Aharonov-Anandan phase and persistent currents in a mesoscopic ring

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The exact solution for a mesoscopic ring in the presence of a classical, static, inhomogeneous magnetic field is found and used to derive the formulas for persistent currents. From the derivation, we can see the close relation of the Aharonov-Anandan phase to the persistent currents. In the adiabatic limit, our results are shown to be identical with those of Loss *et al.*, who explored the connection between the persistent currents and the Berry phase.

The problem of persistent currents in mesoscopic rings has attracted much attention for years.[1-11] In recent experiments,[6] Levy *et al.* found the evidence for a flux-periodic persistent current in a mesoscopic ring threaded by a magnetic flux. Subsequently, some interesting interpretations of the experiments were proposed.[6-10] On the other hand, there has been much interest in the adiabatic geometric phase—Berry phase[12] and its nonadiabatic generalization—Aharonov-Anandan (AA) phase.[13] Recently, Loss, Goldbart, and Balatsdy[1] studied the persistent currents in a mesoscopic ring in the presence of a classical, static, inhomogeneous magnetic field by examining the coupling between spin and orbital motion through the Zeeman term. With the help of a path-integral approach to decouple the spin and orbital motion in the adiabatic approximation, they first demonstrated the deep connection between the Berry phase and the persistent currents and discussed possible experimental verification. The purpose of this paper is to find the exact solution[14] and corresponding persistent currents for the system studied by Loss, Goldbart, and Balatsdy without using the adiabatic approximation.[15] In our approach, the connection between the AA phase and persistent currents is exhibited. In the adiabatic limit, the AA phase becomes the Berry phase. With the help of it, the results in Ref. 1 are reobtained. The condition for the validity of the adiabatic approximation is then discussed.[15]

The Hamiltonian H for an electron of mass m, charge e, and spin $\frac{1}{2}$, confined to a ring of radius a, in the presence of an inhomogeneous magnetic field $B\hat{n}(\theta)=\nabla\times\mathbf{A}$, is as follows:[1]

$$H=[P_\theta-eaA/c]^2/(2ma^2)-ge\hbar B\hat{n}(\theta)\cdot\boldsymbol{\sigma}/(4mc)\ , \quad (1)$$

with $P_\theta=-i\hbar\partial/\partial\theta$ and $\hat{n}(\theta)=\hat{e}_r\sin\chi+\hat{e}_z\cos\chi$. The tilt angle χ describes the deviation of the magnetic field $B\hat{n}(\theta)$ from the z axis. Here, the ring lies in the x-y plane with its center at the origin, r, θ, and z are the usual cylindrical coordinates, and $\hat{e}_r$ and $\hat{e}_z$ are radial and axial unit vectors located at the point θ on the ring for the cylindrical-coordinate system.

We now turn to the exact solution of the stationary-state Schrödinger equation with Hamiltonian (1). Because of the cylindrical symmetry, the system possesses a conserved quantity, $P_\theta+\hbar\sigma^3/2$. From the periodic condition for the wave function in a ring, it is easily seen that the common eigenstates of $P_\theta+\hbar\sigma^3/2$ and H are of the form

$$\Psi_{n,+}(\theta)=e^{in\theta}/(2\pi)^{1/2}\begin{bmatrix}\cos(\beta_n/2)\\ e^{i\theta}\sin(\beta_n/2)\end{bmatrix}, \quad (2a)$$

$$\Psi_{n,-}(\theta)=e^{in\theta}/(2\pi)^{1/2}\begin{bmatrix}-\sin(\beta_n/2)\\ e^{i\theta}\cos(\beta_n/2)\end{bmatrix}, \quad (2b)$$

where n is an integer and β_n is a θ-independent parameter. By calculating the expectation values of $\boldsymbol{\sigma}$ as a function of θ, it is readily seen that β_n is the angle by which the spin deviates from the z axis. The stationary-state Schrödinger equation can then be written as

$$\begin{aligned}H\Psi_{n,\mu}&=\hbar^2[n-eaA/(\hbar c)]^2/(2ma^2)\Psi_{n,\mu}\\ &\quad+\left[\hbar w_n\begin{bmatrix}0&0\\0&1\end{bmatrix}+\hbar w_B\begin{bmatrix}\cos\chi & e^{-i\theta}\sin\chi\\ e^{i\theta}\sin\chi & \cos\chi\end{bmatrix}\right]\Psi_{n,\mu}\\ &=E_{n,\mu}\Psi_{n,\mu}\quad(\mu=+,-)\ ,\end{aligned} \quad (3)$$

with $w_n=\hbar[n+1/2-eaA/(\hbar c)]/(ma^2)$ and $w_B=-geB/(4mc)$. In order to get the exact solution, we have to determine β_n. In the following, we show that the problem of determining β_n is equivalent to finding the AA phase and dynamical phase for a time-dependent system.

Considering the time-dependent Schrödinger equation,

$$(i\hbar\partial/\partial t)\Phi=H_s(t)\Phi\ ,\quad H_s(t)=\hbar w_B\hat{n}(t)\cdot\boldsymbol{\sigma}\ , \quad (4)$$

for the spin motion of an electron in a time-varying magnetic field $B\hat{n}(t)=B\ [\sin\chi\cos(w_n t),\sin\chi\sin(w_n t),\cos\chi]$. It is easy to see that there exists an invariant $I_s(t)$ satisfying $\partial I_s(t)/\partial t - i\hbar^{-1}[I_s,H_s]=0$,

$$I_s(t)=\sin\chi_n\cos(w_n t)\sigma' + \sin\chi_n\sin(w_n t)\sigma^2+\cos\chi_n\sigma^3\ , \tag{5}$$

with $\chi_n = tg^{-1}[\sin\chi/(\cos\chi - w_n/2w_B)]$. The eigenvalue equation of $I_s(t)$ is

$$I_s(t)\Phi_{n,\mu}(t)=\mu\Phi_{n,\mu}(t)\quad(\mu=+,-) \tag{6}$$

with $\Phi_{n,+}(t)=[\cos\chi_n/2,\exp(iw_n t)\sin\chi_n/2]^T$ and $\Phi_{n,-}(t)=[-\sin\chi_n/2,\exp(iw_n t)\cos\chi_n/2]^T$. It is easily found[16] that the exact solution of Eq. (4) is

$$\exp\left[i\int_0^t[-w_n(1-\mu\cos\chi_n)/2 \\ -\mu w_B\cos(\chi_n-\chi)]dt'\right]\Phi_{v,\mu}(t)\ ,$$

which satisfies the cyclic condition[13] for the time interval $[0,2\pi/w_n]$. The AA phase and dynamical phase associated with this cyclic evolution are

$$\int_0^{2\pi/w_n}\Phi^\dagger_{n,\mu}(t)(i\partial/\partial t)\Phi_{n,\mu}(t)dt=-\pi(1-\mu\cos\chi_n)$$

and

$$\int_0^{2\pi/w_n}\Phi^\dagger_{n,\mu}(t)[-\hbar^{-1}H_s(t)]\Phi_{n,\mu}(t)dt \\ =-2\mu\pi w_B\cos(\chi_n-\chi)/w_n\ ,$$

respectively. According to the invariant theory,[16] $-i\hbar\partial/\partial t+H_s(t)$ is diagonal in the representation with the bases $\Phi_{n,\mu}(t)$ ($\mu=+,-$),

$$[-i\hbar\partial/\partial t+H_s(t)]\Phi_{n,\mu}(t)=\left[\hbar w_n\begin{bmatrix}0&0\\0&1\end{bmatrix}+\hbar w_B\begin{bmatrix}\cos\chi & \exp(-iw_n t)\sin\chi\\ \exp(iw_n t)\sin\chi & \cos\chi\end{bmatrix}\right]\Phi_{n,\mu}(t) \\ =[\hbar w_n(1-\mu\cos\chi_n)/2+\mu\hbar w_B\cos(\chi_n-\chi)]\Phi_{n,\mu}(t)\ , \tag{7}$$

where the term $\hbar w_n(1-\mu\cos\chi_n)/2$ is related to the AA phase and the term $\mu\hbar w_B\cos(\chi_n-\chi)$ is related to the dynamical phase. By comparing the stationary-state Schrödinger equation (3) and Eq. (7) satisfied by the instantaneous eigenstates of invariant (5), the β_n is determined and the corresponding exact eigenfunction $\Psi_{n,\mu}$ and eigenvalue $E_{n,\mu}$ obtained

$$\beta_n=\chi_n \\ =\tan^{-1}\{\sin\chi/[\cos\chi-w_n/(2w_B)]\ , \tag{8}$$

$$\Psi^{(\theta)}_{n,\mu}=\exp(in\theta)/(2\pi)^{1/2}\Phi_{n,\mu}(t=\theta/w_n)\ , \tag{9}$$

$$E_{n,\mu}=\hbar^2[n-eaA/(\hbar c)]^2/(2ma^2)+\hbar w_n(1-\mu\cos\beta_n)/2 \\ +\mu\hbar w_B\cos(\beta_n-\chi)\ . \tag{10}$$

The equilibrium expectation values of the dimensionless persistent charge and spin currents, i.e., $\langle J^0\rangle$ and $\langle J^i\rangle$ ($i=1,2,3$), in canonical ensemble are defined as follows:[1]

$$\langle J^\rho\rangle=Z^{-1}\mathrm{tr}[\exp(-\beta H)(P_\theta-eaA/c)\sigma^\rho]/\hbar \\ (\rho=0,1,2,3)\ ,$$

where σ^0 is the identity operator in spin space and Z is the partition function at temperature $T=1/(\beta\kappa)$. With the help of the exact eigenvalues of H, we calculate the expectation values $\langle J^\rho\rangle$ for a ring having one electron only,

$$\langle J^0\rangle=Z^{-1}\sum_{n,\mu}[n-eaA/(\hbar c) \\ +(1-\mu\cos\beta_n)/2]\exp[-\beta E_{n,\mu}]\ , \tag{11}$$

$$\langle J'\rangle=\langle J^2\rangle=0\ , \tag{12}$$

$$\langle J^3\rangle=Z^{-1}\sum_{n,\mu}\{\mu\cos\beta_n[n-eaA/(\hbar c)+(1-\mu\cos\beta_n)/2] \\ -(\sin^2\beta_n)/2\}\exp[-\beta E_{n,\mu}]\ . \tag{13}$$

It is apparent that they are applicable to a ring containing a noninteracting Fermi gas.

Now we turn to the adiabatic limit of the exact solution (8). It should be pointed out that the original problem is that of solving the stationary Schrödinger equation (3). In this problem, the usual adiabatic approximation is without meaning. In Ref. 1, the adiabatic approximation is introduced in a path-integral approach to the problem. In the present paper, solving Eq. (3) reduces to solving Eq. (4) which is a time-dependent Schrödinger equation. In this time-dependent problem, the usual adiabatic approximation is meaningful. The condition for the validity of the usual adiabatic approximation is $\sin\chi(w_n/w_B)\to 0$, which can also be expressed as $(\sin\chi/aw_B)(V_{n,\mu}+\mu\hbar\cos\beta_n/2ma)\to 0$, where $V_{n,\mu}$ is the expectation value of the electron's velocity for the state $\Psi_{n,\mu}$. Apparently, this condition requires that B be sufficiently large. Under the condition $\sin\chi(w_n/w_B)\to 0$, we have

$$\beta_n=\chi\ , \tag{14}$$

$$E_{n,+}=\hbar^2[n-eaA/(\hbar c)]^2/(2ma^2) \\ +\hbar w_n(1-\cos\chi)/2+\hbar w_B \\ =\varepsilon_{n,+}+\hbar^2\sin^2\chi/(8ma^2)\ , \tag{15a}$$

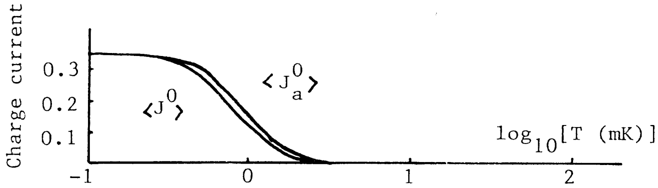

FIG. 1. Charge current $\langle J^0 \rangle$ and its adiabatic approximation $\langle J_a^0 \rangle$ as functions of temperature T (mK) for $\chi=\pi/6$, $B=100$ Gs, $a=3000$ Å, and $-eaA/(\hbar c)-[-eaA/(\hbar c)]=0.4$.

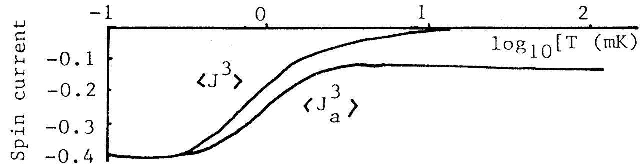

FIG. 2. Spin current $\langle J^3 \rangle$ and its adiabatic approximation $\langle J_a^3 \rangle$ as functions of temperature T (mK) for $\chi=\pi/6$, $B=100$ Gs, $a=3000$ Å, and $-eaA/(\hbar c)-[-eaA/(\hbar c)]=0.4$.

$$
\begin{aligned}
E_{n,-} &= \hbar^2[n-eaA/(\hbar c)]^2/(2ma^2) \\
&\quad +\hbar w_n(1+\cos\chi)/2-\hbar w_B \\
&= \varepsilon_{n+1,-}+\hbar^2\sin^2\chi/(8ma^2) ,
\end{aligned} \tag{15b}
$$

where $\varepsilon_{n,\mu}=[\hbar^2/(2ma^2)][n-eaA/(\hbar c)-\mu(\cos\chi-1)/2]^2+\mu\hbar w_B$ $(\mu=+,-)$ are the eigenvalues obtained in Ref. 1 in the adiabatic limit there. It is worth pointing out that $E_{n,\mu}$ and $\varepsilon_{n,\mu}$ lead to the same partition function since the difference between $E_{n,\mu}$ and $\varepsilon_{n,\mu}$ is independent of n and μ. This is to say that the adiabatic limit here is in agreement with that in Ref. 1. In the adiabatic limit, the partition function and persistent currents become

$$
Z_a=\sum_{n,\mu}\exp(-\beta\varepsilon_{n,\mu}) , \tag{16}
$$

$$
\begin{aligned}
\langle J_a^0 \rangle = Z_a^{-1}\sum_{n,\mu}[n-eaA/(\hbar c) \\
+\mu(1-\cos\chi)/2]\exp(-\beta\varepsilon_{n,\mu}) ,
\end{aligned} \tag{17}
$$

$$
\begin{aligned}
\langle J_a^3 \rangle = Z_a^{-1}\sum_{n,\mu}\mu[n-eaA/(\hbar c)+\mu(1-\cos\chi)/2] \\
\times\cos\chi\exp(-\beta\varepsilon_{n,\mu})-(\sin^2\chi)/2 .
\end{aligned} \tag{18}
$$

With a fixed B, since w_n depends on the quantum number n, even if the adiabatic approximation is valid for the energy eigenstates near the ground state, it is to be violated as $|n-eaA/(\hbar c)|$ becomes large enough. This is to say that the deviation of the adiabatic approximations from the exact values for the persistent current increases as temperature increases. It is clearly seen that the temperature plays an important role in determining whether or not the adiabatic approximation can be used.

The dependence of the charge and spin currents on temperature are depicted in Figs. 1 and 2, from which it can be seen that the exact spin current $\langle J^3 \rangle$ is significantly different from its adiabatic approximation $\langle J_a^3 \rangle$ for the parameters chosen there, as temperature is sufficiently high. Note that phase coherence is preserved in the ring with circumference $L\simeq 2$ μm as temperature is kept below 200 mK.[6]

We now turn to the discussion of the validity of the adiabatic approximation in the low-temperature limit indicated by the subscript zero. It is easy to obtain

$$
\langle J^0 \rangle_0=[n_g-eaA/(\hbar c)+(1-\mu_g\cos\beta_{n_g})/2]
$$

and

$$
\begin{aligned}
\langle J^3 \rangle_0 &= \mu_g\cos\beta_{n_g}[n_g-eaA/(\hbar c)+\tfrac{1}{2}(1-\mu_g\cos\beta_{n_g})] \\
&\quad -(\sin^2\beta_{n_g})/2 ,
\end{aligned}
$$

where n_g and μ_g are the quantum numbers corresponding to the ground state. Here, we want to point out that, even in the low-temperature limit, the condition for the validity of the adiabatic approximation may still be violated if the parameters (χ,B,a) are not properly chosen.

From the expression $\langle J^0\sigma^3 \rangle_0=\langle J^0 \rangle_0\langle \sigma^3 \rangle_0-(\sin^2\beta_{n_g})/2$ ($\langle \sigma^3 \rangle_0=\mu_g\cos\beta_{n_g}$ is the magnetization) for the low-temperature limit of the spin current, we can see that the term "$-(\sin^2\beta_{n_g})/2$" indicates the correlation of σ^3 and $P_\theta-eaA/c$, reflecting the effect of the spin-orbit coupling in an inhomogeneous magnetic field. These correlations between spin and orbital momentum were thoroughly discussed in Ref. 1, and also in Ref. 15, by Loss, Goldbart, and Balatsdy.

As concluding remarks, it is worthwhile to emphasize the following: (i) Our method of finding the exact solution is more complicated than a straightforward diagonalization of the eigenvalue equation.[14,15] This method, however, stresses the role of the AA phase and its adiabatic limit, and is therefore interesting; (ii) the method in the present paper can only be used to study the special case of cylindrically symmetrical fields while the path-integral method[1] allows the study of arbitrary textures; and (iii) the experiments proposed by Loss, Goldbart, and Balatsdy can also be used to verify the results (especially the nonadiabatic effect) obtained in the present paper.

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