Response functions and spectrum of collective excitations of fractional-quantum-Hall-effect systems

Ana Lopez and Eduardo Fradkin
Department of Physics, University of Illinois at Urbana-Champaign, 1110 West Green Street, Urbana, Illinois 61801
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We calculate the electromagnetic response functions of a fractional-quantum-Hall-effect (FQHE) system within the framework of the fermion Chern-Simons theory for the FQHE, which we developed before. Our results are valid in a semiclassical expansion around the average-field approximation (AFA). We reexamine the AFA and the role of fluctuations. We argue that, order-by-order in the semiclassical expansion, the response functions obey the correct symmetry properties required by Galilean and gauge invariance and by the incompressibility of the fluid. In particular, we find that the low-momentum limit of the semiclassical approximation to the response functions is exact and that it saturates the f-sum rule. We obtain the spectrum of collective excitations of FQHE systems in the low-momentum limit. We find a rich spectrum of modes which includes a host of quasiparticle-quasihole bound states and, in general, two collective modes coalescing at the cyclotron frequency. The Hall conductance is obtained from the current-density correlation function, and it has the correct value already at the semiclassical level. We applied these results to the problem of the screening of external charges and fluxes by the electron fluid, and obtained asymptotic expressions of the charge and current-density profiles, for different types of interactions. Finally, we reconsider the anyon superfluid within our scheme and derive the spectrum of collective modes for interacting hard-core bosons and semions. In addition to the gapless phase mode, we find a set of gapped collective modes.

I. INTRODUCTION

The physical systems which exhibit the fractional quantum Hall effect (FQHE) present a very rich response to external electromagnetic perturbations. While some of the observed phenomena, such as cyclotron resonance, can be understood in terms of simple global motions of the center of mass under the combined influence of electric and magnetic fields, the spectrum of collective excitations is certainly determined by the interactions. Given the unusual features of the Laughlin states and its generalizations, it is expected that some features should largely determine the behavior of the collective modes also. However, in spite of the great progress that has been made in the understanding of the ground state, a general theory of the electromagnetic response functions and of the spectrum of collective modes, valid for all the incompressible states, has been lacking. This is the main motivation and goal of this paper.

Various theoretical approaches have been proposed to explain the FQHE. The Laughlin-Haldane-Halperin[1–3] approach is based on the Laughlin variational ansatz for the ground-state wave function. The Laughlin wave function gives the correct value for the Hall conductance, and yields an excellent ground-state energy.[1,2] Later on, Halperin[3] realized that the quasiparticles supported by this state exhibit not only fractional charge but that they are *anyons*, particles with fractional statistics.[4] A hierarchy of *daughter* states at other fractions different from the fundamental fractions, i.e., $\nu=1/m$, can be constructed by considering a Laughlin-type ground state of the fractionally charged quasiparticles defined relative to the parent state one step up in the hierarchy. The higher-order FQHE states occur at a sequence of rational filling fractions.

Related to this approach is the composite fermion theory of the FQHE developed by Jain.[5] He found that the low-energy states of the FQHE can be described in terms of weakly interacting composite fermions, where a composite fermion is an electron bound to an even number of vortices. He also proposed simple Jastrow-Slater trial wave functions for the incompressible FQHE states as well as for their low-energy excitations. The validity of these wave functions was confirmed by calculating numerically their overlap with the true Coulomb states for systems with small number of particles.

Another approach consists of an effective Landau-Ginzburg field theory for the FQHE.[6,7] It was shown that the mean-field solutions and the small fluctuations of the Landau-Ginzburg effective action give a correct qualitative description of the physics of the low-energy degrees of freedom. This approach has revealed the existence of a deep connection between the phenomenon of superfluidity and the FQHE.

Although a lot of progress has been made in the understanding of the FQHE, the electromagnetic response functions for a generic incompressible state have not been calculated and, similarly, the spectrum of collective excitations has not been determined for a general state. In a previous work,[8] we presented a theory of the FQHE based on a second-quantized fermion path-integral approach. There we showed that the problem of interacting electrons moving on a plane in the presence of an external magnetic field is equivalent to a family of systems of fermions bound to an even number of fluxes described by a Chern-Simons gauge field. The semiclassical approxi-

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mation of this system has solutions that describe incompressible-liquid states, Wigner crystals, and soliton-like defects. We studied the Gaussian fluctuations around the liquidlike solution.

In this paper we use the fermion field theory of Ref. 8 to study the collective excitations of the fully polarized FQHE states in the sequence $1/\nu=1/p+2s$ (where p and s are two positive integers) recently introduced by Jain.[5] In 1986 Girvin, MacDonald, and Platzman[9] used the single mode approximation to obtain the lowest collective mode in the lowest Landau level (the *intra* mode) for the states in the Laughlin sequence[1] ($p=1$ in our notation). Our results include, in addition to the *intra* mode, the *inter* mode. In fact we find that in general there is a rich spectrum of collective modes. We also find that there are two modes converging to the cyclotron frequency, and that in general these two modes have different spectral weights in the density correlation function. We discuss in detail the form of the dispersion curves and the spectral weights of the various modes for different types of pair interactions. It should be possible to observe these modes in resonant Raman scattering experiments. Recently, the magnetoplasmon modes of integer Hall states have been observed in light scattering experiments.[10]

The semiclassical approach of Ref. 8 is closely related in spirit to the theories of anyon superfluidity. In both cases an argument is given by which a system of interacting particles (electrons in the case of the FQHE) is seen to be exactly equivalent to a system of fermions coupled to a Chern-Simons gauge field with a properly chosen coupling constant. The mean-field theory then strips the fluxes from the fermions, to which they are locally bound, and replaces the fluxes by an average. While this approximation is certainly very appealing, it has the serious problem that it breaks a number of space-time symmetries quite explicitly. In particular, it breaks both Galilean and magnetic invariance. It turns out that the leading quantum fluctuations around this state, i.e., the collective modes, restore these symmetries, in the uniform $\mathbf{Q}\to 0$ limit, already at the Gaussian level. Indeed, we find that the quadratic or Gaussian level of the semiclassical expansion gives the correct value of the Hall conductance of the system. Also, at this level, we verify that the leading order of the density correlation function saturates the f-sum rule. This is an essential result to show that the absolute value squared of the wave function of all the (incompressible) liquid states has the Laughlin form at very long distances, in the thermodynamic limit.[11] As an application of our results, we derive the form of the response to external test charges and fluxes. Using the same methods, we study the problem of an interacting gas of anyons, and we find the spectrum of collective excitations.

The paper is organized as follows. In Sec. II we review the fermion Chern-Simons theory for the FQHE developed in Ref. 8. We discuss the problem of the violation of Galilean invariance by the mean-field solution, and its restoration after the fluctuations are considered at the Gaussian level. In Sec. III we calculate explicitly the electromagnetic response functions, discuss their analytic properties, the spectrum of collective excitations, and some experimentally observable consequences of our results. We also calculate the Hall response of the system, and verify the saturation of the f-sum rule. In Sec. IV we study the response of the system to external charges and fluxes. We give explicit expressions for the asymptotic long distance form of the induced charge and current density profiles, for different types of pair interactions. In Sec. V we apply our methods to the problem of the interacting anyon gas, and derive their spectrum of collective modes. Section VI is devoted to the conclusions.

II. REVIEW OF THE CHERN-SIMONS FIELD THEORY FOR THE FQHE

In this section we review the Chern-Simons field theory for the FQHE that we developed in Ref. 8. Our work was motivated by the following argument by Jain.[5] The Laughlin wave function,

$$\Psi(z_1,\dots,z_N)=\prod_{i<j}(z_i-z_j)^m\exp\left[-\sum_{i=1}^{N}\frac{|z_i|^2}{4l^2}\right], \tag{2.1}$$

can be factorized as follows:

$$\Psi(z_1,\dots,z_N)=\prod_{i<j}(z_i-z_j)^{m-1}\chi_1(z_1,\dots,z_N), \tag{2.2}$$

where χ_1 is the wave function for a completely filled lowest Landau level

$$\chi_1(z_1,\dots,z_N)=\prod_{i<j}(z_i-z_j)\exp\left[-\sum_{i=1}^{N}\frac{|z_i|^2}{4l^2}\right]. \tag{2.3}$$

Here the set of labels $\{z_i\}$ $(i=1,\dots,N)$ are the coordinates of the N electrons in complex notation $(z=x+iy)$ and l is the cyclotron radius. The *odd* integer m is equal to the inverse of the filling fraction $\nu=N/N_\phi=1/m$ of the lowest Landau level, where N_ϕ is the total number of flux quanta going through a sample of linear size L, $N_\phi=(1/2\pi)[BL^2/(\hbar c/e)]$. In analogy with Laughlin's construction of the quasihole wave functions, Jain observed that the phases associated with the first factor in Eq. (2.2) can be thought to represent an even number $(m-1)$ of fluxes which are attached to each coordinate z_i where an electron is present. Since the electrons bind to an even number of flux quanta, they retain their fermion character.

This observation suggests the possibility of studying the FQHE as an integer quantum Hall effect (IQHE) of bound states, i.e., composites of electrons attached to an even number of fluxes, filling up an integer number of Landau levels of the unscreened part of the field. In order to do so, we need a theory where particles and fluxes are bound together. That is precisely what the Chern-Simons gauge theory does. In 1982 Wilczek[4] pointed out that a particle current coupled to a Chern-Simons (CS) gauge field produces states with fractional statistics through the binding of particles to fluxes. Therefore, if we want to get the Laughlin wave function by attaching $m-1$ fluxes to each electron, it is reasonable to think that the theory should contain fermions coupled to a Chern-Simons gauge field with an appropriate value of the Chern-Simons coupling constant θ.

Following these ideas, in Ref. 8 we studied the problem of a system of interacting electrons moving on a plane in the presence of an external uniform magnetic field B perpendicular to it. In the second quantized language, the action for this system is given by

$$\mathcal{S}=\int d^3z\left\{\psi^*(z)[iD_0+\mu]\psi(z)-\frac{1}{2M}|\mathbf{D}\psi(z)|^2\right\}-\tfrac{1}{2}\int d^3z\int d^3z'(|\psi(z)|^2-\bar{\rho})V(|\mathbf{z}-\mathbf{z}'|)(|\psi(z')|^2-\bar{\rho}) , \tag{2.4}$$

where $\bar{\rho}$ is the average particle density, $\psi(z)$ is a second quantized Fermi field, μ is the chemical potential, and D_μ is the covariant derivative which couples the fermions to the external electromagnetic field A_μ. The electrons are assumed to have an interparticle interaction governed by the pair potential $V(|\mathbf{r}|)$. In what follows we will assume that the pair potential has either the Coulomb form, i.e., $V(|\mathbf{r}|)=q^2/|\mathbf{r}|$, or that it represents a short-range interaction such that in momentum space it satisfies that $\tilde{V}(\mathbf{Q})\mathbf{Q}^2$ vanishes at zero momentum. This includes the case of ultralocal potentials (i.e., with a range smaller or of the same order as the cyclotron length l), in which case we can set $\tilde{V}(0)=0$, or short-range potentials with a range longer than l such as a Yukawa interaction.

In Ref. 8 we showed that this system is equivalent to a system of interacting electrons coupled to an additional statistical vector potential a_μ ($\mu=0,1,2$) whose dynamics is governed by the Chern-Simons action

$$\mathcal{S}_{\rm CS}=\int d^3x\frac{\theta}{4}\epsilon_{\mu\nu\lambda}a^\mu\mathcal{F}^{\nu\lambda} , \tag{2.5}$$

provided that the CS coupling constant satisfies $\theta=(1/2\pi)/1/2s$, where s is an arbitrary integer. In Eq. (2.5) x_0, x_1, and x_2 represent the time and the space coordinates of the electrons, respectively, and $\mathcal{F}^{\nu\lambda}$ is the field tensor for the statistical gauge field, $\mathcal{F}^{\nu\lambda}=\partial^\nu a^\lambda-\partial^\lambda a^\nu$. In the equivalent theory the covariant derivative given by

$$D_\mu=\partial_\mu+i\frac{e}{c}A_\mu+ia_\mu \tag{2.6}$$

couples the fermions to the statistical gauge field and to the external electromagnetic field. For arbitrary values of θ, the system is a set of *anyons* with statistical angle $\delta=1/2\theta$, measured with respect to Fermi statistics. On the other hand, if $\theta=(1/2\pi)/1/2x$, then $\delta=2\pi s$, and the system still represents fermions.

The Chern-Simons action implies a constraint for the particle density $j_0(\mathbf{x})$ and the statistical flux $\mathcal{B}$, given by $j_0(\mathbf{x})=\theta\mathcal{B}(\mathbf{x})$. This relation states that the electrons coupled to a statistical gauge field with Chern-Simons coupling constant θ see a statistical flux per particle of $1/\theta$. Hence, for $\theta=(1/2\pi)/1/2s$, each fermion picks up a statistical flux equal to $1/\theta=2\pi(2s)$, i.e., an *even* number of flux quanta ($2s$) is attached to each particle. Hence, if the coefficient of the Chern-Simons term is chosen in such a way that an even number of flux quanta get attached to each electron, all the physical amplitudes calculated in this theory are identical to the amplitudes calculated in the standard theory, in which the Chern-Simons field is absent. Of course, this is true provided that the dynamics of the statistical gauge fields is taken into account exactly.

In the scheme that we presented in Ref. 8, the dynamics of the Chern-Simons gauge fields is taken into account in a semiclassical expansion, which is a sequence of well-controlled approximations. In practice, we consider the leading and next-to-leading order in the semiclassical approximation. Using the constraint enforced by the Chern-Simons term, the action becomes (in units in which $e=c=\hbar=1$)

$$\mathcal{S}_\theta=\int d^3z\left\{\psi^*(z)[iD_0+\mu]\psi(z)-\frac{1}{2M}|\mathbf{D}\psi(z)|^2+\frac{\theta}{4}\epsilon_{\mu\nu\lambda}a^\mu\mathcal{F}^{\nu\lambda}\right\}-\tfrac{1}{2}\int d^3z\int d^3z'[\theta\mathcal{B}(z)-\bar{\rho}]V(|\mathbf{z}-\mathbf{z}'|)[\theta\mathcal{B}(z')-\bar{\rho}] . \tag{2.7}$$

The quantum partition function for this problem is, at zero temperature,

$$Z=\int \mathcal{D}\psi^*\mathcal{D}\psi\mathcal{D}a_\mu\exp(iS_\theta) . \tag{2.8}$$

Since the action is quadratic in the fermions, they can be integrated out. The effective action ($\mathcal{S}_{\rm eff}$) is given by the sum of the fermion contribution to the effective action (the logarithm of the fermion determinant, which represents the fermionic fluctuations), the Chern-Simons terms, and the interaction term. The resulting theory can be treated within the saddle-point expansion. The external electromagnetic field can be written as a sum of two terms, one representing the uniform magnetic field B, and a small fluctuating term $\tilde{A}_\mu$ whose average vanishes everywhere. The latter will be used to probe the electromagnetic response of the system.

The path integral Z can be approximated by expanding its degrees of freedom in powers of the fluctuations, around stationary configurations of $\mathcal{S}_{\rm eff}$. This requirement yields the classical equations of motion. These equations have many possible solutions, i.e., fluid states, Wigner crystals, and nonuniform states with vortexlike configurations. We will only consider solutions with uniform particle density, i.e., the liquid phase solution. This is the *average-field-approximation* (AFA), which can be regarded as a mean-field approximation. At the mean-field level the electrons see a total flux B_{eff}, equal to the external magnetic flux partially screened by the average Chern-Simons flux, i.e., $B_{\text{eff}}=B+\langle\mathcal{B}\rangle=B-\bar{\rho}/\theta$. It is easy to see that the uniform saddle-point state has a gap only if the effective field B_{eff} is such that the fermions fill exactly an integer number p of the effective Landau levels, i.e., those defined by B_{eff}. In other words, the AFA to this theory yields a state with an energy gap if the filling fraction satisfies $1/\nu=1/p+2s$, where p and s are arbitrary positive integers. Only for incompressible states (i.e., with a gap) the perturbative expansion is meaningful. If the system is compressible, i.e., gapless, the perturbative expansion breaks down. The breakdown is signaled by infrared divergencies at low temperatures. This is what happens if an effective Landau level is not completely filled. The case of a half-filled effective Landau level was analyzed recently by Halperin, Lee, and Read.[12]

Thus, at the mean-field level, the FQHE of a gas of fermions in a uniform magnetic field is equivalent to an IQHE for fermions bound to an even number of flux quanta in the presence of a partially screened external magnetic field. Now we consider the Gaussian (or semiclassical) fluctuations of the statistical vector potential $\tilde{a}_\mu$ around the mean-field state. Unlike other mean-field approaches (such as Hartree-Fock), the Gaussian corrections *must* alter the *qualitative* properties of the state described by the AFA. The reason is that the AFA violates explicitly space-time symmetries, such as Galilean invariance (more generally, *magnetic invariance*) which, for translationally invariant systems, must remain unbroken and unchanged. Thus the center of mass of the system must execute a cyclotronlike motion at, exactly, the cyclotron frequency of noninteracting electrons in the full external magnetic field, as demanded by Kohn's theorem.[13] A naïve application of the AFA would suggest that the cyclotron frequency is renormalized downwards since the effective field seen by the composite fermions is smaller than the external field B. Hence, the *magnetic algebra* may appear to have changed. We will see below that the Gaussian fluctuations yield the correct cyclotron frequency and, thus, restore the correct magnetic algebra.

We will now review the semiclassical expansion for this system.[8] At the Gaussian level, the effective action for $\tilde{a}_\mu$ is

$$S^{(2)}(\tilde{a}^\mu,\tilde{A}^\mu)=\tfrac{1}{2}\int d^3x\, d^3y\, \tilde{a}^\mu(x)\,\Pi^{(0)}_{\mu\nu}(x,y)\tilde{a}^\nu(y)+\frac{\theta}{4}S_{\text{CS}}(\tilde{a}_\mu-\tilde{A}_\mu)$$
$$-\frac{\theta^2}{2}\int d^3x\, d^3y[\tilde{\mathcal{B}}(x)-\tilde{B}(x)]V(x-y)[\tilde{\mathcal{B}}(y)-\tilde{B}(y)]\ . \tag{2.9}$$

Equation (2.9) holds provided that the nonquadratic dependence in the fluctuating part of the statistical vector potentials $\tilde{a}_\mu$ is small. Recall that these nonquadratic terms result from expanding the (logarithm) of the fermion determinant in powers of the fluctuations around the average-field approximation. The kernels that enter in the expressions for these terms are (connected) current correlation functions (or response functions) of the mean-field theory. Thus, the tensor $\Pi^{(0)}_{\mu\nu}$ is the *polarization tensor* of the equivalent fermion problem at the mean-field level, and it is obtained by expanding the fermion determinant up to quadratic order in the statistical gauge field. It was shown in Ref. 8 that this tensor is transverse (as a result of gauge invariance), analytic in $\mathbf{Q}^2/B_{\text{eff}}$ and that it has simple poles at $\omega=k\bar{\omega}_c$ (with k an integer different from zero), where $\bar{\omega}_c\equiv\omega_c/(2sp+1)$ is the cyclotron frequency associated with the effective magnetic field B_{eff}. As a result, $\Pi^{(0)}_{\mu\nu}$ has a gradient expansion in powers of the inverse of the effective magnetic field $1/B_{\text{eff}}$, or equivalently, in powers of the inverse of the external magnetic field $1/B$. In fact, the dimensionless parameter of this expansion is $\mathbf{Q}^2/B$ (we are working in a system of units such that $\hbar=c=e=1$). It also turns out that, within this approximation, the limits of $B\to\infty$ and $M\to 0$ are not equivalent (see the explicit form of $\Pi^{(0)}_{\mu\nu}$ given in Appendix B of Ref. 8).

The nonquadratic terms in $\tilde{a}_\mu$ in the effective action are of the form

$$S_{\text{eff}}=S^{(2)}(\tilde{a}^\mu,\tilde{A}^\mu)+\frac{1}{3!}\int d^3x_1d^3x_2d^3x_3\tilde{a}^\mu(x_1)\tilde{a}^\nu(x_2)\tilde{a}^\lambda(x_3)\Pi^{(0)}_{\mu\nu\lambda}(x_1,x_2,x_3)+\cdots\ , \tag{2.10}$$

where the kernel $\Pi^{(0)}_{\mu\nu\lambda}(x_1,x_2,x_3)$ represents a three-point current correlation function in the mean-field theory. Thus, in the language of Feynman diagrams, while $\Pi^{(0)}_{\mu\nu}(x_1,x_2)$ can be viewed as a fermion bubble with two amputated external collective mode lines, $\Pi^{(0)}_{\mu\nu\lambda}(x_1,x_2,x_3)$ again has one fermion loop tied to three amputated external collective model lines $\tilde{a}_\mu$. Each one of these nonquadratic kernels have the same gauge invariance (i.e., transversality) and analytic properties as the Gaussian [or random-phase approximation (RPA)] kernel. In particular, this means that, in momentum and frequency space, these kernels must be a linear combination of tensors (of the appropriate rank) which have the correct transversality properties, times a set of functions which are analytic in $\mathbf{Q}^2$ and have poles at frequencies equal to an integer multiple of the effective cyclotron frequency. Therefore, the nonquadratic terms necessarily have powers of $\mathbf{Q}^2/B_{\text{eff}}$ (for each one of the external momenta and frequency entering the fermion loop) which are higher than the ones found at the quadratic level. Since the mean-field theory has an integer number of filled Landau levels, the energy denominators of the kernels do not change this counting in powers of $\mathbf{Q}^2/B_{\text{eff}}$. In conclusion, the expansion of the fermion determinant, and hence of the effective action, is actually an expansion in powers of $\mathbf{Q}^2/B_{\text{eff}}$, or equivalently, in powers of $\mathbf{Q}^2/B$. However, an expansion in powers of $\mathbf{Q}^2/B$ is also a gradient expansion. Thus, the gradient expansion and the semiclassical expansion mix and are not independent from each other.

The semiclassical expansion is obtained according to the following rules. The propagator for the fluctuations, which represent collective modes, is the inverse of the kernel of the Gaussian action. Since the pair potential enters only through the propagator for the fluctuations, the perturbation theory is not an expansion in the powers of the pair interaction. From this point of view, this expansion is very different from conventional expansions around the Hartree and Hartree-Fock approximations. The vertices of the expansion are the kernels for the nonquadratic terms. This expansion lacks a natural small parameter (i.e., a coupling constant) and it should be regarded, like all semiclassical expansions, as an expansion in the number of fermion loops (i.e., RPA plus corrections). One should keep in mind, however, our previous discussion on its exactness in powers of $\mathbf{Q}^2/B$. In what follows we will make extensive use of the formal properties of this expansion.

The semiclassical expansion has many features in common with the perturbation theory that Fetter, Hanna, and Laughlin[14] (FHL) have developed for the treatment of the anyon gas. Although, superficially, they look very different, it is easy to check that it is in fact the same procedure. The starting point, in both cases, is the average-field approximation. However, when FHL study the fluctuations, they fix the Coulomb gauge $\nabla\cdot\mathbf{a}=0$. In this gauge, and by making use of the Chern-Simons constraint $\rho(x)=-\theta\mathcal{B}(x)$, one can eliminate the gauge field altogether and the resulting Hamiltonian contains nonlocal three and four fermion interactions. In FHL, these nonquadratic terms are dealt within a Hartree and Hartree-Fock with various improvements adopted to keep track of gauge invariance. Our functional methods keep track of gauge invariance automatically. The expansions look different only because of the different choice of gauge. Our propagators at the Gaussian (or semiclassical level) are the RPA propagators (although in a different gauge). While being equivalent, our approach is conceptually simpler and the computations are more direct.

Up to this point we have reviewed the main features of the theory introduced in Ref. 8. In the next section we will use the effective action of Eq. (2.9) to calculate the full electromagnetic response functions of this theory.

III. ELECTROMAGNETIC RESPONSE FUNCTIONS FOR THE FQHE

Since the effective action $S^{(2)}$, Eq. (2.9), is quadratic in $\tilde{a}_\mu$, we can integrate out this field and obtain the effective action for the electromagnetic fluctuations $\tilde{A}_\mu$, $S^{\text{em}}_{\text{eff}}(\tilde{A}_\mu)$. We will use this effective action to calculate the full electromagnetic response functions at the Gaussian level. Since this calculation is based on a one-loop effective action for the fermions (i.e., a sum of fermion bubble diagrams), this approximation amounts to a random-phase correction to the average-field approximation.

In order to integrate out the statistical gauge field $\tilde{a}_\mu$ we must fix the gauge. The electromagnetic effective action, being gauge invariant, is independent of the choice of gauge for the statistical gauge fields in the path integral. We fix the gauge $\partial_\mu\tilde{a}^\mu=\alpha$ (where α is an arbitrary real number) using the standard Faddeev-Popov procedure (see Ref. 15). The result is explicitly gauge invariant and all dependence on the parameter α cancels out. At the one-loop level [governed by the effective action of Eq. (2.9)] we need to know the inverse of the polarization tensor of the equivalent fermion problem, $\Pi^{(0)}_{\mu\nu}$. In Ref. 8, we showed that $\Pi^{(0)}_{\mu\nu}$ can be written in terms of three gauge invariant tensors, an $\mathbf{E}^2$ term, a $\mathbf{B}^2$ term, and a Chern-Simons term. These three tensors plus $B\nabla\cdot\mathbf{E}$ and a gauge fixing term [such as $(1/2\alpha)(\partial_\mu\tilde{a}^\mu)^2$ which corresponds to the Landau-Lorentz gauge if $\alpha\to 0$] close an algebra that can be used to invert the polarization tensor and to calculate explicitly the electromagnetic response functions.

After integrating out the statistical gauge field in Eq. (2.9), the effective action for the electromagnetic fluctuations $\tilde{A}_\mu$ turns out to be

$$S^{\text{em}}_{\text{eff}}(\tilde{A}_\mu)=\tfrac{1}{2}\int d^3x\int d^3y\;\tilde{A}_\mu(x)K^{\mu\nu}(x,y)\tilde{A}_\nu(y)\;. \tag{3.1}$$

Here $K^{\mu\nu}$ is the electromagnetic polarization tensor. It measures the linear response of the system to a weak electromagnetic perturbation. Its components can be written in momentum space as follows

$$\begin{aligned}
K_{00}&=\mathbf{Q}^2K_0(\omega,\mathbf{Q})\;,\\
K_{0j}&=\omega Q_jK_0(\omega,\mathbf{Q})+i\epsilon_{jk}Q_kK_1(\omega,\mathbf{Q})\;,\\
K_{j0}&=\omega Q_jK_0(\omega,\mathbf{Q})-i\epsilon_{jk}Q_kK_1(\omega,\mathbf{Q})\;,\\
K_{ij}&=\omega^2\delta_{ij}K_0(\omega,\mathbf{Q})-i\epsilon_{ij}\omega K_1(\omega,\mathbf{Q})\\
&\quad+(\mathbf{Q}^2\delta_{ij}-Q_iQ_j)K_2(\omega,\mathbf{Q})\;,
\end{aligned} \tag{3.2}$$

where the functions $K_l(\omega,\mathbf{Q})(l=0,1,2)$ are given by

$$K_0(\omega,\mathbf{Q})=-\theta^2\frac{\Pi_0}{D(\omega,\mathbf{Q})}\;, \tag{3.3}$$

$$K_1(\omega,\mathbf{Q})=\theta+\theta^2\frac{(\theta+\Pi_1)}{D(\omega,\mathbf{Q})}+\theta^3\tilde{V}(\mathbf{Q})\mathbf{Q}^2\frac{\Pi_0}{D(\omega,\mathbf{Q})}\;, \tag{3.4}$$

$$K_2(\omega,\mathbf{Q})=-\theta^2\frac{\Pi_2+\tilde{V}(\mathbf{Q})(\omega^2\Pi_0^2-\Pi_1^2+\mathbf{Q}^2\Pi_0\Pi_2)}{D(\omega,\mathbf{Q})}\;, \tag{3.5}$$

and

$$D(\omega,\mathbf{Q})=\Pi_0^2\omega^2-(\Pi_1+\theta)^2+\Pi_0[\Pi_2-\theta^2\tilde{V}(\mathbf{Q})]\mathbf{Q}^2 \ . \quad (3.6)$$

The coefficients Π_l ($l=0,1,2$) are functions of ω and $\mathbf{Q}$, and are given explicitly in Appendix B of Ref. 8. $\tilde{V}(\mathbf{Q})$ is the Fourier transform of the interparticle pair potential. As we mentioned before, we needed to include a gauge fixing term to be able to compute the functional integral in Eq. (2.9). But at the end of the calculation all the terms which contain the gauge fixing coefficient (α) cancel each other and the final result for the response functions is, as it must be, gauge invariant. The other tensor that we have introduced to make the calculations, $B\nabla\cdot\mathbf{E}$, is not present in the final answer either.

We want to stress here that the thermodynamic limit is crucial for the accuracy of our results. Notice first that in the electromagnetic effective action of Eq. (3.1) we are neglecting higher-order response functions, i.e., correlation functions of three or more currents or densities. We have shown in Sec. II that these higher-order correlation functions have higher-order powers of $\mathbf{Q}^2/B$ than the quadratic term. Strictly speaking, these terms are not negligible for a finite system because, in this case, there is a minimum value that the momentum can take, determined by the linear size of the system L, i.e., $|\mathbf{Q}|>1/L$. But in the thermodynamic limit, $L\rightarrow\infty$ and the momentum can go to zero. In other words, only for an infinite system one is allowed to keep only the quadratic term in the electromagnetic action, Eq. (3.1), and to neglect the higher-order correlation functions.

The electromagnetic response functions determined by $K_{\mu\nu}$ have the following properties.

(i) We saw in Ref. 8 that the polarization tensor at mean-field level $\Pi_{\mu\nu}^{(0)}$, has poles at every value of the effective cyclotron frequency ($\bar{\omega}_c\equiv B_{\text{eff}}/M$). This corresponds to the physical picture, at mean-field level, of an IQHE of the bound states in the presence of a partially screened external magnetic field, B_{eff}. Once we take into account the Gaussian fluctuations, it is easy to prove that all the poles that are present in the numerator and the denominator of the $K_{\mu\nu}$ components through the Π_j's cancel out, and the poles of the response functions are determined only by the zeros of their denominator, $D(\omega,\mathbf{Q})$. In other words, the collective excitations of this system will be determined only by the zeros of $D(\omega,\mathbf{Q})$.

(ii) The leading order term in $\mathbf{Q}^2$ of the K_{00} component of the polarization tensor saturates the f-sum rule.

(iii) The Gaussian fluctuations of the statistical gauge field are responsible for the FQHE. In particular, the Gaussian corrections yield the exact value for the Hall conductance. In the remainder of this section we will discuss these properties and their experimentally accessible consequences in detail.

A. The spectrum of collective excitations

For simplicity, we have studied the zeros of $D(\omega,\mathbf{Q})$ in two cases, when the number of effective Landau levels filled is $p=1$ and $p=2$. The (more tedious) case of general p can be studied by straightforward application of the same methods. We have looked for solutions of the form $\omega^2=(k\bar{\omega}_c)^2+\beta(\mathbf{Q}^2/2B_{\text{eff}})\gamma$, where β and γ are two constants to be determined. Thus, we substitute this expression into the functions $\Pi_i(\omega,\mathbf{Q})$ which appear in $D(\omega,\mathbf{Q})$, and expand both the numerators and the denominators in powers of $\mathbf{Q}$. Looking at the coefficients of the leading and subleading terms of this expansion, we are able to determine the values of β and γ for al the proposed solutions. This procedure is quite straightforward to carry out. Only the modes with $k=1,m$ require special care.

1. Case $p=1$

In this case the filling fraction is $\nu=1/m$, where $m=1+2s$, i.e., the Laughlin sequence. We find that there is a family of collective modes whose zero-momentum gap is $k\bar{\omega}_c$, where k is an integer number different from 1 and m, and whose dispersion curve $\omega_k(\mathbf{Q})$ is

$$\omega_k(\mathbf{Q})=\left[(k\bar{\omega}_c)^2+\left[\frac{\mathbf{Q}^2}{2B_{\text{eff}}}\right]^{k-1}\bar{\omega}_c^2\frac{2k(m-1)(k-1)}{(k-1)!(k-m)}\right]^{1/2} . \quad (3.7)$$

The residue in K_{00} corresponding to this pole is

$$\text{Res}(K_{00},\omega_k(\mathbf{Q}))=-\mathbf{Q}^2\omega_c\frac{\nu}{2\pi}\left[\frac{\mathbf{Q}^2}{2B_{\text{eff}}}\right]^{k-1}\times\frac{2k(m-1)(k-1)}{(k-1)!(k-m)(k^2-m^2)} . \quad (3.8)$$

The cases $k=1,m$ have to be treated separately. In general, we find that there is no mode with a zero-momentum gap at $\bar{\omega}_c$. Instead, at $\mathbf{Q}=0$, there is a doubly degenerate mode with a gap at ω_c. This degenerate cyclotron mode can be viewed as the mixing of the modes with $k=1$ and with $k=m$. Thus, the mode with $k=1$ has been "pushed up" to the cyclotron frequency (at $\mathbf{Q}=0$). Halperin, Lee, and Read[12] have recently found a similar result. For $\mathbf{Q}\neq 0$, the degeneracy is lifted and these two modes have different dispersion curves.

For the special case of $\nu=\frac{1}{3}$, i.e., $m=3$, this effect is particularly important. The dispersion relations for the cyclotron modes are given by

$$\omega\pm(\mathbf{Q})=\left[\omega_c^2+\left[\frac{\mathbf{Q}^2}{2B_{\text{eff}}}\right]\frac{\bar{\omega}_c^2}{2}\alpha\pm\right]^{1/2} , \quad (3.9)$$

where

$$\alpha\pm=8+\frac{2M\tilde{V}(0)}{2\pi}\pm\left[\left[8+\frac{2M\tilde{V}(0)}{2\pi}\right]^2+288\right]^{1/2} . \quad (3.10)$$

The residues corresponding to these poles are

$$\text{Res}(K_{00},\omega\pm(\mathbf{Q}))=-\mathbf{Q}^2\omega_c\frac{\nu}{2\pi}\left[1+\frac{288}{\alpha_\pm^2}\right]^{-1} . \quad (3.11)$$

For $\nu=1/m$, $m\geq 5$, the collective modes whose zero-momentum gap is the cyclotron frequency, ω_c, are

$$\omega_+(\mathbf{Q})=\left[\omega_c^2+\left[\frac{\mathbf{Q}^2}{2B_{\text{eff}}}\right]\bar{\omega}_c^2\left[4\frac{(m-1)}{(m-2)}+\frac{2M\tilde{V}(0)}{2\pi}\right]\right]^{1/2} \tag{3.12}$$

with residue

$$\text{Res}(K_{00},\omega_+(\mathbf{Q}))=-\mathbf{Q}^2\omega_c\frac{\nu}{2\pi}\;. \tag{3.13}$$

The other cyclotron mode has the dispersion

$$\omega_-(\mathbf{Q})=\left[\omega_c^2-\left[\frac{\mathbf{Q}^2}{2B_{\text{eff}}}\right]^{m-2}\bar{\omega}_c^2\frac{4m^2(m-1)^2}{(m-1)!}\left[4\frac{(m-1)}{(m-2)}+\frac{2M\tilde{V}(0)}{2\pi}\right]^{-1}\right]^{1/2} \tag{3.14}$$

with residue

$$\text{Res}(K_{00},\omega_-(\mathbf{Q}))=-\mathbf{Q}^2\omega_c\frac{\nu}{2\pi}\left[\frac{\mathbf{Q}^2}{2B_{\text{eff}}}\right]^{m-3}\frac{4m^2(m-1)^2}{(m-1)!}\left[4\frac{(m-1)}{(m-2)}+\frac{2M\tilde{V}(0)}{2\pi}\right]^{-2}\;. \tag{3.15}$$

The above results are valid only if the pair potential $\tilde{V}(\mathbf{Q})$, has a gradient expansion in powers of $\mathbf{Q}$, i.e., for short-range interactions. $\tilde{V}(0)$ stands for the leading order term in that expansion.

If the pair potential has the Coulomb form, i.e., $\tilde{V}(\mathbf{Q})=2\pi q^2/|\mathbf{Q}|$ in two spatial dimensions, both, the dispersion relations with zero-momentum gap at the cyclotron frequency and their residues get modified. The expressions valid in this case are, for any allowed value of m

$$\omega_+(\mathbf{Q})=\left[\omega_c^2+\frac{|\mathbf{Q}|}{2B_{\text{eff}}}\bar{\omega}_c^2 2Mq^2\right]^{1/2} \tag{3.16}$$

with the same residue given by Eq. (3.13), and

$$\omega_-(\mathbf{Q})=\left[\omega_c^2-\frac{|\mathbf{Q}|^{2m-3}}{(2B_{\text{eff}})^{m-2}}\bar{\omega}_c^2\frac{4m^2(m-1)^2}{2Mq^2(m-1)!}\right]^{1/2} \tag{3.17}$$

with residue

$$\text{Res}(K_{00},\omega_-(\mathbf{Q}))=-\mathbf{Q}^2\omega_c\frac{\nu}{2\pi}\frac{|\mathbf{Q}|^{2(m-2)}}{(2B_{\text{eff}})^{m-3}}\times\frac{4m^2(m-1)^2}{(2Mq^2)^2(m-1)!}\;. \tag{3.18}$$

2. *Case* $p=2$

In this case the filling fraction is $\nu=2/m$ where $m=1+4s$. The same remarks about the pair potential are valid in this case. If the pair potential has a gradient expansion in powers of $\mathbf{Q}$ the following results hold.

We find that there is a family of collective modes whose zero-momentum gap is $k\bar{\omega}_c$, with $k\neq 1,m$, and whose dispersion curve $\omega_k(\mathbf{Q})$ is

$$\omega_k(\mathbf{Q})=\left[(k\bar{\omega}_c)^2+\left[\frac{\mathbf{Q}^2}{2B_{\text{eff}}}\right]^{k-1}\bar{\omega}_c^2\frac{(m-1)(k-1)k(k+2)}{(k-m)(k-1)!}\right]^{1/2}\;. \tag{3.19}$$

The residue corresponding to this pole in K_{00} is

$$\text{Res}(K_{00},\omega_k(\mathbf{Q}))=-\mathbf{Q}^2\omega_c\frac{\nu}{2\pi}\left[\frac{\mathbf{Q}^2}{2B_{\text{eff}}}\right]^{k-1}\frac{(k+2)(k+1)k(k-1)^2}{(k-1)!(k-m)(k^2-m^2)(m+1)}\;. \tag{3.20}$$

The collective modes whose zero-momentum gap is the cyclotron frequency, ω_c, are

$$\omega_+(\mathbf{Q})=\left[\omega_c^2+\left[\frac{\mathbf{Q}^2}{2B_{\text{eff}}}\right]\bar{\omega}_c^2 2\left[4\frac{(m-1)}{(m-2)}+\frac{2M\tilde{V}(0)}{2\pi}\right]\right]^{1/2} \tag{3.21}$$

with residue

$$\text{Res}(K_{00},\omega_+(\mathbf{Q}))=-\mathbf{Q}^2\omega_c\frac{\nu}{2\pi} \tag{3.22}$$

and

$$\omega_-(\mathbf{Q})=\left[\omega_c^2-\left[\frac{\mathbf{Q}^2}{2B_{\text{eff}}}\right]^{m-2}\bar{\omega}_c^2\frac{m^2(m-1)^2(m+2)}{(m-1)!}\left[4\frac{(m-1)}{(m-2)}+\frac{2M\tilde{V}(0)}{2\pi}\right]^{-1}\right]^{1/2} \tag{3.23}$$

with residue

$$\mathrm{Res}(K_{00},\omega_-(\mathbf{Q}))=-\mathbf{Q}^2\omega_c\frac{\nu}{2\pi}\left[\frac{\mathbf{Q}^2}{2B_{\mathrm{eff}}}\right]^{m-3}\frac{m^2(m-1)^2(m+2)}{2(m-1)!}\left[4\frac{(m-1)}{(m-2)}+\frac{2M\tilde{V}(0)}{2\pi}\right]^{-2} . \tag{3.24}$$

If the pair potential has the Coulomb form, the dispersion relations with zero-momentum gap at the cyclotron frequency become

$$\omega_+(\mathbf{Q})=\left[\omega_c^2+\frac{|\mathbf{Q}|}{2B_{\mathrm{eff}}}\bar{\omega}_c^2 4Mq^2\right]^{1/2} \tag{3.25}$$

with residue given by Eq. (3.22), and

$$\omega_-(\mathbf{Q})=\left[\omega_c^2-\frac{|\mathbf{Q}|^{2m-3}}{(2B_{\mathrm{eff}})^{m-2}}\bar{\omega}_c^2\frac{m^2(m-1)^2(m+2)}{2Mq^2(m-1)!}\right]^{1/2} \tag{3.26}$$

with residue

$$\mathrm{Res}(K_{00},\omega_-(\mathbf{Q}))=-\mathbf{Q}^2\omega_c\frac{\nu}{2\pi}\frac{|\mathbf{Q}|^{2(m-3)}}{(2B_{\mathrm{eff}})^{m-3}}\frac{m^2(m-1)^2(m+2)}{2(2Mq^2)^2(m-1)!} . \tag{3.27}$$

In this section we have found the spectrum of collective excitations for some values of the filling fraction. Our results are a generalization of the work of Kallin and Halperin[16] who studied the spectrum of collective modes for the *integer* quantum Hall effect within the RPA. We find a family of collective modes with dispersion relations whose zero-momentum gap is $k\bar{\omega}_c$, where k is an integer number different from 1 and m. When $k=m$, i.e., the zero-momentum gap is the cyclotron frequency, there is a splitting in the dispersion relation for finite wave vector. One of these solutions, ω_-, has negative slope for small values of $\mathbf{Q}$. Therefore, there must be a roton minimum at some finite value of the wave vector. Since our results are accurate only for small $\mathbf{Q}$, our dispersion curves do not apply close to the roton minimum. Nevertheless, this mode is expected to become damped due to nonquadratic interactions among the collective modes. On the other hand, the collective mode with lowest energy has $k=2$, is stable (at least for reasonably small wave vectors) and, at small wave vectors, it disperses downwards in energy. This behavior suggests that there should be a magnetoroton minimum for this mode. This result is consistent with the work of Girvin, MacDonald, and Platzman.[9]

The splitting of the cyclotron mode for $\nu=\frac{1}{3}$ is a little puzzling. It only happens for $\nu=\frac{1}{3}$ and for short-range interactions. In all other cases, only the residue for one of the two cyclotron modes is proportional to $\mathbf{Q}^2$. Standard lore has it that Kohn's theorem demands that there should be one and only one mode converging to the cyclotron frequency as $\mathbf{Q}^2\to 0$ with residue proportional to $\mathbf{Q}^2$. Zhang has emphasized this point recently.[7] It is generally assumed that Kohn's theorem is valid even at nonzero wave vectors and that it requires the existence of only one mode with residue proportional to $\mathbf{Q}^2$ converging to ω_c. However, at nonzero wave vectors, these arguments make the unstated assumption of the analyticity of the current operators on the wave vectors. While this may well be correct, it is an additional assumption and it does deserve closer scrutiny. The results from our theory do indeed predict the existence of only one mode at ω_c with residue proportional to $\mathbf{Q}^2$, which is the statement of Kohn's theorem. And, also, for all filling fractions and for all pair potentials (except $\nu=\frac{1}{3}$, and short-range interactions) we do find only one mode with residue $\mathbf{Q}^2$ even at nonzero wave vectors. The case $\nu=\frac{1}{3}$ and short-range interactions appears to be exceptional in that we find two modes which coalesce at the cyclotron frequency as $\mathbf{Q}^2\to 0$. But both of these modes have residue proportional to $\mathbf{Q}^2$, with different amplitude, and together they satisfy the sum rule.[17] While it is possible that the non-Gaussian corrections may change this result since, in a sense, these are subleading pieces in $\mathbf{Q}^2$, these non-Gaussian corrections are expected to be very small at small wave vectors.

We close this section with a few comments on the validity of this spectrum of collective modes beyond the semiclassical approximation. Primarily we have to consider the physics at moderately large wave vectors and the (expected) effects of non-Gaussian corrections. At the Gaussian (RPA) level we found a family of collective modes which, for sufficiently small momentum, are infinitely long lived (i.e., the response functions have δ-function sharp poles at their location). These modes represent charge-neutral bound states. It is in principle possible that, for $\mathbf{Q}$ sufficiently large, these modes should become damped. The threshold should occur when the energy of the collective mode becomes equal to the energy necessary to create the lowest available two-particle state: a quasiparticle-quasihole pair. In the AFA, the energy of a pair is equal to $\bar{\omega}_c$. Gaussian fluctuations are expected to renormalize this energy upwards and to give it a momentum dependence. This is in principle calculable with methods of this paper but this result is not available at the present time. Non-Gaussian corrections to the RPA are also expected to give a finite width to (presumably) all the collective modes but the lowest one. This is so because the corrections to the semiclassical approximations are due to effective vertices (due to virtual quasiparticle-quasihole pairs) which couple the various collective modes and, thus, induce the higher-energy modes to decay down into the lower modes. However, by gauge invariance, these vertices have a momentum dependence and should vanish as $\mathbf{Q}\to 0$. Thus, the width of the higher-energy modes goes to zero as $\mathbf{Q}\to 0$ and these modes only become sharp at $\mathbf{Q}=0$. But at $\mathbf{Q}=0$ the only accessible mode is the cyclotron mode (the other modes have a vanishingly small spectral weight). These arguments strongly suggest that the only truly sharp mode, at $\mathbf{Q}=0$, is the cyclotron mode, which is required to be stable by Kohn's theorem.[13] Since the modes with zero-momentum gap at $k\bar{\omega}_c, k\geq 3$, are not the collective modes with lowest energy, it is possible that at finite wave vectors they may also decay into the collective mode with lowest energy (the mode with $k=2$, which has a gap at $\bar{\omega}_c$). These issues remain to be invertigated.

B. Experimental consequences

In this section we discuss the experimental consequences of the results that we have just derived.

The density correlation function can be probed by optical absorption and by Raman-scattering experiments.

In the first case, the optical absorption is proportional to the imaginary part of the density correlation function. We predict that there will be absorption peaks at a discrete set of frequencies which, for $\mathbf{Q}\to 0$ converge to $\omega=k\bar{\omega}_c$, where k is an integer number greater than two. Since the spectral weight of these modes vanishes as $\mathbf{Q}\to 0$, the associated absorption peaks are, for a strictly translationally invariant system, only observable at nonzero momentum.

In the case of the Raman scattering, the geometry must be such that there is a component of the incident light wave vector in the plane of the sample. The Raman spectrum, $I(\omega)$, is also proportional to the imaginary part of the density correlation function.[8]

We have seen that in the limit $|\mathbf{Q}| \ll l^{-1}$, where l is the magnetic length, most of the weight of $K_{00}(\omega,\mathbf{Q})$ is in one of the cyclotron modes. The pole in $K_{00}(\omega,\mathbf{Q})$ for the lowest excitation frequency, ω_k with $k=2$, has a residue which is proportional to $|\mathbf{Q}|^4$, i.e., it is smaller by a factor of $|\mathbf{Q}|^2$ than the residue at the highest weighted mode at the cyclotron frequency.

We have also found that there is a splitting in the cyclotron modes. If the pair potential has a gradient expansion in $|\mathbf{Q}|$, i.e., short-range interaction, the pole at ω_- [Eqs. (3.14) and (3.23)], has a residue that is smaller by a factor of $|\mathbf{Q}|^{2(m-3)}$ than the residue of ω_+ [Eqs. (3.12) and (3.21)]. The relative Raman intensity, $I(\omega_+)/I(\omega_-)$, is proportional to $(2B_{\text{eff}}/\mathbf{Q}^2)^{(m-3)}$ which is a big number within our approximation. If the filling fraction is $\nu=\frac{1}{3}$, both modes have the same $\mathbf{Q}^2$ dependence in their spectral weight, but the relative intensity is ≈ 2.5 provided that $\tilde{V}(0)=0$. Except for $\nu=\frac{1}{3}$, the splitting between the two modes at the cyclotron frequency satisfies, at leading order in $|\mathbf{Q}|$, $\Delta\omega^2=\omega_+^2-\omega_-^2=\omega_+^2-\omega_c^2$, which is proportional to $|\mathbf{Q}|^2$. Up to this order, experimentally one should observe one mode dispersing ω_+ [Eqs. (3.12) or (3.21)], and the other as $\omega=\omega_c$. For $\nu=\frac{1}{3}$ the splitting is also proportional to $|\mathbf{Q}|^2$. In this case one should observe both modes [ω_+ and ω_-, Eq. (3.9)], but with different intensities.

If the pair potential has the Coulomb form, the residue of ω_- [Eqs. (3.18) and (3.27)] is smaller by a factor of $|\mathbf{Q}|^{2(m-2)}$ than the residue of ω_+ [Eqs. (3.16) and (3.25)], and this is valid for all the values of the filling fraction that we have studied. The splitting between these two modes satisfies, at leading order in $|\mathbf{Q}|, \Delta\omega^2=\omega_+^2-\omega_c^2$, which is proportional to $|\mathbf{Q}|$. For ν different from $\frac{1}{3}$, the relative intensity between the two modes is proportional to $(2B_{\text{eff}}/\mathbf{Q}^2)^{(m-3)}M\tilde{V}(\mathbf{Q})$, which is bigger than 1 within our approximation. For $\nu=\frac{1}{3}$, the relative intensity is proportional to $M\tilde{V}(\mathbf{Q})$. This factor can be written in terms of the magnetic length and the cyclotron energy as follows $[\tilde{V}(\mathbf{Q})/l]/\omega_c$. Since our approximation is only valid in the limit $1/|\mathbf{Q}| \gg l$, the numerator satisfies $\tilde{V}(\mathbf{Q})/l \gg 2\pi q^2/l$. The second term in this inequality is the Coulomb energy at the magnetic length, which is typically of the same order of magnitude as the cyclotron energy. Therefore, $[\tilde{V}(\mathbf{Q})/l]/\omega_c \gg (2\pi q^2/l)/\omega_c \approx 1$. In other words, the relative intensity for $\nu=\frac{1}{3}$ is also bigger than one.

C. Saturation of the f-sum rule

We show now that the long wavelength form of K_{00}, found at this semiclassical level, saturates the f-sum rule. This result implies that the non-Gaussian corrections do not contribute at very small momentum. In a separate publication[11] we have used this result to show that the absolute value squared wave function of all the (incompressible) liquid states has the Laughlin form at very long distances, in the thermodynamic limit.

The retarded density and current correlation functions of this theory are, by definition

$$D^R_{\mu\nu}(x,y)=-i\theta(x_0-y_0)\langle G|[J_\mu(x),J_\nu(y)]|G\rangle , \qquad (3.28)$$

where J_μ ($\mu=0,1,2$) are the conserved currents of the theory defined by Eq. (2.4), and $|G\rangle$ is the ground state of the system. Using this definition and the commutation relations between the currents, one can derive the f-sum rule for the retarded density correlation function D^R_{00}. In units in which $e=c=\hbar=1$, it states that

$$\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}i\omega D^R_{00}(\omega,\mathbf{Q})=\frac{\bar{\rho}}{M}\mathbf{Q}^2 . \qquad (3.29)$$

On the other hand, it is easy to show (see, for instance, Ref. 19) that the polarization tensor $K_{\mu\nu}$ and the density and current correlation functions $D_{\mu\nu}$ satisfy the following identity

$$K_{\mu\nu}(x,y)=-D_{\mu\nu}(x,y)+\left\langle\frac{\delta J_\mu(x)}{\delta A_\nu(y)}\right\rangle . \qquad (3.30)$$

From Eqs. (3.2) and (3.3) we see that the leading order term in $\mathbf{Q}^2$ of the zero-zero component of the electromagnetic response is given by

$$K_{00}=-\frac{\bar{\rho}}{M}\frac{\mathbf{Q}^2}{\omega^2-\omega_c^2+i\epsilon} , \qquad (3.31)$$

where we have used that $\bar{\rho}/B=\nu/2\pi$.

The correlation functions that we derive from the path integral formalism are time ordered. Therefore, if we use the relation between time-ordered and retarded Green's functions, and Eqs. (3.30) and (3.31), we see that the leading order term of K_{00} saturates the f-sum rule, Eq. (3.29).

It is important to remark that the coefficient of the leading order term of K_{00} cannot be renormalized by higher-order terms in the gradient expansion, nor in the semiclassical expansion. In the case of the gradient expansion, it is clear that higher-order terms have higher-order powers of $\mathbf{Q}^2$, and then, do not modify the leading order term. In the case of the corrections to K_{00} originating in higher-order terms in the semiclassical expansion, they also come with higher-order powers of $\mathbf{Q}^2$. The reason of that is essentially the gauge invariance of the system. This implies that the higher-order correlation functions must be transverse in real space, or equivalently they have higher-order powers of $\mathbf{Q}^2$ in momentum space. Being higher-order terms in the $\mathbf{Q}^2$ expansion they cannot change the leading order term.

As we have already mentioned, these results hold for any model Hamiltonian for the two-dimensional electron gas (2DEG) with reasonably local interactions, i.e., with pair interactions that obey $\mathbf{Q}^2\tilde{V}(Q)\to 0$ as $\mathbf{Q}^2\to 0$.

D. Hall conductance

We show now that, already within our approximations, this state does exhibit the fractional Hall effect with the exact value of the Hall conductance. We have previously shown elsewhere[8,19] that this is the case using the effective action of the statistical gauge fields. Here we show that, as expected, the electromagnetic response functions exhibit the correct FQHE.

In order to do so, we will calculate the Hall conductance of the whole system. Since we are only interested in the leading long-distance behavior, it is sufficient to keep only with those terms in Eq. (3.1) which have the smallest number of derivatives, or in momentum space, the smallest number of powers of $\mathbf{Q}$. Therefore, from Eqs. (3.1) and (3.2), we see that the leading long-distance behavior (i.e., small momentum) of the effective action for the electromagnetic field is governed by the Chern-Simons term. In this limit Eq. (3.1) turns out to be

$$S^{\rm em}_{\rm eff}(\tilde{A}_\mu)\approx -\frac{i}{2}\int\frac{d^2Q}{(2\pi)^2}\int\frac{d\omega}{(2\pi)}\tilde{A}_\mu(-\omega,-\mathbf{Q})K_1(\omega,\mathbf{Q})\epsilon_{\mu\nu\lambda}Q^\lambda\tilde{A}_\nu(\omega,\mathbf{Q})\ , \tag{3.32}$$

where $Q^0=\omega$ and $Q^i=-Q_i$ according with the convention that we have used in Ref. 8.

To study the Hall response of the system, we will now consider the limit of small ω and small $\mathbf{Q}$. We have checked that in this theory the two limits commute

$$\lim_{\mathbf{Q}\to 0}\lim_{\omega\to 0}K_1(\omega,\mathbf{Q})=\frac{\Pi_1(0,0)}{\theta+\Pi_1(0,0)}\ , \tag{3.33}$$

$$\lim_{\omega\to 0}\lim_{\mathbf{Q}\to 0}K_1(\omega,\mathbf{Q})=\frac{\Pi_1(0,0)}{\theta+\Pi_1(0,0)}\ . \tag{3.34}$$

This is a consequence of the incompressibility of the ground state. Since $\theta=1/2\pi 2s$ and $\Pi_1(0,0)=p/2\pi$,

$$K_1\to\frac{\nu}{2\pi}\equiv\theta_{\rm eff}\ , \tag{3.35}$$

where ν is the filling fraction. The electromagnetic current J_μ induced in the system is obtained by differentiating the effective action $S_{\rm eff}(\tilde{A}_\mu)$ with respect to the electromagnetic vector potential. The current is $J_\mu=(\theta_{\rm eff}/2)\epsilon_{\mu\nu\lambda}\tilde{F}^{\nu\lambda}$. Thus, if a weak external electric field $\tilde{E}_j$ is applied, the induced current is $J_k=\theta_{\rm eff}\epsilon_{lk}\tilde{E}_l$. We can then identify the coefficient $\theta_{\rm eff}$ with the *actual* Hall conductance of the system σ_{xy} and get

$$\sigma_{xy}\equiv\theta_{\rm eff}=\frac{\nu}{2\pi} \tag{3.36}$$

which is a *fractional* multiple of e^2/h (in units in which $e=\hbar=1$). Thus, the uniform states exhibit a fractional quantum Hall effect with the correct value of the Hall conductance.

IV. ELECTROMAGNETIC RESPONSE TO AN EXTERNAL CHARGE AND FLUX

In this section we will study the linear response of the system to a static charge and a static flux.

Consider the case of a static probe of electric charge $\tilde{q}$, located at the origin. The electromagnetic vector potential can be written as

$$\tilde{A}_0(\mathbf{x},t)=\frac{\tilde{q}}{|\mathbf{x}|}\ ,\qquad \tilde{A}_j(\mathbf{x},t)=0, j=1,2\ , \tag{4.1}$$

or in momentum space

$$\tilde{A}_0(\omega,\mathbf{Q})=(2\pi)^2\delta(\omega)\frac{\tilde{q}}{|\mathbf{Q}|}\ . \tag{4.2}$$

The electromagnetic current induced in the system J_μ can be calculated by differentiating the effective action, Eq. (3.1), with respect to the electromagnetic vector potential. In momentum space the induced current is

$$J_\mu(\omega,\mathbf{Q})=\tfrac{1}{2}\tilde{A}_\nu(-\omega,-\mathbf{Q})[K_{\mu\nu}(-\omega,-\mathbf{Q})+K_{\nu\mu}(\omega,\mathbf{Q})]\ . \tag{4.3}$$

In particular, the charge and the current density induced by the external perturbation, Eq. (4.1) are

$$J_0(\omega,\mathbf{Q})=\tilde{A}_0(-\omega,-\mathbf{Q})K_{00}(\omega,\mathbf{Q})\ , \tag{4.4}$$

$$J_j(\omega,\mathbf{Q})=\tfrac{1}{2}\tilde{A}_0(-\omega,-\mathbf{Q})[K_{j0}(-\omega,-\mathbf{Q})+K_{0j}(\omega,\mathbf{Q})]\ , \tag{4.5}$$

or in real space

$$J_0(\mathbf{x},t)=2\pi\tilde{q}\int\frac{d^2Q}{(2\pi)^2}|\mathbf{Q}|K_0(0,\mathbf{Q})e^{i\mathbf{Q}\mathbf{x}}\ , \tag{4.6}$$

$$J_j(\mathbf{x},t)=2\pi\tilde{q}\epsilon_{jk}\partial_k\int\frac{d^2Q}{(2\pi)^2}\frac{K_1(0,\mathbf{Q})}{|\mathbf{Q}|}e^{i\mathbf{Q}\mathbf{x}}\ . \tag{4.7}$$

Using the expression (3.3) for K_0, the leading order term of the induced charge becomes

$$J_0(\mathbf{x},t)=-\frac{\bar{\rho}\tilde{q}}{M\omega_c^2}\frac{1}{|\mathbf{x}|^3}=-\sigma_{xy}\frac{\tilde{q}}{\omega_c}\frac{1}{|\mathbf{x}|^3}\ . \tag{4.8}$$

Since the external perturbation generates an electric field in the radial direction, the induced current given by Eq. (4.7) has only components in the azimuthal direction ($\hat{\varphi}$). The leading order term is

$$J_\varphi(\mathbf{x},t)=\sigma_{xy}\tilde{q}\frac{1}{|\mathbf{x}|^2}\ . \tag{4.9}$$

These results coincide with those obtained by Sondhi and Kivelson,[20] who calculated the current induced by the presence of a quasiparticle, within the framework of the Chern-Simons Landau-Ginzburg theory for the FQHE.

The expressions obtained above are formally exact in the limit of infinite magnetic field. Their corrections can be calculated by taking into account higher-order terms in the gradient expansions of the functions K_0 and K_1, and in the semiclassical expansion. These results hold if the pair potential $\tilde{V}(\mathbf{Q})$ is such that $\mathbf{Q}^2\tilde{V}(\mathbf{Q})$ vanishes when $\mathbf{Q}\to 0$. In both cases, for a short-range potential or for the Coulomb potential, the corrections to Eqs. (4.8) and (4.9) will go as $1/|\mathbf{x}|^5$ and $1/|\mathbf{x}|^4$, respectively. In principle, the corresponding coefficients might be renormalized by non-Gaussian fluctuations of the statistical gauge field.

We now calculate the linear response of the system in the presence of a static magnetic flux located at the origin. We will consider here an infinitesimally thin flux tube with intensity Φ_0, such that the system remains in its ground state even in the presence of the flux. If the flux through the solenoid gets to be big enough, the system will be able to lower its energy by creating quasiparticles or quasiholes (i.e., moving into an excited state), which eventually might screen the flux. Hence, we expect that the response to an external infinitesimally thin solenoid of flux Φ_0 should be a periodic function of Φ_0 with period equal to one flux quantum. However, this problem cannot be studied within the mean-field solution that we have chosen, because there is no way to go perturbatively from the uniform or liquidlike ground-state solution, to a solution which represents an excited state with one or more quasiparticles present. To recover the expected periodic behavior of the induced current as a function of the external flux, we would have to study not only the uniform solution of the saddle-point equations, but also the vortexlike solutions, and sum over all of the saddle points to obtain the full, periodic, response to an external arbitrary flux. In this work we will only consider solenoids with flux Φ_0 much smaller than the flux quantum and, thus, we will only consider the response of the uniform state.

The electromagnetic vector potential is in this case

$$\tilde{A}_0(\mathbf{x},t)=0\ ,\qquad \tilde{A}_j(\mathbf{x},t)=\frac{\Phi_0}{2\pi|\mathbf{x}|}\frac{-\epsilon_{jk}x_k}{|\mathbf{x}|}\ , \tag{4.10}$$

or in momentum space

$$\tilde{A}_j(\omega,\mathbf{Q})=i2\pi\delta(\omega)\Phi_0\frac{\epsilon_{jk}Q_k}{|\mathbf{Q}|^2}\ . \tag{4.11}$$

According to Eq. (4.3), the charge and current density induced by this perturbation are, respectively,

$$J_0(\omega,\mathbf{Q})=\tfrac{1}{2}\tilde{A}_i(-\omega,-\mathbf{Q})[K_{0i}(-\omega,-\mathbf{Q})+K_{i0}(\omega,\mathbf{Q})]\ , \tag{4.12}$$

$$J_j(\omega,\mathbf{Q})=\tfrac{1}{2}\tilde{A}_i(-\omega,-\mathbf{Q})[K_{ji}(-\omega,-\mathbf{Q})+K_{ij}(\omega,\mathbf{Q})]\ . \tag{4.13}$$

Substituting in these equations the explicit form of the external probe [Eq. (4.11)], and transforming back to real space, the induced charge and current are given by

$$J_0(\mathbf{x},t)=-\Phi_0\int\frac{d^2Q}{(2\pi)^2}K_1(0,\mathbf{Q})e^{i\mathbf{Q}\mathbf{x}}\ , \tag{4.14}$$

$$J_j(\mathbf{x},t)=-\Phi_0\epsilon_{jk}\partial_k\int\frac{d^2Q}{(2\pi)^2}K_2(0,\mathbf{Q})e^{i\mathbf{Q}\mathbf{x}}\ . \tag{4.15}$$

Keeping only the leading order terms in $K_1(0,\mathbf{Q})$, Eq. (3.4), the induced charge becomes

$$J_0(\mathbf{x},t)=-K_1(0,0)\epsilon_{ik}\partial_i A_k(\mathbf{x})\ , \tag{4.16}$$

where $K_1(0,0)$ is evaluated at zero frequency and momentum. Using that $K_1(0,0)=\nu/2\pi$, we get

$$J_0(\mathbf{x},t)=-\frac{\nu}{2\pi}\mathcal{B}(\mathbf{x})\ , \tag{4.17}$$

where $\mathcal{B}=\Phi_0\delta^2(\mathbf{x})$ is the magnetic field associated to the external flux, Eq. (4.10). It is important to remark that Eq. (4.17) is strictly valid in the limit in which the external uniform magnetic field goes to infinity. Otherwise, we find that the induced charge has Gaussian factors that in the limit of infinite magnetic field become δ functions which combine to reproduce exactly the magnetic field produced by the external perturbation [Eq. (4.10)].

The total charge $\tilde{Q}$ induced by the external perturbation is obtained by integrating Eq. (4.17) over the area of the system. The result is

$$\tilde{Q}=-\nu\frac{\Phi_0}{2\pi}\ . \tag{4.18}$$

Since the induced charge $\tilde{Q}$ has been determined from linear response theory, it may seem that Eq. (4.18) should only hold if the flux Φ_0 is small relative to the flux quanta. Equation (4.18) is, however, exact. This follows from the fact that the leading behavior at small momentum of the response functions saturates the sum rules and, in consequence, coefficients such as $K_1(0,0)$ are given exactly by the linear response result. For instance, if $\Phi_0=2\pi$, the induced charge is just the filling fraction of the system. In particular, for $\nu=1/m$ the induced charge is $-1/m$. This result agrees with Laughlin's *gedanken experiment* argument for the construction of the quasihole.

Finally we consider the current induced by the solenoid. If the interparticle pair potential is short range, being its range much smaller than the magnetic length (l), the induced current density, Eq. (4.15), is

$$J_{\varphi}(\mathbf{x},t)=\left[\frac{\nu}{2\pi}\right]^2 \Phi_0 B_{\text{eff}}\varpi_c|\mathbf{x}|e^{-B_{\text{eff}}|\mathbf{x}|^2/2} . \tag{4.19}$$

In particular, in the limit of $B\to\infty$, the above expression becomes

$$J_{\varphi}(\mathbf{x},t)=-\frac{\nu^2}{2\pi M}\Phi_0\frac{\partial}{\partial|\mathbf{x}|}\delta^2(|\mathbf{x}|) \tag{4.20}$$

which is ultralocal.

If the pair potential has the Coulomb form, the leading order term in the azimuthal component of the induced current density is given by

$$J_{\varphi}(\mathbf{x},t)=\left[\frac{\nu}{2\pi}\right]^2 \Phi_0\frac{q^2}{|\mathbf{x}|^2} \tag{4.21}$$

which has a long-range, power-law tail.

We close this section with a remark. The sum rule arguments tell us that the asymptotic long-distance behaviors for the charge and current density profiles that we derived in this section, are exact. However, at shorter distances they are expected to pick up corrections. For instance, except for the case of the Coulomb potential, our results show no dependence on the strength of the short-range potentials. This is so since we are considering *ultralocal* pair interactions, i.e., with a range comparable or smaller than the cyclotron radius. If, for example, we consider a Yukawa-like interaction with a range a much larger than l, but still finite, we should expect a somewhat different behavior at distances $R\approx a$. In fact, for $R\gg a\gg l$ we find that the profiles decay exponentially fast with range a (i.e., not Gaussian). For the regime $a\gg R\gg l$ we find, for the current density profile, a $1/R$ power-law decay crossing over to Gaussian behavior at the scale of l. In other words, we only expect changes either at the cyclotron scale or at any new length scales introduced by the interaction.

V. APPLICATION TO THE INTERACTING ANYON GAS

In this section we consider a system of interacting anyons in the absence of an external magnetic field. This problem was previously discussed by many authors.[14,21–25] All of these works deal with anyons which, apart from a hard core, are not interacting.

Here we find the spectrum of collective modes for different types of pair interactions and we rederived some of the previously known results on the electromagnetic response functions within the framework of our theory.

For this problem, we can also expand the path integral $\mathcal{Z}$ around stationary configurations of the effective action. There are many possible solutions for the classical equations of motion, but we only study that one with uniform particle density. At mean-field level the anyons see a total flux B_{eff}, which coincides with the average Chern-Simons flux, i.e., $B_{\text{eff}}=-\bar{\rho}/\theta$. In order for this theory to have a gap in the single-particle spectrum, an integer number p of effective Landau levels defined by B_{eff} must be completely filled. This requirement implies a relation between p and the statistics parameter given by $p/2\pi=-\theta$. Provided that this identity holds, the coefficient of the Chern-Simons terms vanishes and the system has a gapless collective mode.

The next step is to take into account the Gaussian fluctuations. We can use the results derived in Sec. III for the electromagnetic response functions, but with $B_{\text{eff}}=-\bar{\rho}/\theta$ and $p/2\pi=-\theta$. For the system of anyons we have also studied the spectrum of collective modes only for the cases $p=1$ and $p=2$. The general case can be analyzed by using the same methods.

1. Case $p=1$

Here $\theta=-1/2\pi$. Therefore the statistical angle is $\delta=\pi$ and we are dealing with a system of interacting bosons.

We find that there is a family of collective modes whose zero-momentum gap is $k\varpi_c$, where k is an integer number different from 1, and whose dispersion curve $\omega_k(\mathbf{Q})$ is

$$\omega_k(\mathbf{Q})=\left[(k\varpi_c)^2-\left[\frac{\mathbf{Q}^2}{2B_{\text{eff}}}\right]^{k-1}\varpi_c^2\frac{2}{(k-2)!}\right]^{1/2} . \tag{5.1}$$

The residue in K_{00} corresponding to this pole is

$$\text{Res}(K_{00},\omega_k(\mathbf{Q}))=\mathbf{Q}^2\frac{\varpi_c}{2\pi}\left[\frac{\mathbf{Q}^2}{2B_{\text{eff}}}\right]^{k-1}\frac{2}{k^2(k-2)!} \tag{5.2}$$

In particular, if $k=2$, the above dispersion relation becomes

$$\omega_2(\mathbf{Q})=\left[\left[\frac{4\pi\bar{\rho}}{M}\right]^2-v_0^2\mathbf{Q}^2\right]^{1/2} . \tag{5.3}$$

To obtain this expression we have used that $\varpi_c/2\pi=\bar{\rho}/Mp$. This mode appears to correspond to the "density mode" of the Bose gas. Notice that all of the modes with $k\geq 2$ are expected to become damped due to the nonquadratic interaction terms which induce decays into the gapless $k=1$ modes. The modes with larger values of k, presumably, should get damped more quickly than the mode at $k=2$.

The mode with $k=1$ is "pulled down" and it becomes gapless (at $\mathbf{Q}=0$). Its dispersion curve is

$$\omega(\mathbf{Q})=v_0|\mathbf{Q}| , \tag{5.4}$$

where

$$v_0^2=\frac{2\pi\bar{\rho}}{M^2} . \tag{5.5}$$

Equation (5.4) is valid provided that the pair potential is short range, and that $\tilde{V}(0)=0$. If the pair potential is $\tilde{V}(\mathbf{Q})=2\pi q^2/|\mathbf{Q}|$, the gapless mode has the form

$$\omega^2(\mathbf{Q})=v_0^2 M q^2|\mathbf{Q}| \ . \tag{5.6}$$

In both cases the residue is

$$\text{Res}(K_{00},\omega(\mathbf{Q}))=-\mathbf{Q}^2\frac{\bar{\rho}}{M} \ . \tag{5.7}$$

These results agree with the general discussion by Fetter[26] of plasmons in two-dimensional compressible fluids. This behavior is a consequence of the compressibility and two-dimensionality and it is independent of the statistics of the particles.

2. *Case $p=2$*

Here $\theta=-2/2\pi$. This is the case of semions. Again we find that there is a family of collective modes whose zero-momentum gap is $k\bar{\omega}_c$, where k is an integer number different from 1, and whose dispersion curve $\omega_k(\mathbf{Q})$ is

$$\omega_k(\mathbf{Q})=\left[(k\bar{\omega}_c)^2-\left[\frac{\mathbf{Q}^2}{2B_{\text{eff}}}\right]^{k-1}\bar{\omega}_c^2\frac{k+2}{(k-2)!}\right]^{1/2} . \tag{5.8}$$

The residue in K_{00} corresponding to this pole is

$$\text{Res}(K_{00},\omega_k(\mathbf{Q}))=\mathbf{Q}^2\frac{\bar{\omega}_c}{2\pi}\left[\frac{\mathbf{Q}^2}{2B_{\text{eff}}}\right]^{k-1}\frac{(k+2)}{k^2(k-2)!} . \tag{5.9}$$

In particular, if $k=2$, the above dispersion curve coincides with Eq. (5.3). Again, the mode at $k=2$ can be regarded as a "density mode." All of these modes will also become damped by interaction effects.

The mode with $k=1$ is "pulled down" and it becomes gapless (at $\mathbf{Q}=0$). Its dispersion curves and residue are the same as in the case $p=1$, Eqs. (5.4), (5.6), and (5.7), respectively.

We have further checked that for any other value of p, the gapless mode and the mode with zero momentum gap at $2\bar{\omega}_c$ have the same form as those for $p=1$, Eqs. (5.3), (5.4), and (5.6).

We have seen then that the electromagnetic response functions that we find at the semiclassical level have a gapless collective excitation. We can see also that there is a restoration of parity in the uniform limit, and that the system exhibits Meissner effect. These two last results are already well known, but we reproduce them here for completeness.

For short-ranged pair interactions, the density correlation function is, to leading order in $\mathbf{Q}^2$,

$$K_{00}(\omega,\mathbf{Q})=-\frac{\bar{\rho}}{M}\frac{\mathbf{Q}^2}{\omega^2-v_0^2\mathbf{Q}^2} \ . \tag{5.10}$$

Therefore, K_{00} has a massless pole which corresponds to a gapless collective mode whose velocity is given by v_0. In this sense we can say that the Gaussian fluctuations *restore* the compressibility of the system. This coincides with the results of Refs. 22 and 25.

The coefficient of the Chern-Simons term, Eq. (3.4), has the properties

$$\lim_{\omega\to 0}K_1(\omega,\mathbf{Q})=-\frac{p}{8\pi} \tag{5.11}$$

if the limit $\omega\to 0$ is taken first. However, if $Q^2\to 0$ first, we find

$$\lim_{\mathbf{Q}\to 0}K_1(\omega,\mathbf{Q})=0 \ . \tag{5.12}$$

Equation (5.12) simply means that the Hall conductance of the anyon gas is zero. Halperin, March-Russell, and Wilczek[27] have argued that this result is a consequence of Galilean invariance. Equation (5.11) can be thought of as a static response of the ground state to a periodic modulation of the charge density with wave vector $\mathbf{Q}$ which induces a periodic arrangement of currents. These currents give rise to an orbital magnetic moment. Equation (5.11) is the static (or equilibrium) orbital susceptibility. It is hard to believe that for the case of bosons, which do not have any violation of time reversal, there should be any orbital currents. These results coincide with those of Hanna, Laughlin, and Ferrer[14] and Chen *et al*,[22] at the Hartree level. Dai *et al.*[14] have shown recently that non-Gaussian fluctuations (beyond the RPA) yield a limiting value of zero for $K_1(\omega,\mathbf{Q})$ (as $\omega\to 0$) for bosons but not for semions.

Finally, we will show that the system exhibits Meissner effect. In the limit of long distances, and for short-range interactions, the coefficient K_2, Eq. (3.5), can be written as

$$K_2(\omega,\mathbf{Q})=-\frac{2\pi\bar{\rho}}{M^2}K_0(\omega,\mathbf{Q}) \ , \tag{5.13}$$

where according to Eqs. (3.2) and (5.10) K_0 is

$$K_0(\omega,\mathbf{Q})=-\frac{\bar{\rho}}{M}\frac{1}{\omega^2-v_0^2\mathbf{Q}^2} \ . \tag{5.14}$$

The electromagnetic current induced in the system because of the presence of a magnetic field is $J_k=K_{kj}A_j$. Using Eqs. (3.2) and (5.12) we get

$$J_k=[\omega^2K_0(\omega,\mathbf{Q})+\mathbf{Q}^2K_2(\omega,\mathbf{Q})]A_k - Q_kQ_jK_2(\omega,\mathbf{Q})A_j \ . \tag{5.15}$$

Therefore, the curl of the current is, in momentum space

$$\epsilon_{lk}Q_lJ_k=-\frac{\bar{\rho}}{M}\epsilon_{lk}Q_lA_k \ . \tag{5.16}$$

This is precisely the London equation, where the London penetration depth is $1/\lambda^2=4\pi\bar{\rho}/M$.

VI. CONCLUSIONS

In Ref. 8 we developed a Chern-Simons theory for the FQHE based on a second-quantized fermion path-integral approach. In this paper we have calculated the electromagnetic response functions of the fractional-quantum-Hall-effect system within the framework of that theory. We made a semiclassical expansion and we worked around the average-field approximation. As we have already mentioned above, the mean-field solution violates explicitly Galilean invariance. At this level of the approximation, the center of mass of the system executes a cyclotronlike motion at the effective cyclotron frequency. In this sense the Gaussian (or semiclassical) fluctuations are essential to restore the original symmetries of the problem. We saw that order by order in the semiclassical expansion the response functions obey the correct symmetry properties required by Galilean and gauge invariance, and by the incompressibility of the fluid. We showed that, already at the semiclassical or Gaussian level, the low-momentum limit of the density correlation function saturates the f-sum rule, and in that sense this result is exact, i.e., it cannot be renormalized by higher-order corrections. We calculated the Hall conductance out of the density-current correlation functions, and we found that it has the correct value at this order of the approximation. We obtained the spectrum of collective excitations in the low-momentum limit for short-range and for Coulomb interparticle pair potential. We found that there is a family of collective modes whose zero-momentum gaps are integer multiples of the effective cyclotron frequency. In particular, there are two modes merging at the cyclotron frequency at zero momentum, but with different intensities, i.e., different weights in their residues in the density correlation function. We argued that all of these modes, except the one with least energy, will be damped once the higher-order terms in the semiclassical expansion are taken into account. We also calculated the linear response of the system to external charges and fluxes, and found expressions for the asymptotic form of the charge and current-density profiles. We found that the responses to an external charge always show profiles with universal power-law decays. In contrast, the responses to external infinitesimally thin solenoids exhibit a variety of behaviors which depend on the nature of the interactions.

Finally, we reconsidered the anyon superfluid within our scheme and derived the spectrum of collective excitations for interacting hard-core bosons and semions. In addition to the gapless phase mode, we found a set of gapped collective modes.

There are still many questions left open. The theory presented here provides a good description of the uniform FQHE ground state, of its collective excitations, and of its linear electromagnetic response. We have made a number of predictions about the existence of a family of collective modes. The observability of them depends not only on their intensities, but also on their lifetimes. We have not addressed this problem here. Another open point of interest is the study of this theory starting from another solution of the saddle-point equations, as the Wigner crystals and nonuniform states with vortexlike configurations.

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