Quantum Hall liquid, Josephson effect, and hierarchy in a double-layer electron system

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Based on a bosonic Chern-Simons gauge theory, we present a microscopic theory of a double-layer electron system in which even-denominator fractional quantum Hall (FQH) states have been observed in a strong external magnetic field. In our approach electrons belonging to different layers are interpreted as two distinguishable anyons with appropriate statistics. Neglecting the Zeeman energy, we find in semiclassical approximation that Hall states are realized as condensed states of the bosonized electrons in both of the layers. We calculate the Coulomb energy of the Hall states as well as some excitation spectrum as a function of the distance between the two layers. We also derive the ground-state wave function, which is found to coincide with that of Halperin for a nonvanishing interlayer distance. We find a superfluidity mode in FQH states at the filling $\nu = 1, \frac{1}{3}, \frac{1}{5}, \ldots$, in which the interlayer tunneling acts as a Josephson junction in the states. It is manifest in our formalism that the Josephson current is given by $J = \Delta_{\rm SAS}\rho_0 \sin(eVt + \delta_0)$ for a constant voltage V applied between the two layers, where $\Delta_{\rm SAS}$ is the gap energy between the symmetric and antisymmetric single-particle states and ρ_0 is the average electron density in one layer. The observation of the Josephson effect constitutes a direct experimental test of the existence of condensation of bosonized electrons in FQH states. A hierarchy of the FQH states is also analyzed by using an effective-field theory of vortex solitons (quasiparticles).

I. INTRODUCTION

The two-dimensional electron system has a rich ground-state structure. A most prominent discovery is the fractional quantum Hall (FQH) effect in a strong magnetic field.[1] The FQH effect at odd-denominator filling factor is well known. Recently, it has been observed also at even-denominator filling factor in a double-layer electron system.[2] To understand these phenomena, one usually makes a variational analysis with use of trial functions[3,4] or a numerical study on a few-electron system.[5–7] Landau-Ginzburg models have also been proposed for the odd-denominator FQH effect in a single-layer system.[8–11] See also Ref. 12.

Recently, we have formulated a microscopic theory of the planar electron system using the Chern-Simons (CS) gauge theory,[13,14] where the electron is represented by a hard-core boson obeying the Pauli exclusion principle. When the hard-core boson is expressed in terms of the ordinary boson, a renormalization counter term appears. This term gives a repulsive contact interaction, and makes a distinctive feature of our formalism. Due to the term we can explicitly see the degeneracy of the states at the lowest Landau level in the absence of the Coulomb interaction as well as the role of the Coulomb interaction which resolves this degeneracy. Furthermore, only with this term we obtain a finite quantum correction to the ground-state energy. The short-range characteristic behavior of the Laughlin wave function can also be derived due to this term. Thus, some of essential physical features of the FQH effect can only be revealed with this term.

The aim of this paper is to analyze various aspects of the double-layer electron system based on this field-theoretical formalism. To simplify the problem, it is assumed that the system contains no impurity and that all spins are completely polarized. It is also assumed that the interlayer tunneling is small enough. Then, we have two sets of electrons, the spin-polarized electrons in one layer and the spin-polarized electrons in the other layer. This double-layer system may be simulated by a single-layer system containing two distinguishable electrons;[6,7] here, each electron is labeled by the layer index, which we call pseudospin. Electrons interact with each other via the Coulomb interaction depending on the interlayer distance. It is found at the filling $\nu = 1, \frac{1}{3}, \frac{1}{5}, \ldots$, that the tunneling acts as if a Josephson junction and that the Josephson effect is to be observed.

In our approach planar electrons are regarded as anyons[15] and we use the CS gauge theory to describe them. We are free to represent the anyon by the boson field or the fermion field, and for our purpose it is convenient to use the boson field. The present system contains two sets of electrons. Hence, we consider two distinguishable anyons with their own statistics parameters α, β and their relative statistics parameter γ; here, α/π and β/π are odd integers while γ/π is an integer. Each anyon carries two types of fluxes associated with two types of CS gauge fields. Thus, in this scheme an electron is decomposed into a boson and two types of flux quanta. The fact $\gamma \neq 0$ implies that there is a statistical interaction between electrons on the two different layers. Physically,

this would be due to the interlayer Coulomb interaction and an overlap of their three-dimensional wave functions. In the present two-dimensional simulation the latter effect may be taken into account by way of tunneling between the two sets of electrons. For the sake of simplicity we suppress the tunneling in most cases. This is justified in the case when its physical effect is negligibly small. However, in some Hall states it plays an important role of deriving the Josephson effect.

The contents of this paper are as follows. In Sec. II, we present a microscopic Hamiltonian describing the double-layer electron system without impurity. We analyze this Hamiltonian in the semiclassical approximation. In the semiclassical approximation we first determine the ground state by minimizing the classical energy of the system: it is a solution of the classical field equations of the Hamiltonian. Thus, we call it the classical ground state or the mean-field ground state. We then consider quantum fluctuations around the classical ground state using the Gaussian approximation.

In Sec. III, we analyze the classical ground states by switching off and on the Coulomb interaction. It is found that the essential features of the ground state are determined by the classical solutions, which describe ensembles of vortex solitons (quasiparticles) in general. However, when and only when the filling factor takes a magic number

$$\nu = \frac{\alpha+\beta-2\gamma}{\alpha\beta-\gamma^2}\pi, \tag{1}$$

additionally exists a constant solution, which describes a Hall liquid, that is a uniform condensed state of bosonized electrons (electrons bound to fluxes). Using these classical solutions we explain why the system is always compressible in the absence of the Coulomb interaction. We also explain why the system remains compressible even in the presence of the Coulomb interaction when the filling factor does not take a magic number (1). It is argued that the compressible ground state will be given by a Wigner crystal made of vortex solitons at the vicinity of the filling factor (1). We present a schematic phase diagram of the system as a function of the filling factor and these statistics parameters.

In Sec. IV, we take account of quantum fluctuations around the classical ground state when the filling factor is given by (1). In this way we evaluate the Coulomb energy of the ground state as well as some excitation spectra as a function of the distance d between the two layers. We find that the ground state at (1) is an incompressible FQH state unless $\alpha=\beta=\gamma$.

In Sec. V, we analyze the FQH state at $\alpha=\beta=\gamma$, whose filling factor is

$$\nu = \frac{\pi}{\alpha} = 1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots. \tag{2}$$

Here, a new feature appears peculiar to the double-layer electron system. Namely, there arises a zero-energy mode in the pseudospin density fluctuation. (Physically, the energy of the state does not change even if we move electrons from one layer to the other layer in this Hall state.) This zero-energy mode has already been known at some filling factors in literature.[16–18] We point out that our analysis exhausts all possible cases. When the small tunneling is allowed, the zero-energy mode leads to the Josephson current associated with the phase difference between the two layers.[19,20] For a constant voltage V applied, the Josephson current is given by

$$J = \Delta_{\rm SAS}\rho_0 \sin(eVt+\delta_0), \tag{3}$$

where $\Delta_{\rm SAS}$ is the gap energy between the symmetric and antisymmetric single-electron states; ρ_0 is the average electron density in each layer and δ_0 is an arbitrary phase.

In Sec. VI, the wave function of the Hall state is derived, which coincides with the one suggested by Halperin.[4]

In Sec. VII, constructing an effective-field theory of vortex solitons, we analyze the hierarchy of the FQH states in the double-layer system.

We have prepared an appendix for some detailed calculations. In this paper we use the unit such that $c=1$ and $\hbar=1$.

II. HAMILTONIAN OF DOUBLE-LAYER ELECTRONS

We consider a double-layer electron system. Our strategy is to regard electrons as anyons. Thus, we consider a system containing two types of distinguishable anyons in external magnetic field B, i.e., anyons with up ($\uparrow$) pseudospin and down ($\downarrow$) pseudospin. Physically, anyons with up (down) pseudospin stand for the spin-polarized electrons in the up (down) layers. We represent anyons in terms of boson fields $\psi^\uparrow$ and $\psi^\downarrow$ with the aid of the CS gauge fields $a_k^\uparrow$ and $a_k^\downarrow$. In a previous paper[13] we have derived a Hamiltonian containing two sets of anyons, which we used to describe electrons and vortices in the single-layer system. This Hamiltonian may also describe the system containing spin-polarized electrons in different layers by introducing the interlayer distance d into the Coulomb potential. For the sake of simplicity we suppress the interlayer tunneling. We analyze the effects of the tunneling in Sec. V.

Our microscopic Hamiltonian reads[13]

$$\mathcal{H} = \mathcal{H}_B + \Delta\mathcal{H}, \tag{4}$$

where

$$\mathcal{H}_B = \frac{1}{2M}|D_k^\uparrow\psi^\uparrow|^2 + \frac{1}{2M}|D_k^\downarrow\psi^\downarrow|^2 + \mathcal{V}[\psi], \tag{5}$$

and

$$\Delta\mathcal{H} = \frac{\alpha}{M}:|\psi^\uparrow|^4: + \frac{\beta}{M}:|\psi^\downarrow|^4: + \frac{2\gamma}{M}:|\psi^\uparrow\psi^\downarrow|^2:, \tag{6}$$

with $iD_k^{\uparrow\downarrow} = i\partial_k + a_k^{\uparrow\downarrow} - eA_k$. The Hamiltonian $\mathcal{H}_B$ is the standard one, while ΔH is the renormalization counter term on which we comment later. The CS gauge fields are defined by the constraint equations

$$\begin{aligned} \varepsilon_{ij}\partial_i a_j^\uparrow &= 2\alpha|\psi^\uparrow|^2 + 2\gamma|\psi^\downarrow|^2, \\ \varepsilon_{ij}\partial_i a_j^\downarrow &= 2\gamma|\psi^\uparrow|^2 + 2\beta|\psi^\downarrow|^2. \end{aligned} \tag{7}$$

Anyons have the mass M and the electric charge $-e$. They are interacting via the Coulomb potential $\mathcal{V}[\psi]$. The external electromagnetic potential is taken in the symmetric gauge as $A_k = -\frac{1}{2}eB\varepsilon_{kj}x^j$. We choose $\alpha > 0$, $\beta > 0$, and $B > 0$ without loss of generality. It is understood that the precise ordering of the operator $|\mathcal{O}|^2$ is $\mathcal{O}^\dagger\mathcal{O}$; e.g., $|D_k\psi|^2$ stands for $(D_k\psi)^\dagger(D_k\psi)$. This convention is always understood in what follows.

It is clear from (7) that two types of statistical fluxes are attached to each anyon: the anyon with up (down) pseudospin carries the $a^\uparrow$ flux with strength 2α (2γ) and the $a^\downarrow$ flux with strength 2γ (2β). These fluxes make the anyons feel statistical forces of the other anyons. We call γ the relative statistics.[13] See also Ref. 21.

The Coulomb potential $\mathcal{V}[\psi]$ is taken such that

$$\mathcal{V}[\psi] = \mathcal{V}^{\uparrow\uparrow}[\psi] + \mathcal{V}^{\downarrow\downarrow}[\psi] + \mathcal{V}^{\uparrow\downarrow}[\psi], \tag{8}$$

with

$$\mathcal{V}^{\uparrow\uparrow}[\psi] = \frac{e^2}{2\varepsilon}\int d^2x\, d^2y : \{\psi^{\uparrow\dagger}\psi^\uparrow(x) - \rho^\uparrow\} \times \frac{1}{|x-y|}\{\psi^{\uparrow\dagger}\psi^\uparrow(y) - \rho^\uparrow\} :, \tag{9a}$$

$$\mathcal{V}^{\downarrow\downarrow}[\psi] = \frac{e^2}{2\varepsilon}\int d^2x\, d^2y : \{\psi^{\downarrow\dagger}\psi^\downarrow(x) - \rho^\downarrow\} \times \frac{1}{|x-y|}\{\psi^{\downarrow\dagger}\psi^\downarrow(y) - \rho^\downarrow\} :, \tag{9b}$$

$$\mathcal{V}^{\uparrow\downarrow}[\psi] = \frac{e^2}{\varepsilon}\int d^2x\, d^2y : \{\psi^{\uparrow\dagger}\psi^\uparrow(x) - \rho^\uparrow\} \times \frac{1}{\sqrt{(x-y)^2 + d^2}} \times \{\psi^{\downarrow\dagger}\psi^\downarrow(y) - \rho^\downarrow\} :, \tag{9c}$$

where ε is the dielectric constant and we have added uniform background charges $e\rho^\uparrow$ and $e\rho^\downarrow$ to the Coulomb term $\mathcal{V}$ for charge neutrality. (The average density of each type of anyons is denoted by $\rho^\uparrow$ and $\rho^\downarrow$.) The parameter d in the Coulomb term $\mathcal{V}^{\uparrow\downarrow}$ stands for the distance between the two layers. In the system with $d \neq 0$ the conserved quantities are given by the total number of electrons and the z component of the pseudospin. Physically, they are the numbers of electrons in each layers. In Sec. V we introduce the interlayer tunneling, in which case only the conserved quantity is the total number of electrons.

We now comment on the term $\Delta\mathcal{H}$ in (6). When we express the anyon in terms of the boson, the boson should satisfy the hard-core condition. (They are called hard-core bosons.) Namely, the wave function should vanish when two of the hard-core bosons come to the same point. However, it is very difficult to use the hard-core bosons in the semiclassical quantization. Thus, it is convenient to express the Hamiltonian in terms of the ordinary boson field on which the hard-core condition is not imposed. This is possible. The resulting Hamiltonian, which we call the modified Hamiltonian, contains an explicit repulsive contact interaction. It is determined so as to reproduce the correct short-distance behavior and the correct energy spectrum by making a perturbation expansion around the ordinary boson states.[22] Let us briefly review the essence.

The necessity of an explicit repulsive interaction can be understood quantum mechanically as follows.[23] We consider two anyons of the same type in an external magnetic field. In general, anyon wave functions vanish with a fractional power of r as $\psi^{(\uparrow\uparrow)}(r) \to r^{\alpha/\pi}$ when two anyons come close. However, unperturbed boson states ($\alpha = 0$) do not involve this fractional power of r; their wave functions vanish like r^ℓ with ℓ being angular momentum. Hence, we cannot expand anyon states in terms of the unperturbed boson states which have the standard partial waves. Instead of doing so, we may expand $r^{-\alpha/\pi}\psi^{(\uparrow\uparrow)}$ by using the standard partial waves. Schematically the expansion looks like

$$\psi^{(\uparrow\uparrow)} = r^{\alpha/\pi}\left(\sum W_p\right), \tag{10}$$

where W_p are partial waves. This expansion leads to a modification of the perturbation series of α/π. Especially, in the first order of α/π, we have to add to the naive Hamiltonian a δ-function type repulsive force with the strength $2\alpha/M$, which leads to the $(\alpha/M) : |\psi^\uparrow|^4 :$ term in (6) field theoretically.

The above quantum-mechanical result is also derived by examining the energy spectrum.[22] We solve the Hamiltonian (5) perturbatively with respect to the statistics parameters α/π, β/π, and γ/π. We expand the Hamiltonian, the eigenvalue, and the eigenfunction as follows:

$$\begin{aligned} H_B &= H_0 + H_1 + \cdots, \\ E &= E_0 + E_1 + \cdots, \\ \Psi &= \Psi_0 + \sum_i C_1^i \Psi_i + \cdots . \end{aligned} \tag{11}$$

Here, the unperturbed quantities H_0, E_0, and Ψ_i are those of the ordinary bosons without the hard-core condition; $H_0|\Psi_0\rangle = E_0|\Psi_0\rangle$; the quantities H_1, E_1, and C_1^i represent the first-order corrections in these parameters, and the ellipses stand for the higher order corrections.

Now, the two-anyon system can be exactly solved quantum mechanically. We also know some spectra of the many-anyon system.[24] Hence, we know what is the correct first-order energy E_1. By an explicit calculation we can prove that

$$E_1 = \langle\Psi_0|(H_1 + \Delta H)|\Psi_0\rangle, \tag{12}$$

where the Hamiltonian density for ΔH is given by (6). Consequently, in performing a perturbation expansion with the unperturbed states being those of the ordinary bosons, we need to modify the Hamiltonian (5) by adding the repulsive term ΔH.

The contact term (5) is nothing but a renormalization counter term which occurs in the Hamiltonian when the "bare" field (hard-core boson) is expressed in terms of the "physical" field (ordinary boson).

Adding the renormalization term (6) to (5), we obtain (4) as the second-quantized Hamiltonian of the bosonic CS gauge theory describing two sets of anyons with their own statistics α, β and the relative statistics γ. Then,

using the Bogomol'nyi decomposition of the type

$$|D_k\psi|^2 = |(D_1 - iD_2)\psi|^2 - \varepsilon_{jk}\partial_j(\psi^\dagger iD_k\psi) - :\omega_{12}\psi^\dagger\psi:, \tag{13}$$

with $\omega_k = a_k - eA_k$ and $\omega_{12} = \partial_1\omega_2 - \partial_2\omega_1$, we can rewrite the modified Hamiltonian density precisely as

$$\mathcal{H} = \frac{1}{2M}|(D_1^\uparrow - iD_2^\uparrow)\psi^\uparrow|^2 + \tfrac{1}{2}\omega_c\psi^{\uparrow\dagger}\psi^\uparrow + \frac{1}{2M}|(D_1^\downarrow - iD_2^\downarrow)\psi^\downarrow|^2 + \tfrac{1}{2}\omega_c\psi^{\downarrow\dagger}\psi^\downarrow + \mathcal{V}[\psi], \tag{14}$$

where a surface term has been neglected and ω_c is the cyclotron frequency $\omega_c = eB/M$.

The modified Hamiltonian (14) has been derived in the first order of the statistics parameters. However, as we shall see, it is also valid for any values of the statistics parameters in the semiclassical approximation. Namely, the correct energy and the wave function of the ground state are reproduced for any values of them in the approximation. Therefore, we do not assume that α/π, β/π, and γ/π are small quantities in what follows.

III. GROUND STATES IN CLASSICAL ANALYSIS

We analyze the Hamiltonian (14) in the semiclassical approximation. For definiteness we use the notation $\widehat{\psi}^{\uparrow\downarrow}$ for the quantum field and $\psi^{\uparrow\downarrow}$ for the classical field in what follows.

Let us assume that the system consists of $N^\uparrow$ electrons with up pseudospin and $N^\downarrow$ electrons with down pseudospin. In this section we analyze the mean-field ground states, that is, the classical ground states described by the c-number functions $\psi^{\uparrow\downarrow}$ minimizing the classical energy of the system. Using these classical fields we show how we can understand the ground-state degeneracy and its removal depending on the absence and the presence of the Coulomb interaction.

A. Hall states and vortex excitations

In this subsection we neglect the Coulomb term $\mathcal{V}$. Then, the classical ground state is given by solving the self-dual equations

$$(D_1^\uparrow - iD_2^\uparrow)\psi^\uparrow = 0, \qquad (D_1^\downarrow - iD_2^\downarrow)\psi^\downarrow = 0, \tag{15}$$

since it minimizes the classical energy. All the solutions have the same energy; thus the corresponding states are degenerate. It is clear in the Hamiltonian that the "classical" energy gives precisely the exact result

$$E_N = \tfrac{1}{2}\hbar\omega_c(N^\uparrow + N^\downarrow), \tag{16}$$

which is the kinetic energy of electrons, when all electrons are in the lowest Landau level. Here, we have made explicit the $\hbar$ dependence of the energy. The formula (16) suggests that there should be no quantum corrections to the ground-state energy in the absence of the Coulomb interaction, which we shall confirm in the next section.

When the external magnetic field is absent and when there exists only one type of anyons, Jackiw and Pi[25,26] have analyzed the self-dual equation in detail. In this case, it is reduced to the Liouville equation, and all the solutions can be obtained analytically. They are interpreted to represent nontopological vortices.

When the external magnetic field is present and when there are two types of anyons, it is impossible to solve all the solutions explicitly. Nevertheless, we can easily derive some of essential features of the solutions without knowing their explicit forms.

First of all, let us show that a constant solution exists only at a specific filling factor $\nu \equiv 2\pi\rho/eB$. We substitute

$$\psi^\uparrow = \sqrt{\rho^\uparrow}e^{i\theta^\uparrow}, \qquad \psi^\downarrow = \sqrt{\rho^\downarrow}e^{i\theta^\downarrow}, \qquad \rho = \rho^\uparrow + \rho^\downarrow, \tag{17a}$$

with constant phases and amplitudes into the self-dual equation (15), and find that

$$a_k^\uparrow = a_k^\downarrow = eA_k. \tag{17b}$$

From the constraint equations it follows that

$$\begin{aligned} eB &= 2\alpha\rho^\uparrow + 2\gamma\rho^\downarrow, \\ eB &= 2\gamma\rho^\uparrow + 2\beta\rho^\downarrow, \end{aligned} \tag{18}$$

which are solved as

$$\begin{aligned} \frac{N^\uparrow}{V} &= \rho^\uparrow = \frac{\beta - \gamma}{\alpha + \beta - 2\gamma}\rho, \\ \frac{N^\downarrow}{V} &= \rho^\downarrow = \frac{\alpha - \gamma}{\alpha + \beta - 2\gamma}\rho, \end{aligned} \tag{19}$$

where V is the volume of the system. Then, the filling factor is uniquely determined as

$$\nu^{(0)} \equiv \frac{\alpha + \beta - 2\gamma}{\alpha\beta - \gamma^2}\pi. \tag{20}$$

Equation (17b) means that the statistical fields of electrons and the external magnetic field cancel each others precisely. Physically, this is the condition for the condensed phase of bosonized electrons (Hall liquid) to exist.[13]

Second, there are topological vortex solutions in addition to nontopological vortex solutions. The nontopological vortex is characterized by the asymptotic behavior $\psi^{\uparrow\downarrow} \to 0$, and it exists at any filling factor ν. On the other hand, the topological vortex is defined by the asymptotic behaviors such that

$$\psi^\uparrow \simeq \sqrt{\rho^\uparrow}e^{ip\theta}, \qquad \psi^\downarrow \simeq \sqrt{\rho^\downarrow}e^{iq\theta}, \tag{21}$$

with

$$a_k^\uparrow \simeq p\partial_k\theta, \qquad a_k^\downarrow \simeq q\partial_k\theta, \tag{22}$$

as $r \to \infty$, where (r, θ) is the polar coordinate, and p and q are integers. This vortex carries (p, q) units of statisti-

cal fluxes associated with the CS gauge fields $(a_k^{\uparrow}, a_k^{\downarrow})$. It exists only at specific filling factors. Topological vortices play important roles as quasiparticles.

Vortex solutions are explicitly analyzed by substituting

$$\psi^{\uparrow}(x) = \sqrt{\rho}\exp[u_1(x) + ip\theta], \qquad \psi^{\downarrow}(x) = \sqrt{\rho}\exp[u_2(x) + iq\theta], \tag{23}$$

into the self-dual equations, which yield the modified Toda equations:

$$(\partial_1^2 + \partial_2^2)u_i(x) + eB = 2\rho\sum_j K_{ij}e^{2u_j(x)}, \tag{24}$$

where $K_{11} = \alpha$, $K_{22} = \beta$, and $K_{12} = K_{21} = \gamma$. Analyzing the equations, it is easy to see[13,27] that only antivortices (vortices with $p < 0$ or $q < 0$) exist as classical solutions. We conjecture that vortices with $p > 0$ or $q > 0$ appear not as solitons but as bound states of electrons and antivortex solitons. This difference in their characters causes a discontinuity in the chemical potential at the magic filling factor (20).[13]

Vortices with the minimum flux will be physically relevant: When $\alpha\beta \neq \gamma^2$, the antivortex with $(p, q) = (-1, 0)$ or $(0, -1)$ is the quasihole, while the vortex with $(1, 0)$ or $(0, 1)$ is the quasiparticle. In what follows we consider these vortices.

From the constraint equation (7) the pseudospin components of the vortex soliton are obtained:

$$S^{\uparrow} = \frac{\beta p\pi - \gamma q\pi}{\alpha\beta - \gamma^2}, \qquad S^{\downarrow} = \frac{\alpha q\pi - \gamma p\pi}{\alpha\beta - \gamma^2}. \tag{25}$$

The quantities $-eS^{\uparrow}$ and $-eS^{\downarrow}$ are the electric charges of the vortex soliton in the up and down layers, respectively; recall that the electron carries the charge $-e$. It is interesting that one vortex has necessarily the electric charges on both of the layers. Thus, the vortex in the double-layer system may be viewed as a dipole-like object with the electric charge $-eS^{\uparrow}$ on one layer and the electric charge $-eS^{\downarrow}$ on the other layer. The total electric charge is given by $eQ_v = -e(S^{\uparrow} + S^{\downarrow})$, which we denote as $eQ^{\uparrow}$ and $eQ^{\downarrow}$ for the vortex with $(p, q) = (-1, 0)$ and $(0, -1)$, respectively, where

$$Q^{\uparrow} = \frac{(\beta - \gamma)\pi}{\alpha\beta - \gamma^2}, \qquad Q^{\downarrow} = \frac{(\alpha - \gamma)\pi}{\alpha\beta - \gamma^2}. \tag{26}$$

The vortex may be considered to be composed of $Q^{\uparrow}$ or $Q^{\downarrow}$ electrons, and hence it is concluded that its mass is given by

$$M_v^{\uparrow\downarrow} = M|Q^{\uparrow\downarrow}|, \tag{27}$$

as in the case of the single-layer system.[13] In particular, when $\alpha = \beta$ we find that

$$Q^{\uparrow} = Q^{\downarrow} = \tfrac{1}{2}\nu \tag{28}$$

at the filling factor ν; namely, the total electric charge is $\frac{1}{2}\nu e$.

For instance, the even-denominator filling factor $\nu = \frac{1}{2}$ is realized with the choice of $\alpha = \beta = 3\pi$ and $\gamma = \pi$. In this case, the total electric charge of the vortex with $(-1, 0)$ or $(0, -1)$ is $Q = \frac{1}{4}e$, which consists of $\frac{3}{8}e$ on one layer and $-\frac{1}{8}e$ on the other layer. We note that four of the vortices have precisely the opposite quantum number of one electron in this Hall state.

On the other hand, when $\alpha = \beta = \gamma$, the quasihole is the vortex with $(p, q) = (-1, -1)$, which may be viewed as a diatomic molecule with the electric charge $\frac{1}{2}\nu e$ on each layer: thus, the total electric charge is νe.

B. Coulomb interaction and removal of degeneracy

By switching off the Coulomb interaction we have shown that there are many nonuniform solutions to the self-dual equations. All these nonuniform solutions have at least two zero-energy modes associated with the translation of the system itself. Furthermore, at $\nu = \nu^{(0)}$ there is the uniform (constant) solution in addition to nonuniform solutions. They are all degenerate. Therefore, the system is compressible in the absence of the Coulomb interaction.

Now, we switch on the Coulomb interaction between electrons. When the filling factor does not take the magic number (20), the constant solution does not exist. Thus, a nonuniform solution with the minimum Coulomb energy is a ground state. It should be noted that the zero-energy modes necessarily exist associated with the translational invariance of the system itself, and hence the system is still compressible at $\nu \neq \nu^{(0)}$. A candidate of the ground state is given by a Wigner crystal made of vortex solitons at the vicinity of the magic filling factor (20), as we shall discuss soon after.

The situation is entirely different when the filling factor takes the magic number (20). In this case the constant solution exists and minimizes the Coulomb energy. Hence, the corresponding state is the ground state. To find the gap energy, we should consider perturbative excitations and nonperturbative excitations separately. Nonperturbative excitations are given by topological vortices and described by a nonuniform classical solution, and hence their creation costs a Coulomb energy. Actually the gap energy is given by the creation energy of a pair of a vortex and an antivortex. At $\nu = \frac{1}{2}$ a numerical estimation[28] of such a Coulomb energy reads $\langle\mathcal{V}\rangle \approx 0.046e^2/\varepsilon\ell_B$ for the antivortex with unit flux when the interlayer distance is taken as $d \approx 2.5\ell_B$; here, $\ell_B = 1/\sqrt{eB}$ is the magnetic length. The gap energy is approximately given by the creation energy of four antivortices,[13] which is of the order of $4\langle\mathcal{V}\rangle$ at $\nu = \frac{1}{2}$. On the other hand, the perturbative excitations are given by Gaussian fluctuations around the constant solution, which we analyze in the next section, where we find that the ground state is incompressible at the magic number (20) unless $\alpha = \beta = \gamma$.

The case $\alpha = \beta = \gamma$ needs a separate treatment since

the classical ground state is not uniquely determined as in (19). The form of the solution (17a) and (17b) is still correct but $\rho^{\uparrow}$ and $\rho^{\downarrow}$ are not determined as (19); they are arbitrary constants satisfying $\rho = \rho^{\uparrow} + \rho^{\downarrow}$. The ground state is compressible and shows superfluidity even if the Coulomb interaction exists, as we show in Sec. V.

Let us make a physical argument of the ground state at the vicinity of the magic filling factor. We have already argued that the mean-field ground state is described by a nonuniform solution of the classical field equation. As such a nonuniform solution there are topological vortex solutions. We have noticed that four of the antivortices have the opposite quantum number of one electron in the instance of the $\nu = \frac{1}{2}$ FQH state; see (26). Thus, when we remove one electron from the FQH state, some antivortices would appear. As more electrons are removed, more antivortices are created. Similarly, vortices appear when electrons are added to the FQH state. (Alternatively, to create vortices or antivortices, we may decrease or increase the external magnetic field B with the electron density kept unchanged.) At the vicinity of the magic filling factor the mean-field ground state must be described by an ensemble of these topological solitons; these solitons would make a Wigner crystal to minimize the Coulomb energy. The relation between the filling factor ν and the average vortex density ρ_v is easily derived from the constraint equation (7):

$$\nu \equiv \frac{2\pi N}{eBV} = \frac{\pi}{\alpha}\left(1 \pm \frac{\pi}{\alpha}\frac{\rho_v}{\rho}\right). \tag{29}$$

Detailed analysis of this Wigner crystal is given elsewhere.[14,27]

We have depicted in Fig. 1 a schematic phase diagram of the double-layer anyon system as a function of the statistics parameter α and the inverse filling factor $\nu^{-1} \equiv eB/2\pi\rho$ for the cases $\alpha = \beta$ with γ integers. Note that a similar diagram has been given for the single-layer anyon system by Wilczek.[29] The FQH states are realized on the hyperplane defined by (20), i.e., on the lines in the figure. At a point not on the hyperplane but in its vicinity, the ground state is given by a Wigner crystal. In the analogy with the single-layer system[29] anyon superconductivity is expected to arise at the point $\nu^{-1} = 0$ or $B = 0$, which is also indicated in Fig. 1. In particular, the double-layer semion superconductor is realized at $\alpha = \beta = -\gamma = \frac{1}{2}\pi$. We discuss anyon superconductivity in a separate paper.[30]

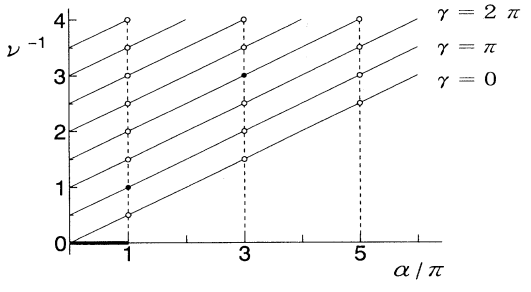

FIG. 1. Phase diagram of the double-layer anyon system for the cases $\alpha = \beta$ with γ integers. The points denoted by white circles are Hall states of electrons, which we denote by $(\alpha/\pi, \beta/\pi, \gamma/\pi)$. Typical points are as follows. The $\nu = \frac{1}{2}$ Hall state is given by $(3,3,1)$ on the line $\gamma = \pi$. At $\nu = \frac{2}{5}$ there are three Hall states $(3,3,2)$, $(5,5,0)$, and $(1,1,4)$. The Josephson effect is expected in the states $(1,1,1)$ at $\nu = 1$ and $(3,3,3)$ at $\nu = \frac{1}{3}$ as indicated with black circles. Anyon superconductivity may arise in the region $0 \le \alpha = -\gamma < 1$ at $\nu^{-1} = 0$.

IV. COULOMB ENERGY OF HALL STATES

In this section we analyze the case when the filling factor takes the magic number (20). As we have argued in the previous section, there exists the constant solution $(\psi^{\uparrow\downarrow} = \sqrt{\rho^{\uparrow\downarrow}})$ with the minimum Coulomb energy. (We may set $\theta^{\uparrow\downarrow} = 0$ without loss of generality.) We now consider small quantum fluctuations around it to evaluate the Coulomb energy of the ground state as well as excitation spectra. The results of this section are also used in Sec. VI to derive the ground-state wave function.

We substitute

$$\widehat{\psi}^{\uparrow} = \sqrt{\rho^{\uparrow}} + \eta^{\uparrow}, \qquad \widehat{\psi}^{\downarrow} = \sqrt{\rho^{\downarrow}} + \eta^{\downarrow}, \tag{30}$$

into the Hamiltonian density, where

$$\eta^{\uparrow} = \frac{1}{\sqrt{V}}\sum_{\mathbf{p}\neq 0} a_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}},$$
$$\eta^{\downarrow} = \frac{1}{\sqrt{V}}\sum_{\mathbf{p}\neq 0} b_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}}, \tag{31}$$

with V being the volume of the system, and

$$[a_{\mathbf{p}}, a^{\dagger}_{\mathbf{q}}] = \delta_{\mathbf{pq}}, \qquad [a_{\mathbf{p}}, a_{\mathbf{q}}] = [a^{\dagger}_{\mathbf{p}}, a^{\dagger}_{\mathbf{q}}] = 0,$$
$$[b_{\mathbf{p}}, b^{\dagger}_{\mathbf{q}}] = \delta_{\mathbf{pq}}, \qquad [b_{\mathbf{p}}, b_{\mathbf{q}}] = [b^{\dagger}_{\mathbf{p}}, b^{\dagger}_{\mathbf{q}}] = 0. \tag{32}$$

We then diagonalize the Hamiltonian in the Gaussian approximation by way of the Bogoljubov transformation

$$a_{\mathbf{p}} = \Gamma_{\mathbf{p}} A_{\mathbf{p}} + \Lambda_{\mathbf{p}} A^{\dagger}_{-\mathbf{p}} + \Omega_{\mathbf{p}} B_{\mathbf{p}} + \Upsilon_{\mathbf{p}} B^{\dagger}_{-\mathbf{p}},$$
$$b_{\mathbf{p}} = \Gamma'_{\mathbf{p}} A_{\mathbf{p}} + \Lambda'_{\mathbf{p}} A^{\dagger}_{-\mathbf{p}} + \Omega'_{\mathbf{p}} B_{\mathbf{p}} + \Upsilon'_{\mathbf{p}} B^{\dagger}_{-\mathbf{p}}, \tag{33}$$

with appropriate coefficients $\Gamma_{\mathbf{p}}$, $\Lambda_{\mathbf{p}}$, $\Omega_{\mathbf{p}}$, $\Upsilon_{\mathbf{p}}$ and their primed quantities: see the Appendix for details. We then find that

$$H = \frac{\omega_c}{2}N + E_{\eta} + \sum_{\mathbf{p}\neq 0}(E_{\mathbf{p}} A^{\dagger}_{\mathbf{p}} A_{\mathbf{p}} + E'_{\mathbf{p}} B^{\dagger}_{\mathbf{p}} B_{\mathbf{p}}), \tag{34}$$

with

$$[A_{\mathbf{p}}, A^{\dagger}_{\mathbf{q}}] = \delta_{\mathbf{pq}}, \qquad [A_{\mathbf{p}}, A_{\mathbf{q}}] = [A^{\dagger}_{\mathbf{p}}, A^{\dagger}_{\mathbf{q}}] = 0,$$
$$[B_{\mathbf{p}}, B^{\dagger}_{\mathbf{q}}] = \delta_{\mathbf{pq}}, \qquad [B_{\mathbf{p}}, B_{\mathbf{q}}] = [B^{\dagger}_{\mathbf{p}}, B^{\dagger}_{\mathbf{q}}] = 0. \tag{35}$$

The ground state is defined by

$$A_{\mathbf{p}}|g\rangle = B_{\mathbf{p}}|g\rangle = 0. \tag{36}$$

Here, excitation spectra are calculated as a function of the layer separation d as

$$E_{\mathbf{p}}^2 = \left(\frac{\mathbf{p}^2}{2M}+\omega_c\right)^2 + \frac{\pi e^2\rho|\mathbf{p}|}{\varepsilon M}(1+L+Ke^{-|\mathbf{p}|d}),$$
$$E_{\mathbf{p}}'^2 = \left(\frac{\mathbf{p}^2}{2M}+\omega_c'\right)^2 + \frac{\pi e^2\rho|\mathbf{p}|}{\varepsilon M}(1-L-Ke^{-|\mathbf{p}|d}), \tag{37}$$

with

$$\omega_c \equiv \frac{eB}{M} = \frac{2(\alpha\beta-\gamma^2)\rho}{(\alpha+\beta-2\gamma)M},$$
$$\omega_c' = \frac{2(\alpha-\gamma)(\beta-\gamma)\rho}{(\alpha+\beta-2\gamma)M}, \tag{38}$$

while the ground-state energy is

$$E_\eta = -\frac{e^2\pi N}{2\sqrt{2}\varepsilon\ell_B}\left(\frac{1}{2}(1+L)+\frac{1}{2}\sqrt{\frac{\omega_c'}{\omega_c}}(1-L)+\frac{K}{\pi}\left\{\sin(d_1)\mathrm{ci}(d_1)-\cos(d_1)\mathrm{si}(d_1)\right.\right.$$
$$\left.\left.-\sqrt{\frac{\omega_c'}{\omega_c}}\left[\sin(d_2)\mathrm{ci}(d_2)-\cos(d_2)\mathrm{si}(d_2)\right]\right\}\right)+O(e^4), \tag{39}$$

where $\ell_B = 1/\sqrt{eB}$ is the magnetic length, and

$$d_1 \equiv d\sqrt{2M\omega_c}, \qquad d_2 \equiv d\sqrt{2M\omega_c'}. \tag{40}$$

The functions $\mathrm{ci}(x)$ and $\mathrm{si}(x)$ are the integral cosine and sine functions:

$$\mathrm{ci}(x) = -\int_x^\infty dt\frac{\cos(t)}{t},$$
$$\mathrm{si}(x) = -\int_x^\infty dt\frac{\sin(t)}{t}.$$

In (37) and (39) the parameters K and L are defined by

$$K = \frac{4(\alpha-\gamma)(\beta-\gamma)}{(\alpha+\beta-2\gamma)^2},$$
$$L = \frac{(\alpha-\beta)^2}{(\alpha+\beta-2\gamma)^2}. \tag{41}$$

Here, $1 \pm [L + K\exp(-|\mathbf{p}|d)] \geq 0$ for any values of the statistics parameters; hence, $E_{\mathbf{p}}^2 \geq 0$ and $E_{\mathbf{p}}'^2 \geq 0$. Note that $K = 1$ and $L = 0$ when $\alpha = \beta$.

We have obtained two excitation modes $E_{\mathbf{p}}$ and $E_{\mathbf{p}}'$ with the gaps ω_c and ω_c'. It is important to recognize that there are no zero-energy modes in these quantum fluctuations around the constant solution as far as $\alpha\beta - \gamma^2 \neq 0$ and $\alpha \neq \gamma \neq \beta$. Therefore, in this case the system is incompressible. We make a detailed analysis of the zero-energy mode $E_{\mathbf{p}}'$ that appears for $\alpha = \beta = \gamma$ in the next section.

In the previous section we have argued that, when the Coulomb interaction is neglected, the uniform state is degenerate with many nonuniform states containing topological vortices. One may wonder where all the associated zero-energy modes have gone because no reminiscence of these modes is found in the energy spectra (37). As we have noticed in the previous section, these modes cannot be generated by (perturbative) quantum fluctuations. It occurs for instance by exciting topological vortices.

The quantum correction due to the fluctuation η to the ground-state energy is given by E_η, which vanishes in the absence of the Coulomb interaction ($e^2 \to 0$). The result has been anticipated in the previous section. Namely, the energy of the system should be given by $\frac{1}{2}\omega_c N$ when $e^2 \to 0$; see (16). It indicates that our Hamiltonian (14) is correct for any values of the statistic parameters in the semiclassical approximation. We also comment that if we did the above calculation without the renormalization counter term $\Delta\mathcal{H}$ in (5) we would obtain a meaningless divergence for the ground-state energy E_η even in the absence of the Coulomb interaction.

The ground-state energy E_η depends on the statistics parameters and the interlayer distance d. It is easy to see that $E_\eta(d)$ is a monotoneously increasing function of d for $\gamma \neq 0$ while it does not depend on d for $\gamma = 0$. As a typical example we have depicted in Fig. 2(a) the function $E_\eta(d)$ at $\nu = \frac{2}{5}$, where there are three FQH states characterized by the statistics parameters: (a) $\alpha = \beta = 3\pi$ and $\gamma = 2\pi$; (b) $\alpha = \beta = 5\pi$ and $\gamma = 0$; (c) $\alpha = \beta = \pi$ and $\gamma = 4\pi$. They are called the $(3,3,2)$ state, the $(5,5,0)$ state, and $(1,1,4)$ state; see Fig. 1. [We only consider the cases of (a) and (b) for simplicity.] It is notable that the Coulomb energy of the $(5,5,0)$ state is always smaller than that of the $(3,3,2)$ state for $d \neq 0$. (They are equal at $d = 0$.) This fact may be understood physically as follows. First, it is rather natural that the Coulomb energy is an increasing function of d since the Coulomb interaction between electrons belonging to different layers decreases as d increases. (Note that the Coulomb energy is negative definite.) On the other hand, the case $\gamma = 0$ is very special. In Sec. VI we derive the wave function of the FQH state, which is the Halperin wave function[4] as is expected; see (83). When $\gamma = 0$ the Halperin wave function is just a product of the two Laughlin wave functions describing two independent single-layer systems. With such a wave function the d dependence of the Coulomb energy is washed away. Indeed, in our formalism the ground-state energy of the double-layer system with $\gamma = 0$ is the sum of the energy of the corresponding single-layer systems.[13]

It is reasonable that the $(5,5,0)$ state is realized at the large value of d because the double-layer system should approach two single-layer systems as $d \to \infty$, where it is necessary to have $\gamma = 0$. However, our result naively

indicates that the $(5,5,0)$ state is always realized. This conclusion is different from the one given in Ref. 6, where the overlapping between the Halperin wave function and the ground state of a few-electron system has been calculated numerically. They suggest that there is a phase transition at $d \approx \ell_B$ and that the $(3,3,2)$ state would be realized at small d while the $(5,5,0)$ state at large d. Let us discuss this paradox.

Our above results have been obtained without including the contributions of topological vortices. Let us guess their contributions. Recall that the quantum correction E_η due to the fluctuation η to the ground-state energy is essentially the Coulomb energy of the η quanta and roughly speaking is proportional to the square of their electric charge. We conjecture that the quantum correction to the ground-state energy due to the topological fluctuations has a similar property. Now, the vortex soliton with unit flux has the charge $\frac{1}{5}e$ at $\nu = \frac{2}{5}$; see (28). On one hand, in the $(5,5,0)$ state this charge is induced only on one of the two layers. The Coulomb energy would be proportional to $-(1/25)e^2$. On the other hand, in the $(3,3,2)$ state this charge is distributed on both of the two layers, $\frac{3}{5}e$ on one layer and $-\frac{2}{5}e$ on the other; see (25). The Coulomb energy would be proportional to $-(13/25)e^2$. Then, their contribution to the ground-state energy would be bigger in the $(3,3,2)$ state. Taking account of them, we expect that the ground-state energy would be modified as in Fig. 2(b). Hence, the $(3,3,2)$ state would realize at small d while the $(5,5,0)$ state at large d. In this way we may reconcile our formalism with the result of a numerical computation.[6]

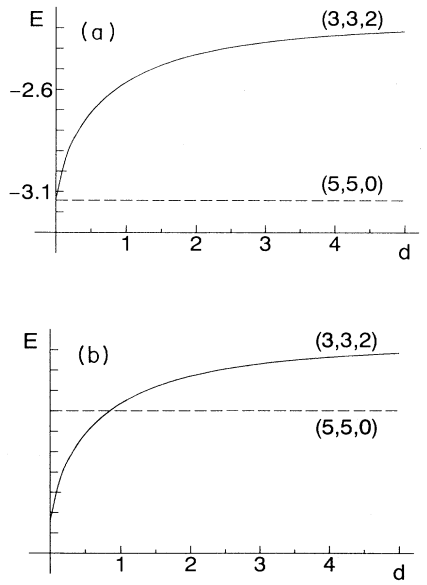

FIG. 2. The ground-state energy E as a function of the interlayer distance d for the (3,3,2) and (5,5,0) states. The distance is in unit of the magnetic length ℓ_B. (a) The ground-state energy E_η without the effects of topological vortices in unit of $e^2N/(2\sqrt{2}\varepsilon\ell_B)$. (b) An expected ground-state energy with the effects of topological vortices in an appropriate unit.

V. INTERLAYER TUNNELING AND JOSEPHSON EFFECT

The excitation spectrum (37) can have a zero-energy mode, which is the case if $\alpha = \gamma$ or $\beta = \gamma$. Actually, only the case $\alpha = \beta = \gamma$ is physically interesting. For instance, if $\alpha = \gamma$ but $\beta \neq \gamma$, we obtain $\rho^\uparrow = \rho$ and $\rho^\downarrow = 0$ in (19). Namely, in one of the layers there is no condensation and the corresponding CS gauge symmetry is not spontaneously broken. In what follows let us assume that $\alpha = \beta = \gamma$, at which (38) reads $\omega_c = 2\rho/M$ and $\omega_c' = 0$. The gapless FQH state appears at the odd-denominator filling factor $\nu = \pi/\alpha$; see Fig. 1.

First we analyze the origin of the zero-energy mode (37). In the double-layer electron system, there are the $U(1) \times U(1)$ symmetry associated with the phase transformation of the two fields $\psi^{\uparrow\downarrow}$. This symmetry is spontaneously broken by the condensation of the bosonized electrons ($\langle\psi^{\uparrow\downarrow}\rangle \neq 0$) , producing two Goldstone modes. However, for a general value of the statistics parameters, these Goldstone modes are absorbed into the two CS gauge fields and there are no zero-energy modes associated with the $U(1) \times U(1)$ symmetry. This is the Anderson-Higgs mechanism, and what happens in the case of the incompressible FQH state. However, when $\alpha = \beta = \gamma$, one of the Goldstone modes survives. Indeed, in this case it is obvious from the constraint condition (7) that

$$\varepsilon_{ij}\partial_i(a_j^\uparrow - a_j^\downarrow) = 0. \tag{42}$$

Therefore, one of the CS gauge fields is redundant, and hence one of the Goldstone modes survives. Without loss of generality we may set $a_j^\uparrow = a_j^\downarrow$. This explains why there is one Goldstone mode left over, as is indicated by (37).

Consequently, the basic Hamiltonian density is given by (14) with

$$iD_k^\uparrow = iD_k^\downarrow = i\partial_k + a_k - eA_k, \tag{43}$$

together with a single constraint condition

$$\varepsilon_{ij}\partial_i a_j = 2\alpha\left(|\psi^\uparrow|^2 + |\psi^\downarrow|^2\right), \tag{44}$$

where we have set $a_i \equiv a_i^\uparrow = a_i^\downarrow$. It is found that the CS gauge field a_k couples to the charge density current J_k but not to the pseudospin current I_k, where

$$\begin{aligned} J_k &= i\psi^{\uparrow\dagger} \overleftrightarrow{\partial}_k \psi^\uparrow + i\psi^{\downarrow\dagger} \overleftrightarrow{\partial}_k \psi^\downarrow, \\ I_k &= i\psi^{\uparrow\dagger} \overleftrightarrow{\partial}_k \psi^\uparrow - i\psi^{\downarrow\dagger} \overleftrightarrow{\partial}_k \psi^\downarrow. \end{aligned} \tag{45}$$

Therefore, there is the Goldstone mode only in the pseudospin fluctuation as in (37).

The constant classical solution representing a Hall liquid is found as before. Substituting

$$\psi^{\uparrow}=\sqrt{\rho^{\uparrow}}e^{i\theta^{\uparrow}}, \qquad \psi^{\downarrow}=\sqrt{\rho^{\downarrow}}e^{i\theta^{\downarrow}}, \qquad \rho=\rho^{\uparrow}+\rho^{\downarrow}, \tag{46}$$

into the field equation we find that

$$a_k = eA_k. \tag{47}$$

Then, the constraint equation (44) yields

$$eB = 2\alpha(\rho^{\uparrow}+\rho^{\downarrow}), \tag{48}$$

which determines the filling factor uniquely as

$$\nu = \frac{\pi}{\alpha}. \tag{49}$$

Because there is only one constraint equation, the density of the electron in each layer is not fixed uniquely. This ambiguity is physically the origin of the zero-energy mode in the excitation spectrum. Namely, we can move electrons $\delta\rho$ in one layer to the other layer without cost of energy.

The existence of the gapless mode has been known at some filling factors in the literature.[16–18] Our analysis exhausts all possible cases at which this gapless mode appears. At $\mathbf{p}\approx 0$, (37) implies

$$E'_{\mathbf{p}} = \sqrt{\frac{\pi e^2\rho d}{\varepsilon M}}|\mathbf{p}|, \tag{50}$$

for $d\neq 0$. This behavior is consistent with the previous results obtained in different ways.[16–18] The gapless mode implies that the ground state is a superfluid. We emphasize that to get the linear dispersion relation (50) the existence of the Coulomb interaction ($e^2\neq 0$) is essential.

The zero-energy mode has actually appeared because interlayer tunneling has been suppressed. In a recent paper in collaboration with Wu[20] we have analyzed the effect of the interlayer tunneling. The tunneling is easily incorporated by including a term $\lambda(\psi^{\uparrow\dagger}\psi^{\downarrow}+\psi^{\downarrow\dagger}\psi^{\uparrow})$ into the Hamiltonian:

$$\mathcal{H}_T = \mathcal{H} - \lambda(\psi^{\uparrow\dagger}\psi^{\downarrow}+\psi^{\downarrow\dagger}\psi^{\uparrow}), \tag{51}$$

where $\mathcal{H}$ is given by (14) with (43). For the consistency of our two-dimensional simulation, the tunneling is assumed small relative to the creation energy of topological vortices. To see the effect of the tunneling to the above Hall state we diagonalize the tunneling term. Let us introduce the symmetric and antisymmetric field operators by

$$\begin{aligned}\psi_S &= \frac{1}{\sqrt{2}}(\psi^{\uparrow}+\psi^{\downarrow}),\\ \psi_{\mathrm{AS}} &= \frac{1}{\sqrt{2}}(\psi^{\uparrow}-\psi^{\downarrow}).\end{aligned} \tag{52}$$

The Hamiltonian reads

$$\begin{aligned}\mathcal{H}_T = &\frac{1}{2M}|(D_1-iD_2)\psi_S|^2+\frac{1}{2M}|(D_1-iD_2)\psi_{\mathrm{AS}}|^2\\ &+\tfrac{1}{2}\omega_c(|\psi_S|^2+|\psi_{\mathrm{AS}}|^2)+\mathcal{V}[\psi_S,\psi_{\mathrm{AS}}]\\ &-\lambda(|\psi_S|^2-|\psi_{\mathrm{AS}}|^2),\end{aligned} \tag{53}$$

where the Coulomb energy $\mathcal{V}[\psi_S,\psi_{\mathrm{AS}}]$ is a symmetric function of ψ_S and ψ_{AS}. It follows that the energy gap Δ_{SAS} between the symmetric state (ψ_S) and antisymmetric state (ψ_{AS}) is given by

$$\Delta_{\mathrm{SAS}} = 2\lambda. \tag{54}$$

The energy gap[2] Δ_{SAS} is of order of 1 K.

Due to the tunneling effect all electrons occupy the symmetric state, $|\psi_S|=\sqrt{\rho}$ with $\theta^{\uparrow}=\theta^{\downarrow}$. Thus, $|\psi_{\mathrm{AS}}|=\frac{1}{\sqrt{2}}|\sqrt{\rho^{\uparrow}}-\sqrt{\rho^{\downarrow}}|=0$, or $\rho^{\uparrow}=\rho^{\downarrow}$. It is reasonable that the density of the electrons in each layer becomes equal because of the tunneling.

We next see how the energy spectrum (37) is modified by the effect of tunneling. It is straightforward to see that the dispersion relation now reads

$$E'_{\mathbf{p}} = 2\lambda \tag{55}$$

near $\mathbf{p}\approx 0$, instead of (50). Thus, the gapless mode disappears and the ground state becomes incompressible.

Although the ground state is incompressible its property is very different from that of the other FQH states where $\alpha\neq\gamma\neq\beta$. In these cases the Hall-state conditions uniquely determine $\rho^{\uparrow}$ and $\rho^{\downarrow}$ in a condensed state as in (19): Namely, the state not satisfying (19) is no longer a Hall liquid. On the other hand, in the present case where $\alpha=\beta=\gamma$, even if $\rho^{\uparrow}\neq\rho^{\downarrow}$, the state can still be a Hall liquid with the bose condensation as long as the filling factor takes the magic number (49). This characteristic feature originates in the presence of the zero mode for $\lambda=0$, or the looseness of the Hall-state condition (48). Therefore, in the present Hall state, if we supply an external voltage between the layers, electrons $\delta\rho$ move from one layer to the other layer while keeping the Hall-state conditions. Thus, we expect the Josephson current[31] to be induced as a result of tunneling.

To study the problem we introduce the electric potentials associated with the voltage:

$$\mu^{\uparrow}=\frac{eV}{2}, \qquad \mu^{\downarrow}=-\frac{eV}{2}. \tag{56}$$

The Hamiltonian is modified by adding a term $(\mu^{\uparrow}|\psi^{\uparrow}|^2+\mu^{\downarrow}|\psi^{\downarrow}|^2)$ to (53). The Lagrangian reads

$$\begin{aligned}\mathcal{L}_T = &\psi^{\uparrow\dagger}(i\partial_0+a_0)\psi^{\uparrow}-\frac{1}{2M}|(D_1-iD_2)\psi^{\uparrow}|^2+\psi^{\downarrow\dagger}(i\partial_0+a_0)\psi^{\downarrow}-\frac{1}{2M}|(D_1-iD_2)\psi^{\downarrow}|^2\\ &-\tfrac{1}{2}\omega_c(|\psi^{\uparrow}|^2+|\psi^{\downarrow}|^2)-\mathcal{V}[\psi]+\lambda(\psi^{\uparrow\dagger}\psi^{\downarrow}+\psi^{\downarrow\dagger}\psi^{\uparrow})-\mu^{\uparrow}|\psi^{\uparrow}|^2-\mu^{\downarrow}|\psi^{\downarrow}|^2-\frac{1}{4\alpha}\varepsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho,\end{aligned} \tag{57}$$

which reproduces the Hamiltonian and the constraint equation. Here, $iD_k=i\partial_k+a_k-eA_k$. The field equations are given by

$$(i\partial_0 + a_0)\psi^\uparrow = -\frac{1}{2M}(D_1 + iD_2)(D_1 - iD_2)\psi^\uparrow + \left(\frac{\omega_c}{2} + \mu^\uparrow\right)\psi^\uparrow - \lambda\psi^\downarrow + \frac{\partial \mathcal{V}}{\partial \psi^{\uparrow\dagger}},$$
$$(i\partial_0 + a_0)\psi^\downarrow = -\frac{1}{2M}(D_1 + iD_2)(D_1 - iD_2)\psi^\downarrow + \left(\frac{\omega_c}{2} + \mu^\downarrow\right)\psi^\downarrow - \lambda\psi^\uparrow + \frac{\partial \mathcal{V}}{\partial \psi^{\downarrow\dagger}}. \tag{58}$$

and

$$\varepsilon^{kj}\partial_j a_0 = -\frac{2\alpha}{M}(\psi^{\uparrow\dagger} i D_k \psi^\uparrow + \psi^{\downarrow\dagger} i D_k \psi^\downarrow) + \varepsilon^{kj}\partial_0 a_j, \tag{59}$$

together with the constraint equation (44).

We solve Eqs. (58) together with the constraint equation (44) to obtain a uniform but a time-dependent solution. Here, we require $a_k = eA_k$ so that the ground state is still a Hall liquid. Then, the equations are reduced to

$$i\partial_0\psi^\uparrow = \frac{eV}{2}\psi^\uparrow - \lambda\psi^\downarrow,$$
$$i\partial_0\psi^\downarrow = -\frac{eV}{2}\psi^\downarrow - \lambda\psi^\uparrow, \tag{60}$$

by neglecting the Coulomb correction. Here, we have solved a_0 as $a_0 = -\frac{1}{2}\omega_c$, which is consistent with (59). The constraint equation (44) states simply that the filling factor is given by (49).

The set of equations (60) is precisely the one determining the Josephson current when two superconductors are jointed with a thin insulator; see Eq. (21.40) of the text book of Feynman.[32] Substituting

$$\psi^{\uparrow\downarrow} = \sqrt{\rho^{\uparrow\downarrow}(t)}e^{i\theta^{\uparrow\downarrow}(t)}, \tag{61}$$

into (60) we obtain

$$\partial_0\rho^\uparrow(t) = -2\lambda\sqrt{\rho^\uparrow\rho^\downarrow}\sin\delta(t), \tag{62a}$$
$$\partial_0\rho^\downarrow(t) = 2\lambda\sqrt{\rho^\uparrow\rho^\downarrow}\sin\delta(t), \tag{62b}$$
$$\partial_0\delta(t) = -eV(t) + \lambda\frac{\rho^\uparrow - \rho^\downarrow}{\sqrt{\rho^\uparrow\rho^\downarrow}}\cos\delta(t), \tag{62c}$$

where $\delta(t)$ is the phase difference across the interlayer separation, $\delta(t) \equiv \theta^\uparrow(t) - \theta^\downarrow(t)$.

Before applying the external voltage, the system is in the ground state where $\rho^\uparrow = \rho^\downarrow = \rho_0$ and $\delta(0) = 0$. When it is applied, the phase difference starts to develop as $\delta(t) = -eV(t)\Delta t$, and the Josephson current

$$J \equiv \partial_0\rho^\uparrow = -2\lambda\rho_0\sin\delta(t), \tag{63}$$

begins to flow. The resulting density difference will be immediately compensated by the external supply. Namely, $\rho^\uparrow = \rho^\downarrow$ in the actual system. Thus, in the presence of the external supply, we can neglect the last term of (62c) and integrate it as

$$\delta(t) \equiv \theta^\uparrow(t) - \theta^\downarrow(t) = e\int_0^t dt' V(t'). \tag{64}$$

The Josephson current is given by (63) with (64). When we apply a dc voltage we get $\delta(t) = eVt$. However, the net current is zero because the $\sin\delta(t)$ oscillates rather rapidly. On the other hand, when the high-frequency voltage is applied in addition to a dc voltage V_0 such as

$$V = V_0 + v\cos\omega t, \tag{65}$$

with $v \ll V_0$, the dc supercurrent is observed at

$$\omega = eV_0. \tag{66}$$

Similarly in the system with current feed, the voltage between the layers does not appear until the current exceeds a critical value J_c given by

$$J_c = 2\lambda\rho_0 = \Delta_{\mathrm{SAS}}\rho_0. \tag{67}$$

Here, the Josephson current is generated by the phase different δ_0 as $J = J_c\sin\delta_0$. The Josephson effect in the double-layer electron system is within the reach of present experimental technology.

In the absence of the external voltage, we cannot neglect the last term in (62c). Let us set $V = 0$ and study a small fluctuation of the phase difference δ. For $\delta \ll 1$, we may solve the coupled equations (62a)–(62c) as

$$\rho^\uparrow(t) = \rho_0(1 - c\sin(2\lambda t)),$$
$$\delta(t) = c\sin(2\lambda t), \tag{68}$$

where $c \ll 1$. This is the coherent mode in (55) which gives the gap energy 2λ due to the tunneling effect. This coherent mode is a very distinct one characterizing the Hall state with the Josephson current.

Finally, we mention that some of the results on the tunneling and the Josephson current have also been argued by Wen and Zee in their heuristic work.[19] Our formalism of the double-layer system gives a systematic treatment of this phenomenon. It is as clear as in the case of the ordinary Josephson junction in the superconductor.[32] The check of the Josephson current gives a justification of our formalism of the FQH effect based on the bosonic CS gauge theory.

VI. WAVE FUNCTIONS OF HALL STATES

We proceed to derive the wave function corresponding to the classical ground state given by the constant solution (17a). The ground-state wave function is defined by

$$\Psi_g(x_1^\uparrow, \ldots, x_{N^\uparrow}^\uparrow; x_1^\downarrow, \ldots, x_{N^\downarrow}^\downarrow) \equiv \langle 0|\hat{\psi}^\uparrow(x_1^\uparrow)\cdots\hat{\psi}^\uparrow(x_{N^\uparrow}^\uparrow)\hat{\psi}^\downarrow(x_1^\downarrow)\cdots\hat{\psi}^\downarrow(x_{N^\downarrow}^\downarrow)|g\rangle, \tag{69}$$

with $\widehat{\psi}(x^{\uparrow\downarrow})|0\rangle = 0$. The wave function of electrons is constructed by performing a singular gauge transformation to the wave function Ψ_g of the bosonized ones.

We first note that the ground-state wave functions of the general N electron system are exactly solvable when the Coulomb interaction is switched off. The ground state $|g\rangle$ is determined by the eigenvalue problem

$$H|g\rangle = \tfrac{1}{2}\omega_c(N^{\uparrow} + N^{\downarrow})|g\rangle, \tag{70}$$

when all electrons are in the lowest Landau level, which is equivalent to solving the self-dual equations

$$(D_1^{\uparrow} - iD_2^{\uparrow})\widehat{\psi}^{\uparrow}|g\rangle = 0, \qquad (D_1^{\downarrow} - iD_2^{\downarrow})\widehat{\psi}^{\downarrow}|g\rangle = 0. \tag{71}$$

We introduce the complex coordinate $z^{\uparrow}$ and $z^{\downarrow}$ for each electron with up and down pseudospin, $z^{\uparrow\downarrow} = x^{\uparrow\downarrow} + iy^{\uparrow\downarrow}$. Then, the self-dual equations yield the following equations in terms of the wave function defined by (69):

$$\left(\frac{\partial}{\partial z_r^{\uparrow}} + \frac{eB}{4}\bar{z}_r^{\uparrow} - \frac{\alpha}{2\pi}\sum_{s\neq r}\frac{1}{z_r^{\uparrow} - z_s^{\uparrow}} - \frac{\gamma}{\pi}\sum_{s,r}\frac{1}{z_r^{\uparrow} - z_s^{\downarrow}}\right)\Psi_g = 0,$$
$$\left(\frac{\partial}{\partial z_r^{\downarrow}} + \frac{eB}{4}\bar{z}_r^{\downarrow} - \frac{\beta}{2\pi}\sum_{s\neq r}\frac{1}{z_r^{\downarrow} - z_s^{\downarrow}} - \frac{\gamma}{\pi}\sum_{s,r}\frac{1}{z_r^{\uparrow} - z_s^{\downarrow}}\right)\Psi_g = 0. \tag{72}$$

It is easy to solve these equations, and we obtain

$$\Psi_g(z^{\uparrow}; z^{\downarrow}) = f(\bar{z}^{\uparrow}; \bar{z}^{\downarrow})\prod_{r>s}|z_r^{\uparrow} - z_s^{\uparrow}|^{\alpha/\pi}\prod_{t>u}|z_t^{\downarrow} - z_u^{\downarrow}|^{\beta/\pi}\prod_{r,t}|z_r^{\uparrow} - z_t^{\downarrow}|^{\gamma/\pi}\exp\Big[-\frac{eB}{4}\Big(\sum|z_r^{\uparrow}|^2 + \sum|z_t^{\downarrow}|^2\Big)\Big], \tag{73}$$

where $f(\bar{z}^{\uparrow}; \bar{z}^{\downarrow})$ is an arbitrary entire function of the $\bar{z}_r^{\uparrow}$ and $\bar{z}_t^{\downarrow}$.

The electron wave function is obtained by using a singular gauge transformation:

$$\Psi_g(z^{\uparrow}; z^{\downarrow}) = f(\bar{z}^{\uparrow}; \bar{z}^{\downarrow})\prod_{r>s}(\bar{z}_r^{\uparrow} - \bar{z}_s^{\uparrow})^{\alpha/\pi}\prod_{t>u}(\bar{z}_t^{\downarrow} - \bar{z}_u^{\downarrow})^{\beta/\pi}\prod_{r,t}(\bar{z}_r^{\uparrow} - \bar{z}_t^{\downarrow})^{\gamma/\pi}\exp\left[-\frac{eB}{4}\left(\sum|z_r^{\uparrow}|^2 + \sum|z_t^{\downarrow}|^2\right)\right], \tag{74}$$

which has a Laughlin-type expression. In our convention the conjugate coordinate $\bar{z}$ appears instead of z.

The above ground-state wave functions are the most general ones. They just describe an ensemble of two sets of electrons making cyclotron motions independently in the lowest Landau level. The fact that f is an entire function implies that all electrons are in the lowest Landau level, and the fact that it is an arbitrary function implies that the ground state is highly degenerate.

In the presence of the Coulomb interaction, it is impossible to obtain the ground-state wave function exactly. However, we can do it in the semiclassical approximation. We make the following reasoning. Because the Coulomb interaction $\mathcal{V}$ removes the degeneracy, a certain function $F(\bar{z}^{\uparrow}, z^{\uparrow}; \bar{z}^{\downarrow}, z^{\downarrow})$ must be fixed uniquely instead of $f(\bar{z}^{\uparrow}; \bar{z}^{\downarrow})$ in (73). In the limit of vanishing Coulomb interaction ($e^2 \to 0$), F is to approach one of the entire functions f in (73), which we denote by f_g. We then have $F = f_g + e^2G + O(e^4)$. Our idea is to determine the function f_g semiclassicaly. This method was first used in the single-layer electron system, where the first-order correction term G is also calculated;[33] here, the derivation is refined and the statistics parameters are not assumed to be small. In what follows let us assume $\alpha = \beta$ to simplify calculations. Then, $\rho^{\uparrow} = \rho^{\downarrow} = \frac{1}{2}\rho \equiv \rho_0$.

We decompose the field operator into the real part and the imaginary part:

$$\widehat{\psi}^{\uparrow\downarrow}(x) = \sqrt{\rho_0 + \delta\rho^{\uparrow\downarrow}(x)}e^{i\theta^{\uparrow\downarrow}(x)} \approx \sqrt{\rho_0}e^{\eta^{\uparrow\downarrow}(x)/\sqrt{\rho_0}} \tag{75}$$

with

$$\eta^{\uparrow\downarrow}(x) = \frac{\delta\rho^{\uparrow\downarrow}}{2} + i\theta^{\uparrow\downarrow}\sqrt{\rho_0}, \tag{76}$$

where $\eta^{\uparrow\downarrow}$ is defined in Sec. IV. Substituting (75) into the definition (69), we find

$$\begin{aligned}\Psi_g &= \rho_0^{N/2}\langle 0|\prod_r \exp\Big(\frac{\eta^{\uparrow}(x_r^{\uparrow})}{\sqrt{\rho_0}}\Big)\prod_s \exp\Big(\frac{\eta^{\downarrow}(x_s^{\downarrow})}{\sqrt{\rho_0}}\Big)|g\rangle \\ &= \rho_0^{N/2}\langle 0|g\rangle\prod_{r<s}\exp\Big(\frac{\langle 0|\eta^{\uparrow}(x_r^{\uparrow})\eta^{\uparrow}(x_s^{\uparrow})|g\rangle}{\rho_0\langle 0|g\rangle}\Big)\prod_{r<s}\exp\Big(\frac{\langle 0|\eta^{\downarrow}(x_r^{\downarrow})\eta^{\downarrow}(x_s^{\downarrow})|g\rangle}{\rho_0\langle 0|g\rangle}\Big)\prod_{r,s}\exp\Big(\frac{\langle 0|\eta^{\uparrow}(x_r^{\uparrow})\eta^{\downarrow}(x_s^{\downarrow})|g\rangle}{\rho_0\langle 0|g\rangle}\Big) \\ &= \rho_0^{N/2}\langle 0|g\rangle\Big(\prod_{r<s}e^{g_1(x_r^{\uparrow} - x_s^{\uparrow})/\rho_0}\prod_{r<s}e^{g_2(x_r^{\downarrow} - x_s^{\downarrow})/\rho_0}\prod_{r,s}e^{g_3(x_r^{\uparrow} - x_s^{\downarrow})/\rho_0}\Big).\end{aligned} \tag{77}$$

within the Gaussian approximation, where

$$g_1(x) \equiv \frac{\langle 0|\eta^\uparrow(x)\eta^\uparrow(0)|g\rangle}{\langle 0|g\rangle},$$
$$g_2(x) \equiv \frac{\langle 0|\eta^\downarrow(x)\eta^\downarrow(0)|g\rangle}{\langle 0|g\rangle}, \tag{78}$$
$$g_3(x) \equiv \frac{\langle 0|\eta^\uparrow(x)\eta^\downarrow(0)|g\rangle}{\langle 0|g\rangle}.$$

We calculate functions $g_i(x)$ in the vanishing limit of the Coulomb interaction ($e^2 \to 0$), which is enough to determine the function f_g. Using the definition (36) of the ground state $|g\rangle$ and the explicit forms (33) of the Bogoljubov transformation, we can explicitly evaluate them as

$$\frac{g_1(x)}{\rho_0} = -\frac{\alpha+\gamma}{2\pi}K_0(|x|\sqrt{eB}) - \frac{\alpha-\gamma}{2\pi}K_0(|x|\sqrt{eB'}),$$
$$\frac{g_2(x)}{\rho_0} = -\frac{\alpha+\gamma}{2\pi}K_0(|x|\sqrt{eB}) - \frac{\alpha-\gamma}{2\pi}K_0(|x|\sqrt{eB'}), \tag{79}$$
$$\frac{g_3(x)}{\rho_0} = -\frac{\alpha+\gamma}{2\pi}K_0(|x|\sqrt{eB}) + \frac{\alpha-\gamma}{2\pi}K_0(|x|\sqrt{eB'}),$$

where $B' \equiv (\alpha-\gamma)/(\alpha+\gamma)B$, and $K_0(y)$ is the modified Bessel function. See the Appendix for their derivation.

Because the Gaussian approximation is valid for describing the short-distance behavior of the system where the kinetic term $\mathbf{p}^2/2M$ dominates, the wave function (77) is correct in this region. Hence, taking the short-distance limit ($|x| \ll \ell_B$), we get

$$\frac{g_1(x)}{\rho_0} \approx -\frac{\alpha}{\pi}\ln(C_1|x|),$$
$$\frac{g_2(x)}{\rho_0} \approx -\frac{\alpha}{\pi}\ln(C_1|x|), \tag{80}$$
$$\frac{g_3(x)}{\rho_0} \approx -\frac{\gamma}{\pi}\ln(C_3|x|),$$

with C_i being a constant, or

$$\Psi_g = \prod_{r>s}|z_r^\uparrow - z_s^\uparrow|^{\alpha/\pi}\prod_{t>u}|z_t^\downarrow - z_u^\downarrow|^{\alpha/\pi}\prod_{r,t}|z_r^\uparrow - z_t^\downarrow|^{\gamma/\pi}, \tag{81}$$

apart from an overall constant factor. To see the long-distance behavior we have to go beyond the Gaussian approximation. The wave function (81) is to be compared with the wave function (73).

We comment on the exponential factor in (73), which may be rewritten as

$$\prod_r \exp\Big[-\frac{eB}{4}|x_r|^2\Big] = \exp\Big[-\frac{eB}{4}\Big(\sum_r \frac{x_r}{\sqrt{N}}\Big)^2 - \frac{eB}{4N}\sum_{r>s}|x_r - x_s|^2\Big], \tag{82}$$

where x stands for $x^{\uparrow\downarrow}$, and we have separated the coordinate of the center of mass from those of the relative motions. Our wave function does not contain the term depending on the center-of-mass coordinate because of the translational invariance in our semiclassical approximation. Furthermore, the remaining terms $\frac{eB}{4N}|x_r - x_s|$ do not appear in the short distance.

Consequently, we have found that the ground-state wave function (69) is given by (73) with $f = \text{const}$, or

$$\Psi_g(z^\uparrow; z^\downarrow) = \prod_{r>s}(\bar{z}_r^\uparrow - \bar{z}_s^\uparrow)^{\alpha/\pi}\prod_{t>u}(\bar{z}_t^\downarrow - \bar{z}_u^\downarrow)^{\beta/\pi}\prod_{r,t}(\bar{z}_r^\uparrow - \bar{z}_t^\downarrow)^{\gamma/\pi}\exp\Big[-\frac{eB}{4}\Big(\sum|z_r^\uparrow|^2 + \sum|z_t^\downarrow|^2\Big)\Big]. \tag{83}$$

This wave function has the correct property of the anyon statistics. Namely, when one anyon moves around another anyon, the correct phase factor is generated. For the normalizability of the wave function it is necessary that $\gamma > -\pi$. The wave function (83) is precisely the one suggested by Halperin.[4]

Finally, we wish to remark that the origin of the power behaviors of the ground-state wave function (83) is traced back to the terms $\alpha\rho^\uparrow/M$, $\beta\rho^\downarrow/M$, and $2\gamma\sqrt{\rho^\uparrow\rho^\downarrow}/M$ which dominate as $\mathbf{p}^2 \to \infty$ in (A2) in the Appendix. But, these terms have come from the renormalization term (6). Thus, the renormalization term is crucial to derive the correct behavior of the wave function. This is quite reasonable because the term has been introduced so as to reproduce the correct short-distance behavior of the wave function in the perturbation theory with respect to the statistics parameters.

VII. CONDENSATION OF VORTICES AND HIERARCHY

We have noticed in Sec. III that there are topological excitations of vortices (quasiparticles) on the FQH state at the magic filling factor. When the external magnetic field is increased (decreased), antivortices (vortices) must appear to balance the magnetic flux and the statistical flux. We have argued that the ground state at the vicinity of the magic filling factor must be described by such an ensemble of the vortices and that they would form a Wigner crystal in order to minimize the Coulomb energy. When vortices are sufficiently created, they will eventually condense. We would then obtain a new condensed phase of electrons and vortices (new Hall liquid).

In order to describe this new phase, it is convenient to

construct an effective-field theory of vortices. In previous papers[13,12] we have proposed such a scheme in the single-layer system, where a field-theoretical realization of the Haldane-Halperin hierarchy[3] is presented. Because the present theory is its simple generalization we briefly describe how to derive the effective-field theory of vortices. Using it we discuss the hierarchy structure of the FQH states in the double-layer electron system.

We start with the Lagrangian describing the double-layer electron system. It is given by

$$\mathcal{L}_T = \psi^{\uparrow\dagger}(i\partial_0 + a_0^{\uparrow})\psi^{\uparrow} - \frac{1}{2M}|(D_1^{\uparrow} - iD_2^{\uparrow})\psi^{\uparrow}|^2 + \psi^{\downarrow\dagger}(i\partial_0 + a_0^{\downarrow})\psi^{\downarrow} - \frac{1}{2M}|(D_1^{\downarrow} - iD_2^{\downarrow})\psi^{\downarrow}|^2 - \tfrac{1}{2}\omega_c(|\psi^{\uparrow}|^2 + |\psi^{\downarrow}|^2) - \mathcal{V}[\psi]$$

$$-\frac{1}{4X}\varepsilon^{\mu\nu\lambda}a_\mu^{\uparrow}\partial_\nu a_\lambda^{\uparrow} - \frac{1}{4Y}\varepsilon^{\mu\nu\lambda}a_\mu^{\downarrow}\partial_\nu a_\lambda^{\downarrow} - \frac{1}{4Z}\varepsilon^{\mu\nu\lambda}a_\mu^{\uparrow}\partial_\nu a_\lambda^{\downarrow} - \frac{1}{4Z}\varepsilon^{\mu\nu\lambda}a_\mu^{\downarrow}\partial_\nu a_\lambda^{\uparrow}. \tag{84}$$

Here, parameters X, Y, and Z are defined by

$$X = \frac{\alpha\beta - \gamma^2}{\beta}, \qquad Y = \frac{\alpha\beta - \gamma^2}{\alpha}, \qquad Z = \frac{\gamma^2 - \alpha\beta}{\gamma}. \tag{85}$$

The Lagrangian has been constructed so as to reproduce the Hamiltonian (14) and the constraint equation (7). Notice that X^{-1}, Y^{-1}, and Z^{-1} are singular for $\alpha = \beta = \gamma$ and we should use the Lagrangian (57) in this case.

We would like to construct a local field theory of vortex solitons with vorticity $n = \pm 1$. Although vortex solitons are extended objects, we take the pointlike limit. Then, their local field theory describes correctly the effects whose scale is larger than the scale ℓ_B of the vortex soliton. For definiteness, we treat the case of antivortices ($n = -1$) explicitly but the case of vortices ($n = 1$) can be similarly discussed.

A pointlike vortex is a flux concentrated in a small domain. Such a flux is easily introduced by considering a singular gauge transformation. To create them at $z_r^{\uparrow}$ ($z_r^{\downarrow}$) in the upper (lower) layer, we perform a singular gauge transformation

$$\psi^{\uparrow\downarrow} \to e^{if^{\uparrow\downarrow}}\psi^{\uparrow\downarrow}, \qquad a_\mu^{\uparrow\downarrow} \to a_\mu^{\uparrow\downarrow} + \partial_\mu f^{\uparrow\downarrow}, \tag{86}$$

where $f^{\uparrow\downarrow}(x) = -\sum_r \theta(x - z_r^{\uparrow\downarrow})$, and $\theta(x - z_r^{\uparrow\downarrow})$ is the azimuthal angle. (We set the minus sign here since the vortices are actually antivortices. For vortices we should set the positive sign.) We then get

$$\varepsilon_{ij}\partial_i a_j^{\uparrow\downarrow} \to \varepsilon_{ij}\partial_i a_j^{\uparrow\downarrow} - 2\pi\rho_v^{\uparrow\downarrow}(x), \tag{87}$$

where

$$\rho_v^{\uparrow\downarrow}(x) = \sum_r \delta^2(x - z_r^{\uparrow\downarrow}) \tag{88}$$

describes a set of local vortices sitting at $x^k = z_r^{k\uparrow\downarrow}(t)$ in the two-dimensional space at time t, where r labels the vortices.

Applying this singular gauge transformation to the Lagrangian (84), we obtain

$$\mathcal{L} \to \mathcal{L} + \Delta\mathcal{L}^{\text{vortex}}, \tag{89}$$

with

$$\Delta\mathcal{L}^{\text{vortex}} = \frac{\pi}{X}a_\mu^{\uparrow}K^{\uparrow\mu} + \frac{\pi}{Z}a_\mu^{\downarrow}K^{\uparrow\mu} + \frac{\pi}{Y}a_\mu^{\downarrow}K^{\downarrow\mu} + \frac{\pi}{Z}a_\mu^{\uparrow}K^{\downarrow\mu} + \widehat{\alpha}G^{\uparrow} + \widehat{\beta}G^{\downarrow} + \widehat{\gamma}G. \tag{90}$$

Here,

$$K^{\uparrow\downarrow\mu} = \sum \dot{z}_r^{\uparrow\downarrow\mu}\delta^2(x - z_r^{\uparrow\downarrow}), \tag{91}$$

which represents world lines of the vortices; $z^{\uparrow\downarrow\mu} = (t, z^{\uparrow\downarrow k})$. On the other hand,

$$G^{\uparrow\downarrow} = (1/2\pi)\sum_s K^{\uparrow\downarrow\mu}\partial_\mu\theta(x - z_s^{\uparrow\downarrow}),$$
$$G = (1/4\pi)\sum_{r,s}[K^{\uparrow\mu}\partial_\mu\theta(x - z_s^{\downarrow}) + K^{\downarrow\mu}\partial_\mu\theta(x - z_s^{\uparrow})]. \tag{92}$$

These quantities describe the linking of world lines of vortices. We have created vortices on the two layers with $\widehat{\alpha}$ and $\widehat{\beta}$ representing their own statistic and $\widehat{\gamma}$ their relative statistics. They are given by the generalized reciprocal relation[12]

$$\widehat{\alpha} = -\frac{\pi^2}{X} - 2\pi p, \qquad \widehat{\beta} = -\frac{\pi^2}{Y} - 2\pi q, \qquad \widehat{\gamma} = -\frac{\pi^2}{Z} - 2\pi r, \tag{93}$$

with p, q, r being integers.

Integrating $\Delta\mathcal{L}^{\text{vortex}}$ over the two-dimensional space and adding the kinetic terms of the vortices we get

$$\mathcal{L}^{\text{vortex}} = \sum \frac{M_v^{\uparrow}}{2}\Big(\frac{dz_r^{\uparrow k}}{dt}\Big)^2 + \sum \frac{M_v^{\downarrow}}{2}\Big(\frac{dz_r^{\downarrow k}}{dt}\Big)^2 + \int d^2x \Delta\mathcal{L}^{\text{vortex}}$$
$$= \sum\Big[\frac{M_v^{\uparrow}}{2}\Big(\frac{dz_r^{\uparrow k}}{dt}\Big)^2 + \Big(\frac{\pi}{X}a_k^{\uparrow} + \frac{\pi}{Z}a_k^{\downarrow}\Big)\frac{dz_r^{\uparrow k}}{dt} + \Big(\frac{\pi}{X}a_0^{\uparrow} + \frac{\pi}{Z}a_0^{\downarrow}\Big)\Big]$$
$$+ \sum\Big[\frac{M_v^{\downarrow}}{2}\Big(\frac{dz_r^{\downarrow k}}{dt}\Big)^2 + \Big(\frac{\pi}{Y}a_k^{\downarrow} + \frac{\pi}{Z}a_k^{\uparrow}\Big)\frac{dz_r^{\downarrow k}}{dt} + \Big(\frac{\pi}{X}a_0^{\downarrow} + \frac{\pi}{Z}a_0^{\uparrow}\Big)\Big]$$
$$+ \frac{\widehat{\alpha}}{\pi}\sum_{r<s}\frac{d}{dt}\theta(z_r^{\uparrow} - z_s^{\uparrow}) + \frac{\widehat{\beta}}{\pi}\sum_{r<s}\frac{d}{dt}\theta(z_r^{\downarrow} - z_s^{\downarrow}) + \frac{\widehat{\gamma}}{\pi}\sum_{r,s}\frac{d}{dt}\theta(z_r^{\uparrow} - z_s^{\downarrow}). \tag{94}$$

The mass of the vortex is given by (27). This describes the particle mechanics of vortices interacting with the background CS gauge fields $a_k^{\uparrow\downarrow}$ and among themselves. Because these vortices live on the condensate of bosonized electrons, we substitute the mean-field value in (17b) for $a_k^{\uparrow\downarrow}$, i.e., $a_k^{\uparrow\downarrow} = eA_k$.

From the Lagrangian (94), the particle-mechanical Hamiltonian is derived:[13]

$$H^{\text{vortex}} = \frac{1}{2M_v^{\uparrow}}\sum[p_r^{\uparrow k} - Q^{\uparrow}eA_k - c_k^{\uparrow}(z_r)]^2$$
$$+ \frac{1}{2M_v^{\downarrow}}\sum[p_r^{\downarrow k} - Q^{\downarrow}eA_k - c_k^{\downarrow}(z_r)]^2, \tag{95}$$

together with the constraint equations:

$$\varepsilon_{ij}\partial_i c_j^{\uparrow} = 2\widehat{\alpha}\rho_v^{\uparrow} + 2\widehat{\gamma}\rho_v^{\downarrow},$$
$$\varepsilon_{ij}\partial_i c_j^{\downarrow} = 2\widehat{\gamma}\rho_v^{\uparrow} + 2\widehat{\beta}\rho_v^{\downarrow}, \tag{96}$$

where $\rho_v^{\uparrow\downarrow}$ is defined by (88). In (95) we have set $Q^{\uparrow} = \pi/X + \pi/Z$ and $Q^{\downarrow} = \pi/Y + \pi/Z$. The quantity $eQ^{\uparrow\downarrow}$ is the electric charge of the vortex. Using the definition (85) for X, Y, and Z, we find that $Q^{\uparrow\downarrow}$ is precisely the same one as (26) derived for the classical topological soliton.

To avoid unnecessary complications let us assume $\alpha = \beta$ and $\widehat{\alpha} = \widehat{\beta}$ in what follows. Then,

$$Q^{\uparrow} = Q^{\downarrow} \equiv Q = \frac{\pi}{\alpha + \gamma}. \tag{97}$$

We also have $M_v^{\uparrow} = M_v^{\downarrow} \equiv M_v = MQ$.

Second quantization proceeds straightforwardly once the quantum-mechanical Hamiltonian is given. We introduce field operators $\phi^{\uparrow\downarrow}$ which annihilate vortices. Since the vortices are anyons, we need to add appropriate renormalization counter terms like (6). Then, the second quantized Hamiltonian density reads[13]

$$\mathcal{H}^{\text{vortex}} = \frac{1}{2M_v}|(D_1^{\uparrow(1)} - iD_2^{\uparrow(1)})\phi^{\uparrow}|^2$$
$$+ \frac{1}{2M_v}|(D_1^{\downarrow(1)} - iD_2^{\downarrow(1)})\phi^{\downarrow}|^2 + \mathcal{V}_v[\phi], \tag{98}$$

with an appropriate Coulomb interaction term for vortices and

$$iD_k^{\uparrow\downarrow(1)} = i\partial_k + c_k^{\uparrow\downarrow} + \frac{\pi}{\alpha + \gamma}eA_k. \tag{99}$$

Here, $c_k^{\uparrow\downarrow}$ are subject to the constraint equations

$$\varepsilon_{ij}\partial_i c_j^{\uparrow} = 2\widehat{\alpha}|\phi^{\uparrow}|^2 + 2\widehat{\gamma}|\phi^{\downarrow}|^2,$$
$$\varepsilon_{ij}\partial_i c_j^{\downarrow} = 2\widehat{\gamma}|\phi^{\uparrow}|^2 + 2\widehat{\alpha}|\phi^{\downarrow}|^2, \tag{100}$$

which follow from (95) and (96) by the correspondence principle. Compare these with the Hamiltonian density (14) and the constraint equations (7) of the electrons. They are formally identical except that there is no term $\frac{1}{2}\omega_c|\phi|^2$ representing the cyclotron motion of vortices. This is physically correct because vortices are really collective modes of electrons in the lowest Landau level.

The derivation of the Hamiltonian (98) is by no means rigorous. We have included the kinetic term of the vortices by hand. It may well be that there are self-interaction terms of $\phi^{\uparrow\downarrow}$ which may result, e.g., from the finite size of actual vortices. However, the form of the Hamiltonian is severely restricted by the condition that vortices are really collective modes of electrons in the lowest Landau level. Our effective Hamiltonian (98) would be the simplest one that satisfies this criterion. In this sense it is plausible and it seems to be practically correct.

Because of the formal similarity of the vortex Hamiltonian and the electron Hamiltonian, the ground-state structure can easily be derived without any further calculations. Thus, the uniform condensed state of vortices exists only when the vortex-filling factor $\nu_v \equiv 2\pi\rho_v/(-eQB)$ takes a magic number

$$\nu_v \equiv \frac{2\pi}{\widehat{\alpha} + \widehat{\gamma}}, \tag{101}$$

as corresponds to (20). (Note that we have assumed $\widehat{\alpha} = \widehat{\beta}$.) It is necessary to relate this to the electron-filling factor $\nu \equiv 2\pi\rho/eB$. After the singular transformation (87) generating vortices, the constraint equations (7) read

$$\varepsilon_{ij}\partial_i a_j^{\uparrow} = 2\alpha|\psi^{\uparrow}|^2 + 2\gamma|\psi^{\downarrow}|^2 + 2\pi\rho_v^{\uparrow},$$
$$\varepsilon_{ij}\partial_i a_j^{\downarrow} = 2\gamma|\psi^{\uparrow}|^2 + 2\alpha|\psi^{\downarrow}|^2 + 2\pi\rho_v^{\downarrow}. \tag{102}$$

Using the mean-field values we find that

$$\nu = \frac{2\pi}{\alpha + \gamma} - \Big(\frac{2\pi}{\alpha + \gamma}\Big)^2\nu_v = \frac{1}{k + \dfrac{1}{4(p + r)}}, \tag{103}$$

where $k \equiv (\alpha + \gamma)/2\pi$. Note the similarity of this formula to the corresponding one in the single-layer case: see Eq. (8.2) in Ref. 13. Repeating the same procedure

we can produce the full hierarchy of the FQH states in the double-layer electron system.

VIII. CONCLUSION

In this paper we have analyzed the double-layer electron system based on the bosonic CS gauge theory, where the FQH effect has been experimentally observed at even-denominator filling factors. We have used the semiclassical approximation. At the magic filling factor (1), Hall liquids are identified with the states in the condensed phase of bosonized electrons. Furthermore, we have shown that the wave functions of the Hall states are precisely the ones suggested by Halperin. In their derivation the renormalization counter term (6) has played a crucial role.

An interesting prediction of our formalism is the Josephson effect realized at the filling factor $\nu = 1, \frac{1}{3}, \frac{1}{5}, \ldots$. Although this effect has also been pointed out by Wen and Zee,[19] their argument is heuristic and rather *ad hoc*. On the contrary, our argument is as clear as in the case of the ordinary Josephson junction in the superconductor.[32]

In our formalism quasiparticles (quasiholes) are identified with vortex (antivortex) solitons. A characteristic feature is that a soliton has the electric charges on both of the layers. For instance at $\nu = \frac{1}{2}$ the electric charge of the antivortex with unit flux is $\frac{1}{4}e$, which is the sum of the charges $\frac{3}{8}e$ and $-\frac{1}{8}e$ on each layer. In this example the gap energy of the Hall state is roughly given by 4 times the Coulomb energy of a single antivortex,[13] which is numerically estimated[28] as $0.18 \times (e^2/\varepsilon\ell_B)$. Recall that in the single-layer case the Coulomb energy of the antivortex is precisely proportional to the square of its electric charge, and hence proportional to the filling factor. Therefore, the gap energy is estimated[13] as $0.40\nu \times (e^2/\varepsilon\ell_B)$. There is no such a simple relation in the double-layer Hall state with $\gamma \neq 0$. This is because the Coulomb energy of the soliton is a sum of three terms, that is, each Coulomb energy associated with the charge in each layer and the relative Coulomb energy between the two charges in the two layers.

We have also calculated the Coulomb energy of the Hall state as a function of the interlayer distance d. However, the calculation is not sufficient because the contribution of vortex solitons has not been taken into account in our semiclassical approximation. This problem is presently under study.

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APPENDIX

In this appendix we give some detailed formulas which we used in Secs. IV and VI. Substituting (30) into the Hamiltonian (14), and taking only terms up to quadratic in η, we obtain

$$\begin{aligned} H = \sum_{\mathbf{p}\neq 0}\Big(& (\varepsilon_{\mathbf{p}} + 2U_{\mathbf{p}}^{\uparrow})a_{\mathbf{p}}^{\dagger}a_{\mathbf{p}} + U_{\mathbf{p}}^{\uparrow}(a_{\mathbf{p}}^{\dagger}a_{-\mathbf{p}}^{\dagger} + a_{\mathbf{p}}a_{-\mathbf{p}}) \\ & + (\varepsilon_{\mathbf{p}} + 2U_{\mathbf{p}}^{\downarrow})b_{\mathbf{p}}^{\dagger}b_{\mathbf{p}} + U_{\mathbf{p}}^{\downarrow}(b_{\mathbf{p}}^{\dagger}b_{-\mathbf{p}}^{\dagger} + b_{\mathbf{p}}b_{-\mathbf{p}}) \\ & + V_{\mathbf{p}}(a_{\mathbf{p}}^{\dagger}b_{-\mathbf{p}}^{\dagger} + a_{\mathbf{p}}b_{-\mathbf{p}} + a_{\mathbf{p}}^{\dagger}b_{\mathbf{p}} + b_{\mathbf{p}}^{\dagger}a_{\mathbf{p}}) \\ & + \tfrac{1}{2}\omega_c N + U_{\mathbf{p}}^{\uparrow} + U_{\mathbf{p}}^{\downarrow} - \frac{\alpha\rho^{\uparrow} + \beta\rho^{\downarrow}}{M} - \frac{\pi e^2\rho}{|\mathbf{p}|}\Big), \end{aligned} \tag{A1}$$

where

$$\begin{aligned} U_{\mathbf{p}}^{\uparrow} &= \frac{2\alpha^2\rho^{\uparrow 2} + 2\gamma^2\rho^{\uparrow}\rho^{\downarrow}}{M\mathbf{p}^2} + \frac{\alpha\rho^{\uparrow}}{M} + \frac{\pi e^2\rho^{\uparrow}}{|\mathbf{p}|}, \\ U_{\mathbf{p}}^{\downarrow} &= \frac{2\beta^2\rho^{\downarrow 2} + 2\gamma^2\rho^{\uparrow}\rho^{\downarrow}}{M\mathbf{p}^2} + \frac{\beta\rho^{\downarrow}}{M} + \frac{\pi e^2\rho^{\downarrow}}{|\mathbf{p}|}, \\ V_{\mathbf{p}} &= \frac{4\gamma\sqrt{\rho^{\uparrow}\rho^{\downarrow}}(\alpha\rho^{\uparrow} + \beta\rho^{\downarrow})}{M\mathbf{p}^2} + \frac{2\gamma\sqrt{\rho^{\uparrow}\rho^{\downarrow}}}{M} \\ &\quad + \frac{2\pi e^2\sqrt{\rho^{\uparrow}\rho^{\downarrow}}}{|\mathbf{p}|}e^{-|\mathbf{p}|d}. \end{aligned} \tag{A2}$$

In these expressions, $\sqrt{\rho^{\uparrow\downarrow}} = \langle\widehat{\psi}^{\uparrow\downarrow}\rangle$ are functions of α, β, and γ as given by (19).

Let us first diagonalize (A1) except for the term proportional to $V_{\mathbf{p}}$ by way of the standard Bogoljubov transformations. Thus, into the Hamiltonian (A1) we substitute

$$\begin{aligned} a_{\mathbf{p}} &= g_{\mathbf{p}}^{\uparrow}a'_{\mathbf{p}} - h_{\mathbf{p}}^{\uparrow}a'^{\dagger}_{-\mathbf{p}}, \\ b_{\mathbf{p}} &= g_{\mathbf{p}}^{\downarrow}b'_{\mathbf{p}} - h_{\mathbf{p}}^{\downarrow}b'^{\dagger}_{-\mathbf{p}}, \end{aligned} \tag{A3}$$

with $(h_{\mathbf{p}}^{\uparrow\downarrow})^2 = (g_{\mathbf{p}}^{\uparrow\downarrow})^2 - 1$ and

$$(g_{\mathbf{p}}^{\uparrow\downarrow})^2 = \frac{1}{2}\left(\frac{\varepsilon_{\mathbf{p}} + 2U_{\mathbf{p}}^{\uparrow\downarrow}}{E_{\mathbf{p}}^{\uparrow\downarrow}} + 1\right). \tag{A4}$$

Here, $g_{\mathbf{p}}^{\uparrow\downarrow} = g_{-\mathbf{p}}^{\uparrow\downarrow}$, $h_{\mathbf{p}}^{\uparrow\downarrow} = h_{-\mathbf{p}}^{\uparrow\downarrow}$, and

$$\varepsilon_{\mathbf{p}} = \frac{\mathbf{p}^2}{2m}, \qquad E_{\mathbf{p}}^{\uparrow\downarrow} = \sqrt{\varepsilon_{\mathbf{p}}^2 + 4\varepsilon_{\mathbf{p}}U_{\mathbf{p}}^{\uparrow\downarrow}}. \tag{A5}$$

The resulting Hamiltonian is

$$\begin{aligned} H = \sum_{\mathbf{p}\neq 0}\Big(& E_{\mathbf{p}}^{\uparrow}a'^{\dagger}_{\mathbf{p}}a'_{\mathbf{p}} + E_{\mathbf{p}}^{\downarrow}b'^{\dagger}_{\mathbf{p}}b'_{\mathbf{p}} \\ & + \widehat{V}_{\mathbf{p}}(a'^{\dagger}_{\mathbf{p}}b'^{\dagger}_{-\mathbf{p}} + a'_{\mathbf{p}}b'_{-\mathbf{p}} + a'^{\dagger}_{\mathbf{p}}b'_{\mathbf{p}} + b'^{\dagger}_{\mathbf{p}}a'_{\mathbf{p}}) \\ & + \tfrac{1}{2}\omega_c N + \tfrac{1}{2}(E_{\mathbf{p}}^{\uparrow} + E_{\mathbf{p}}^{\downarrow}) - \varepsilon_{\mathbf{p}} \\ & - \frac{\alpha\rho^{\uparrow} + \beta\rho^{\downarrow}}{M} - \frac{\pi e^2\rho}{|\mathbf{p}|}\Big), \end{aligned} \tag{A6}$$

with

$$\widehat{V}_{\mathbf{p}} = V_{\mathbf{p}}(g_{\mathbf{p}}^{\uparrow} - h_{\mathbf{p}}^{\uparrow})(g_{\mathbf{p}}^{\downarrow} - h_{\mathbf{p}}^{\downarrow}). \tag{A7}$$

We next substitute

$$a'_{\mathbf{p}} = G_{\mathbf{p}} A_{\mathbf{p}} + H_{\mathbf{p}} A^{\dagger}_{-\mathbf{p}} + I_{\mathbf{p}} B_{\mathbf{p}} + J_{\mathbf{p}} B^{\dagger}_{-\mathbf{p}},$$
$$b'_{\mathbf{p}} = G'_{\mathbf{p}} A_{\mathbf{p}} + H'_{\mathbf{p}} A^{\dagger}_{-\mathbf{p}} + I'_{\mathbf{p}} B_{\mathbf{p}} + J'_{\mathbf{p}} B^{\dagger}_{-\mathbf{p}}. \tag{A8}$$

into the Hamiltonian (A6). With appropriate coefficients $G_{\mathbf{p}}$, $H_{\mathbf{p}}$, $I_{\mathbf{p}}$, $J_{\mathbf{p}}$ and their primed ones, it is tedious but straightforward to show that the Hamiltonian is diagonalized as in (34).

The explicit forms of the coefficients in (A8) are very complicated. Here, for simplicity we present them only in the case of $\alpha = \beta$ and the vanishing limit of the Coulomb interaction ($e^2 \to 0$):

$$G_{\mathbf{p}} = \frac{1}{2} \frac{\widehat{E}_{\mathbf{P}} + \varepsilon_{\mathbf{p}} + \omega_c}{\sqrt{2(\varepsilon_{\mathbf{p}} + \omega_c)\widehat{E}_{\mathbf{p}}}},$$
$$H_{\mathbf{p}} = \frac{1}{2} \frac{\widehat{E}_{\mathbf{P}} - \varepsilon_{\mathbf{p}} - \omega_c}{\sqrt{2(\varepsilon_{\mathbf{p}} + \omega_c)\widehat{E}_{\mathbf{p}}}},$$
$$I_{\mathbf{p}} = \frac{1}{2} \frac{\widehat{E}_{\mathbf{P}} + \varepsilon_{\mathbf{p}} + \omega'_c}{\sqrt{2(\varepsilon_{\mathbf{p}} + \omega'_c)\widehat{E}_{\mathbf{p}}}}, \tag{A9}$$
$$J_{\mathbf{p}} = \frac{1}{2} \frac{\widehat{E}_{\mathbf{P}} - \varepsilon_{\mathbf{p}} - \omega'_c}{\sqrt{2(\varepsilon_{\mathbf{p}} + \omega'_c)\widehat{E}_{\mathbf{p}}}},$$

where $\widehat{E}_{\mathbf{p}} \equiv E^{\uparrow}_{\mathbf{p}} = E^{\downarrow}_{\mathbf{p}}$, with $G'_{\mathbf{p}} = G_{\mathbf{p}}$, $H'_{\mathbf{p}} = H_{\mathbf{p}}$, $I'_{\mathbf{p}} = -I_{\mathbf{p}}$, and $J'_{\mathbf{p}} = -J_{\mathbf{p}}$. Combinations of these coefficients with $g^{\uparrow\downarrow}$ and $h^{\uparrow\downarrow}$ in (A3) give the coefficients in (33) as

$$\Gamma_{\mathbf{p}} = g_{\mathbf{p}} G_{\mathbf{p}} - h_{\mathbf{p}} H_{\mathbf{p}},$$
$$\Lambda_{\mathbf{p}} = g_{\mathbf{p}} H_{\mathbf{p}} - h_{\mathbf{p}} G_{\mathbf{p}},$$
$$\Omega_{\mathbf{p}} = g_{\mathbf{p}} I_{\mathbf{p}} - h_{\mathbf{p}} J_{\mathbf{p}}, \tag{A10}$$
$$\Upsilon_{\mathbf{p}} = g_{\mathbf{p}} J_{\mathbf{p}} - h_{\mathbf{p}} I_{\mathbf{p}}.$$

with $\Gamma' = \Gamma$, $\Lambda' = \Lambda$, $\Omega' = -\Omega$, $\Upsilon' = -\Upsilon$.

In order to obtain the ground-state wave function, we note that the ground state is defined by (36), where $A_{\mathbf{p}}$ and $B_{\mathbf{p}}$ are related to the original operators $a_{\mathbf{p}}$ and $b_{\mathbf{p}}$ as in (33). It follows from these equations that

$$\langle 0|a_{\mathbf{p}} a_{\mathbf{q}}|g\rangle = \frac{\Lambda_{\mathbf{p}}\Omega_{\mathbf{p}} + \Gamma_{\mathbf{p}}\Upsilon_{\mathbf{p}}}{2\Gamma_{\mathbf{p}}\Omega_{\mathbf{p}}} \delta_{\mathbf{p},-\mathbf{q}} \langle 0|g\rangle,$$
$$\langle 0|b_{\mathbf{p}} b_{\mathbf{q}}|g\rangle = \frac{\Lambda_{\mathbf{p}}\Omega_{\mathbf{p}} + \Gamma_{\mathbf{p}}\Upsilon_{\mathbf{p}}}{2\Gamma_{\mathbf{p}}\Omega_{\mathbf{p}}} \delta_{\mathbf{p},-\mathbf{q}} \langle 0|g\rangle, \tag{A11}$$
$$\langle 0|a_{\mathbf{p}} b_{\mathbf{q}}|g\rangle = \frac{\Lambda_{\mathbf{p}}\Omega_{\mathbf{p}} - \Gamma_{\mathbf{p}}\Upsilon_{\mathbf{p}}}{2\Gamma_{\mathbf{p}}\Omega_{\mathbf{p}}} \delta_{\mathbf{p},-\mathbf{q}} \langle 0|g\rangle,$$

which give

$$\langle 0|a_{\mathbf{p}} a_{\mathbf{q}}|g\rangle = \frac{1}{2}\Big(\frac{-eB}{\mathbf{p}^2 + eB} + \frac{-eB'}{\mathbf{p}^2 + eB'}\Big) \delta_{\mathbf{p},-\mathbf{q}} \langle 0|g\rangle,$$
$$\langle 0|b_{\mathbf{p}} b_{\mathbf{q}}|g\rangle = \frac{1}{2}\Big(\frac{-eB}{\mathbf{p}^2 + eB} + \frac{-eB'}{\mathbf{p}^2 + eB'}\Big) \delta_{\mathbf{p},-\mathbf{q}} \langle 0|g\rangle, \tag{A12}$$
$$\langle 0|a_{\mathbf{p}} b_{\mathbf{q}}|g\rangle = \Big(\frac{-eB}{\mathbf{p}^2 + eB} + \frac{eB'}{\mathbf{p}^2 + eB'}\Big) \delta_{\mathbf{p},-\mathbf{q}} \langle 0|g\rangle,$$

where use was made of (A9). Now, since

$$\frac{1}{2\pi} \int d^2p \frac{1}{\mathbf{p}^2 + eB} e^{i\mathbf{p}\cdot\mathbf{x}} = K_0(|x|\sqrt{eB}),$$

it is easy to derive (80) in the text.

[1] For a review, see *The Quantum Hall Effect*, 2nd ed., edited by S. Girvin and R. Prange (Springer-Verlag, New York, 1990).
[2] Y.W. Suen *et al.*, Phys. Rev. Lett. **68**, 1379 (1992); J.P. Eisenstein *et al.*, *ibid.* **68**, 1383 (1992).
[3] R.B. Laughlin, Phys. Rev. B **23**, 3383 (1983); F.D.M. Haldane, Phys. Rev. Lett. **51**, 605 (1983); B.I. Halperin, *ibid.* **52**, 1583 (1984).
[4] B.I. Halperin, Helv. Phys. Acta **56**, 75 (1983).
[5] D. Yoshioka, B.I. Halperin, and P.A. Lee, Phys. Rev. Lett. **50**, 1219 (1983).
[6] D. Yoshioka, A.H. MacDonald, and S.M. Girvin, Phys. Rev. B **39**, 1932 (1989).
[7] S. He, X.C. Xie. S.D. Sarma, and F.C. Zhang, Phys. Rev. B **43**, 9339 (1991).
[8] S.M. Girvin and A.H. McDonald, Phys. Rev. Lett. **58**, 1252 (1987).
[9] S.C. Zhang, T.H. Hansen, and S. Kivelson, Phys. Rev. Lett. **62**, 82 (1989).
[10] N. Read, Phys. Rev. Lett. **62**, 86 (1989).
[11] D.H. Lee and M.P.A. Fisher, Phys. Rev. Lett. **63**, 903 (1989).
[12] Z.F. Ezawa and A. Iwazaki, Phys. Rev. B **43**, 2637 (1991).
[13] Z.F. Ezawa, M. Hotta, and A. Iwazaki, Phys. Rev. B **46**, 7765 (1992).
[14] Z.F. Ezawa and A. Iwazaki, J. Phys. Soc. Jpn. **61**, 4133 (1992).
[15] J.M. Leinaas and J. Myrheim, Nuovo Cimento B **37**, 1 (1977); F. Wilczek, Phys. Rev. Lett. **49**, 957 (1982); F. Wilczek and A. Zee, *ibid.* **51**, 2250 (1983).
[16] M. Rasolt and A.H. MacDonald, Phys. Rev. B **34**, 5530 (1986).
[17] H. A. Fertig, Phys. Rev. B **40**, 1087 (1989).
[18] A.H. MacDonald, P.M. Platzman, and G.S. Boebinger, Phys. Rev. Lett. **65**, 775 (1990).
[19] X.G. Wen and A. Zee, Phys. Rev. B (to be published).
[20] Z.F. Ezawa, A. Iwazaki, and Y.S. Wu (unpublished).
[21] F. Wilczek, Phys. Rev. Lett. **69**, 132 (1992).
[22] Z.F. Ezawa and A. Iwazaki (unpublished).
[23] R. Sen, Nucl. Phys. **B360**, 397 (1991).
[24] K.H. Cho and C. Rim, Ann. Phys. (N.Y.) **213**, 295 (1992); G. Dunne, A. Lerda, S. Sciuto, and C.A. Trugenberger, Nucl. Phys. **B370**, 601 (1992).
[25] R. Jackiw and S-Y. Pi, Phys. Rev. Lett. **64**, 2969 (1990).
[26] R. Jackiw and S-Y. Pi, Phys. Rev. D **42**, 3500 (1990).
[27] Z.F. Ezawa and A. Iwazaki, Int. J. Mod. Phys. B **6**, 3205 (1992).
[28] We would like to thank Y. Shimizu for carrying out numer-

ical computations.
[29]F. Wilczek, *Fractional Statistics and Anyon Superconductivity* (World Scientific, Singapore, 1990).
[30]Z.F. Ezawa and A. Iwazaki, Mod. Phys. Lett. B (to be published).
[31]B.D. Josephson, Phys. Lett. **1**, 251 (1962).
[32]R.P. Feynman, *The Feynman Lectures on Physics* (Addison-Wesley, Reading, MA, 1989), Vol. III.
[33]Z.F. Ezawa and A. Iwazaki, Mod. Phys. Lett. B **6**, 737 (1992).