Mechanics of disordered solids. I. Percolation on elastic networks with central forces

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Bond and site percolation on two- and three-dimensional elastic and superelastic percolation networks with central forces are studied using large-scale Monte Carlo simulations and finite-size scaling analysis. A highly accurate method of estimating the elastic percolation threshold p_{ce} is proposed. For bond percolation (BP) on a triangular network we find $p_{ce} \simeq 0.641 \pm 0.001$, for site percolation (SP), $p_{ce} \simeq 0.713 \pm 0.002$, and for BP on a bcc network we obtain, $p_{ce} \simeq 0.737 \pm 0.002$. We calculate the force distribution (FD) in these networks, i.e., the distribution of forces that the intact bonds of the networks suffer, near and away from p_{ce}. Far from p_{ce} the FD is unimodal, but as p_{ce} is approached, it becomes bimodal. We find that for BP on the triangular network near p_{ce}, the zeroth and second moments of the FD belong to the universality class of bond-bending models discussed in paper II. However, this is not the case for SP on the triangular network and BP on the bcc network. In particular, for the bcc network we find $f/\nu_e \simeq 2.1$, where f and ν_e are the critical exponents of the elastic moduli and the correlation length of the system, respectively. This value of f/ν_e is distinctly different from that of bond-bending models, which is about 4.3.

I. INTRODUCTION

The relationships between the microstructure and effective properties of materials, e.g., their transport (electrical and thermal conductivities), mechanical (elastic moduli), and failure properties (the fracture strength distribution and toughness) have been studied for a long time.[1] The effect of *disorder* on transport and mechanical properties has been studied and a better understanding has been developed over the last two decades by combining a variety of techniques and ideas such as percolation and scaling concepts,[2] effective-medium approximation (EMA),[3] renormalization-group methods,[4] as well as large-scale simulation, and well-controlled experiments on model systems.[5,6]

In this paper and its sequels (hereafter referred to as papers II and III), we study vector transport and failure properties of disordered materials. As the model of disordered materials we use percolation networks in which each bond represents an elastic element, or a spring, with an elastic constant e which can take on values from a probability density function $H(e)$. In most cases we use the simple binary distribution

$$H(e)=p\delta(e-a)+(1-p)\delta(e-b)\ , \tag{1}$$

i.e., e takes the values a and b with the probabilities p and $1-p$, respectively. In a few cases, we consider a more general distribution, $H(e)=pf_1(e)+(1-p)f_2(e)$, where f_1 and f_2 are two continuous and normalized probability density functions. If a is finite and $b=0$, we obtain an *elastic percolation network* (EPN). Another case of interest is when $a=\infty$ and b is finite, i.e., a fraction p of the springs are totally *rigid* and the rest are *soft*. We call this a superelastic percolation network (SEPN). As the percolation threshold p_{ce} of an EPN is approached from *above*, all elastic moduli G of the system *vanish*. Near p_{ce}, the elastic moduli obey the following scaling law:

$$G \sim (p-p_{ce})^f\ , \tag{2}$$

whereas, in a SEPN all elastic moduli *diverge* as p_{ce} is approached from *below* according to

$$G \sim (p_{ce}-p)^{-\tau}\ . \tag{3}$$

The elastic percolation threshold p_{ce} and the critical exponents f and τ may depend on the microscopic force laws between the sites of the network. One of the main goals of this paper and paper II is to estimate p_{ce}, f, and τ for a variety of two- and three-dimensional networks. In this paper we study EPN's and SEPN's with only central or stretching forces. Such networks are of both theoretical and practical interest. For example, in the absence of friction between the particles of a granular packing, which is a reasonable model of sedimentary rocks, the mechanical behavior of the packing is similar to those of EPN's that we study in this paper. Paper II contains our results for percolation networks with both central and bond-bending (BB) forces, while in paper III we study fracture and failure properties of disordered materials. The preliminary results of our investigations have been reported before.[7-10] In the present papers, we give full details of our previous results for elastic and superelastic percolation networks, present a number of results, and provide a detailed analysis of the relevant experimental data and their comparison with our results.

II. NUMBER OF INDEPENDENT ELASTIC MODULI

Before describing EPN's with central forces (CF's), it is necessary to discuss the number of elastic moduli that one needs for describing a two-dimensional (2D) or three-dimensional (3D) percolation network, since most

of the networks that we use in our papers are *not* elastically isotropic. The constitutive equations for a linear elastic solid relate the strain and stress tensors through the expression[11]

$$\epsilon_{ij}=S_{ijk_m}\sigma_{km} , \tag{4}$$

where ϵ_{ij} and σ_{km} are the strain and stress tensors entries, respectively, and S_{ijk_m} denotes the *compliance* tensor. The above equation, the generalized Hooke's law, can be written in an alternative form relating the stress and strain tensors through the *stiffness* or the elastic moduli tensor, C_{ijk_m}

$$\sigma_{ij}=C_{ijk_m}\epsilon_{km} . \tag{5}$$

In this equation C_{ijk_m} has 81 entries or components which, due to the symmetry of both the stress and strain tensors, can be reduced to 36 distinct elastic moduli. The double-indexed system of stress and strain components is often replaced by a single-indexed system having a range of 6, which is usually called the Voigt notation. Hence, the generalized Hooke's law may be rewritten as

$$\sigma_K=C_{KM}\epsilon_M \quad (K,M=1-6) , \tag{6}$$

where C_{KM} is a 6×6 square matrix containing elements C_{11} through C_{66}. If a strain energy function exists for the solid body, $C_{KM}=C_{MK}$, and the 36 independent moduli are reduced to 21. If a material further possesses three mutually perpendicular planes of elastic symmetry (*orthotropic* material), the number of independent elastic moduli reduces to 9.

For isotropic materials, the number of independent elastic moduli reduces to 2, and the elastic matrix is symmetric regardless of the existence of a strain energy function. The usual elastic moduli, namely, Young's modulus Y, shear modulus μ, and bulk modulus K, are given by

$$\mu=C_{44} , \tag{7}$$

$$Y=4\mu(1-\mu/C_{11}) , \tag{8}$$

$$K=C_{11}-\mu . \tag{9}$$

The triangular network, which is extensively used in our papers, is an isotropic lattice and, therefore, the above expressions directly apply to the determination of its elastic moduli. We also use a square network in 2D which is not isotropic. In 3D, almost all regular lattices are anisotropic.

For the cubic family of lattices [square, simple-cubic, and body-centered (bcc) networks] that we use in our papers, it is easy to show that $\mu=C_{44}$, as before, and

$$Y=\frac{(C_{11}-C_{12})(C_{11}+2C_{12})}{C_{11}+C_{12}} , \tag{10}$$

$$K=\tfrac{1}{3}(C_{11}+2C_{12}) , \tag{11}$$

so that one, in general, needs *three* independent elastic moduli for characterizing the network. Moreover, Y and μ are *not* isotropic for the cubic lattices[12] and the above formulas for Y and μ refer to the moduli in the principal directions, whereas K is isotropic for a cubic system. Finally, Garboczi[13] showed that, within numerical uncertainty, a bond-disordered triangular network with CF's obeys the Cauchy relation, $C_{12}=C_{66}$.

III. PERCOLATION NETWORKS WITH CENTRAL FORCES

The elastic energy of a percolation network with CF's is given by

$$E=\tfrac{1}{2}\sum_{\langle ij\rangle}[(\mathbf{u}_i-\mathbf{u}_j)\cdot\mathbf{R}_{ij}]^2 e_{ij} , \tag{12}$$

where $\mathbf{u}_i$ is the displacement of site i, and $\mathbf{R}_{ij}$ a unit vector from site i to j. It has been established[14,15] that the bond percolation threshold p_{ce}^B of this system is much larger than p_c^B, the connectivity threshold of scalar percolation. For example, for a d-dimensional cubic network one has $p_{ce}^B=p_{ce}^S=1$, where p_{ce}^S is the site percolation threshold of the system. Therefore, a meaningful study of CF networks is restricted to certain lattices, e.g., the triangular and bcc networks. Earlier simulations of the triangular network[14,16,17] gave $p_{ce}^B\simeq0.65$, as compared with $p_c^B=0.347$. For the CF model a correlation length ξ_e can be defined that diverges as p_{ce} is approached

$$\xi_e\sim(p-p_{ce})^{-\nu_e} , \tag{13}$$

where earlier simulations[16,17] yielded $\nu_e\simeq1.1$ for bond percolation (BP) in 2D CF networks. An EMA predicts[14,18–20] that for a d-dimensional network of coordination number Z, one has

$$p_{ce}^B=2d/Z . \tag{14}$$

According to Eq. (14), $p_{ce}^B Z=2d$ is the *average* coordination number of a d-dimensional network *at* p_{ce}^B, which implies that the coordination number of a d-dimensional CF network must be *greater* than $2d$ in order for it to have nonzero elastic moduli. Previous studies of CF networks were restricted to 2D, and no extensive study of 3D CF models has been undertaken so far.

Before presenting our results, let us briefly discuss how our Monte Carlo calculations are done. To calculate elastic properties of a given model, we minimize its elastic energy E with respect to $\mathbf{u}_i$, $\partial E/\partial\mathbf{u}_i=0$. Writing down this equation for every interior node of the network results in a dN simultaneous equations for nodal displacement $\mathbf{u}_i$ of a d-dimensional network of N internal nodes. The boundary conditions depend on the quantity that we would like to calculate. For example, to calculate the elastic constant C_{11} we stretch two opposite faces of the network by a given strain S_b, and impose periodic boundary conditions in the other directions. The resulting set of linear equations is solved by either Gaussian elimination (in 2D), or by the adaptive accelerated Jacobi-conjugate gradient method (in 3D), that uses an acceleration parameter which is optimized for each iteration. The convergence criterion is that for *all* sites i, $|\mathbf{u}_i^{(k)}-\mathbf{u}_i^{(k-1)}|/|\mathbf{u}_i^{(k-1)}|<\epsilon$, where $\mathbf{u}_i^{(k)}$ is the displacement of site i after the kth iteration, and $\epsilon=10^{-5}$. Unless

specified otherwise, these solution methods and boundary conditions were used for all of our computations discussed in this paper and papers II and III. From the solution of the set of equations for nodal displacements we calculate E and hence the elastic modulus $C_{11}=2E/S_b^2$. All of our computations were carried out on the Crays X-MP and Y-MP of the San Diego Supercomputer Center.

IV. RESULTS

We now present our results for 2D and 3D CF networks. We restrict our attention to the triangular and bcc networks.

A. Results for the triangular network

As discussed above, earlier simulations[16,17] had indicated that for BP on a CF triangular network one has $p_{ce}^B \simeq 0.65$ and $f/\nu_e \simeq 1.45$. The result for f/ν_e was interesting because it differed from the critical exponent of conductivity g of percolation networks defined by $g \sim (p-p_c)^t$, for which $t/\nu \simeq 0.973$, where ν is the correlation-length exponent of scalar (connectivity) percolation ($\nu=\frac{4}{3}$ in 2D), and from the elasticity exponent of BB models in 2D for which[21] $f/\nu \simeq 2.96$. In a previous paper,[7] we found that for site percolation (SP), $f/\nu_e \simeq 1.14$. However, in a series of papers, Roux and Hansen[22] proposed that the CF model on the triangular network belongs to the universality class of the BB model. Using a transfer-matrix method,[17] they obtained for BP on the triangular network $f/\nu_e \simeq 3$, compatible with that of the BB model.[21] According to Roux and Hansen,[22] the reason for the discrepancy between their result and the earlier ones[7,16,17] is that the scaling properties of the CF model are extremely sensitive to the value of p_{ce}, and for the triangular network they obtained $p_{ce}^B \simeq 0.642$. Although this is only 1.2% less than the earlier estimates,[16,17] it apparently can cause such a dramatic shift in the value of f/ν_e. However, Roux and Hansen[22] did not study SP on the triangular network to check the universality of f with respect to the type of percolation process.

To further check these results, and to study SP more accurately on the triangular network and BP on the bcc network, we first studied the force distribution (FD) in the network, which is the distribution of forces that the intact bonds of an EPN suffer. (For the analogous problem of current distribution in random resistor networks see de Arcangelis, Redner, and Coniglio.[23]) In order to calculate the FD, we impose a unit macroscopic strain on the network and determine the nodal displacements $\mathbf{u}_i$ from which the total force F_α, exerted on a bond α, is calculated. We then calculate the FD for various values of p. Of particular interest are the moments of the FD defined by

$$M(k)=\sum_\alpha n(F_\alpha)F_\alpha^k , \tag{15}$$

when $n(F_\alpha)$ is the number of bonds that suffer a force whose *magnitude* is F_α. Near p_{ce}, the moments $M(k)$ obey the scaling law

$$M(k) \sim (p-p_{ce})^{q(k)} . \tag{16}$$

To estimate any critical exponent such as $q(k)$, we use finite-size scaling analysis (FSSA) according to which, for a network of linear size L at the percolation threshold, one has

$$M(k) \sim L^{-\hat{q}(k)}[a_1+a_2 g_1(L)+a_3 g_2(L)] , \tag{17}$$

where $g_1(L)$ and $g_2(L)$ represent, respectively, the leading nonanalytical and analytical correction-to-scaling terms, which are particularly important for small to moderate values of L, and $\hat{q}(k)=q(k)/\nu_e$. A similar equation can be used for estimating f. We only include nonzero values of F_α in Eq. (15) and, therefore, $M(0)$ is simply the total number of bonds that suffer a nonzero stress, i.e., the total number of bonds in the backbone of the EPN and, hence, $-\hat{q}(0)$ is simply the fractal dimension d_{BB} of the backbone. On the other hand, $M(1)$ is the average force that a bond suffers, and $M(2)$ is proportional to the elastic modulus of the network [hence, $q(2)=f$]. For all of our computations discussed here and in papers II and III, we considered various functional forms for $g_1(L)$ and $g_2(L)$ in order to find the most accurate fit to the data. Some of the functional forms that were used were $L^{-\Omega_1}$, $(\ln L)^{-\Omega_2}$, and $\ln(cL)$, where Ω_1 and Ω_2 are correction-to-scaling exponents, and c is a constant. In order to compare the quality of the fits obtained with various functional forms for $g_1(L)$ and $g_2(L)$, we calculated the quantity Q,

$$Q=\sum_{\text{data}}[(SD-PD)/SD]^2 , \tag{18}$$

where SD represents the data obtained from our simulations and PD represents the predictions of Eq. (17) after it has been fitted to the simulation data. We selected those functional forms for $g_1(L)$ and $g_2(L)$ which yielded the smallest Q. For *all* cases discussed here and in papers II and III, we found that the most accurate fits (with the smallest Q) are provided by[24]

$$g_1(L)=(\ln L)^{-1} , \tag{19}$$

$$g_2(L)=L^{-1} . \tag{20}$$

TABLE I. Number of percolating realizations N for each network size L for calculating the moments of the FD in BP on the triangular network.

L	5	10	15	20	25	30	35	40	45	50
$N(p_{ce}\simeq 0.641)$			800	600	500	500	500	500	500	500
$N(p_{ce}\simeq 0.65)$	800	500	400	300	300	250	200	120	100	

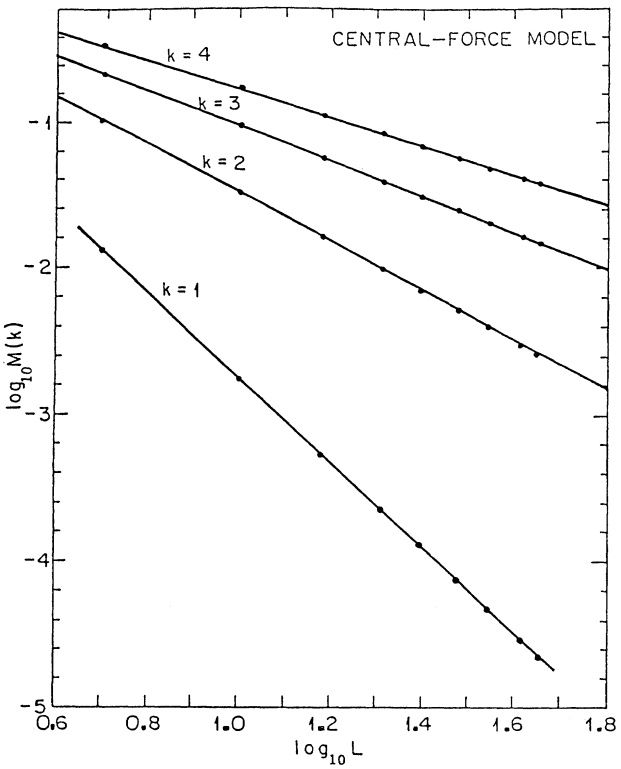

FIG. 1. Moments $M(k)$ of the FD vs network size L for BP on the triangular network at $p_{ce}^B \simeq 0.65$.

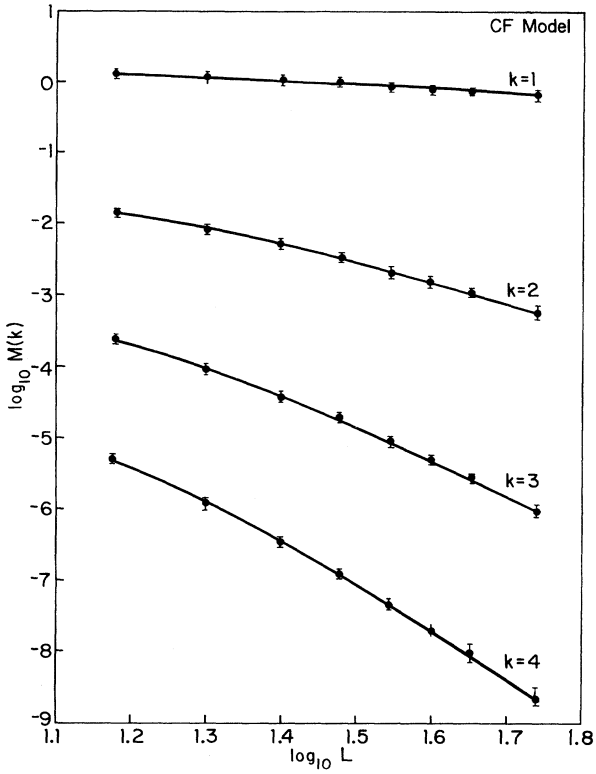

FIG. 2. Moments $M(k)$ of the FD vs network size L for BP on the triangular network at $p_{ce}^B \simeq 0.641$.

We first determined $M(k)$ for $k=0-4$ at p_{ce}^B of the triangular network, both at $p_{ce}^B \simeq 0.642$, which is the estimate of Roux and Hansen,[22] and at $p_{ce}^B \simeq 0.65$ which is the earlier estimate.[16,17] In Table I we present the statistics of our simulations. Nonpercolating clusters were discarded from our simulations, and the numbers in Table I refer to the number of percolating networks. Figure 1 shows the results for $p_{ce}^B \simeq 0.65$, while Fig. 2 presents those at $p_{ce}^B \simeq 0.642$. As Fig. 1 indicates, there is no evidence for significant correction to scaling and all curves appear to be straight lines, even for small values of L.

In Table II we present the estimated $\hat{q}(k)$ for both simulations. The results for $M(0)$ and $M(2)$ at $p_{ce}^B \simeq 0.65$ are consistent with the older estimates,[7,16,17] while those at $p_{ce}^B \simeq 0.642$ are in agreement with the results of Roux and Hansen[22] and the BB model (see paper II). In particular, the result, $-\hat{q}(0) \simeq 1.62$, obtained at $p_{ce}^B \simeq 0.642$, is in agreement with the fractal dimension of 2D percolation backbones,[25] $d_{\rm BB} \simeq 1.64$. Note also that the results at $p_{ce}^B \simeq 0.642$ indicate that for $k \geq 2$

$$\hat{q}(k) = \hat{q}(k-1) + 2 , \tag{21}$$

i.e., there is a constant gap between $\hat{q}(k-1)$ and $\hat{q}(k)$. However, the values of $\hat{q}(k)$ at $p_{ce}^B \simeq 0.642$, for $k \neq 0$ and 2, are *not* in agreement with those of the BB model (see paper II).

We would like to emphasize the importance of the correction-to-scaling terms $g_1(L)$ and $g_2(L)$, especially when L is relatively small. If, for example, we neglect such terms, then, for $p_{ce}^B \simeq 0.642$ we obtain $\hat{q}(2) = f/\nu_e \simeq 2.2$, instead of 2.95 (see Table II), which is also the most accurate estimate for 2D networks.[21] Thus, it is only by including such correction terms that the accuracy of our results becomes comparable with those of the previous most accurate estimates. Of course, if L is large, then the magnitude of such correction terms is small and insignificant. Including such correction terms also gives us confidence that our results are reliable.

Since the results for BP showed extreme sensitivity to the value of p_{ce}^B, we felt compelled to check our earlier results[7] for SP on the triangular network. To do this, we need to develop a new method of estimating p_{ce}. We hypothesize that if we calculate $r = M(2)/M(1)$ for various values of L and p, then, at the true p_{ce}, a plot of $\ln r$ versus

TABLE II. Values of the critical exponents $\hat{q}(k)$ of the moments of the FD in BP on a triangular network.

k	0	1	2	3	4
$\hat{q}(k)(p_{ce} \simeq 0.641)$	-1.62 ± 0.06	0.85 ± 0.05	2.95 ± 0.25	4.94 ± 0.45	7.05 ± 0.75
$\hat{q}(k)(p_{ce} \simeq 0.65)$	-1.95 ± 0.05	0.112 ± 0.006	1.46 ± 0.10	2.96 ± 3.0	4.32 ± 0.50

$\ln L$ would be a straight line, since, according to Eq. (17), the contributions of correction-to-scaling terms to both moments would be of the same order of magnitude and would cancel one another. To test this idea we used network sizes $L=25$, 35, 45, and 55, and calculated r at various values of p ranging from 0.636 to 0.65. Number of realizations varied from 1500 for $L=25$ to 200 for $L=55$. We found that a plot of $\ln r$ versus $\ln L$ is a straight line only if $0.640<p<0.642$; see Fig. 3. Thus, we estimate that

$$p_{ce}^{B}\simeq 0.641\pm 0.001 , \tag{22}$$

which is in excellent agreement with the result of Roux and Hansen.[22] We then used this method of estimating p_{ce} for SP on the triangular network. We used network sizes $L=45$, 55, 65, and 75 and varied p between 0.69 and 0.73. Number of realizations varied from 400 for $L=45$ to 100 for $L=75$. Once again, we obtained a straight line for the plot of $\ln r$ versus $\ln L$ if $0.711<p<0.715$, and, therefore,

$$p_{ce}^{S}\simeq 0.713\pm 0.002 , \tag{23}$$

which is slightly higher and more precise than, but consistent with, our previous estimate,[7] $p_{ce}^{S}\simeq 0.71\pm 0.01$. We then used this new estimate of p_{ce}^{S} and FSSA to estimate f/ν_e for SP on the triangular network. The results are shown in Fig. 4, from which we obtain

$$f/\nu_e\simeq 1.12\pm 0.05 , \tag{24}$$

which is completely consistent with our earlier result[7] mentioned above, but is very different from $\hat{q}(2)=f/\nu_e\simeq 2.95$ obtained for BP (see Table II).

At this point, two questions must be addressed. (i) Why do the scalings of $M(0)$ and $M(2)$ for BP on the triangular network appear to be consistent with those of the BB model (discussed in paper II)? Roux and Hansen[22] argued that this is due to the *lever arm* effect which is caused by a force coupling at the two ends of a connected cluster and, hence, it is similar to the BB forces. To check this we calculated the FD at various p's. At $p=0.9$ we obtained a unimodal distribution which is relatively narrow. As p is lowered to 0.8, the FD becomes broader, and a small second maximum to the left of the main one also appears. At $p=0.65$, the distribution, shown in Fig. 5, is quite broad and bimodal. The magnitude of the smaller maximum is about 44% of the larger one. However, a small shift from $p=0.65$ to $p=0.641$ causes the smaller maximum to become even larger and, as shown in Fig. 6, its magnitude becomes as much as 80% of the larger maximum. The smaller maximum is completely absent when the system is well connected, and appears only when the system loses its connectivity and approaches p_{ce}. This bimodal FD of the CF model at p_{ce}^{B} is completely similar to that of the BB model discussed in paper II. In other words, if the contributions of CF's and lever arm effect are *comparable,* the system effectively behaves like an EPN with BB forces. (ii) Why is SP not behaving like BP? If we calculate the FD for SP on the triangular network for $0.71\leq p\leq 0.72$, although we still obtain a bimodal FD, the magnitude of the smaller maximum is always much smaller than the large one, which means that the lever arm effect for SP on the triangular

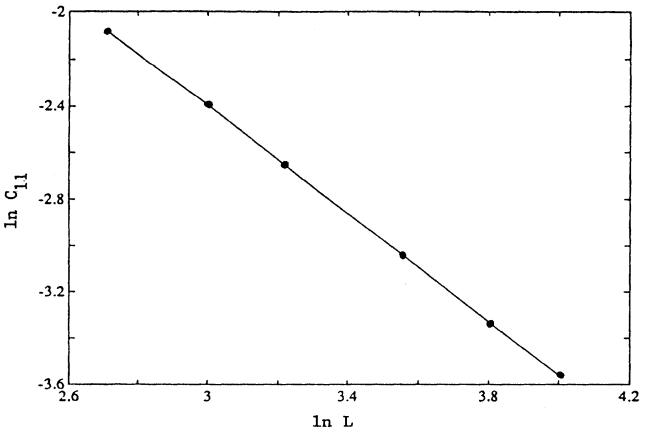

FIG. 4. Variations of elastic modulus C_{11} with size L of the triangular network at $p_{ce}^{S}\simeq 0.713$.

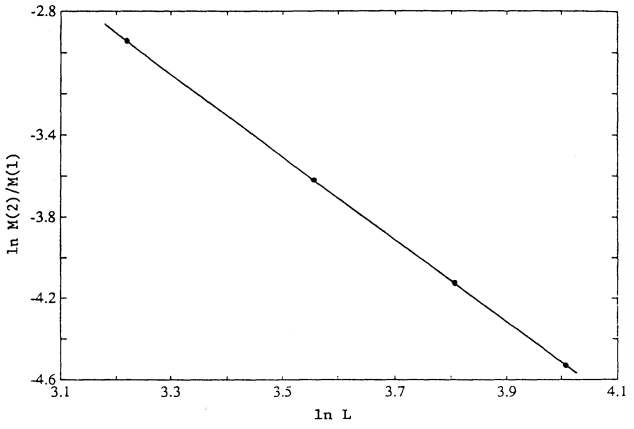

FIG. 3. Variations of $M(2)/M(1)$ with size L of a triangular network for BP at $p_{ce}^{B}\simeq 0.641$.

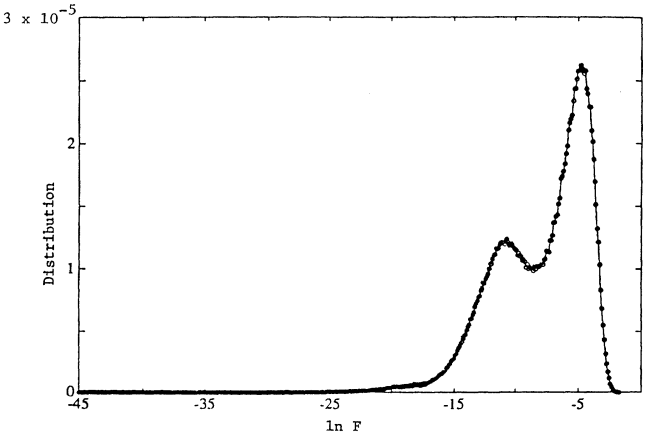

FIG. 5. FD for BP on the triangular network at $p=0.65$.

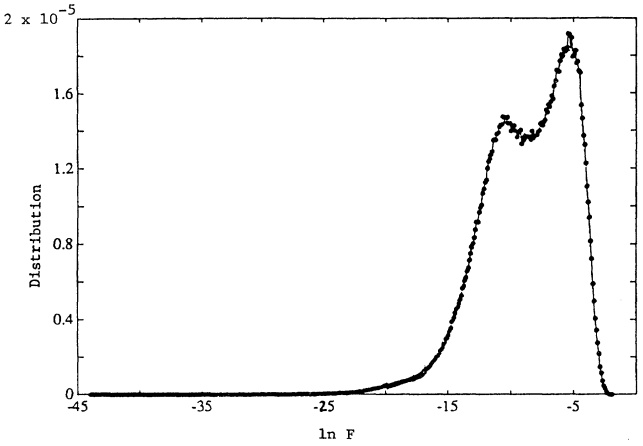

FIG. 6. FD for BP on the triangular network at $p_{ce}^B \simeq 0.641$.

network is much weaker than that for BP, and the contribution of CF's dominates that of the lever arm effect. This might explain why SP does not seem to be in the universality class of BP on the CF triangular network.

Next, we study the superelastic percolation problem on the triangular network,[16] which has been studied by several authors,[22,26,27] using various techniques. For BP on a superelastic CF triangular network, they all obtained $\tau/\nu_e \simeq 0.98$. We studied the same problem for both BP and SP on a triangular network using FSSA. Figure 7 presents our results for BP, from which we obtain

$$\tau/\nu_e \simeq 0.92 \pm 0.02 , \tag{25}$$

which is in agreement with that of the BP model in 2D (see paper II). It is also consistent with the fact that BP of the CF triangular network seems to belong to the universality class of the BB model. However, when we studied the same problem in SP, we obtained, $\tau/\nu_e \simeq 1.05 \pm 0.03$, which does not agree with (25). This again might indicate that BP and SP on the CF triangular network belong to two different universality classes.

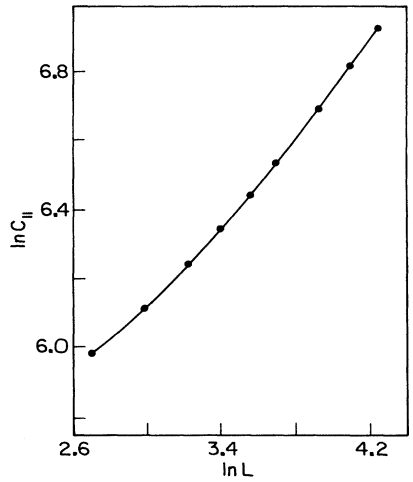

FIG. 7. Variations of elastic modulus C_{11} with size L of a superelastic triangular network at $p_{ce}^B \simeq 0.641$.

B. Results for the bcc network

We now turn our attention to 3D CF networks, which have not been studied before, except within an EMA (Refs. 18 and 19) with limited numerical simulations. We studied BP on a bcc network, and calculated the FD and the critical exponent f. We first estimated p_{ce}^B in order to calculate f. Since EMA's usually overestimate p_{ce}^B, and because Eq. (14) predicts that $p_{ce}^B = 0.75$, we can say that $p_{ce}^B \leq 0.75$. We can also obtain a lower bound for p_{ce}^B by calculating an elastic modulus of a large bcc network and locating the fraction of intact bonds at which the elastic modulus vanishes. Using this idea and an $L = 16$ network, we estimated that $p_{ce}^B > 0.72$. Next, we used our moment ratio method described above for estimating p_{ce}^B. We calculated $M(1)$ and $M(2)$ for various network sizes L in the range $0.72 \leq p \leq 0.75$. The number of realizations varied between 2000 for $L = 6$ to 200 for $L = 16$. A plot of $\ln[M(2)/M(1)]$ versus $\ln L$ for various values of p indicated that the data would make a straight line if $p_{ce}^B \simeq 0.737$, but for $p < 0.735$ or $p > 0.739$ it would show distinct curvature. Therefore, our final estimate is

$$p_{ce}^B \simeq 0.737 \pm 0.002 , \tag{26}$$

which is about 1.7% less than the EMA prediction, but much larger than $p_c^B \simeq 0.1795$ for scalar BP on the bcc network.

Having determined p_{ce}^B, we now can calculate the FD and its moments. For the bcc network the second moment $M(2)$ is exactly equal to the corresponding elastic modulus. We first calculated the FD at $p = 0.95$, 0.85, and 0.737, using a network of size $L = 16$ with 200 different realizations for each p. At $p = 0.95$, we obtained a unimodal FD. However, similar to the triangular network, as $p \to p_{ce}^B$, a second maximum appears to the left of the main maximum. At $p = p_{ce}^B = 0.737$, the FD shown in Fig. 8 is bimodal, but the magnitude of the smaller

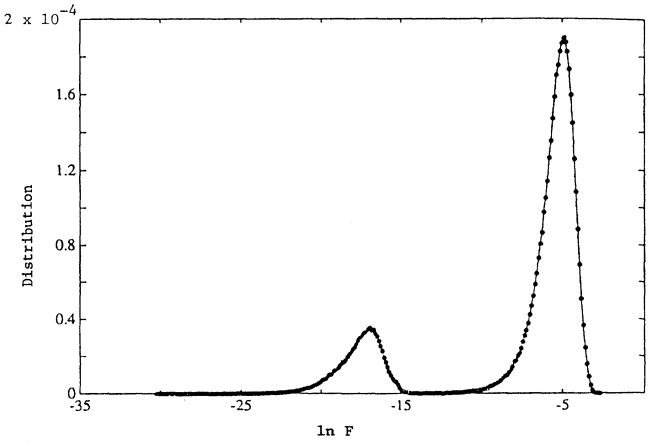

FIG. 8. FD for BP on a bcc network at $p_{ce}^B \simeq 0.737$.

TABLE III. Number of percolating realizations N for each network size L for calculating the moments of the FD in BP on the bcc network at $p_{ce}^B \simeq 0.737$.

L	6	8	10	12	14	16
N	2000	1000	600	400	300	200

maximum, which is presumably due to the lever arm effect that mimics the effect of BB forces, is only about 20% of the main maximum. This is similar to the FD in SP on the triangular network, and once again indicates that as the connectivity of the system improves, the lever arm effect diminishes. Thus, we speculate that the CF model on the bcc network might not belong to the universality class of 3D BB models. To check this, we calculated the moments $M(k)$ for various sizes L at p_{ce}^B. The statistics of our simulations are shown in Table III; the results for $M(k)$ are presented in Fig. 9, with the estimated $\hat{q}(k)$ given in Table IV, which confirm our assertion. As Table IV indicates, $\hat{q}(2)=f/\nu_e \simeq 2.1\pm0.2$, which is completely different from $f/\nu \simeq 4.3\pm0.1$ found for 3D BB models (see paper II). Note that the values of $\hat{q}(k)$ given in Table IV satisfy the scaling relation (21). Note also that the value of $f/\nu_e \simeq 2.1\pm0.2$ is close to $t/\nu \simeq 2.27\pm0.01$, found recently by Gingold and Lobb.[28] Moreover, $-\hat{q}(0)=d_{\rm BB} \simeq 2.5$, which is the same as the fractal dimension of the largest percolation cluster at p_c in the scalar percolation problem. The reason may be that p_{ce}^B of the bcc network is so much larger than p_c^B that the backbone of the elastic network is essentially identical with the percolating cluster in the scalar problem. Next, we studied the superelasticity problem on the bcc network using FSSA, with the result that, $\tau/\nu_e \simeq 0.80\pm0.03$, somewhat larger than the corresponding value for 3D BB models, $\tau/\nu \simeq 0.74$ (see paper II).

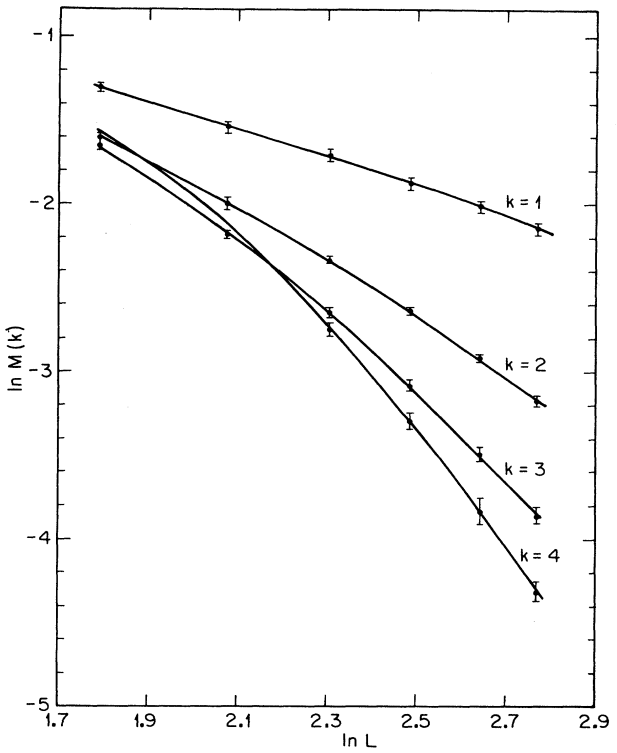

FIG. 9. Moments $M(k)$ of the FD vs linear size L of a bcc network at $p_{ce}^B \simeq 0.737$.

V. SUMMARY

We studied bond and site percolation processes on 2D and 3D elastic percolation networks with CF's, and developed a method for estimating the percolation threshold p_{ce} of such networks that appears to be highly accurate. In 2D we found that the zeroth and second moments of the FD in BP on the triangular network belong to the universality class of the BB model, while the same may not be true for SP on the same network. In 3D, we found that BP on the bcc network is not in the universality class of the BB models. Instead, the critical exponent f/ν_e appears to be close to t/ν, the critical exponent for the conductivity of percolation networks. The same is true about the scaling properties of SEPN's in both 2D and 3D. We should mention that Roux and Hansen[22] proposed that

$$\hat{q}(k)=\hat{q}_c(k)+k \ , \tag{27}$$

where $\hat{q}_c(k)$ is the critical exponent of the kth moment of current distribution in a percolating random resistor network.[23] Our results for BP on the triangular network agree with this equation only for $k=0$ and 2, for which $\hat{q}_c(0)=-1.64$ [recall that $-\hat{q}_c(0)=d_{\rm BB}$], and $\hat{q}_c(2)=t/\nu \simeq 0.97$.

After completion of this work, Knackstedt and Sahimi[29] used a two-parameter position-space renormalization group to study the universality of the CF problem on a triangular network. They found that while the geometrical properties of CF networks in BP and SP may belong to the same universality class, the elasticity critical exponents f of the two systems are very different.

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TABLE IV. Values of the critical exponents $\hat{q}(k)$ of the moments of the FD for BP on the bcc network at $p_{ce}^B \simeq 0.737$.

k	0	1	2	3	4
$\hat{q}(k)$	-2.5 ± 0.3	1.0 ± 0.1	2.1 ± 0.2	3.97 ± 0.2	5.91 ± 0.35

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