphasize the fact that there are no Fermi-liquid corrections to K_{11} in the long-wavelength limit. If we also neglect the recoil energy of the fermion, so that we set $\xi_{\mathbf{k}+\mathbf{q}}=\xi_{\mathbf{k}}$, then we obtain a contribution to $\Sigma(k,\varepsilon)$ of the form

$$\delta\Sigma(k,\varepsilon)\approx\frac{\pi n_e\tilde{\phi}^2}{m_b}\ln(Rq_{\max})\frac{(\xi_k-\varepsilon)}{\omega_c-(\varepsilon-\xi_k)\mathrm{sgn}\xi_k}\,,\quad(6.25)$$

where R is the radius of the system, and $q_{\max}$ is a maximum wave vector which may depend on ε or ξ_k. We see that this leads to a reduction in the residue z at the quasiparticle pole by a factor which diverges in the limit of an infinite system size. Since the contribution $\delta\Sigma$ vanishes for $\varepsilon=\xi_k$, however, it does not lead to a renormalization of m^* or v_F^*.

The physical origin of term (6.25) is not difficult to establish. Let us consider the mean energy cost to instantaneously remove a fermion from the system, which we define as

$$\bar{E}=\frac{\langle\psi^+(\mathbf{r})H\psi(\mathbf{r})\rangle}{\langle\psi^+(\mathbf{r})\psi(\mathbf{r})\rangle}\,.\quad(6.26)$$

If one thinks in terms of electron operators, rather than the transformed fermions, one sees that the operator $\psi(\mathbf{r})$ removes an electron from the ground state at point $\mathbf{r}$, and multiplies the electron wave function by a phase factor that depends on the positions of all the other electrons according to Eq. (2.3). If we employ first-quantized notation in terms of electron coordinates $\mathbf{r}_i$, the phase factor is equal to $\Pi_i e^{i\tilde{\phi}\arg(\mathbf{r}_i-\mathbf{r})}$. Choosing $\mathbf{r}$ to lie at the origin, we see that all the electrons near a point $\mathbf{r}'$ far from the origin are given a velocity boost by an amount $\tilde{\phi}/(m_b r')$ in the direction perpendicular to $\mathbf{r}'$. This will clearly lead to an energy cost

$$E_0=\frac{n_e\tilde{\phi}^2}{2m_b}\int\frac{d^2_{\mathbf{r}'}}{|r'|^2}\approx\frac{\pi n_e\tilde{\phi}^2}{m_b}\ln(Rq_{\max})\ , \tag{6.27}$$

which appears as a contribution to the mean energy $\bar{E}$. A similar positive-energy contribution appears in the mean energy to *add* a fermion to the system with the operator ψ^+.

If the operator $\psi(0)$ is applied to the ground state at time $t=0$, the induced velocity field may be thought of as arising from an impulse electric field in the azimuthal direction, which falls off as $1/r$, and which may be considered as slowly varying in space for distances r which are large compared to the magnetic length l_0. The result of this impulse force is simply to excite oscillations at the cyclotron frequency ω_c. The mean number of quanta excited is equal to E_0/ω_c. Using the relation $\omega_c=2\pi\tilde{\phi}n_e/m_b$, we find

$$\frac{E_0}{\omega_c}=\frac{\tilde{\phi}}{2}\ln(Rq_{\max})\ . \tag{6.28}$$

Although the mean number of excited quanta is given by (6.28), there will actually be a distribution of quanta excited and for a finite system there will be a finite probability that there are no quanta of excitations at the cyclotron frequency. This probability is given by

$$P_0\approx e^{-E_0/\omega_c}\ . \tag{6.29}$$

(This may be thought of as a Debye-Waller factor, which is given by the square of the overlap integral between the harmonic-oscillator ground state and a displaced harmonic-oscillator ground state with mean energy E_0.) If we ignore the fact that E_0 is large, and simply expand P_0^{-1} in powers of E_0, we find

$$P_0\approx(1+E_0/\omega_c)^{-1}\ . \tag{6.30}$$

This coincides precisely with the reduction in the residue z at the quasiparticle pole that one obtains from the first-order contribution to the self-energy $\delta\Sigma$, given by Eq. (6.25). [The more accurate estimate of the reduction of z, given by P_0 in Eq. (6.29), can be obtained formally from a diagrammatic expansion, if one sums up contributions to $\Sigma(k,\varepsilon)$ in which an arbitrary number of a_0 fluctuations are emitted and absorbed, using the approximation (6.24) for D_{00}, and ignoring the recoil energy of the fermion in all intermediate states.]

The vanishing residue at the quasiparticle pole means that in the limit of an infinite system there is vanishing overlap between the state where a bare fermion is added or removed and the quasiparticle states whose properties we really wish to study. Nevertheless, we believe that if the limit of $R\to\infty$ is studied with sufficient care, it should be possible to extract from the fermion Green function G various properties of the quasiparticle states. Moreover, we believe that the singular contribution to the fermion self-energy arising from scattering by transverse gauge fluctuations, which we found in the case of short-range or $1/r$ interactions, has a physical significance for the quasiparticle propagation which must be taken into account in any Fermi-liquid-like description.

D. Proposed form of the renormalized theory

At the present time, a complete analysis of the effects of gauge fluctuations is not available. Nevertheless, we are able to propose a modified Fermi-liquid description which seems at least to be self-consistent, and which we believe to be a good candidate for the correct description of the Landau level at $\nu=\frac{1}{2}$ even for the case of short-range interactions.

For the case of short-range interactions, the essential features of our proposed description are the following.

We assume that there exists a well-defined Fermi wave vector $k_F=(4\pi n_e)^{1/2}$. For wave vectors $\mathbf{k}$ with magnitude close to k_F, there exist quasiparticle or quasihole excitations with energy spectrum

$$\varepsilon_k\propto|k-k_F|^{3/2}\ . \tag{6.31}$$

This means that there is a renormalized effective mass m^* of the form

$$m^*\propto|k-k_F|^{-1/2}\propto|\varepsilon_k|^{-1/3}\ . \tag{6.32}$$

The scattering rate, at $T=0$, for quasiparticles close to the Fermi surface is also of order $|k-k_F|^{3/2}$, so that there is an imaginary part to ε_k which is proportional to ε_k for $\varepsilon_k\to0$. Fermi-liquid parameters such as u_0 and u_1, defined in Sec. V, remain finite for $k\to k_F$.

The long-wavelength behavior of the density-current response function $K_{\mu\nu}(\mathbf{q},\omega)$ and the gauge field propagator $D_{\mu\nu}(\mathbf{q},\omega)$ are essentially the same as in the RPA. In particular, we believe that the characteristic relaxation rate ω_q of density fluctuations remains proportional to q^3, while the static response functions $K_{00}(q,\omega=0)$ and $K_{11}(q,\omega=0)$ remain finite in the limit $q\to0$. This is plausible because the effective mass m^* disappears from the low-frequency part of the RPA formulas (2.22), (2.30), (2.31), and (6.24), in the limit $m^*\to\infty$, as long as the potential $v(q=0)$ (or more correctly the Fermi-liquid parameter u_0) is assumed to be finite. Of course, there also remains a pole in the response functions at the bare cyclotron frequency ω_c. (See also Sec. VI E, below.)

If the behavior of $D_{\mu\nu}$ indeed remains unchanged in the limit $q\to0$, $\omega\to0$, we can make the following self-consistency argument in support of Eqs. (6.31) and (6.32).

Fluctuations which renormalize m^* will also have an

effect on the vertex for interaction between a quasiparticle and a fluctuation in the transverse gauge field. Since the form of the coupling is essentially determined by the requirements of gauge invariance, however, it seems reasonable to assume that the matrix element for scattering a quasiparticle from momentum state $\mathbf{k}$ to momentum state $\mathbf{k}+\mathbf{q}$, where $\mathbf{k}$ is close to the Fermi surface, and where $\mathbf{q}$ is small and nearly orthogonal to $\mathbf{k}$, is given at least approximately by the form $i\mathbf{k}\cdot\mathbf{a}(\mathbf{q})/m^*$, where m^* is the renormalized mass at wave vector $\mathbf{k}$. Then the scattering rate for quasiparticles should be given by an expression identical in form to the imaginary part of expression (6.18) for the lowest-order self-energy $\Sigma(k,\varepsilon)$, except that the energies $\xi_{\mathbf{k}+\mathbf{q}}$ of the particle in the intermediate state should be replaced by the renormalized energy spectrum $\varepsilon_{\mathbf{k}+\mathbf{q}}$ given by (6.31). This gives rise to a scattering rate of form $C\varepsilon^{2/3}/m^*$, similar to the right-hand side of (6.22a), except that now the mass m^* should be evaluated at the energy ε, assuming that $\mathbf{k}$ is close to the Fermi surface and ε is the same order as ε_k. Since the expression for $\mathrm{Im}\Sigma$ depends primarily on ε and is not a singular function of $(\varepsilon-\varepsilon_k)$, it is reasonable to assume that a Kramers-Kronig relation applies to the dependence on ε, and that the real part of the energy shift should have a similar singularity as the imaginary part in the neighborhood of the Fermi surface. Thus we see that the assumption (6.31) is self-consistent in that it leads to a scattering rate and a shift in energy for a quasiparticle on the mass shell with $\varepsilon_k \propto |k-k_F|^{3/2}$, which are both of order ε_k.

The fact that the quasiparticle decay rate is found to vary as ε_k at $T=0$ suggests that, at finite temperatures, the decay rate for a thermally excited quasiparticle should itself be of order T. However, the dominant terms arise from scattering by a gauge fluctuation whose energy is itself of order T, and thus whose wave vector $\mathbf{q}$ is of order $T^{1/3}$. Thus the scattering is only small-angle scattering in the limit $T\to 0$, and is therefore not effective in relaxing a distortion of the Fermi surface. To make this more precise, let us define

$$n_m \equiv \int_0^\infty k\,dk \int_0^{2\pi} d\phi\, e^{im\phi}\delta n(\mathbf{k}) , \tag{6.33}$$

where $\delta n(\mathbf{k})$ is the change in occupation of the quasiparticle state at momentum $\mathbf{k}$ relative to the occupation in the ground state at $T=0$, and ϕ is the orientation of $\mathbf{k}$ relative to the x axis. The coefficients n_m characterize distortions of the shape in the Fermi surface, being sensitive to the direction of the wave vector of the quasiparticle, but not being sensitive to the value of the energy. The coefficients n_m can also be defined separately for different points $\mathbf{r}$ in space, within an accuracy that is limited by the Heisenberg uncertainty principle.

The lowest coefficient n_0 measures the change in the area enclosed by the Fermi surface. If n_0 is independent of $\mathbf{r}$, it must remain constant in time, since the total number of quasiparticles is conserved in the system. The coefficients $n_{\pm 1}$ determine the total momentum of the quasiparticles. In the absence of impurities, this quantity should also be conserved, and thus should not relax by quasiparticle scattering. (The value of $n_{\pm 1}$ can oscillate at the cyclotron frequency, however, through coherent interactions with the Chern-Simons field at $\mathbf{q}=0$.) If any of the higher coefficients n_m is slightly different from zero, at initial time $t=0$, however, it will relax due to the scattering mechanism with a rate of order

$$\frac{1}{\tau_m} \approx \frac{m^2}{2k_F^2}\int r(q)q^2 dq , \tag{6.34}$$

where $r(q)$ is the scattering rate for wave-vector transfer q. If the scattering rate $r(q)$ is peaked at $q\approx T^{1/3}$, and the total scattering rate is proportional to T, then for any fixed m we have

$$\frac{1}{\tau_m} \propto m^2 T^{5/3} \tag{6.35}$$

for $T\to 0$. Thus we find that relaxation of the shape of the Fermi surface is slow compared to T in our model, just as in the case of an ordinary Fermi liquid (where $1/\tau_m \propto T^2$).

A greater difference from the case of an ordinary Fermi liquid occurs if we consider the quantity

$$\varepsilon_m \equiv \int_0^\infty k\,dk \int_0^{2\pi} d\phi\, e^{im\phi}\varepsilon_{\mathbf{k}}\delta n(\mathbf{k}) . \tag{6.36}$$

We may interpret the coefficients ε_m for $m\neq 0$ as describing nonuniformities in the energy or temperature of the quasiparticles and quasiholes on various portions of the Fermi surface. In a conventional Fermi liquid the quantities ε_m will relax slowly compared to the temperature T, for small distortions, just as the coefficients n_m. In the present case, however, we expect that ε_m will relax at a rate proportional to T for even values of m. The reason for this is that the scattering of quasiparticles via the gauge field is an efficient mechanism for exchanging energy between quasiparticles at diametrically opposed regions of the Fermi surface, but is much less efficient otherwise. In particular, a low-energy gauge fluctuation with wave vector $\mathbf{q}\ll k_F$ is made up predominantly of particle-hole excitations where the particle wave vector $\mathbf{k}$ is nearly perpendicular to the wave vector $\mathbf{q}$. For a given direction of $\mathbf{q}$, this occurs at the two points on the Fermi surface which are perpendicular to $\mathbf{q}$. When a quasiparticle with wave vector $\mathbf{k}$ decays by emitting a gauge fluctuation, the gauge fluctuation will have predominantly wave vector $\mathbf{q}\perp\mathbf{k}$, and will carry away on average about half of the quasiparticle energy. This energy is then redistributed among other quasiparticles and quasiholes with $\mathbf{k}'\perp\mathbf{q}$. We see that the energy is thus concentrated at points with $\mathbf{k}'$ parallel or antiparallel to $\mathbf{k}$.

The analysis given above applies to the case of short-range interactions $v(r)$. For the Coulomb interaction $v(r)\propto 1/r$, the modifications are relatively straightforward. Equations (6.31) and (6.32) are replaced by

$$\varepsilon_k \propto (k-k_F)|\ln(k-k_F)| , \tag{6.37}$$

$$m^* \propto |\ln(k-k_F)| \propto |\ln\varepsilon_k| . \tag{6.38}$$

The characteristic relaxation rate for density fluctuations remains of the form $\omega_q \propto q^2$, as in the RPA, and the dominant scattering mechanism for quasiparticles of energy

ε_k will come from long-wavelength gauge fluctuations with $\mathbf{q}\perp\mathbf{k}$ and $q\propto\varepsilon_k^2$. The imaginary part of ε_k is expected to be slightly smaller than the real part by a logarithmic factor; the relaxation rate of the coefficients n_m which describes distortions of the Fermi surface should vary as $\approx m^2T^2$ for $T\to 0$, while the relaxation of the coefficients ε_m should occur at rate $\propto T$ for even values of m.

We may note that in both the short-range case and the Coulomb-interaction case, the quasiparticle contribution to the specific heat, which we obtain with the renormalized mass m^* given by (6.32) or (6.38), has the same dependence on temperature for $T\to 0$ as the collective-mode contribution computed in Sec. VI B.

In all cases, the weight of the quasiparticle pole in the fermion Green function G will be vanishingly small for an infinite system. The behavior of the electron Green function has not been studied, but we also expect that there will be no contribution to it from a single quasiparticle excitation.

The properties of the modified Fermi-liquid theory outlined above bear a certain resemblance to theories of a "marginal Fermi liquid" or of a "tomographic Luttinger liquid" which have been suggested by various authors in the context of two-dimensional models of high-temperature superconductivity.[61,62] It is not clear to us whether this resemblance is other than superficial, however.

E. Additional remarks

Our expectation that the long-wavelength behavior of $K_{\mu\nu}$ is unaffected by the fluctuations responsible for divergences in m^* is given further support by the work of Ioffe and Kalmeyer,[63] who have studied the problem of a system of bosons or fermions coupled to a fluctuating transverse gauge field. Ioffe and Kalmeyer compute in particular the particular the diamagnetic susceptibility χ for the boson problem. They find that, while the divergences encountered in the Green function appear in individual diagrams, they are canceled for the final result. Using a static approximation for the gauge-field fluctuations, they find a correction of the form

$$\delta\chi\sim T\frac{\partial n_B}{\partial\varepsilon}\bigg|_{\varepsilon=0}, \tag{6.39}$$

where $n_B(\varepsilon)=1/(e^{\beta(\varepsilon-\mu)}-1)$ is the free boson occupation function. For the fermion problem, they find that within the static approximation the answer is the same as Eq. (6.39) except that n_B is replaced by $n_F=1/(e^{\beta(\varepsilon-\varepsilon_F)}+1)$. Thus the correction is exponentially small in ε_F/T. It is expected that if dynamical fluctuations in the gauge field are taken into account, a power-law correction in T would arise. This is indeed the case for the imaginary part of K_{11}, where it is found that[64]

$$\mathrm{Im}K_{11}\approx\frac{\omega}{v_Fq}\left[1+\left[\frac{\max(T,\omega)}{\varepsilon_F}\right]^{2/3}\right]. \tag{6.40}$$

Thus the higher order correction to K_{11} appears to vanish for low temperatures and frequencies.

F. Fractional quantized Hall states at $\nu=p/(2p+1)$

Fluctuations in the transverse gauge field, which can lead to a divergent renormalization of m^* at $\nu=\frac{1}{2}$, should also have an effect on the size of the energy gap at the nearby fractional quantized Hall states, where $\nu=p/(2p+1)$. Neglecting these effects, we found in Secs. III and IV that, for large p, the energy gap $E_g^{(\nu)}$ should be proportional to $(m^*|2p+1|)^{-1}$. If interactions with gauge fluctuations are properly taken into account, we would expect that m^* in this formula should be replaced by its value at the energy gap $E_g^{(\nu)}$, i.e., for short-range interactions,

$$m^*\sim\mathrm{const}(E_g)^{-1/3}, \tag{6.41}$$

while for Coulomb interactions,

$$m^*\sim\mathrm{const}|\ln E_g|. \tag{6.42}$$

This leads to the result, for short-range interactions,

$$E_g^{(\nu)}\sim\frac{\mathrm{const}}{|2p+1|^{3/2}}, \tag{6.43}$$

while for Coulomb interactions,

$$E_g^{(\nu)}\sim\frac{Ce^2}{\varepsilon l_0}\frac{1}{|2p+1|}\frac{1}{C'+\ln|2p+1|}, \tag{6.44}$$

where C and C' are constants. A more careful analysis of the logarithmic divergences suggests that the correct value for C in (6.44) is $C=\tilde{\phi}^2/\pi=1.27$.

As was noted in Sec. IV, the numerical data on the energy gap for Coulomb interactions at $\nu=\frac{1}{3}$, $\frac{2}{5}$, and $\frac{3}{7}$ can be well fit without use of the logarithmic correction to the effective mass. Equation (6.44) does not fit as well; if we choose $C'=3.0$ to fit the gap at $\nu=\frac{1}{3}$, then the value at $\nu=\frac{2}{5}$ is 13% too small.

Gros and MacDonald[38] have performed numerical calculations of the energy eigenvalues for finite-size systems of particles in the lowest Landau level with a "hard-core interaction," in which only the pseudopotential coefficient for the lowest angular momentum state is different from zero. These calculations allow crude estimates of the energy gap at $\nu=\frac{1}{3}$, $\frac{2}{5}$, and $\frac{3}{7}$ which also appear quite consistent with the simple formula $E_g^{(\nu)}\propto|2p+1|^{-1}$. The values of p are sufficiently small, however, and the uncertainties of extrapolation to infinite system size are sufficiently large that the asymptotic behavior of Eq. (6.43) must also be considered compatible with the data.

It is possible to envision a diagrammatic calculation at $\nu=p/(2p+1)$ in which one self-consistently calculates the contribution to the fermion propagator arising from the diagram in Fig. 2, and one calculates $D_{\mu\nu}(q,\omega)$ or $K_{\mu\nu}(q,\omega)$ in an approximation which includes, in addition to the RPA diagrams discussed in Sec. III, corrections to $K^0_{\mu\nu}$ arising from exchange of a transverse gauge fluctuation such as was taken into account by Ioffe and

Kalmeyer[63] in the problem they considered. Hopefully, in such a calculation one could recover explicitly the renormalization of $E_g^{(\nu)}$ that we have predicted in (6.43) and (6.44). Hopefully, also, one would find that the lowest excitation mode of $K_{\mu\nu}$ occurs as a simple pole at an excitation frequency $\omega_1(q)$ which approaches $E_g^{(\nu)}$ for $q\to\infty$, and remains >0 for all values of q.

We note that the diverging reduction of the quasiparticle residue z in the one-fermion Green function, arising from long-wavelength excitations at the bare cyclotron frequency ω_c, which we found for $\nu=\frac{1}{2}$, also occurs in the quantized Hall states at $\nu=p/(2p+1)$. Thus proper care must be taken in letting the system size go to infinity, for this case as well as for $\nu=\frac{1}{2}$.

G. The disordered case

We consider briefly the effects of exchange of a gauge fluctuation on the conductivity tensor σ_{ij} for the system at $\nu=\frac{1}{2}$. We are interested here in the limit $q=0$, $\omega\to 0$ of the conductivity tensor $\sigma_{ij}(\mathbf{q},\omega)$ defined in Appendix B. Since the pure system obeys Galilean invariance, we must consider the disordered case to get nontrivial results.

From Appendix B, we find that the physical resistivity tensor ρ_{xx} and ρ_{xy} is given by

$$\rho=\begin{bmatrix}\tilde{\rho}_{xx} & -(2\pi\hbar/e^2)\tilde{\phi}\\ (2\pi\hbar/e^2)\tilde{\phi} & \tilde{\rho}_{xx}\end{bmatrix}, \tag{6.45}$$

where $\tilde{\rho}_{xx}=\tilde{\sigma}_{xx}^{-1}$ and $\tilde{\sigma}_{xx}$ is the conductivity for a Fermi system scattered by gauge field fluctuations and by impurities. In general, $\tilde{\sigma}$ consists of a residual zero-temperature part $\tilde{\sigma}_0$ and a temperature-dependent correction $\delta\sigma(T)$. The term $\tilde{\sigma}_0$ is due to impurity scattering and its inverse is given by Eq. (5.10). The temperature-dependent part has contributions from (i) inelastic scattering from the gauge-field fluctuations, (ii) weak-localization corrections, and (iii) interaction effects in the presence of disorder. In (iii), the interaction is mediated by the gauge field propagator $D_{\mu\nu}$. As pointed out in Sec. V, the leading weak-localization effect is suppressed because time-reversal symmetry is broken in the impurity-scattering process. The importance of (i) and (iii) depends on whether we have long-range Coulomb interaction or not and is discussed briefly below.

We first consider the case of short-range interactions. First consider process (i). Using the disordered form of Eq. (6.3), we find that D_{11} has a denominator which vanishes as q^2 and $-i\omega$ for small q and ω. We can estimate the transport scattering rate in the same way as the calculation of Σ'' in Eq. (6.20), except that an extra factor of q^2/k_F^2 must be inserted in the integrand to account for the momentum relaxation. We find that the singular behavior of D_{11} leads to a temperature-dependent contribution to $\tau_{\rm tr}^{-1}$ varying as $T^{3/2}$, which is enhanced compared with the standard Fermi-liquid result of T^2. Assuming that the impurity-scattering rate is large enough so that the momentum of the gauge field relaxes to the laboratory frame before it is returned to the fermions, we can assume Mathiesson's rule operates and we simply add the static impurity and the dynamic scattering rates. In this case the $T^{3/2}$ correction is the leading temperature-dependent term in the weak disorder limit ($k_F l\gg 1$, where $l=v_F\tau_{\rm tr}$).

For process (iii), it is known that, in ordinary metals, the combined effect of short-range interaction and disorder leads to a correction[51,53]

$$\delta\sigma\sim\frac{\lambda e^2}{2\pi\hbar}\ln(T\tau_{\rm tr}), \tag{6.46}$$

where $\lambda>0$ is a dimensionless coupling constant. This is a consequence of dressing the density vertex which couples the fermion to D_{00} by a ladder of impurity scattering. We note that for $k_F l\gg 1$ this is a small correction to

$$\tilde{\sigma}_0\approx\frac{e^2}{4\pi\hbar}(k_F l)\quad\text{for } k_F l\gg 1. \tag{6.47}$$

However, the present problem is more complicated in that the fermions also interact by exchanging D_{11} and D_{01} gauge fields. Due to the vanishing denominator in D_{11} in the small q,ω limit mentioned earlier, these processes also lead to logarithmic corrections to $\delta\sigma$. The coefficient and sign of the leading logarithmic correction requires a detailed calculation which has not been undertaken at this point.

Next we discuss the case of *Coulomb interactions.* In this case $D_{11}(q,\omega)\approx[-\omega\gamma_q+\tilde{\chi}'(q)q^2]^{-1}$ and the denominator is of the form $-i\omega+|q|$, which is less singular than the short-range case. As a result, we find that process (i) yields only a standard T^2 correction to σ. Similarly, for process (iii) the exchange of D_{11} and D_{01} is not sufficiently singular to lead to logarithmic corrections. Thus the only logarithmic correction comes from an exchange of D_{00} and we recover the same result as the exchange of screened Coulomb interaction in a conventional metal except that our fermions are spinless. In this case,[51,53]

$$\delta\sigma=\frac{e^2}{4\pi^2\hbar}\ln(T\tau_{\rm tr}). \tag{6.48}$$

This term reduces the conductivity and the system scales toward strong coupling, at which point the perturbation expansion in $(k_F l)^{-1}$ breaks down. (Presumably this occurs when $\bar{\sigma}$ becomes of order e^2/h.) In conventional metals this is interpreted as the transition to an insulating state. For the present problem, we do not know what the strong-coupling fixed point should be. However, we may speculate that the effect is to drive the system into a regime where impurity scattering dominates over the electron-electron interaction, and the system looks at least qualitatively like a system of noninteracting electrons near the center of a Landau level in the presence of disorder. In this case there is only a single value of B for which there is an extended state at the Fermi level at $T=0$. For any value of B in the neighborhood of this critical value, the states at the Fermi level are localized, and therefore $\sigma_{xx}=0$ at $T=0$. The value of σ_{xy} will be 0 or 1, depending on whether the value of B is less than or

greater than the critical B_c.[26–28,45]

If the initial value of $\tilde{\sigma}_0$ given by (6.47) is large, corresponding to a large initial value of the mean free path l, then one would have to go to exceedingly low temperatures before $\tilde{\sigma}$ becomes of order e^2/h. If (6.48) is correct, this should occur at a very low temperature,

$$T_0 \propto \tau_{\rm tr}^{-1} e^{-\pi k_F l} . \tag{6.49}$$

For temperatures larger than T_0, impurity effects are presumably not dominant, and there should be a metallic region near $\nu=\frac{1}{2}$, over a finite range of B, where our theory should be applicable.

Finally, we remark that in the usual case of interacting fermions in a disordered potential there are anomalous magnetoresistance effects which set in at low magnetic fields and have a logarithmic field dependence.[51] Formally the role of magnetic field is played by ΔB in our problem. However, we do not expect the usual anomalous magnetoresistance in our case. The reason is that the anomalous magnetoresistance originates from two sources: (1) the sensitivity of the particle-particle ladder to the breaking of time-reversal breaking by a magnetic field; and (2) spin-splitting effects. In our case, time-reversal symmetry is already broken and we assume full spin polarization, so that both known sources of anomalous magnetoresistance are absent.

VII. POSSIBLE EXPERIMENTAL CONSEQUENCES

A. Fermi-surface effects

We have already indicated several implications of the existence of a Fermi surface at $\nu=\frac{1}{2}$, which should have direct experimental consequences if the impurity scattering can be made sufficiently small. The most readily observable of these consequences, the existence of a set of most prominent series of Hall plateaus at filling fraction $\nu=p/(2p+1)$, has indeed been observed experimentally, and in fact has been noted previously by various authors.[23,38] A more stringent test of the validity of our modified Fermi-liquid theory would come from an experimental verification of the prediction of Eq. (6.44) for the relative sizes of energy gaps in the series $\nu=p/(2p+1)$. We have seen that these formulas work moderately well for the first few fractional Hall states in the series, based on numerical calculations on finite systems. However, the small size of these systems makes it difficult to obtain reliable results for energy gaps for the larger values of p, which would be necessary to properly test the theory.

Unfortunately, experience has shown that the presence of impurity scattering in actual samples also makes it difficult to extract from experimental data useful information about the energy gap for an ideal system without impurity scattering. Even in the case of the largest fractional quantized Hall gaps, at $\nu=\frac{1}{3}$ and $\frac{2}{3}$, it is only quite recently, in samples of extremely high mobility, that an energy gap approaching the ideal energy gap has been observed.[65] Since our estimate of the effects of impurity scattering on the transport properties near $\nu=\frac{1}{2}$ gives scattering rates larger than those that are actually observed in the best samples, we are not in a good position to speculate on the possible improvements that might result from new techniques of sample preparation.

The formulas of Sec. III lead to predictions for the size of the de Haas–van Alphen oscillations and for oscillations in the compressibility $dn/d\mu$ as a function of a filling fraction ν in an ideal system at $T=0$. These formulas can be readily generalized to finite temperatures as well. Again, however, it is difficult to make contact with experimental results because of our poor understanding of the effects of impurities. The effects of impurities on these measurements may be particularly strong, since it is known that localized electron states caused by impurities can contribute to the thermodynamic properties near a quantized Hall plateau, even though they do not contribute to the transport properties. In fact, even for the integer quantized Hall effect, the density of states in the energy gap, measured by various thermodynamic methods, is typically found to be a substantial fraction of the density of states near the center of a Landau level.[66]

We have already argued that the amplitude of the Kohn anomaly in response functions near $2k_F$ is likely to be quite small, and difficult to observe even in the absence of impurity scattering. In the presence of impurities, these effects would presumably be reduced still further.

B. Magnetoresistance oscillations in a modulated system

A possible means of observing Fermi-surface effects might be a variation of the experiments reported by Gerhardts, Weiss, and von Klitzing,[67] and by Winkler, Kotthaus, and Ploog in 1989.[68] These authors observed the magnetoresistance of a two-dimensional electron gas modulated by a one-dimensional superlattice potential with period a in the range of 0.5 μm. They observed an oscillatory structure in the magnetoresistance which corresponds to the condition that the classical cyclotron diameter $2R_c$ is a multiple of the period a. For example, Gerhardts, Weiss, and von Klitzing report maxima and minima in the resistivity ρ_{xx} perpendicular to the lines of the grating which are given by

$$\frac{\hbar}{eB} = \frac{a}{2k_F}(n+\phi) , \tag{7.1}$$

where n is an integer, and the phase shift $\phi \cong 0.17$ for maxima and $\phi \approx -0.25$ for minima. Prominent oscillations were observed in the field range 0.15–0.6 T, for a sample with an electron density $n_e \approx 3\times 10^{11}$ cm^{-2}.

If there exists a Fermi surface at $\nu=\frac{1}{2}$, then, at least in principle, one ought to see magnetoresistance oscillations in a modulated sample near $\nu=\frac{1}{2}$, with the quantity ΔB substituted for B in (7.1). In particular, in the absence of impurity scattering, the quasiparticles should move in a circular orbit with a radius

$$R_c^* = \frac{B}{\Delta B k_F} . \tag{7.2}$$

Thus there should be maxima or minima in the resistivity when $2R_c^*$ is a multiple of a, or

$$\frac{e\Delta B}{\hbar} \equiv \frac{eB}{\hbar} - 4\pi n_e = \frac{4(\pi n_e)^{1/2}}{a(n+\phi)} \ . \tag{7.3}$$

The values of ΔB corresponding to these oscillations will be much smaller than the values which correspond to the de Haas–Shubnikov oscillations or fractional quantized Hall states, predicted by (1.3), because a superlattice period a in the range of several tenths of a micrometer is large compared to the quantity $2(\pi n_e)^{-1/2}$.

We must also inquire what are the conditions on impurity scattering that must be satisfied if this effect is to be seen. A necessary condition is presumably that $e\Delta B/m^*c$ be small compared to the transport scattering rate $1/\tau_{\rm tr}$. This condition might be difficult to satisfy if the scattering rate is as large as we have estimated theoretically in this section, but the condition should not be so difficult to satisfy in practice if one uses the transport scattering rate implied by experimental values of ρ_{xx}, which are significantly smaller than our estimate.

C. Specific heat

Another prediction of our model is the existence of a nearly linear electronic specific heat at low temperatures for $\nu=\frac{1}{2}$. If one considers only the contribution from quasiparticle and quasihole excitations close to the Fermi surface, then one expects a specific-heat contribution

$$C_{\rm qp} = \frac{\pi}{6} m^* k_B^2 T \ , \tag{7.4}$$

where we have estimated m^* in the mean-field theory, for Coulomb interactions, in Sec. IV as $m^* \approx 3.3 k_F \varepsilon/e^2$. When gauge fluctuations were taken into account, we found logarithmic corrections to this result [cf. Eqs. (6.13), (6.15), and (6.38)]. However, the coefficient of the logarithm may be relatively small; e.g., the coefficient $(C'/4\pi)$ in (6.13) is smaller than the coefficient in (7.4) by a factor ≈ 5.

The electronic specific heat of a two-dimensional electron system is not easily measured experimentally, but comparisons with calculations in finite-size systems should be possible.

D. Surface acoustic wave propagation

Perhaps the most interesting experimental results concerning the two-dimensional electron system at $\nu=\frac{1}{2}$ are the recent observation by Willett and co-workers[32,33] of an anomaly in the propagation of surface acoustic waves (SAW's) in a magnetic field near this point. Our Fermi-liquid description of the $\nu=\frac{1}{2}$ system predicts an anomaly in the SAW propagation which is in good qualitative agreement with these observations.

The effect of the 2D electron gas on the SAW propagation is essentially determined by the value of the density-response function $K_{00}(q,\omega)$ at the wave vector $\mathbf{q}$ and frequency $\omega = v_s q$ of the sound wave.

The experiments are conducted at a wavelength sufficiently large that the static compressibility of the electron system is dominated by the Coulomb interaction $v(q)$. (At $\nu=\frac{1}{2}$, this condition should be satisfied as long as $q \ll k_F$.) Also, the sound velocity v_s is small compared to v_F^* so that $\omega \ll v_F^* q$. Under these conditions, we may use Eq. (B6), ignoring the second term on the right-hand side, to write

$$K_{00}(q,\omega=v_s q) \approx \frac{\varepsilon q}{2\pi e^2[1 - i\sigma_m/\sigma_{xx}(q)]} \ , \tag{7.5}$$

where $\sigma_{xx}(\mathbf{q})$ is the longitudinal wave-vector-dependent conductivity, evaluated at low frequencies as discussed in Appendix B. We have assumed $\mathbf{q} \| \hat{x}$, and

$$\sigma_m \equiv \frac{v_s \varepsilon}{2\pi} \ . \tag{7.6}$$

For a heterostructure near the surface of GaAs, using ε as a weighted average of the dielectric constant in the vacuum and the medium, which depends on the distance to the surface and the wave vector q,[69] the value of σ_m corresponds to $\approx 6\times 10^{-7}\ \Omega^{-1}$.[33] If (7.5) is used in the equations for SAW propagation, one obtains a velocity shift Δv_s and an attenuation rate κ for the SAW amplitude of the form

$$\frac{\Delta v_s}{v_s} = \frac{\alpha^2}{2} \frac{1}{1+[\sigma_{xx}(q)/\sigma_m]^2} \ , \tag{7.7}$$

$$\kappa = \frac{q\alpha^2}{2} \frac{[\sigma_{xx}(q)/\sigma_m]}{1+[\sigma_{xx}(q)/\sigma_m]^2} \ , \tag{7.8}$$

where α is a constant proportional to the piezoelectric coupling in GaAs. In the limit of sufficiently long wavelengths, $\sigma_{xx}(q)$ becomes the macroscopic dc conductivity σ_{xx}, in which case Eqs. (7.7) and (7.8) reduce to the standard formulas employed to describe the effects of a 2D electron system on SAW propagation at low frequencies.[32,69] (Note that the convention for the velocity shift is such that $\Delta v_s = 0$ when $\sigma_{xx} = \infty$; i.e., when the electron layer behaves like a perfectly conducting plane.)

In analyzing their experiments, Willett *et al.*[33] found that the variations of both Δv_s and κ could be fit with the same choice of $\sigma_{xx}(q)$ in (7.7) and (7.8). To achieve this, however, and to maintain consistency with dc conductivity measurements, it was necessary to use a value of σ_m approximately four times larger than is obtained from (7.6) with a reasonable value of ε.

In order to discuss the SAW propagation at $\nu=\frac{1}{2}$, we wish to substitute for $K_{00}(q,\omega)$ the RPA results obtained in Secs. III and VI. The formula for $\sigma_{xx}(q)$ may then be rewritten as

$$\sigma_{xx}(q) = \frac{\rho_{yy}(q)}{\rho_{xy}^2} \ , \tag{7.9}$$

where $\rho_{xy} = 4\pi\hbar/e^2$, and

$$\frac{1}{\rho_{yy}(q)} = e^2 \lim_{\omega\to 0} \frac{1}{\omega} {\rm Im} K_{11}^0(q,\omega) \ . \tag{7.10}$$

(We work in a regime where $\rho_{xx}\rho_{yy} \ll \rho_{xy}^2$.) From our previous discussion, we have

$$\rho_{yy}(q)=\frac{2\pi}{k_F}q\frac{\hbar}{e^2} \quad \text{for } q\gg\frac{2}{l}\ , \tag{7.11a}$$

$$\rho_{yy}(q)=\frac{4\pi}{k_F l}\frac{\hbar}{e^2} \quad \text{for } q<\frac{2}{l}\ , \tag{7.11b}$$

where l is the transport mean free path at $\nu=\frac{1}{2}$ [cf. Eqs. (2.28), (6.2), and (6.4)].

A linear dependence of $\rho_{yy}(q)$ on q, for $q\gg 1/l$ as predicted by Eq. (7.11a), is precisely what is needed to explain the frequency dependence of the SAW anomaly at $\nu=\frac{1}{2}$, according to the analysis of Willett *et al.*[33] However, the absolute value of $\sigma_{xx}(q)$ obtained from (7.9) and (7.11a), which has no adjustable parameters, is approximately a factor of 2 smaller than the values obtained by Willett *et al.*

The physical reason for the wave-vector dependence of the transverse resistivity $\rho_{yy}(q)$, and hence of the SAW anomaly, is not difficult to see. Essentially, $\rho_{yy}(q)$ is determined by the relaxation rate for a transverse current fluctuation at wave vector $\mathbf{q}$ for a collection of noninteracting fermions with effective mass m^*. If q is small compared to l^{-1}, the relaxation rate is determined by scattering from the impurities. If q is large, however, the Fourier component of the current can relax without scattering, due to motion of the fermions in the direction of $\mathbf{q}$. The effective mass m^* drops out of the final results (7.11), because the factor of $(m^*)^{-1}$ in the fermion velocity is canceled by a factor in the magnitude of the current fluctuation in equilibrium. (See Appendix B.)

Our analysis at $\nu=\frac{1}{2}$ can be extended to discuss the behavior of the SAW anomaly when magnetic field B deviates by an amount ΔB from the value at $\nu=\frac{1}{2}$. We can again use Eq. (7.10) to find $\rho_{yy}(q)$, but now the fermions which enter K_{11}^0 are assumed to move in an effective magnetic field ΔB. Then for $ql>1$, the mechanism for relaxation of the transverse current by fermion motion parallel to $\mathbf{q}$ will be cut off when the effective cyclotron radius R_c^*, given by (7.2), becomes shorter than $\approx q^{-1}$. This predicts that for large values of the SAW frequency ω, the range of values ΔB for which the SAW anomaly is seen should increase linearly with ω, roughly according to the formula

$$|\Delta B|\approx\frac{Cqk_F\hbar}{e}=\frac{CBq}{k_F}\ , \tag{7.12}$$

where C is a constant of order unity, and the last equality makes use of the relation between k_F and B at $\nu=\frac{1}{2}$.

A more quantitative estimate of $\sigma_{xx}(q)$ is obtained using the semiclassical analysis of Appendix B. We continue to use Eq. (7.9), but now we must use

$$\rho_{yy}(q)=\frac{\tilde{\sigma}_{xx}}{\tilde{\sigma}_{xx}\tilde{\sigma}_{yy}+\tilde{\sigma}_{xy}^2}\ , \tag{7.13}$$

where $\tilde{\sigma}_{ij}(q)$ is given by Eqs. (B17)–(B22).

We note that the behavior predicted by these formulas is rather complicated, and for $\Delta B\neq 0$, the value of $\sigma_{xx}(q)$ is dependent on l even for $ql\gg 1$. In the limit $ql\gg 1$, the semiclassical approximation predicts oscillations at small values of ΔB, with maxima occurring close to the zeros of $J_1(qR_c^*)$, or roughly at

$$\frac{\Delta B}{B}\approx\frac{q}{\pi(n+1/4)k_F}\ , \tag{7.14}$$

where n is an integer. This behavior is illustrated in Fig. 3, for several values of ql. These effects are analogous to "geometric resonances" which are found in the propagation of acoustic waves in a direction perpendicular to an applied magnetic field in a three-dimensional metal.[70] Observation of such effects near $\nu=\frac{1}{2}$ in a 2D electron system would be a striking demonstration of the existence of a Fermi surface in the system.

Experimental results for the half-width of the SAW anomaly near $\nu=\frac{1}{2}$ are in good agreement with Eq. (7.12), with a value of $C\approx 0.4$. On the other hand, if we try to assign a half-width of the form of (7.12) to the theoretical curves in Fig. 3, we find that the most reasonable choice is $C\approx\frac{1}{3}$, corresponding to the arrows in the figure. Thus we find that the experimental widths and the theoretical ones are in good agreement.

Willett *et al.*[32] have observed that the SAW anomaly decreases with increasing temperature, for temperatures which are still far below the Fermi energy E_F^* that we would obtain using the value of m^* deduced in Sec. V.

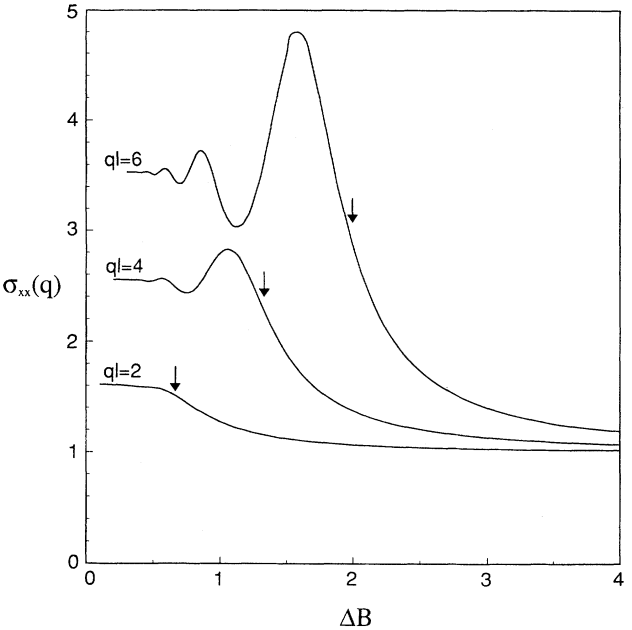

FIG. 3. The wave-vector-dependent conductivity $\sigma_{xx}(\mathbf{q})$, as a function of the deviation ΔB of the magnetic field from the value at $\nu=\frac{1}{2}$, for several values of the wave-vector q, as calculated in the semiclassical approximation using the formula in Appendix B. The conductivity is measured in units of $\sigma_{xx}(0)$, which is independent of ΔB in this approximation, while ΔB is measured in terms of the unit $B_l=\hbar ck_F/(el)$, where l is the transport mean free path of the quasiparticles at $\nu=\frac{1}{2}$. The quantity $\sigma_{xx}(q)$ determines the attenuation and velocity shift in a surface acoustic wave experiment. Arrows correspond to values of ΔB given by Eq. (7.12), with $C=\frac{1}{3}$, or $qR_c^*=3$.

The SAW anomaly was found to persist to higher temperature for larger values of q. We believe that this effect can probably be explained by collisions between thermally excited fermion quasiparticles. Such collisions do not lead to decay of the total momentum, but they should lead to a decrease in $\rho_{yy}(q)$ relative to the value in (7.11a) when the collision-induced mean free path becomes smaller than q^{-1}.

E. Far-infrared absorption and Raman scattering

In a pure system, Kohn's theorem applies, and optical absorption occurs only at the cyclotron frequency ω_c. If translational invariance is broken, however, either because of random impurities or because of an imposed periodic spatial modulation of the electron system, then other degrees of freedom may also be excited.[71,72] In the latter case, the optical absorption is proportional to $\mathrm{Im}K_{00}(q,\omega)$, where ω is the radiation frequency and q the wave vector of the modulation.

The behavior of $K_{00}(q,\omega)$ has been discussed at various points in this paper, primarily in the context of the RPA. A particularly interesting structure may occur in high mobility samples for suitable values of q, and values of B shifted somewhat away from $\nu=\frac{1}{2}$. In addition to possibilities of geometric resonances at small frequencies ω, the results of the RPA or of the semiclassical formulas, given by Eqs. (B6), (B8)–(B11), and (B14)–(B17), predict absorption peaks at a discrete set of frequencies, approximately given by $\omega=n(\Delta\omega_c^*)$. Interaction terms beyond the RPA may cause strong decay of excitations above the lowest few branches of the spectrum, so that the discrete nature of the spectrum may become rapidly washed out at high frequencies, except for the cyclotron mode which remains intact and underdamped at small wave vectors.

The region most easily accessible to experiment is in the regime $qR_c^* \ll 1$, where most of the weight of $K_{00}(q,\omega)$ is in the cyclotron mode. The pole in $K_{00}(\mathbf{q},\omega)$ at the lowest excitation frequency $\omega_1(q)$, which is $\approx 2\Delta\omega_c^*$ for $q\to 0$ in the RPA, has a residue which is smaller by a factor of order $(ql_0)^2$ than the residue at the cyclotron frequency. The residues at the other excitation branches are still smaller in this limit, by a factor of at least $(qR_c^*)^2$.

The density-response function $K_{00}(q,\omega)$ can also be probed by Raman-scattering experiments in a geometry such that there is a substantial scattering wave vector $\mathbf{q}$ parallel to the plane.[73]

VIII. EXTENSION TO OTHER FILLING FRACTIONS WITH EVEN DENOMINATORS

The results we have obtained near filling fraction $\nu=\frac{1}{2}$ can be extended in various ways to other filling fractions with even denominators.

A. Case $\nu=n+\frac{1}{2}$

The most straightforward case to consider is filling fraction $\nu=n+\frac{1}{2}$, where n is a positive integer. In this case, we may hope to describe the ground state as a state where the first n Landau levels are completely full, while level $(n+1)$ is precisely one-half full. The basis states for electrons in any given Landau level are in one-to-one correspondence with states in the lowest Landau level and matrix elements of any centrally symmetric two-body potential can be uniquely expressed in terms of a discrete set of pseudopotential coefficients v_l, for relative angular momenta l. (For spinless fermions, only odd l need be considered.) As before, we assume spinless electrons, or equivalently, that the partly filled Landau level is completely spin polarized. For a given potential $v(\mathbf{r}-\mathbf{r}')$, the pseudopotential coefficients will be different in different Landau levels, but the behavior at large l should be similar in any Landau level. (For the Coulomb interactions, $v_l \propto l^{-1/2}$ for large l.)

If the Fermi surface at $\nu=\frac{1}{2}$ exists and is stable for a large class of potentials, then it should also exist and be stable in any higher Landau level for some other class of potentials, namely those potentials which lead to similar pseudopotential coefficients in the specified Landau level like the ones which lead to the Fermi surface in the lowest Landau level. At present, our theoretical understanding is not sufficient to tell us what are the conditions necessary for a potential to give a stable Fermi surface in a half-filled Landau level, but it seems at least reasonable to hope that the Coulomb potential falls in this class for a spin-polarized electron system in any low-lying Landau level.

We note that for $\nu=n+\frac{1}{2}$, the Fermi wave vector $k_F=1/l_0$ is the same as for the state $\nu=\frac{1}{2}$ at the same value of the *magnetic field*. For $n>0$, however, k_F is related to the total electron density by $k_F=(2\pi n_e/\nu)^{1/2}$, which is no longer the same as for the same density of spinless electrons in zero magnetic field.

B. Case $\nu=n\pm 1/(2k)$

Another straightforward generalization of our analysis is to consider fermions defined as an electron with $\tilde{\phi}$ flux quanta attached, where now $\tilde{\phi}\equiv 2k$ is an even integer greater than 2. The fictitious magnetic field will again cancel the external field if $\nu=1/\tilde{\phi}$, so that in this way we can generate states such as $\frac{1}{4}$, $\frac{1}{6}$, etc. Of course as the value of $\tilde{\phi}$ increases, the magnitude of fluctuations in the Chern-Simons vector potential increases as well, so the validity of an approximation in which one ignores these fluctuations becomes even more questionable.

In fact, there is good theoretical and experimental evidence that, in the vicinity of $\nu=\frac{1}{6}$, the ground state of the 2D electron system, in the absence of impurities, should be a Wigner crystal.[74] We interpret this to mean that, for $\tilde{\phi}=6$, the Fermi surface is actually unstable to the formation of charge-density waves. Of course this should also apply to larger values of $\tilde{\phi}$.

We note that, in the vicinity of $\nu=(2k)^{-1}$, we expect to find de Haas–van Alphen oscillations and conductance

minima when

$$e\Delta B \equiv (eB - 4\pi k n_e) = \frac{2\pi n_e}{p} \tag{8.1}$$

and p is a positive or negative integer. This leads to a series of fractional quantized Hall states with[23]

$$\nu = p/(2kp+1) . \tag{8.2}$$

The Fermi wave vector at $\nu=(2k)^{-1}$ is in each case given by

$$k_F = (4\pi n_e)^{1/2} , \tag{8.3}$$

which is the same as for spinless electrons in zero magnetic field.

States at $\nu=\frac{3}{4}$ and $\frac{5}{6}$ can be obtained from the states at $\nu=\frac{1}{4}$ and $\frac{1}{6}$ by particle-hole conjugation. The Fermi-surface radii k_F are the same for $\nu=(2k)^{-1}$ and $\nu=1-(2k)^{-1}$ in the same value of magnetic field B, but of course these correspond to different electron densities n_e. More generally, starting from a parent state with $\nu_0=n$, we can construct in this way a metallic state at $\nu=n\pm(2k)^{-1}$, with a Fermi-surface radius k_F that is related to n_e by

$$k_F = \left[\frac{4\pi n_e}{2kn \pm 1} \right]^{1/2} . \tag{8.4}$$

C. States descending from a fractional quantized Hall state

The mathematical transformation we have carried out for electrons or holes added to the vacuum state or a state with an integer number of filled Landau levels can also be extended to quasiparticles or quasiholes in a fractional quantized Hall state. For definiteness, let us consider the situation where the parent state is one of the elementary Laughlin states with filling fraction

$$\nu_0 = \frac{1}{2k+1} , \tag{8.5}$$

where k is a non-negative integer. The elementary excitations from this state are quasiparticles and quasiholes, obeying fractional statistics, with statistical angle[9]

$$\theta = \frac{\pi}{2k+1} \tag{8.6}$$

and with charge $-Qe$ where $Q=\pm(2k+1)^{-1}$. We can represent the quasiparticle or quasiholes as fermions, provided that we introduce a Chern-Simons gauge field and we attach to each fermion a fluxtube containing $\tilde{\phi}$ flux quanta, where

$$\tilde{\phi} = 2l + 1 - \frac{\theta}{\pi} \tag{8.7}$$

and l is an arbitrary integer. If the density of quasiparticles or quasiholes is n_q, then each fermion feels an average effective magnetic field B^* given by

$$eQB^* = eQB - 2\pi\tilde{\phi} n_q . \tag{8.8}$$

The total charge density $-en_e$ is obtained by adding the charge density $-eQn_q$ of the quasiparticles to the charge density contained in the parent Hall state $-\nu_0 Be^2/2\pi$. Thus we have

$$n_q = \frac{n_e}{\nu} \frac{(\nu-\nu_0)}{Q} , \tag{8.9}$$

$$eB^* = \frac{2\pi n_e [Q^2 - \tilde{\phi}(\nu-\nu_0)]}{\nu Q^2} . \tag{8.10}$$

If we consider the case of *quasiparticles,* with $Q=(2k+1)^{-1}$, then we can achieve the condition $B^*=0$ by choosing $l \geq 0$ in (8.7) and choosing ν to satisfy

$$\nu = \frac{2l+1}{(2k+1)(2l+1)-1} . \tag{8.11}$$

For the case of *quasiholes* with $Q=-(2k+1)^{-1}$, in order to achieve $B^*=0$ we must choose l such that $\tilde{\phi}$, given by (8.7), is negative and we obtain

$$\nu = \frac{(2l'+1)}{(2k+1)(2l'+1)+1} , \tag{8.12}$$

where $l' \equiv -(l+1) \geq 0$. The Fermi wave vector in either case is given by

$$k_F = (4\pi n_q)^{1/2} . \tag{8.13}$$

As some simple examples, let us consider cases where the starting state is $\nu_0=\frac{1}{3}$, corresponding to $k=1$. If we add quasiparticles we obtain, from (8.9) and (8.11), for $l=0$ and 1,

$$\nu=\tfrac{1}{2}, \quad k_F=(4\pi n_e)^{1/2} \quad \text{for } l=0 , \tag{8.14}$$

$$\nu=\tfrac{3}{8}, \quad k_F=\left[\frac{4\pi n_e}{3}\right]^{1/2} \quad \text{for } l=1 . \tag{8.15}$$

If we add quasiholes instead, we find

$$\nu=\tfrac{1}{4}, \quad k_F=(4\pi n_e)^{1/2} \quad \text{for } l'=0 , \tag{8.16}$$

$$\nu=\tfrac{3}{10}, \quad k_F=\left[\frac{4\pi n_e}{3}\right]^{1/2} \quad \text{for } l'=1 . \tag{8.17}$$

The states at $\nu=\frac{1}{4}$ and $\frac{1}{2}$ are states we have previously encountered, but the states $\nu=\frac{3}{8}$ and $\frac{3}{10}$, and the other states with $l \geq 1$ or $l' \geq 1$ are new fractions with even denominators.

It is natural to inquire whether or not the states $\nu=\frac{1}{2}$ and $\frac{1}{4}$ that we build out of quasiparticle or quasiholes added to the $\nu=\frac{1}{3}$ state are the same as the states we previously constructed from electrons added to the vacuum. We note that the value of k_F given by (8.14) or (8.16) is the same for these two states as the value of k_F given by (8.3), which we obtained using electrons with $\tilde{\phi}=2k$. This is of direct significance since the quantity k_F can be measured, in principle, by looking for a Kohn anomaly in the density-response function at wave vector $2k_F$. It is possible that there are other physical properties which distinguish the alternative constructions of the metallic states. In the absence of other information, however, it seems sensible to work in all cases with the simplest con-

struction, working with integer charges where possible, and more generally choosing the parent state with the largest energy gap.

As another important example, we may consider a situation where the parent state is a fractional quantized Hall state of the form $\nu_0=p/(2p+1)$. In this case the quasiparticles have $Q=1/(2p+1)$ and $\theta=\pi(2p-1)/(2p+1)$.[9] From (8.7), and the condition that the average Chern-Simons field $n_q\tilde{\phi}$ should cancel the effective external magnetic field $-eQB$ seen by the quasiparticles, we find that

$$\tilde{\phi}=2l+\frac{2}{2p+1}\ , \tag{8.18}$$

$$\nu=\frac{2pl+1}{(2l+1)(2p+1)+1-2p}\ . \tag{8.19}$$

If we choose $l=1$ in these formulas, we obtain

$$\nu=\frac{2p+1}{4p+4}\ . \tag{8.20}$$

This gives the series of even-denominator fractions $\nu=\frac{3}{8}$, $\frac{5}{12}$, $\frac{7}{16}$, etc., which sit between the dominant quantized Hall states of $\nu=\frac{1}{3}$, $\frac{2}{5}$, $\frac{3}{7}$, $\cdots$. However, as was noted in Sec. III, we expect that for sufficiently large p the states given by (8.20) will be unstable to the formation of domains with electron densities equal to those of the adjacent quantized Hall states.

Assuming the existence of the fractionally charged "Fermi-liquid" state at the even-denominator fractions ν given by Eq. (8.20), it is natural to ask what are the most prominent odd-denominator quantum Hall states that can be derived from these states in analogy with our construction of the states $\nu=p/(2p+1)$, by adding a field $\Delta B=2\pi n_e/pe$ to the integer-charged Fermi liquid at $\nu=\frac{1}{2}$.

The condition that k Landau levels are filled by a set of fractionally charged fermions is given by

$$Qe|B^*|=\frac{2\pi n_q}{|k|}\ , \tag{8.21}$$

where B^* is the average effective magnetic field seen by the quasiparticle, given by (8.8) or (8.10). Using (8.9), we find that the filling fraction corresponding to (8.21) is

$$\nu_{\pm k}=\nu_0+\frac{Q^2}{\tilde{\phi}\pm k^{-1}}\ . \tag{8.22}$$

This method of constructing additional quantized Hall states with odd denominators was discussed previously by Jain[23] and by Read.[11]

In the case where the parent state is a principal quantized Hall state, with $\nu_0=p/(2p+1)$, we find, for the first two solutions of (8.22),

$$\nu_1=\frac{3p+1}{6p+5}\ , \tag{8.23}$$

$$\nu_{-2}=\frac{3p+2}{6p+7}\ . \tag{8.24}$$

[The ν_{-1} states are excluded because $\nu_{-1}=(p+1)/(2p+3)$ is simply one of the principal quantized Hall states which we have previously encountered.] The states $\nu_{\pm k}$, other than ν_{-1}, are odd-denominator states which do not belong to Jain's principal series, though they do appear as higher-order quantized Hall states in Jain's picture as well as in the original hierarchical scheme.[8,9] For example, in the hierarchical scheme, the state ν_1 is a daughter state of the state ν_0, while ν_{-2} is a daughter state that is obtained by adding quasiholes to the state ν_{-1}, which itself is a daughter of the state ν_0. In the limit $k\to\infty$, the states $\nu_{\pm k}$ given by (8.22) approach the even-denominator fraction ν, given by (8.20), which lies midway between the two principal quantized Hall states ν_0 and ν_{-1}. Thus we may think of the entire series as derived from a Fermi-liquid state at the fraction ν.

As an example, we may choose $p=1$ in the above equations, so that Eq. (8.20) gives $\nu=\frac{3}{8}$, lying midway between $\nu_0=\frac{1}{3}$ and $\nu_{-1}=\frac{2}{5}$. Equations (8.23) and (8.24) then predict quantized Hall states at $\nu_1=\frac{4}{11}$ and $\nu_{-2}=\frac{5}{13}$. Observation of these states might be interpreted as supporting the existence of a fractionally charged Fermi-liquid state at $\nu=\frac{3}{8}$. At present there is no clear evidence of a quantized Hall state at either $\frac{4}{11}$ or $\frac{5}{13}$. However, there is evidence for a quantized Hall state at $\nu=\frac{7}{11}$, which is the particle-hole conjugate of $\frac{4}{11}$ if the system is fully spin polarized.[50,75]

D. Transport properties

We next consider the electrical resistivity of the various even-denominator states considered above. As before, we assume that n_q quasiparticles with charge $-e^*=-Qe$ and statistics θ have been added to a parent incompressible quantum Hall state with filling factor ν_0. By attaching $\tilde{\phi}$ quanta of Chern-Simons flux to each quasiparticle where $\tilde{\phi}$ satisfies Eq. (8.7), we convert them to a collection of fermions, with a kinetic-energy term represented by

$$K\approx\frac{1}{2m^*}\int d^2r\,\psi^+(\mathbf{r})[-i\nabla+e^*\mathbf{A}(\mathbf{r})-\mathbf{a}(\mathbf{r})]^2\psi(\mathbf{r})\ . \tag{8.25}$$

The quasiparticle density n_q and the effective average magnetic field B^* seen by the quasiparticles are given by (8.9) and (8.10). The field B^* will vanish if we choose ν to be the even-denominator fraction given by

$$\nu_l=\nu_0+Q^2[(2l+1)-\theta\pi^{-1}]^{-1}\ . \tag{8.26}$$

We assume that the resulting "Fermi liquid" is decoupled from the parent quantum Hall state, so that, for frequencies small compared to the energy gap of the parent state, and wave vectors which are correspondingly small, we can add the conductivity tensors of the background state and the quasiparticles:

$$\sigma_{ij}(\mathbf{q},\omega)=\sigma_{ij}^{Q}(\mathbf{q},\omega)+\sigma_{ij}^{b}\ , \tag{8.27}$$

where

$$\sigma_{ij}^{b}=\frac{\nu_0e^2}{2\pi}\begin{bmatrix}0&1\\-1&0\end{bmatrix} \tag{8.28}$$

and σ_{ij}^{Q} is the conductivity of the unbound quasiparticles.

In the case where the parent state is an integer-quantized Hall state, there is evidence to support this point of view from experiments on transport in edge channels near the boundary of a sample. In particular, it has been shown that the edge channels corresponding to lower filled Landau levels are effectively decoupled from the last filled Landau level, and a simple decoupled network model has been used successfully to analyze the resistance data.[76]

The quasiparticle conductivity $\sigma_{ij}^{Q}(q,\omega)$ can be estimated using a straightforward extension of the semiclassical approximations of Appendix B and the Born approximation for impurity scattering discussed in Sec. V. In particular, we may write, in analogy with (B8) and (B9), the matrix equation

$$[\sigma^{Q}]^{-1}\equiv\rho^{Q}=\tilde{\rho}+Q^{-2}\rho_{\rm cs}\ , \tag{8.29}$$

where $\rho_{\rm cs}$ is given by (B11), and $\tilde{\rho}$ is approximated by the resistivity tensor for a collection of noninteracting fermions of charge Qe.

Let us first estimate the mean free path of the fermions, using the model of Sec. V, where a density $n_{\rm imp}$ of uncorrelated impurities is assumed to lie in a doping layer a distance d_s away from the electron layer. An impurity at $\mathbf{r}=0$ produces an excess electron density in the layer given by $\rho(q)=e^{-qd_s}$, which is manifested by an excess quasiparticle density $Q^{-1}\rho(q)$. This leads in turn to a static Chern-Simons flux $i\mathbf{q}\times\mathbf{a}(\mathbf{q})=2\pi\tilde{\phi}Q^{-1}e^{-qd_s}$, which scatters the fermions. If we use the first Born approximation to calculate the scattering rate, as in Sec. V, we find that the mean free path l is given by

$$\frac{1}{l}=\frac{m^*}{k_F\tau_{\rm tr}}=\frac{n_{\rm imp}}{k_F}\frac{\pi\tilde{\phi}^2Q^{-2}}{k_Fd_s}\ , \tag{8.30}$$

with k_F given by (8.13). The quasiparticle resistivity, at $q=\omega=0$, is then given by $\rho_{xx}^{Q}=\tilde{\rho}_{xx}=\rho_0$, where

$$\rho_0=\frac{n_{\rm imp}}{n_q}\frac{\pi\tilde{\phi}^2}{e^2Q^4k_Fd_s}\ . \tag{8.31}$$

The physical resistivity, obtained using (8.27)–(8.31) and $\rho=\sigma^{-1}$, is

$$\rho_{xx}=\frac{n_{\rm imp}}{n_e}\frac{n_q}{n_e}\frac{\pi\tilde{\phi}^2}{e^2Q^2k_Fd_s}=\frac{(\nu-\nu_0)^2}{\nu^2}\rho_0\ , \tag{8.32}$$

while the Hall resistance is given by

$$\rho_{yx}=\frac{1}{\nu}\frac{2\pi}{e^2}=\frac{(\nu-\nu_0)}{\nu_0}\rho_{xy}^{Q}\ , \tag{8.33}$$

assuming weak disorder, so that $\tilde{\rho}_{xx}\ll 2\pi/e^2$. Equation (8.32) could also have been derived directly by using the analysis which led from (5.8) to (5.10) in Sec. V, provided that one takes into account the fact that in the present case only a fraction $(\rho_{yx}/\rho_{yx}^{Q})$ of the total current is carried by the quasiparticles and is subject to dissipation.

We can now illustrate these equations with a few examples. Let us first consider the case $\nu=n+\frac{1}{2}$, which we obtain from the above formulas using $\nu_0=n$, $Q=1$, and $n_q=n_e/(2n+1)$. In this case we obtain

$$\rho_{xx}=\frac{n_{\rm imp}}{n_e}\frac{\pi\tilde{\phi}^2}{e^2}\frac{1}{k_Fd_s}\frac{1}{2n+1}\ . \tag{8.34}$$

Since k_F scales as $(2n+1)^{-1/2}$, (8.34) predicts that ρ_{xx} for $\nu=n+\frac{1}{2}$ scales as $(2n+1)^{-1/2}$ or as $B^{1/2}$ for fixed n_e. Experimentally it seems that ρ_{xx} scales linearly with B over a wide range of field strengths,[50] if the regions close to a quantized Hall plateau are ignored. We believe that the main reason for this discrepancy is that the Born approximation is unlikely to be correct for large values of n, as the mean free path predicted by (8.30) will then be smaller than the correlation length ($\approx d_s$) for fluctuations in the static Chern-Simons field. Thus the fact that Eq. (8.34) shows correct qualitative behavior, in that ρ_{xx} decreases with decreasing B, may be as much as one could reasonably hope for.

It is worth contrasting (8.34) with the well-known results of a self-consistent calculation of the resistivity for *noninteracting* electrons due to short-range impurity scattering,[71,77] which yields ρ_{xx} equal to $(4\hbar/e^2)(2n+1)^{-1}$. While the scaling with $(2n+1)^{-1}$ is in better agreement with experiment, the absolute magnitude is wrong by one or two orders of magnitude and is independent of impurity concentration. In our model, the degeneracy of the Landau level is broken by interactions, and a perturbation calculation in disorder is possible, leading to Eq. (8.34).

Our Born-approximation estimates can also be applied to other values of ν. For example, we may consider the state $\nu=\frac{1}{4}$ constructed by binding four flux quanta to an electron. We have $\tilde{\phi}=4$, $Q=1$, and $n_q/n_e=1$. Compared with the $\nu=\frac{1}{2}$ state at the same n_e, we have $\rho_{xx}(\frac{1}{4})/\rho_{xx}(\frac{1}{2})=4$. For the particle-hole conjugate state $\nu=\frac{3}{4}$, we have $\tilde{\phi}=4$, $Q=1$, $\nu_0=1$, $n_q/n_e=\frac{1}{3}$, and we find $\rho_{xx}(\frac{3}{4})/\rho_{xx}(\frac{1}{2})=4/\sqrt{3}$.

Next we consider the states $\nu=(2p+1)/(4p+4)$ constructed by adding particles to the quantum Hall states $\nu_0=p/(2p+1)$ using (8.19) with $l=1$. In this case we have $n_q/n_e=(2p+1)^{-1}$, so that when normalized to the resistivity at $\nu=\frac{1}{2}$, for the same electron density, Eq. (8.33) predicts that

$$\frac{\rho_{xx}(\nu)}{\rho_{xx}(1/2)}=\left[1+\frac{1}{2p+1}\right]^2(2p+1)^{3/2}\ . \tag{8.35}$$

This result is based on the Born approximation for the scattering of the fermions by the gauge field, which is not reliable for reasons mentioned earlier. We do not believe that Eq. (8.35) is applicable to the physical situation, and it is included here only for completeness.

E. Surface acoustic wave propagation

The longitudinal conductivity $\sigma_{xx}(\mathbf{q})$ which determines the anomaly in surface acoustic wave propagation can be readily calculated using (8.27)–(8.29) and the semiclassical approximation discussed in Appendix B.

At the even-denominator fraction ν_l given by (8.26) we find $\sigma_{xx}(q)\approx\rho_{yy}(q)(\nu e^2/2\pi)^2$, where $\rho_{yy}(q)$ is given by

Eq. (B23), using values of ρ_0 and l which are given by (8.30) and (8.31).

We may also calculate $\sigma_{xx}(q)$ for filling factors slightly different from ν_l, using (8.27)–(8.29) and (B18)–(B22), where now we must define R_c^* by

$$R_c^* = \frac{k_F}{QeB^*} , \tag{8.36}$$

with B^* given by (8.8) or (8.10). If we define

$$\Delta B \equiv B - \frac{2\pi n_e}{\nu_l e} , \tag{8.37}$$

then we find

$$B^* = \Delta B \left[1 + \frac{\nu_0 \tilde{\phi}}{Q^2} \right] . \tag{8.38}$$

As an example of these relations, let us consider the case where $\nu_0 = n$, $\tilde{\phi} = \pm 2k$, and $Q = \pm 1$, $l = 1$, so that $\nu_l = n \pm (2k)^{-1}$. [This case includes in particular the cases $\nu_l = n + \frac{1}{2}$ and $\nu_l = (n \pm \frac{1}{4})$.] Then we find $|B^*| = |\Delta B|(2kn \pm 1)$. We also find $k_F = [4\pi n_e / (2kn \pm 1)]^{1/2}$. If we define the width of the anomaly in $\sigma_{xx}(q)$ as the value of $|\Delta B|$ at which $qR_c^* = C^{-1}$, in analogy with Eq. (7.12), then we find, for a fixed value of n_e and q,

$$|\Delta B| \propto (2kn \pm 1)^{-3/2} . \tag{8.39}$$

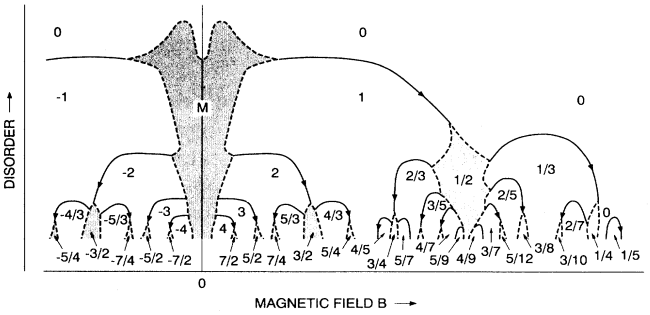

FIG. 4. The hypothesized phase diagram, for spinless electrons, at very low temperatures. The phase diagram is sketched for a fixed electron density, as a function of magnetic field B, and as a function of the disorder strength, ignoring the logarithmic divergence at $T=0$, which arises from "weak localization" or from density fluctuations in the presence of Coulomb interactions and disorder. The unshaded regions, labeled by integers or by fractions with odd denominators, indicate quantized Hall states (or an insulator) where $\sigma_{xx}=0$ at $T=0$, and σ_{xy} has the indicated value, in units of e^2/h. Shaded regions labeled by fractions with even denominators are metallic states where $\sigma_{xx} \neq 0$, as is the region labeled M. The value of ρ_{yx}^{-1} should vary continuously in the metallic regions, passing through the value indicated by the labeling fraction near the center of each region. Further details are explained in the text.

IX. OVERALL PHASE DIAGRAM

The overall phase diagram for a system of two-dimensional electrons in a strong magnetic field at very low temperatures, which follows from our previous discussions, is sketched in Fig. 4. The case of spinless electrons has been considered for simplicity, and we have neglected the effects of logarithmic divergences, such as weak localization, which ultimately must be considered at $T=0$.

The horizontal axis in Fig. 4 is the magnetic field B, while the vertical axis represents the strength of the disorder potential plotted on a logarithmic scale. Unshaded regions, labeled by nonzero integers or by fractions with odd denominators, indicate quantized Hall states where $\sigma_{xx}=0$ at $T=0$, and σ_{xy} has the labeled value in units of e^2/h. The unshaded region labeled 0 is an insulator, with $\sigma_{xx}=\sigma_{yy}=0$. Shaded regions labeled by fractions with even denominators are metallic states where $\sigma_{xx} \neq 0$, and where σ_{xy} has variable value. Near the center of each shaded region, the value of ρ_{yx}^{-1} should pass through the value indicated by the labeling fraction, and for the appropriate value of the magnetic field the metallic state is supposed to exist even in the limit of zero disorder. The shaded region labeled M is the ordinary low-field metal, with $\rho_{yx}=0$ at $B=0$.

The solid lines represent a direct transition driven by disorder between one quantized Hall state and another which is its parent state in the usual hierarchical schemes. These transitions have been discussed by various authors,[26–28,45] and are believed to be perfectly sharp at $T=0$. The downward arrows on the solid curves show the portions of the curves which are "normal" in the sense that the Hall resistance ρ_{yx} increases when the line is crossed in the direction of increasing B. There may also be rising portions of the curves which are reentrant in the sense that ρ_{yx} decreases with increasing B.

The broken curves which separate the metallic regions from the quantized Hall or insulating states are probably not sharp, when weak logarithmic effects are taken into account. Most likely, in limit of $T \to 0$, the metallic regions slowly shrink. If Eq. (6.48) is valid, we might guess that the condition for metallic behavior is that T should be larger than $T_0 \approx \tau_{\rm tr}^{-1} e^{-\pi k_F l}$ given by (6.49), where l is the mean free path before renormalization.

Note that each metallic region other than the normal metal M has, entering near its top, a single solid-line transition on the normal descending portion of the curve. In regions of lower and lower disorder, we find a series of subsidiary quantized Hall transition lines, leaving the metallic region, presumably on the sides of the regions.

The normal metallic region M is similar to the other metallic regions in Fig. 4 except that (1) there is no solid-line quantized Hall transition entering the top of this region, and (2) the effects of weak localization which blur the metal-insulator transition are expected to be stronger in zero magnetic field than in finite B. Weak-localization effects near $B=0$, and the associated negative magnetoresistance, are the reasons why we have indicated a dip in the upper boundary of the metallic region near $B=0$.

The indication of an insulating phase in Fig. 4 between

the metallic phase at $\nu=\frac{1}{4}$ and the quantized Hall state at $\nu=\frac{1}{5}$ is based on experiments which suggest a reentrant Wigner crystal phase in this region.[78]

If electrons with spin had been considered rather than spinless electrons, the phase diagram would have been generally more complicated, as states with varying amounts of spin polarization may compete with each other in the phase diagram, and more possibilities arise for forming quantized Hall states with even denominators, through a mechanism involving formation of spin-singlet pairs. The importance of the spin degree of freedom depends critically on the effective g factor of the carriers, which determines the size of the Zeeman energy relative to the cyclotron energy and to the electron correlation energies at a given filling factor.

For Landau-level filling factors less than 1, and for large enough g factor, all electrons will have their spins aligned with the magnetic field so that the spin degree of freedom can be ignored. For electrons in a GaAs heterostructure, the Zeeman energy is anomalously small relative to the cyclotron energy, because the g factor is small while m_b^{-1} is large, so that complete polarization cannot be taken for granted.[9,79] For example, numerical calculations of small finite systems suggest that the quantized Hall state at $\nu=\frac{2}{5}$ and at the particle-hole conjugate fraction $\nu=\frac{8}{5}$ should be unpolarized at sufficiently low electron densities.[13,79] Experimental evidence that the system is not fully polarized has been found at $\nu=\frac{8}{5}$ for appropriate samples,[80,81] but not in the case of $\nu=\frac{2}{5}$. There also exists experimental evidence for lack of complete spin polarization in the neighborhood of $\nu=\frac{2}{3}$ and $\frac{3}{5}$ for low-density samples.[81–83] (This might account for the difference in the behavior of ρ_{xx} near $\nu=\frac{4}{11}$ and $\frac{7}{11}$, referred to in Sec. VIII.[13]) However, we are not aware of any evidence for lack of complete polarization at $\nu=\frac{1}{2}$ for GaAs samples.

Finally, we caution the reader that the diagram in Fig. 4 is only schematic, intended to emphasize the topology of the phase diagram. Quantitative features such as the relative heights on the diagram of various metallic regions and quantized Hall states should not be taken seriously.

X. CONCLUSIONS

In this paper, we have explored in some detail the consequences of a theory of electrons in a partially filled Landau level, where the starting point is a mathematical transformation to a collection of fermions interacting with a Chern-Simons gauge field. We find that the theory is self-consistent and leads to a variety of predictions which could be tested, at least in principle, by experiments or by exact numerical calculations on finite systems. We find good qualitative agreement with experiments that are currently available, although there exist significant quantitative discrepancies. Since our quantitative calculations are primarily derived from a random-phase approximation, which is itself based on a perturbation expansion in a parameter $\tilde{\phi}$ that is not small in our problem, the quantitative discrepancies are perhaps not surprising. It is clear that further theoretical and experimental work is needed, but we consider the successes of the theory to be quite encouraging and we believe that we have identified the proper starting point for describing properties of a strongly interaction electron system in a Landau level near to half-filling.

From a purely conceptual point of view, the most striking feature of the theory is the implication that, for an ideal sample with no impurity scattering, precisely at $\nu=\frac{1}{2}$ (and at various other filling fractions with even denominators) there exists at $T=0$ a sharp Fermi surface of some kind. As discussed in Sec. VI D, we believe that there must be divergent renormalization of the effective mass m^* at the Fermi surface, whose precise form depends on the assumed behavior of the electron-electron interaction at large distances, and that there must be other modifications of the traditional Fermi-liquid theory as a result of the strong interaction between quasiparticles arising from the coupling to density fluctuations via the Chern-Simons vector potential. Nevertheless, the most important implications of a sharp Fermi surface should survive.

A first success of the theory is that it provides additional insight into Jain's explanation for the existence of prominent fractional quantized Hall states at filling fractions $\nu=p/(2p+1)$. These states are regarded[23] as integer quantized Hall states of the $\nu=\frac{1}{2}$ quasiparticles in an effective magnetic field ΔB which is the deviation of the applied magnetic field from its value at $\nu=\frac{1}{2}$. The asymptotic form of the energy gap at $\nu=p/(2p+1)$ obtained from our mean-field theory fits very well to numerical calculations of the gap for the states $\nu=\frac{1}{3}$, $\frac{2}{5}$, and $\frac{3}{7}$, as discussed in Sec. V. When we include the logarithmic corrections expected from gauge field fluctuations, assuming Coulomb interactions between electrons, we find that the fit is less good, as noted in Sec. VI F. However, there are relatively large uncertainties in the gaps at $\nu=\frac{2}{5}$ and $\frac{3}{7}$, and the values of p may be too small to allow an accurate fit to the asymptotic form (6.44).

A second important success of the theory is the explanation it provides for the observed occurrence of an anomaly in surface acoustic wave propagation in GaAs samples with a two-dimensional electron system near $\nu=\frac{1}{2}$, $\frac{3}{2}$, $\frac{1}{4}$, and $\frac{3}{4}$. Experimental results for both the sound velocity shift and attenuation coefficient are parametrized by a single quantity, which one may identify with the longitudinal conductivity $\sigma_{xx}(\mathbf{q})$ of the system, at the wave vector $\mathbf{q}$ equal to the wave vector of the acoustic wave. We predict that $\sigma_{xx}(\mathbf{q})$ should increase linearly with q, for q larger than the mean free path of the quasiparticles, which is in good agreement with experiment. We also predict that near $\nu=\frac{1}{2}$ the anomalous portion of $\sigma_{xx}(\mathbf{q})$ should occur over a range of magnetic fields ΔB which is proportional to q. This feature also is in good agreement with the experimental results. The absolute value of $\sigma_{xx}(q)$ predicted by our RPA theory is too small, however, by a factor of approximately 2, while the width ΔB is close to the experimental values.

Another area where the theory has a qualitative success but quantitative difficulties is in our predictions for

the quasiparticle mean free path, or for the macroscopic resistivity ρ_{xx}. Our analysis is based on a model in which the source of the impurity scattering is a layer of uncorrelated charged impurities, with a density $n_{\rm imp}$ equal to that of the electrons, set back a distance d_s from the electron layer. Our approximations predict, correctly, that for $k_F d_s \gg 1$, the resistivity at $\nu=\frac{1}{2}$ is small compared to ρ_{yx}, but large compared to the zero-field resistivity. As was discussed in Sec. VIII, the theory also predicts correctly that, for $\nu=n+\frac{1}{2}$, the value of ρ_{xx} decreases with increasing n. However, there is no quantitative agreement between the theory, which is based on the first Born approximation, and experimental measurements of ρ_{xx}.

In addition to resistivity and surface acoustic wave propagation, we discussed in Sec. VII several other possible experiments which could eventually provide evidence for the validity of the theory. The wave-vector- and frequency-dependent conductivity $\rho_{xx}(\mathbf{q},\omega)$ may be probed in principle by infrared absorption in the presence of a grating, or by Raman-scattering measurements with significant in-plane wave vector $\mathbf{q}$. With a suitable choice of $\mathbf{q}$ and of the deviation ΔB of the magnetic field from the value at $\nu=\frac{1}{2}$, it may be possible to see resonant structure at a frequency $\omega_1(q)$ which is of order $2\Delta\omega_c^*=2e\Delta B/m^*c$.

Another possible experiment is to perform transport measurements in a sample with a periodic modulation along one direction in the plane. Our theoretical analysis leads us to predict size-effect oscillations as a function of the field deviation ΔB, the period in $(1/\Delta B)$ determined by k_F and the period a of the modulation according to Eq. (7.3). Similar oscillations are predicted for surface acoustic wave measurements at high frequency, where the period in $(1/\Delta B)$ is now determined by k_F and the acoustic wave vector q, according to Eq. (7.14). Observation of either of these oscillatory effects should provide strong support for the validity of the theory.

Note added in proof

In a recent paper, independent of our work, V. Kalmeyerand and S.-C. Zhang [Phys. Rev. B **46**, 9889 (1992)] have considered the effects of impurities on the electron system near $\nu=\frac{1}{2}$, in an approximation which emphasizes the effects of static gauge fluctuations, as in Sec. V of the present paper. They do not present a quantitative estimate of the mean-free path, and they do not discuss the logarithmic corrections to the conductivity (6.48) which we find arising from the combination of interaction and disorder effects, but their general conclusions, including the overall topology of the phase diagram, are similar to ours.

Recently, R. R. Du, H. L. Stormer, D. C. Tsui, L. N. Pfeiffer, and K. W. West have reported a measurement of the energy gaps at a series of quantized Hall states with $\nu=p/(2p+1)$. They find a linear dependence on $\Delta B=B-B_{1/2}$, in good agreement with the present theory as given by Eqs. (4.11) or (6.44).

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APPENDIX A: LAGRANGIAN FORMULATION

In this appendix we present a Lagrangian formulation which describes fermions tied to flux tubes and includes the long-range Coulomb interaction between the fermions. We write

$$Z=\int D\psi Da_i Da_0 e^{-S}\,, \tag{A1}$$

where, in the imaginary time formalism,

$$S=\int_0^\beta d\tau\, d^2r\, \mathcal{L} \tag{A2}$$

and the Lagrangian density is

$$\mathcal{L}=\psi^*(\partial_\tau-ia_0)\psi-\frac{1}{2m_b}\psi^*(\partial_i-ia_i+eA_i)^2\psi - \frac{ia_0}{2\pi\tilde{\phi}}\varepsilon^{ij}\partial_i a_j-\mu\psi^*\psi\,. \tag{A3}$$

We work in the Coulomb gauge so that $\partial_i a_i=0$. The next to last term in (A3) is the Chern-Simons term introduced by Zhang and co-workers,[29] so that integration over a_0 enforces the constraint that the two-dimensional curl of $\mathbf{a}$ is given by

$$\nabla\times\mathbf{a}\equiv\varepsilon^{ij}\partial_i a_j=2\pi\tilde{\phi}\psi^*(r)\psi(r)\,, \tag{A4}$$

which is in agreement with Eq. (2.6). In addition, we can introduce a Coulomb interaction between the fermions of the form

$$\mathcal{L}_1=\frac{1}{2}\int d^2\mathbf{r}'\,\psi^*(\mathbf{r})\psi(\mathbf{r})v(\mathbf{r}-\mathbf{r}')\psi^*(\mathbf{r}')\psi(\mathbf{r}')\,. \tag{A5a}$$

Using the constraint (A4), this can be written as

$$\mathcal{L}_1=\frac{1}{2}\frac{1}{(2\pi\tilde{\phi})^2}\int[\nabla\times\mathbf{a}(\mathbf{r})]v(\mathbf{r}-\mathbf{r})[\nabla\times\mathbf{a}(\mathbf{r}')]d^2\mathbf{r}'\,. \tag{A5b}$$

The quantum-mechanical problem defined by the Lagrangian $\mathcal{L}+\mathcal{L}_1$ is equivalent to the problem defined by the Hamiltonian $H=K+V$, given in Eqs. (2.5)–(2.8).

The random-phase approximation may be obtained in the Lagrangian formulation by making a saddle-point approximation, where we expand about the point where $a_0=(\mathbf{a}-e\,\mathbf{A})=0$. In the vicinity of this point, we may integrate out the fermion fields, and derive an effective action for the gauge field by writing $Z=Z_0\int Da_0 Da_1\exp(-\int\mathcal{L}_{\rm eff})$, where Z_0 is the partition

function with no fluctuations of the gauge fields. Up to second order in the gauge fields, one finds

$$\mathcal{L}_{\rm eff}=\frac{1}{2}\sum_{q,\omega_n}\sum_{\mu,\nu=0,1} a_\mu^*(\mathbf{q},\omega_n)\mathcal{D}_{\mu\nu}^{-1}(q,\omega_n)a_\nu(\mathbf{q},\omega_n)\ , \tag{A6}$$

where $a_\nu(\mathbf{q},\omega_n)$ is the fluctuation at wave vector $\mathbf{q}$ and Matsubara frequency ω_n,

$$\mathcal{D}_{\mu\nu}^{-1}=\begin{bmatrix}\mathcal{K}_{00}^0 & 0\\ 0 & \mathcal{K}_{11}^0\end{bmatrix}+\begin{bmatrix}0 & \dfrac{iq}{2\pi\tilde{\phi}}\\ \dfrac{-iq}{2\pi\tilde{\phi}} & \dfrac{v(q)q^2}{(2\pi\tilde{\phi})^2}\end{bmatrix}\ , \tag{A7}$$

and $\mathcal{K}_{\mu\nu}^0$ is the sum of the two free-electron bubbles indicated in Fig. 1.

The second term in (A7) is U^{-1}, where U is given by Eq. (2.16). Thus we obtain the RPA result for the gauge-field propagator

$$\mathcal{D}=(\mathcal{K}^0+U^{-1})^{-1}\ . \tag{A8}$$

Finally we note that upon integrating out a_0 and a_1 in $\mathcal{L}_{\rm eff}$ we obtain the correction to the free energy ΔF given in (6.8).

In order to obtain the RPA gauge-field response function $D_{\mu\nu}(q,\omega)$ for frequencies on the real axis, or the corresponding density-current response function $K(q,\omega)\equiv U^{-1}-U^{-1}DU^{-1}$ discussed in Sec. II, we must analytically continue $\mathcal{D}$ to the real frequency axis in the standard way.

APPENDIX B: THE CONDUCTIVITY TENSOR $\sigma_{ij}(\mathbf{q},\omega)$

The conductivity tensor is generally defined as the response to the total electromagnetic field A_μ. By contrast, it is the external field $A_\mu^{\rm ext}$ which appears in Eq. (2.14) defining the density-current response function $K_{\mu\nu}$. In practice, the actual magnetic field generated by a two-dimensional electron system is always very small, so for the vector potential there is no difference between $\mathbf{A}$ and $\mathbf{A}_{\rm ext}$. For the scalar potential, however, the Fourier components $eA_0(\mathbf{q})$ and $eA_0^{\rm ext}(q)$ differ by the Coulomb potential $v(q)j_0(q)$ arising from the electron-density fluctuation $j_0(q)$. [We confine ourselves here to the case of a Coulomb interaction, where $v(q)=2\pi e^2/\varepsilon q$.]

Let us define the 2×2 matrices

$$V=\begin{bmatrix}v(q) & 0\\ 0 & 0\end{bmatrix}\ , \tag{B1}$$

$$C=\frac{1}{2\pi\tilde{\phi}}\begin{bmatrix}0 & iq\\ -iq & 0\end{bmatrix}\ , \tag{B2}$$

so that the interaction matrix U, defined in (2.16), may be written as $U=V+C^{-1}$. Let us also define the 2×2 matrices $\Pi_{\mu\nu}(\mathbf{q},\omega)$ and $\tilde{K}_{\mu\nu}(\mathbf{q},\omega)$ such that

$$K^{-1}=\tilde{K}^{-1}+U \tag{B3a}$$

$$=\Pi^{-1}+V\ . \tag{B3b}$$

Thus Π consists of the sum of all Feynman diagrams for K which are irreducible with respect to the Coulomb interaction V, while $\tilde{K}$ includes only those diagrams which are irreducible with respect to the Chern-Simons interaction C^{-1} as well as to V. The random-phase approximation (2.15) consists of replacing $\tilde{K}$ by the noninteracting fermion response K^0.

For simplicity, we choose the wave vector $\mathbf{q}$ to lie in the x direction. Then we define the conductivity tensor $\sigma_{ij}(\mathbf{q},\omega)$ by[84]

$$\frac{1}{\sigma_{xx}(\mathbf{q},\omega)}=\frac{iq^2}{\omega}\left[\frac{1}{\Pi_{00}(q,\omega)}-\frac{1}{\Pi_{00}(q,0)}\right]\ , \tag{B4a}$$

$$\sigma_{yy}(\mathbf{q},\omega)=\frac{-i}{\omega}[\Pi_{11}(q,\omega)-\Pi_{11}(q,0)]\ , \tag{B4b}$$

$$\sigma_{xy}(\mathbf{q},\omega)=-\sigma_{yx}(\mathbf{q},\omega)=\frac{i}{q}\Pi_{01}(q,\omega)\ . \tag{B4c}$$

Similarly we may define an intrinsic "quasiparticle conductivity tensor" $\tilde{\sigma}_{ij}(\mathbf{q},\omega)$ by

$$\frac{1}{\tilde{\sigma}_{xx}(\mathbf{q},\omega)}=\frac{iq^2}{\omega}\left[\frac{1}{\tilde{K}_{00}(q,\omega)}-\frac{1}{\tilde{K}_{00}(q,0)}\right]\ , \tag{B5a}$$

$$\tilde{\sigma}_{yy}(\mathbf{q},\omega)=\frac{-i}{\omega}[\tilde{K}_{11}(q,\omega)-\tilde{K}_{11}(q,0)]\ , \tag{B5b}$$

$$\tilde{\sigma}_{xy}(\mathbf{q},\omega)=-\tilde{\sigma}_{xy}(\mathbf{q},\omega)=\frac{i}{q}\tilde{K}_{01}(q,\omega)\ . \tag{B5c}$$

In the limit $\mathbf{q}=0$, $\sigma(\mathbf{q},\omega)$ becomes the frequency-dependent conductivity tensor $\sigma_{ij}(\omega)$ which describes the response to a uniform electric field at frequency ω. Also, using Eqs. (B1), (B3b), and (B4a), one may establish the general relation

$$\frac{1}{K_{00}(\mathbf{q},\omega)}=\frac{2\pi e^2}{\varepsilon q}+\frac{1}{\Pi_{00}(\mathbf{q},0)}-\frac{i\omega}{q^2\sigma_{xx}(\mathbf{q},\omega)}\ . \tag{B6}$$

It follows from Eq. (B3) that

$$\Pi^{-1}=\tilde{K}^{-1}+C^{-1}\ . \tag{B7}$$

We shall be interested in situations where the diamagnetic term $\tilde{K}_{11}(q,0)$ in Eq. (B5) is of order q^2 and where the compressibility $\tilde{K}_{00}(q,0)$ is finite in the limit $q\to0$. Therefore, if we limit ourselves to the frequencies $\omega\gg\omega_{\rm min}$, with $\omega_{\rm min}$ roughly of order q^2, the behavior of $\tilde{K}$ is determined by $\tilde{\sigma}(q,\omega)$. Moreover, for $\omega\gg\omega_{\rm min}$ we see that σ and $\tilde{\sigma}$ are related through the following sequence of matrix equations:

$$\sigma\equiv\rho^{-1}\ , \tag{B8}$$

$$\rho\equiv\tilde{\rho}+\rho_{\rm cs}\ , \tag{B9}$$

$$\tilde{\rho}=\tilde{\sigma}^{-1}\ , \tag{B10}$$

$$\rho_{\rm cs}\equiv\frac{2\pi\hbar\tilde{\phi}}{e^2}\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\ . \tag{B11}$$

Note that ρ_{cs} which is associated with the Chern-Simons term in the Lagrangian, is the resistivity tensor that corresponds to a quantized Hall system at $\nu=\frac{1}{2}$, with the choice $\tilde{\phi}=2$. The result (B9) that one should add the resistivity tensors is a general feature of systems where constraints are enforced by gauge fields.[85]

Let us first consider the case where $\tilde{\sigma}_{xy}=\tilde{\sigma}_{yx}=0$, which occurs at $\nu=\frac{1}{2}$, at least in the RPA. Then we have

$$\rho_{xx}(q,\omega)=\frac{1}{\tilde{\sigma}_{xx}(q,\omega)} , \tag{B12a}$$

$$\rho_{yy}(q,\omega)=\frac{1}{\tilde{\sigma}_{yy}(q,\omega)} . \tag{B12b}$$

In the limit $q\rightarrow 0$, if we use the Drude formula for $\tilde{\sigma}(\omega)$, we obtain

$$\rho_{xx}(\omega)=\rho_{yy}(\omega)=\rho_0(1-i\omega\tau_{tr}) , \tag{B13a}$$

$$\rho_{yx}=-\rho_{xy}=\frac{2\pi\hbar}{e^2}\tilde{\phi} , \tag{B13b}$$

with $\rho_0\equiv m^*/(n_e e^2\tau_{tr})$ as in Sec. VI. Taking the limit $\omega\rightarrow 0$, we recover the formulas of Sec. V, if we calculate the transport scattering rate τ_{tr}^{-1} using the Born approximation for the model described in that section. Similarly, if we calculate $\tilde{\sigma}(q,\omega)$ for free fermions in the absence of impurity scattering, we recover the clean limit formulas of Secs. II, VI, and VII.

The case $\nu\neq\frac{1}{2}$ may also be treated if we make a semiclassical approximation for the motion of the fermions at the Fermi surface, in the presence of the effective magnetic field ΔB. Following the standard methods,[70] one obtains

$$\tilde{\sigma}_{ij}(q,\omega)=\frac{2}{\rho_0}\sum_{n=-\infty}^{\infty}\frac{F_i^{(n)}F_j^{(n)*}}{1-i(\omega-n\Delta\omega_c^*)\tau_{tr}} , \tag{B14}$$

$$F_x^{(n)}\equiv\frac{n}{X}J_n(X) , \tag{B15}$$

$$F_y^{(n)}\equiv\frac{idJ_n(X)}{dX} , \tag{B16}$$

$$X\equiv qR_c^* , \tag{B17}$$

where J_n is the Bessel function, $R_c^*=\hbar k_F/e\Delta B$ is the effective cyclotron radius, and $\Delta\omega_c^*=e\Delta B/m^*$ is the effective cyclotron frequency. (We set $c=1$, though we have restored factors of $\hbar$.) In the limit $\omega\rightarrow 0$, these equations reduce to

$$\tilde{\sigma}_{ij}(q)=\frac{2}{\rho_0}\sum_{n=-\infty}^{\infty}\frac{G_{ij}^{(n)}(X)}{1+n^2\lambda^2} , \tag{B18}$$

$$G_{xx}^{(n)}=\frac{n^2}{X^2}[J_n(X)]^2 , \tag{B19}$$

$$G_{yy}^{(n)}=\left[\frac{dJ_n(X)}{dX}\right]^2 , \tag{B20}$$

$$G_{yx}^{(n)}=-G_{xy}^{(n)}=\frac{n^2\lambda}{X}J_n(X)\frac{dJ_n(X)}{dX} , \tag{B21}$$

$$\lambda\equiv l/R_c^* , \tag{B22}$$

where $l=v_F\tau_{tr}$ is the mean free path. It should be noted that Eqs. (B14)–(B22) are actually derived under the assumption of isotropic impurity scattering rather than small-angle scattering, which we expect to be the dominant loss mechanism in the present system.

For the analysis of surface acoustic wave propagation, we wish to know the value of $\sigma_{xx}(q,\omega)$ for $\omega=v_s q$, where v_s is the sound velocity. Since v_s is small compared to v_F, we would like to identify $\sigma_{xx}(q,\omega)$ with a "low-frequency" conductivity $\sigma_{xx}(\mathbf{q})$, which we obtain from (B8)–(B11) using the zero-frequency limit for $\tilde{\sigma}$ given by (B18). This identification is only valid, however, if ω is large compared to the frequency ω_{min}, below which we may have to take into account the static diamagnetic susceptibility and compressibility in (B5). As remarked above, ω_{min} is roughly of order q^2. More accurately, if we use the values of the compressibility and susceptibility appropriate to $\nu=\frac{1}{2}$, we find $\omega_{min}\approx\hbar q^2(m^*k_F l)^{-1}$ or $\omega_{min}\approx\hbar q^3(m^*k_F l)^{-1}$, whichever is larger. The inequality $\omega\gg\omega_{min}$ is in either case well satisfied for the wave vectors employed in the experiments. We note also that, for q of order $1/R_c^*$, the inequality $v_s\ll v_F$ also implies $\omega\ll\Delta\omega_c^*$.

If the field ΔB is chosen so that the system sits at one of the principal quantized Hall states, and if the temperature and impurity scattering rate are sufficiently small compared to the energy gap, then the compressibility and diamagnetism may become very different from the value at $\nu=\frac{1}{2}$, and the inequality $\omega\gg\omega_{min}$ may not be satisfied at long wavelengths. Of course, one cannot use the semiclassical formula to calculate the conductivity tensor under those circumstances.

We note that the semiclassical result (B14) for $\tilde{\sigma}_{ij}(q,\omega)$ at finite frequencies leads to a result for $\tilde{K}(q,\omega)$ with a form that is very similar to the form of K^0, which we discussed in Sec. III for calculating the RPA response at $\nu=p/(2p+1)$, provided that we take the limit $\Delta\omega_c^*\tau_{tr}\rightarrow\infty$. The most important difference is that in the RPA the lowest branch of the spectrum, corresponding to $n=0$ in (B14), is absent at $T=0$, since all the carriers are frozen out in the quantized Hall state.

The semiclassical approximation with short-range scatterers, which led to (B18), can be evaluated quite simply in the case of $\Delta B=0$, giving

$$\rho_{yy}(q)=\frac{1}{\tilde{\sigma}_{yy}(q)}=\frac{\rho_0}{2}\frac{q^2l^2}{(1+q^2l^2)^{1/2}-1} . \tag{B23}$$

This reduces to the results (7.11) in the appropriate limits

of large or small ql. We note that for $\Delta B=0$, one can invert $\tilde{K}_{\mu\nu}(\mathbf{q},\omega)$, trivially, and one can use (B23) to obtain the conductivity $\sigma(q,\omega)$ at arbitrarily small frequency, without regard to the restriction $\omega>\omega_{\min}$.

The formulas for $\rho_{yy}(q)$ obtained from (B18) reduce to (B23) in the limit $\Delta B\to 0$. In the limit $q\to 0$, for any fixed l and ΔB, the values of ρ_{xx} and ρ_{yy} obtained from (B18) become equal to the resistivity ρ_0 at $\Delta B=0$, while the Hall resistance ρ_{xy} takes on its classical value $2\pi\hbar/\nu e^2$.

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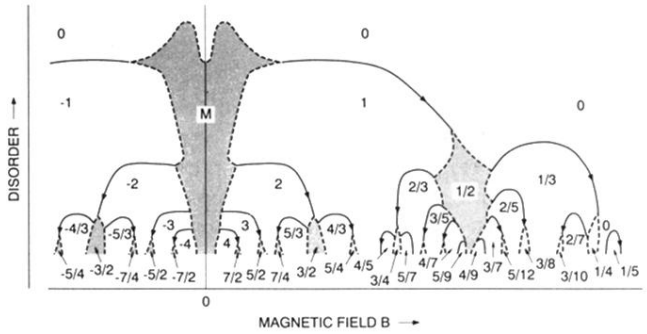

FIG. 4. The hypothesized phase diagram, for spinless electrons, at very low temperatures. The phase diagram is sketched for a fixed electron density, as a function of magnetic field B, and as a function of the disorder strength, ignoring the logarithmic divergence at $T=0$, which arises from "weak localization" or from density fluctuations in the presence of Coulomb interactions and disorder. The unshaded regions, labeled by integers or by fractions with odd denominators, indicate quantized Hall states (or an insulator) where $\sigma_{xx}=0$ at $T=0$, and σ_{xy} has the indicated value, in units of e^2/h. Shaded regions labeled by fractions with even denominators are metallic states where $\sigma_{xx}\neq 0$, as is the region labeled M. The value of ρ_{yx}^{-1} should vary continuously in the metallic regions, passing through the value indicated by the labeling fraction near the center of each region. Further details are explained in the text.