Relativistic band gaps in one-dimensional disordered systems

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(Received 24 March 1992; revised manuscript received 12 October 1992)

Conditions for the existence of band gaps in a one-dimensional disordered array of δ-function potentials possessing short-range order are developed in a relativistic framework. Both Lorentz vector- and scalar-type potentials are treated. The relationship between the energy gaps and the transmission properties of the array are also discussed.

I. INTRODUCTION

As is well known the application of a nonrelativistic one-particle approximation to the problem of an electron in a periodic array of potentials results in an energy spectrum comprising a ladder of allowed energy bands. This phenomenon arises because any particle in the array may tunnel from one of the cells comprising the array, through the potential barrier, into an adjacent cell. This tunneling splits the degenerate single-particle energies into a nondegenerate multiplet of states. In the limit that the array becomes infinite these states form a series of energy bands which are separated by regions containing no allowed states, the forbidden energy regions or the energy gaps.

The energy gaps arise as a consequence of Bragg reflection of the particle by the potential barriers of the array. The resultant standing waves must possess either even or odd parity. This implies that the corresponding wave functions peak at distinct points on the lattice with the potential difference at these points being the basis of the energy gap. The properties of tunneling and the band structure of the particle are therefore intimately related in the nonrelativistic system.[1]

Borland,[2] and Makinson and Roberts,[3] were able to show that the energy gaps which arise for systems possessing perfect periodicity also persist for disordered lattices possessing short-range order. Makinson and Roberts used the Schrödinger equation to analyze the energy spectrum of an electron in a periodic array of δ-function potentials and to show that for certain energies the effect of the δ function is to distort the effective wavelength of the wave function (the wavelength of a Bloch-type solution) so that it coincides with twice the cell width. At the lowest order this would correspond to the nodes of the wave function occurring at the positions of the δ functions, a phenomenon termed locking. This wave function has only one node per cell.

As James and Ginzbarg[4] first observed, the number of nodes of a solution of the Schrödinger equation for a particle of energy E determines how many states of the system have energy less than E. By defining the phase of this solution they were also able to show that if the change in phase over an arbitrary cell for a wave function of energy E is given by $\Delta\phi$, then this is related to the integrated density of states, the number of states per cell with energy less than E, by

$$N(E) \approx \Delta\phi/\pi \,, \tag{1}$$

where the approximate inequality is good to ± 1. Therefore for any range of energies where the wave function is locked to the cell width, $\Delta\phi = 0$ and so $N(E)$ must be constant and consequently the energy range must belong to a forbidden energy region. Energy gaps occur because locking of the wave function to the cell width implies that the Bloch momentum (also known as the crystal momentum in condensed-matter physics) of the solution coincides with that for complete Bragg reflection.

In this paper we will generalize the Borland-Makinson-Roberts results to the one-dimensional Dirac equation. For both Lorentz vector potentials, which transform like the time component of a four-vector, and Lorentz scalar potentials, which transform like a mass, we will show that an energy gap persists for random lattices which possess a short-range order, as is the case in the nonrelativistic domain. An upper bound on the amount of disorder necessary for dissolution of the gap structure will then be determined. In a recent Letter[5] a brief description of this procedure and its results for a δ-function potential of Lorentz vector type was given. In this paper a comprehensive derivation and analysis of those results as well as those for the Lorentz scalar potential is developed. It is possible, although perhaps not too instructive, to determine the energy gaps for a combination of the Lorentz scalar and Lorentz vector potential. These results are quoted in Appendix A.

Recently[6] Roy and Basu investigated the band structure that results for a disordered Kronig-Penney array with finite-width square barriers. However, it should be noted that their derivation was based on an approximation to the one-dimensional Dirac equation which was first developed by Steślicka and Davison[7] while our derivation uses the exact one-dimensional Dirac equation. As a comparison we will also list the results we obtain for finite-width barriers. These results will be given in Appendix B.

The results we obtain, in the case of the scalar potentials, have some relevance to some recent models of

quarks in nuclei.[8] It is immediately apparent that the usual picture of the nucleus, as comprising a fermion fluid of nucleons, is not compatible with the solidlike periodic structures of these models. In the nuclear physics context our results show that one of the features of the periodic model, namely the existence of a band structure, survives the introduction of randomization. However, it should be emphasized that our model is more glassy than liquid in that the potential barriers binding the quarks are not free to move, although they have a randomized spacing.

The case of the vector potential is, of course, the appropriate generalization of the random Kronig-Penney model for the case of a high-Z metal.[9]

II. ENERGY GAPS

We will begin our investigation of the energy structure which results for our potential array by defining the phase of a relativistic solution of the one-dimensional Dirac equation. From this beginning we will examine the phenomenon of locking and its consequences for an aperiodic array of potentials possessing short-range order.

The similarities between this work and that concerned with the study of the spectrum of a one-dimensional system in the two-band approximation is quite pronounced. As examples of this type already appear in the literature[10] we will endeavor to be succinct in the exposition of our methods.

A. Definition of the relativistic phase

The one-dimensional time-independent Dirac equation for a particle of mass m and energy E in a potential $U(x)$ is given by the two-component equation

$$\left[i\sigma_x \frac{d}{dx} + E - \sigma_z m - U(x) \right] \Psi(x) = 0 , \tag{2}$$

where we have chosen to use the well-known Dirac-Pauli representation.[11]

In our model we assume that the potential consists of an infinite array of δ-function potentials which may be of either Lorentz vector or scalar character. Hence we may write

$$U(x) = \begin{cases} X(x) & \text{for a vector potential} \\ \sigma_z X(x) & \text{for a scalar potential,} \end{cases} \tag{3}$$

where $X(x)$ is given by

$$X(x) = \lambda \sum_{i=1}^{\infty} \delta(x - x_i) , \tag{4}$$

for the δ-function strength λ.

The randomness is introduced by the hypothesis that the interpotential distances, $l_j = x_{j+1} - x_j$, are independent random variables distributed in the interval $[l, l+d]$, i.e.,

$$l \le l_j \le l + d . \tag{5}$$

As the Pauli matrices are 2×2 matrices $\Psi(x)$ is a two-component spinor which we write explicitly as

$$\Psi(x) = \begin{pmatrix} \Psi_1(x) \\ \Psi_2(x) \end{pmatrix} . \tag{6}$$

By analogy with the definition of the phase of a solution of the Schrödinger equation we define the phase of a solution of this particular representation of the Dirac equation by

$$\tan \phi(k,x) = -\frac{i\Psi_2(x)}{r\Psi_1(x)} , \tag{7}$$

where $\phi(k,x)$ changes continuously with x, and $r = k/(E+m)$. The definition (7) is equivalent to $\tan \phi(k,x) = -\Psi_1'(x)/[k\Psi_1(x)]$ in a field-free region, but is more convenient in the presence of a potential. This deliberate choice of phase has the utility that over a field-free region of length Δx the change in phase is $k\Delta x$, and it corresponds to the nonrelativistic phase in the appropriate limit. It is readily seen that a change in ϕ of π in an interval (x_1, x_2) indicates that $\Psi_1(x)$ has a node in this interval. As node counting is equivalent to counting the number of states (this also applies in the relativistic theory[12]) we see that the James-Ginzbarg[4] relation remains valid. Therefore the properties of the integrated density of states obtained in the nonrelativistic theory are also directly applicable to the relativistic domain.

The discontinuity in the wave function due to a δ function of strength λ at $x = 0$ is given by[13]

$$\Psi(0+) = \exp(-\sigma_y \lambda)\Psi(0-) , \tag{8}$$

for the scalar potential and

$$\Psi(0+) = \exp(-i\sigma_x \lambda)\Psi(0-) , \tag{9}$$

for the vector potential.[13] From these we obtain the change in phase across the δ function

$$\phi^+ - \phi^- = \mathcal{S}(\phi^-) = \Pr \arctan\left(\frac{f + g\cos 2\phi^-}{1 + g \sin 2\phi^-} \right) , \tag{10}$$

where for the scalar potential the functions f and g are given by

$$f = -\frac{m}{k} \tanh \lambda , \quad g = -\frac{E}{k} \tanh \lambda , \tag{11}$$

while for the vector potential

$$f = -\frac{E}{k} \tan \lambda , \quad g = -\frac{m}{k} \tan \lambda . \tag{12}$$

Applying the nonrelativistic, weak barrier limit to (10), namely keeping only terms to first order in k/m and λ, we obtain the familiar nonrelativistic result

$$\mathcal{S}(\phi^-) = -\frac{2m\lambda}{k} \cos^2 \phi^- , \tag{13}$$

which always has the opposite sign to λ. For $\lambda < 0$ Makinson and Roberts[3] showed the effect of this phase change is to pull the node of the wave function closer to the δ-function location. This suggests that for a small domain of wavelengths just larger than twice the cell width

the pulling effect of the δ function may be sufficient to draw the node of the wave function such that it coincides with the position of the δ function. The presence of only one node per cell implies that the wave function is locked to the cell width and hence an energy gap must exist for the range of wave numbers $kl \in [\pi - \chi(\lambda), \pi]$ where $\chi(\lambda)$ is some function of the δ-function strength. The case $\lambda > 0$ is treated similarly.

In the relativistic case the structure of (10) suggests that, in contrast to the nonrelativistic case, the sign of $\mathcal{S}(\phi)$ may be indeterminate. As we shall see this indeterminacy allows a scalar δ-function potential to both push the node of the wave function away from itself and to pull it toward itself, the choice being dependent on the phase prior to the δ function, for arbitrary δ-function strength. A similar situation also exists for the vector δ-function potential.

Consequently a mechanism exists for the range of wavelengths available for locking to be expanded to include those just smaller than twice the cell width which have their nodes pushed forward by the δ function. This suggests that the value $kl = \pi$, which in the nonrelativistic system is of particular significance in the first band as it forms the upper (lower) bound of the gap for $\lambda < 0\,(\lambda > 0)$, no longer performs such an important function in the relativistic domain.

The change in phase over the jth cell in the reduced phase notation of Makinson and Roberts[3] [i.e., the phase change modulo π, defined to be in the range $(-\pi/2, \pi/2)$], is given by

$$\Delta(\phi_j^-) = \mathcal{S}(\phi_j^-) + J(l_j)\,, \tag{14}$$

where $\mathcal{S}(\phi_j^-)$ represents the shift in the phase due to the δ function while $J(l_j) = kl_j - n\pi$ is a reduced phase from free propagation across the cell, n being chosen so that $\Delta(\phi_j^-)$ lies in the required range.

An energy gap implies a constant integrated density of states and thus a constant reduced phase. The roots $\beta_j^{\pm}$ of the reduced phase function $\Delta(\phi_j^-)$ are given by

$$\tan\beta_j^{\pm} = \frac{g\tan kl_j \pm \mu_j}{g - f - \tan kl_j}\,, \tag{15}$$

for $\mu_j^2 = [(\alpha_0^j)^2 - 1]/\zeta_j^2$ where for the scalar potential

$$\zeta_j = \cos kl_j \cosh\lambda,$$
$$\alpha_0^j = \zeta_j + \frac{m}{k}\sin kl_j \sinh\lambda_j\,, \tag{16}$$

while for the vector potential

$$\zeta_j = \cos kl_j \cos\lambda,$$
$$\alpha_0^j = \zeta_j + \frac{E}{k}\sin kl_j \sin\lambda_j\,. \tag{17}$$

To complete the generalization of the analysis of Roberts and Makinson we note that the root β_j^+ (or β_j^-) lies in the range (β_a^+, β_b^+) [or (β_a^-, β_b^-)] where $\beta_a^{\pm}$ are given by (15) for $l_j = l$, and $\beta_b^{\pm}$ are given by (15) for $l_j = l + d$.

In either case the phases $\beta_a^{\pm}$ and $\beta_b^{\pm}$ are real if and only if $|\alpha_0^j| > 1$, in which case the energies belong to the relativistic Kronig-Penney[14] gap. The energies $|\alpha_0^j| \leq 1$ then obviously constitute the relativistic Kronig-Penney bands. From this point on there is no advantage in considering the two potentials simultaneously and we will begin our analysis by treating the vector potential case first.

B. Gap structure for the Lorentz vector potential

As discussed previously, and in contrast to its nonrelativistic behavior, the change in the phase over a relativistic vector δ function is not a simple function of the δ-function strength. This is particularly evident if we use (10) to determine the roots of $\mathcal{S}(\phi)$:

$$\tan^2\phi_0 = -\frac{1}{r^2}\,, \tag{18}$$

a result which implies that no roots of $\mathcal{S}(\phi)$ exist for real phases. (In the nonrelativistic limit $r \to 0$ and we recover the roots $\phi_0 = \pm\pi/2$.)

As $\mathcal{S}(\phi)$ cannot cross the ϕ axis (except at a branch point) we find that it becomes convenient to choose a branch of the arctan function so that $\mathcal{S}(\phi)$ is continuous, as this will force $\mathcal{S}(\phi)$ to be of definite sign, as is the case in the nonrelativistic treatment, regardless of the phase. It is also convenient to introduce the "reduced strength" of the vector δ-function potential, λ_r, by

$$\lambda_r = \lambda \bmod(2\pi)\,, \tag{19}$$

for $\lambda_r \in (-\pi, \pi]$. [The reason for the choice $\lambda_r = \lambda \bmod(2\pi)$ and not $\lambda_r = \lambda \bmod(\pi)$, although $\mathcal{S}(\phi)$ has period π in λ, is due to the simple form the extrema of $\mathcal{S}(\phi)$ take with our choice.[15]]

It is straightforward to consider the cases $\lambda_r > 0$ and $\lambda_r < 0$ separately. For $\lambda_r > 0$, $\mathcal{S}(\phi)$ is plotted in Fig. 1 [choosing the branch of the arctan function so that $\mathcal{S}(\phi)$ is positive and continuous]. Choosing the integer n so that $J(l) \leq J(l+d) \leq 0$ we can see that the roots $\beta_a^{\pm}$ and $\beta_b^{\pm}$ exist if the following inequalities are satisfied:

$$\mathcal{S}_{\max}(\phi) \geq -J(l)\,, \tag{20}$$

$$\mathcal{S}_{\min}(\phi) \leq -J(l+d)\,. \tag{21}$$

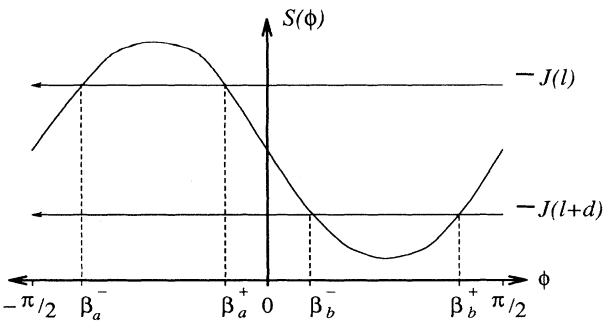

FIG. 1. Phase diagram for the vector potential.

As documented in our previous paper[5] it is a simple matter to demonstrate that the dynamics of the phase, under the constraints on the cell width (5), force any phase in the region (β_a^-, β_b^+) into the stable phase[2] region $[\beta_a^+, \beta_b^-]$, from which there is no escape. The remaining regions, $(-\pi/2, \beta_a^-)$ and $(\beta_b^+, \pi/2)$, which actually constitute a single region in our reduced phase notation and which we will denote by (β_b^+, β_a^-) (also known as the unstable phase[2] region), have the same properties as the stable phase region $[\beta_a^+, \beta_b^-]$ in that the occurrence of any cell of length satisfying (5) is unable to force the phase forwards past β_a^- or backwards past β_b^+. Consequently, as long as the roots $\beta_a^\pm$ and $\beta_b^\pm$ exist there will be a phase-locked region, thus implying a constant integrated density of states and the presence of an energy gap.

Differentiating $\mathcal{S}(\phi)$ from (10) allows the extrema to be determined.[15] For $\lambda_r > 0$ we obtain

$$\mathcal{S}_{\max}(\phi) = 2\,\mathrm{Pr}\arctan\left(\frac{1}{r}\cot\frac{\lambda_r}{2}\right), \tag{22}$$

and

$$\mathcal{S}_{\min}(\phi) = 2\,\mathrm{Pr}\arctan\left(r\cot\frac{\lambda_r}{2}\right). \tag{23}$$

Defining

$$q_t = \frac{2}{\pi}\,\mathrm{Pr}\arctan\left(r^t\tan\frac{|\lambda_r|}{2}\right), \tag{24}$$

so $q_t \in (0,1)$ for $\lambda_r \neq 0, \pm\pi$, we can rewrite the inequalities (20), (21) as

$$\overline{n}\pi + \pi q_{+1} \leq kl \leq \overline{n}\pi + \pi q_{-1} - kd, \tag{25}$$

where $\overline{n} = n - 1 = 0, 1, 2, \ldots$. These regions therefore represent the forbidden energy regions for a vector δ-function strength satisfying $\lambda_r > 0$.

A similar analysis when $\lambda_r < 0$ leads to the condition

$$n\pi - \pi q_{-1} \leq kl \leq n\pi - \pi q_{+1} - kd, \tag{26}$$

with $n = 1, 2, 3, \ldots$, as the condition for the existence of energy gaps.

Equations (25) and (26) are our relativistic generalizations of the Borland-Roberts-Makinson results. To see this we note that, in the nonrelativistic, weak barrier limit, the following conditions result:

$$\overline{n}\pi \leq kl \leq \overline{n}\pi + \pi q - kd, \tag{27}$$

for $\lambda > 0$ and for $\overline{n} = 0, 1, 2, \ldots$, while

$$n\pi - \pi q \leq kl \leq n\pi - kd \tag{28}$$

is found for $\lambda < 0$ and $n = 1, 2, 3, \ldots$ where

$$q = \frac{2}{\pi}\,\mathrm{Pr}\arctan\left(\frac{m|\lambda|}{k}\right), \tag{29}$$

in accordance with their results.

For $\lambda = p\pi$ (p an integer) no band gap exists. This occurs because this condition corresponds precisely to that for destructive interference of multiply reflected wave packets inside the barrier and is a direct consequence of the oscillatory nature of the wave function in this region, the so-called Klein paradox.[16]

Since r is a slowly varying function of k we may utilize this fact to derive the approximate width of the gap from either (25) or (26), from which we find

$$W(k,d) = 2\,\mathrm{Pr}\arctan\left(\frac{m}{k}\sin|\lambda_r|\right) - kd. \tag{30}$$

For $d = 0$, the periodic lattice, the energy gap disappears when $k \to \infty$ and also when $m = 0$. This reason is the same in both cases in that the vector potential is transparent to the particle. However, the causes are quite different. In the infinite energy limit the reasoning is self-evident, while in the massless case the property of conservation of chirality for a vector potential ensures the transparency of the barrier.[17]

Later in this paper we obtain an upper bound for the deviation parameter, d, before dissolution of the gap structure occurs. But first we determine the band structure that results for an array of scalar δ-function potentials.

C. Gap structure for the Lorentz scalar potential

For a scalar potential the roots of $\mathcal{S}(\phi)$ occur where

$$\tan\phi_0 = \pm\frac{1}{r}. \tag{31}$$

Thus $\mathcal{S}(\phi)$ is no longer constrained to be of definite sign. It follows that the corresponding restriction on $J(l_j)$, to be either positive or negative in opposition to the sign of $\mathcal{S}(\phi_j)$ so that we may have $\Delta(\phi) = 0$, may also be relaxed. Consequently, we expect the energy gaps for the scalar potential to straddle the point $kl = \pi$ rather than being confined to either side of it, in contrast to the situation that occurs for both the vector potential and in the nonrelativistic regime.

We consider first the $\lambda > 0$ case, the appropriate diagram for which is shown in Fig. 2 for energies satisfying $|\alpha_0^j| > 1$. The only constraint on the $J(l_j)$ is $J(l+d) > J(l)$, which is automatically satisfied.

The situation is very similar to that found for the vector potential. Initial phases in (β_a^-, β_a^+) and (β_b^+, β_b^-) are forced into the stable region $[\beta_a^+, \beta_b^+]$, from which

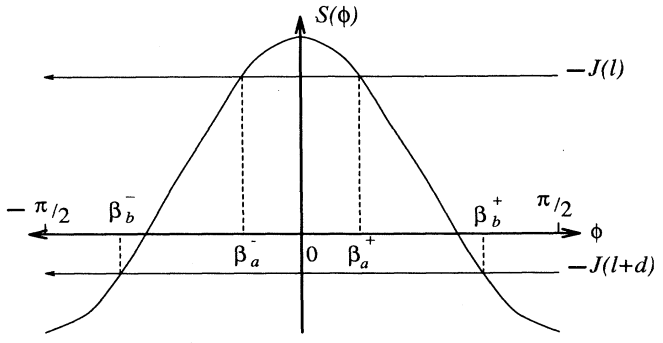

FIG. 2. Phase diagram for the scalar potential.

they cannot escape, while phases in (β_b^-, β_a^-) are similarly trapped. Thus if $\beta_a^\pm$ and $\beta_b^\pm$ exist, as is the case if the inequalities (20) and (21) are satisfied, an energy gap must exist.

From (10) the extrema of $\mathcal{S}(\phi)$ may be determined and they are given by

$$\mathcal{S}_{\max}(\phi) = 2\,\mathrm{Pr}\arctan\left(r\tanh\frac{\lambda}{2}\right), \tag{32}$$

and

$$\mathcal{S}_{\min}(\phi) = -2\,\mathrm{Pr}\arctan\left(\frac{1}{r}\tanh\frac{\lambda}{2}\right). \tag{33}$$

Defining

$$q_t = \frac{2}{\pi}\,\mathrm{Pr}\arctan\left(r^t\tanh\frac{|\lambda|}{2}\right), \tag{34}$$

so $q_t \in (0,1)$ for $\lambda \neq 0$, and utilizing (20) and (21) we obtain the condition for the existence of energy gaps

$$n\pi - \pi q_{+1} \leq kl \leq n\pi + \pi q_{-1} - kd\,, \tag{35}$$

for $n = 1, 2, 3, \ldots$.

In an analogous fashion we can derive similar constraints for the $\lambda < 0$ case from which we obtain

$$n\pi - \pi q_{-1} \leq kl \leq n\pi + \pi q_{+1} - kd\,. \tag{36}$$

Both of these inequalities reduce to their appropriate nonrelativistic, weak barrier limits as given in (27) and (28), respectively, as $q_{+1} \to 0$ and $q_{-1} \to q$.

The width of the gap in kl space is approximately given by

$$W(k,d) = 2\,\mathrm{Pr}\arctan\left(\frac{E}{k}\sinh|\lambda|\right) - kd\,. \tag{37}$$

For a periodic lattice ($d = 0$) the gap width is nonzero for any value of k, for a nonzero strength δ function. We also note that for increasing k the gap width decreases toward a constant value.

For $\lambda \to \infty$ the spectrum becomes discrete as the kl gap width tends to π. Indeed if we put $\lambda \to \infty$, for $d = 0$, into (35) or (36) the discrete energies that result for the lowest band are given by

$$\tan kl + \frac{k}{m} = 0\,, \tag{38}$$

which are the eigenenergies of a one-dimensional bag,[18] as we have shown in a previous paper.[17] This restriction to discrete energies occurs because an infinite scalar δ-function potential is able to completely prohibit the passage of the particle current. For nonzero d these states split into a small band again, with the solutions of (38) providing the upper band edge.

As an aside we note from (34) that for $m = 0$, $q_{-1} = q_{+1}$. Consequently the conditions specifying the gap regions, (35) and (36), become identical; the resulting band structure depends only on the magnitude of the δ-function strength and is independent of its sign. Furthermore, for a periodic lattice ($d = 0$), these gaps form regions which are symmetrical about the values $kl = n\pi$. These two results suggest the existence of some fundamental symmetry of the Dirac equation for a massless particle in a scalar field. This is indeed the case, as we have shown elsewhere.[19]

III. LIMITS ON THE DEVIATION PARAMETER

The conditions for the existence of band gaps derived in the previous sections for both the vector and scalar potentials, namely (25), (26) and (35), (36), respectively, indicate that for increasing d, called the deviation parameter, the gap widths narrow with increasing energy until at some point the gaps disappear entirely. In this section we determine an upper bound for d, the parameter characterizing the degree of disorder in the lattice, before the dissolution of the gap structure occurs.

An analysis of the nonrelativistic disordered Kronig-Penney model by Roy and Basu[20] has previously shown that band gaps exist in such systems if the deviation parameter is bounded above, the exception being that the lowest band gap exists for all finite d however large. (This is true for both repulsive and attractive potentials, as we see below, although this fact is not clear from Ref. 20.)

As we have extended the results of Borland, and Makinson and Roberts to the relativistic Kronig-Penney model we are now in a position to also extend the analysis of Roy and Basu to the relativistic domain.[6]

A. Limits on d for the vector potential

We consider first the $\lambda_r > 0$ case, when the condition for the existence of energy gaps is (25). From the first inequality we obtain

$$kl \geq \overline{n}\pi + \pi q_{+1}\,, \tag{39}$$

whence

$$\frac{kd}{\pi} \geq \frac{d}{l}(\overline{n} + q_{+1})\,, \tag{40}$$

and from the other inequality

$$kd \leq \pi(q_{-1} - q_{+1}) = W(k,0) = 2\,\mathrm{Pr}\arctan\left(\frac{m}{k}\sin|\lambda_r|\right) \leq \pi\,. \tag{41}$$

Combining (40) and the outer inequality in (41) we find that

$$1 \geq \frac{d}{l}(\overline{n} + q_{+1})\,, \tag{42}$$

or

$$\frac{d}{l} \leq \frac{1}{\overline{n} + q_{+1}}\,, \tag{43}$$

for $\overline{n} = 0, 1, 2, \ldots$ is the condition on d for the existence of an energy gap.

A similar argument when $\lambda_r < 0$ gives the condition as

$$\frac{d}{l} \leq \frac{1}{n - q_{-1}} , \tag{44}$$

for $n = 1, 2, 3, \ldots$.

Note that in the nonrelativistic limit $q_{+1} \rightarrow 0$, while $q_{-1} \rightarrow 1$, and we recover the results of Roy and Basu. Note, however, that $q_{+1} > 0$ and $q_{-1} < 1$ due to the relativistic corrections so that we find that the somewhat unphysical absence of a bound on the deviation parameter for the lowest band ($\overline{n} = 0$ or $n = 1$) is removed by relativistic effects.

The extreme relativistic limit is also of some interest. In this case

$$q_{\pm 1} \rightarrow \frac{|\lambda_r|}{\pi} , \tag{45}$$

so that the possibility of an absence of a bound on d for the existence of an energy gap reappears for particular values of the δ-function strength, namely $\lambda_r = 0$ [remembering that $\lambda_r = \lambda \bmod(2\pi)$] for (43) and $\lambda_r = \pm\pi$ for (44). However, as discussed previously, for $\lambda = p\pi$ (p an integer) no energy gaps exist and therefore no such pathology may arise.

B. Limits on d for the scalar potential

Considering the $\lambda > 0$ case first we obtain, from (35), the inequalities

$$kl \geq n\pi - \pi q_{+1} , \tag{46}$$

and

$$kd \leq \pi(q_{+1} + q_{-1}) = W(k, 0) = 2 \Pr \arctan\left(\frac{E}{k} \sinh|\lambda|\right) \leq \pi . \tag{47}$$

Note that, upon the interchange $q_{+1} \rightarrow -q_{+1}$ and $\overline{n} \rightarrow n$, (46) becomes equivalent to (39) while the outer inequality of (47) is the same as that from (41). Consequently, following the same procedure as for the $\lambda_r > 0$ vector potential case we obtain the following:

$$\frac{d}{l} \leq \frac{1}{n - q_{+1}} , \tag{48}$$

for $n = 1, 2, 3, \ldots$.

Similarly for $\lambda < 0$ we obtain

$$\frac{d}{l} \leq \frac{1}{n - q_{-1}} . \tag{49}$$

Again, as $q_{+1} \rightarrow 0$ and $q_{-1} \rightarrow 1$ in the nonrelativistic limit, we recover the results of Roy and Basu. As with the vector potential the constraints $q_{+1} < 1$ and $q_{-1} < 1$ ensure that a bound exists for the deviation parameter even for the lowest band.

IV. CONCLUSION

Our results have shown that, for an array of δ-function potentials of both Lorentz vector and scalar types, the introduction of long-range disorder into the system is insufficient to destroy the relativistic gap structure, provided some degree of short-range order is maintained. Conditions specifying the amount of short-range disorder necessary before dissolution of the gap structure occurred were then developed. From these conditions we were able to show that the parameter representing this degree of disorder, the deviation parameter d, is bounded above for all energy bands. This result differs markedly from those obtained in the nonrelativistic regime where it is found that no upper bound exists on the deviation parameter for the lowest band.

Our major observation, with regard to the relativistic system as a whole, is that the band structure and the transmission properties of a particle in a potential are inextricably linked. This is graphically illustrated in the vector domain by the total absence of band structure for the conditions $m = 0$ and $\lambda = p\pi$, both of which imply a complete absence of Bragg reflection. In the scalar domain, where the potential is more suited to model confinement, we find that energy gaps must occur and indeed will persist even in the ultrarelativistic limit. This is a direct consequence of the fact that the reflection coefficient of a particle for an array of scalar δ-function potentials is never zero;[17] partial Bragg reflection must occur irrespective of the incident energy of the particle.

ACKNOWLEDGMENTS

This work was supported in part by the Australian Research Council. G.J.C. would like to thank the University of Melbourne for financial support.

APPENDIX A: GAP STRUCTURE FOR COMPOSITE LORENTZ POTENTIALS

For an array of δ-function potentials of composite Lorentz scalar and Lorentz vector type we may obtain conditions for the existence of energy gaps in a similar fashion to that used to derive the results in the main body of this paper. Those previous results would then represent the two extremes cases possible ($\lambda_s = 0$ for the vector or $\lambda_v = 0$ for the scalar).

For a composite δ-function potential of vector strength λ_v and scalar strength λ_s we may define the following functions:

$$\gamma = \frac{\lambda_v}{\lambda_s} , \quad \eta = \frac{\lambda_s}{\Lambda} , \quad \text{and} \quad \Lambda^2 = \lambda_s^2 - \lambda_v^2 . \tag{A1}$$

Several cases exist according to the relative magnitudes of the scalar and vector components of the potential.

1. $|\lambda_s| > |\lambda_v|$

For this region (which is similar to the scalar case in the main body) we have $|\gamma| < 1$ and η, γ and $\Lambda \in \Re$. Choosing $\Lambda > 0$ the conditions for the existence of energy gaps are

$$n\pi - \pi q_{+1} \leq kl \leq n\pi + \pi q_{-1} - kd , \tag{A2}$$

for $\lambda_s > 0$ and $n = 1, 2, 3, \ldots$ and

$$n\pi + \pi q_{-1} \leq kl \leq n\pi - \pi q_{+1} - kd , \tag{A3}$$

for $\lambda_s < 0$ where

$$q_t = \frac{2}{\pi} \Pr \arctan \left([r\eta(1-\gamma)]^t \tanh \frac{\Lambda}{2} \right) . \tag{A4}$$

2. $|\lambda_s| < |\lambda_v|$

For this region (which is similar to the vector case in the main body) we have $|\gamma| > 1$ and $\eta, \Lambda \in \Im$. Defining $\overline{\eta} = i\eta$ and $\Gamma = -i\Lambda$ where $\overline{\eta}, \Gamma \in \Re$, as well as the regions $\Gamma_1 = (2p\pi, [2p+1]\pi)$ and $\Gamma_2 = ([2p+1]\pi, [2p+2]\pi)$, for p an integer, we obtain the following results.

For $\lambda_s \geq 0$, $\lambda_v > 0$ and $\Gamma \in \Gamma_1$ or $\lambda_s \leq 0$, $\lambda_v < 0$ and $\Gamma \in \Gamma_2$ we obtain

$$\overline{n}\pi + \pi q_{+1} \leq kl \leq \overline{n}\pi + \pi q_{-1} - kd , \tag{A5}$$

for $\overline{n} = 0, 1, 2, \ldots$ while for $\lambda_s \geq 0$, $\lambda_v > 0$ and $\Gamma \in \Gamma_2$ or $\lambda_s \leq 0$, $\lambda_v < 0$ and $\Gamma \in \Gamma_1$ we obtain

$$n\pi + \pi q_{-1} \leq kl \leq n\pi + \pi q_{+1} - kd , \tag{A6}$$

for $n = 1, 2, 3, \ldots$ where

$$q_t = \frac{2}{\pi} \Pr \arctan \left([r\overline{\eta}(\gamma - 1)]^t \tan \frac{\Gamma}{2} \right) . \tag{A7}$$

For the conditions $\lambda_s > 0$ and $\lambda_v < 0$ or $\lambda_s < 0$ and $\lambda_v > 0$ the situation is not as clearcut, as the extrema of $\mathcal{S}(\phi)$ show an energy dependence with regard to the sign of $\mathcal{S}''(\phi)$. This results in the maxima (minima) of $\mathcal{S}(\phi)$ becoming minima (maxima) when the energy satisfies the constraint

$$E = m|\gamma| . \tag{A8}$$

This energy, which corresponds to complete transmission of the particle through the potential array,[17] necessarily occurs outside of an energy gap.

Consequently for the conditions $\lambda_s > 0$, $\lambda_v < 0$, and $E \leq m|\gamma|$ we obtain (A5) for $\Gamma \in \Gamma_1$ and (A6) for $\Gamma \in \Gamma_2$, while for $E > m|\gamma|$ the situation is reversed. Similarly for $\lambda_s < 0$ and $\lambda_v > 0$ and $E \leq m|\gamma|$ we obtain (A6) for $\Gamma \in \Gamma_1$ and (A5) for $\Gamma \in \Gamma_2$ with the opposite situation occurring for $E > m|\gamma|$.

3. $|\lambda_s| = |\lambda_v|$

For this case we find $\Lambda = 0$. The two major cases are for $\lambda_s = \lambda_v$ and $\lambda_s = -\lambda_v$.

(i) $\lambda_s = \lambda_v$. For $\lambda_s > 0$ we obtain, for $\lambda = \lambda_s + \lambda_v$, the following:

$$\overline{n}\pi \leq kl \leq \overline{n}\pi + \pi q - kd , \tag{A9}$$

for $\overline{n} = 0, 1, 2, \ldots$ while for $\lambda_s < 0$ we obtain

$$n\pi - \pi q \leq kl \leq n\pi - kd , \tag{A10}$$

for $n = 1, 2, 3, \ldots$ where

$$q = \frac{2}{\pi} \Pr \arctan \left(\frac{|\lambda|}{2r} \right) . \tag{A11}$$

The similarities between these results and those for the nonrelativistic system, as quoted in (27)–(29), are quite pronounced and result because of the Schrödinger-like form the Dirac equation assumes for this potential.[21]

(ii) $\lambda_s = -\lambda_v$. For $\lambda_s > 0$, and $\lambda = \lambda_s - \lambda_v = 2\lambda_s$, we obtain

$$n\pi - \pi q \leq kl \leq n\pi - kd , \tag{A12}$$

for $n = 1, 2, 3, \ldots$ while for $\lambda_s < 0$ we find

$$\overline{n}\pi \leq kl \leq \overline{n}\pi + \pi q - kd , \tag{A13}$$

for $\overline{n} = 0, 1, 2, \ldots$ where

$$q = \frac{2}{\pi} \Pr \arctan \left(\frac{r|\lambda|}{2} \right) . \tag{A14}$$

In the nonrelativistic limit these gaps disappear, as we would expect, because in this limit the scalar and vector potentials become indistinguishable and therefore cancel.

APPENDIX B: GAP STRUCTURE FOR FINITE-WIDTH BARRIERS

For this system we replace the δ-function potentials by step potentials of width a and vector and scalar heights V and S, respectively, while the distances between these potential regions, the b_j, are now distributed in the interval $b_j \in [b, b+d]$. We begin by defining

$$K^2 = (E - V)^2 - (m + S)^2 , \tag{B1}$$

the two regions of interest of which are where $K^2 > 0$ and where $K^2 < 0$.

For $K^2 > 0$ we define

$$R_K = \frac{K}{E - V + m + S} = \frac{E - V - m - S}{K} , \tag{B2}$$

while for $K^2 < 0$ we let $\kappa^2 = -K^2 > 0$ and define

$$R_\kappa = \frac{\kappa}{E - V + m + S} = \frac{m + S - E + V}{\kappa} . \tag{B3}$$

Finally, we introduce the function

$$\tau = ES + mV . \tag{B4}$$

With these definitions we can show that for $K^2 < 0$, or for energies satisfying $V - S - m < E < V + S + m$, the energy gaps for $\tau > 0$ are given by

$$n\pi - \pi q_{+1} \leq kb \leq n\pi + \pi q_{-1} - kd , \tag{B5}$$

while for $\tau < 0$ we find

$$n\pi + \pi q_{-1} \leq kb \leq n\pi - \pi q_{+1} - kd , \tag{B6}$$

where

$$q_t = \frac{2}{\pi} \Pr \arctan \left(\left[\frac{r}{R_\kappa} \right]^t \tanh \frac{\kappa a}{2} \right) . \tag{B7}$$

These two equations reduce to their δ-function counterparts of Appendix A in the δ-function limit, V and $S \to \infty$ and $a \to 0$, and where

$$Va \to \lambda_v , \quad Sa \to \lambda_s, \quad \text{and} \quad b \to l , \tag{B8}$$

as expected.

For $K^2 > 0$, or for energies satisfying $E > |m+S| + V$, the situation becomes analogous to that of the dominant

vector potential case of the previous appendix. Consequently, upon defining $\Gamma = Ka$, we can show that for $\tau > 0$ and $\Gamma \in \Gamma_1$ (these regions are defined in the previous appendix), or for $\tau < 0$ and $\Gamma \in \Gamma_2$, the energy gaps are given by

$$n\pi - \pi q_{+1} \leq kb \leq n\pi - \pi q_{-1} - kd \,, \tag{B9}$$

while for $\tau > 0$ and $\Gamma \in \Gamma_2$ or for $\tau < 0$ and $\Gamma \in \Gamma_1$ we find

$$n\pi + \pi q_{-1} \leq kb \leq n\pi + \pi q_{+1} - kd \,, \tag{B10}$$

where

$$q_t = \frac{2}{\pi} \Pr \arctan \left(\left[\frac{r}{R_K} \right]^t \tan \frac{\Gamma}{2} \right) . \tag{B11}$$

For the case where $\tau = 0$ the band gaps disappear entirely, for the same reasons as given in (A8).

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