Mesoscopic rings with finite aspect ratio: Magnetic-field correlation function

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We analytically calculate the ensemble-averaged magnetic-field correlation function of the conductance in a mesoscopic ring. The ring is coupled to external leads and has a finite aspect ratio. To average the correlation function we employ a recently introduced formalism (based on statistical scattering theory and the supersymmetry method) which we extend to include the magnetic field. Our results enable us to discuss the dependence of the correlation function on the length of the connecting disordered leads and on the thickness of the ring. In particular, we study the decay of both aperiodic fluctuations and Aharonov-Bohm oscillations. Our findings are consistent with experimental observation.

I. INTRODUCTION

Measurements of the conductance of small metallic rings in a perpendicular magnetic field were among the key experiments opening the field of mesoscopic physics.[1] Quantum coherence has been identified to be responsible for novel fluctuation phenomena. In ring geometries one expects an interplay between aperiodic fluctuations (the so-called universal conductance fluctuations) and periodic Aharonov-Bohm (AB) oscillations. In principle, the origin of these effects is well understood.[2] However, an analytical quantitative model for a realistic ring geometry is still lacking. At least three features have to be included in a realistic model: First, the mesoscopic ring should—as in the experiment—be coupled to the external world. Otherwise we deal with a closed system and the definition of transport coefficients and their relation to experimentally observed quantities is at best unclear. Second, the magnetic field should actually penetrate the body of the ring and modify the phase relations among different electron paths. It is well known that the aspect ratio (the area of the ring itself as compared to the area enclosed by the ring) decisively determines the observability of the AB effect. Therefore a truly two-dimensional model is called for. Third, it should be possible to investigate the influence of the geometry of the mesoscopic device, both for general reasons (geometry dependence is a main issue in mesoscopic physics) and for comparison with experiment.

The analytical model presented in this paper is the first one to incorporate all three properties just mentioned. Our main result is an analytical expression for the magnetic-field correlation function depending on the thickness of the ring and on a certain geometry parameter ($\tilde{c}$). Qualitatively, $\tilde{c}$ can be interpreted as the ratio between the radius of the ring and the length of the attached disordered leads. In particular, our result enables us to explain the geometry dependence of the AB amplitude and to compare the decay widths of AB oscillations and aperiodic fluctuations both with each other and with experiment. It has been reported[3] that AB oscillations decay approximately twice as fast as aperiodic fluctuations. We find good agreement with this observation. Furthermore, the geometry dependence of the ratio of these decay widths is predicted.

Let us recall why we have to calculate a rather complicated quantity like the magnetic-field correlation function to study the AB effect. According to an ergodic hypothesis[4] averages over the magnetic field (necessary to study statistical properties) can be replaced by averages over an ensemble of random potentials. We will denote these latter averages by $\langle\ \rangle$. The average conductance, $\langle g \rangle$, does *not* exhibit AB oscillations but displays instead so-called AAS (Al'tshuler, Aronov, and Spivak) oscillations[5] having period $\phi_0/2$. There is a simple semiclassical explanation for this phenomenon. The AB effect originates from the interference of two electron paths which together enclose the magnetic flux through the ring. The relative phase of these paths will, however, be randomized upon averaging over an ensemble of random potentials, and the effect vanishes. The only exception is a path which surrounds the whole ring and interferes with its time-reversed counterpart. Electrons following these paths will scatter from identically the same impurities so that ensemble averaging cannot introduce a random phase. Therefore the corresponding oscillations persist. They have period $\phi_0/2$ since the pair of paths encloses two times the actual magnetic flux. These AAS oscillations vanish as the increasing magnetic field breaks time-reversal symmetry and pairs of time-reversed paths cease to exist.

The averaged magnetic-field correlation function $F(B,\Delta B)$ [see Eq. (2.12) for the definition] behaves differently. We again turn to a semiclassical picture. For each pair of paths which encloses the flux through the ring and contributes to $g(B)$ we find a geometrically identical pair contributing to $g(B+\Delta B)$. Again, averaging over the ensemble cannot affect the relative phase which is therefore entirely determined by the flux difference $\Delta\phi=\Delta B A$. Here, A is the area of the ring. The resulting oscillations have period ϕ_0 and do not depend on time-reversal symmetry. Hence, we deal with AB oscillations.

Many theoretical papers have been concerned with the problem of mesoscopic rings in a magnetic field in recent years. At the beginning of the development several works discussed the importance of distinguishing between open and closed systems.[6,7] Using a numerical approach Stone and Imry[8] showed that AAS oscillations are a consequence of ensemble averaging. The authors demonstrate the equivalence of ensemble and energy averaging (another type of ergodic hypothesis) and thus conclude that both AAS and AB oscillations should be observable in single rings at finite temperature. They employed a nearest-neighbor tight-binding model allowing for a magnetic field in the body of the ring. A similar model was investigated by Sawada, Tankei, and Nagaoka.[9] However, these authors restrict themselves to a magnetic flux tube in the annulus of the ring. They find that the variance of the conductance only weakly depends on the system size and the number of open channels. Most interestingly they observe a rather sensitive dependence of the AB amplitude on the system geometry: Increasing the lengths of the disordered leads enhances the amplitude while increasing the radius of the ring suppresses the oscillations. DeVincenzo and Kane[10] consider a ring coupled to four external leads. Again, the magnetic field does not penetrate the ring itself. Using analytical, diagrammatic methods these authors show that the energy correlation lengths for aperiodic fluctuations and AB oscillations differ qualitatively. Aronov and Sharvin[2] derive analytical expressions for the Fourier transform of the magnetic-field correlation function of the conductivity in a ring without external leads. Finally, Feng and Hu[11] consider the relation between AB oscillations and aperiodic fluctuations in the magnetic-field correlation function. However, their analytical procedure, a variational ansatz for the lowest eigenvalue of the diffusion propagator, turns out to be insufficient for quantitative investigations. Therefore the authors restrict themselves to numerical simulations and semiclassical explanations.

In this paper, we present (within a certain approximation) an analytical calculation of $F(B,\Delta B)$ in a ring of finite width coupled to two external disordered leads. We extend a recently developed statistical scattering model[12] based on random matrix theory and the supersymmetry method[13] to include a magnetic field. A similar model for the calculation of the average conductance in a mesoscopic ring can be found in Ref. 14. In our model, the magnetic field penetrates the body of the ring, giving rise to the decay of the correlation function as $\Delta B \rightarrow \infty$. To simplify our calculation we investigate the dependence of $F(B,\Delta B)$ on B and ΔB separately. As a function of B ($\Delta B=0$) we expect a crossover from orthogonal to unitary symmetry reflecting the breaking of time-reversal invariance. It is well known that this effect reduces the variance $F(B,0)$ to one-half of its original value at $B=0$. Nevertheless we discuss the crossover to a certain extent in order to compare this situation to our second case: We assume that time-reversal symmetry is completely broken by a sufficiently strong magnetic field B and consider $F(B,\Delta B)$ to be a function of ΔB only. In this way we derive our analytical expression for the conductance correlation function F.

Our paper is organized as follows. In Sec. II we develop the general formalism with special emphasis on the effect of the magnetic field. Similarities and differences between the two cases just discussed [the crossover $F(B,\Delta B=0)$ and the correlation function $F(B\rightarrow\infty,\Delta B)$] are pointed out on a technical level. In Sec. III we introduce our model geometry and derive an analytical expression for the conductance correlation function as a function of ΔB. Section IV comprises the presentation of the results and their discussion. Appendixes A–C contain some necessary technical considerations. In Appendix D we complete the discussion of the crossover from orthogonal to unitary symmetry.

II. STATISTICAL SCATTERING THEORY AND NONLINEAR SIGMA MODEL

In this section we describe in some detail our approach, which uses a nonlinear σ model in supersymmetric representation. It is actually an extension of previously published models[12,15,16] and we will not repeat every step of the derivation here. But we will be quite explicit about the treatment of the magnetic field. Readers who are not interested in the formal aspects of our method and are acquainted with the representation of the magnetic-field correlation function in terms of diffusion propagators may directly proceed to Sec. III.

A. Model Hamiltonian

For simplicity, we start by considering a rectangle rather than a ring. It will become clear in the course of our calculation that different geometries amount to only minor changes in the formalism. Our model system is depicted in Fig. 1. The disordered region $\{(x,y)|(x,y)\in[0,L_{\parallel}]\times[0,L_{\perp}]\}$ is divided into $K_{\parallel}K_{\perp}$ boxes of linear dimension l, the elastic mean free path for electrons. At $x=0$ and $L_{\parallel}$ we attach ideal, ordered leads. Electron states $|E,a,\kappa\rangle$ in these leads (so-called channel states) are characterized by their total energy E, the channel number a which labels different transverse modes, and an index κ distinguishing between right and left lead. The normalizations of these states are given by

$$\langle E,a,\kappa|E',a',\kappa'\rangle=\delta(E-E')\delta_{aa'}\delta_{\kappa\kappa'} . \qquad (2.1)$$

With ϵ_a the energy of transverse mode a, k the electron momentum along the lead, and m_e^* the effective mass of the electrons we have

$$E=\epsilon_a+\frac{\hbar^2k^2}{2m_e^*} . \qquad (2.2)$$

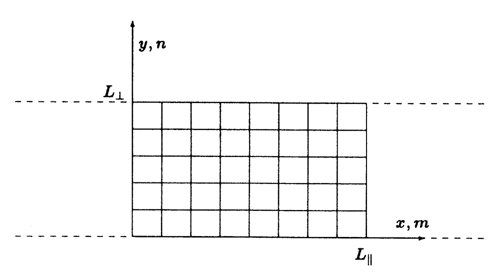

FIG. 1. The rectangular model system. The disordered region is divided into boxes of linear dimension l. Dashed lines indicate the ideal leads.

At zero temperature only electrons at the Fermi surface are relevant for transport properties, so $E=E_F$ in all that follows. The number of different transverse modes for fixed E (the channel number) is typically given by $\Lambda \approx 10^3$. This parameter will later serve to construct a perturbation series. In each box which we identify by its coordinates (m,n) we introduce N mutually orthogonal electron states $|(m,n)\mu\rangle$ $(\mu=1,\ldots,N)$. The model Hamiltonian is then constructed by defining its matrix elements. Writing $H=H_0+V$ we first define the box- (or site-) diagonal part:

$$\langle (m,n)\mu|H_0|(k,l)\nu\rangle=(H_0)^{mn}_{\mu\nu}\delta^{mk}\delta^{nl} . \tag{2.3}$$

The quantities $(H_0)^{mn}_{\mu\nu}$ are chosen to be the elements of a random matrix. For the time being, we consider alternatively the Gaussian unitary ensemble (GUE) and the Gaussian orthogonal ensemble (GOE) defined by

$$\langle (H_0)^{mn}_{\mu\nu}(H_0)^{kl}_{\mu'\nu'}\rangle=\frac{\lambda^2}{N}\delta_{\mu\nu'}\delta_{\nu\mu'}\delta^{mk}\delta^{nl} \quad (\text{GUE}) ,$$
$$\langle (H_0)^{mn}_{\mu\nu}(H_0)^{kl}_{\mu'\nu'}\rangle=\frac{\lambda^2}{N}(\delta_{\mu\mu'}\delta_{\nu\nu'}+\delta_{\mu\nu'}\delta_{\nu\mu'})\delta^{mk}\delta^{nl} \quad (\text{GOE}) , \tag{2.4}$$

respectively.

The magnetic field is introduced by means of the Bethe-Peierls substitution.[17] In this way, the matrix elements of V, defining nearest-neighbor hopping between sites, acquire flux-dependent phase factors:

$$\langle (m,n)\mu|V|(k,l)\nu\rangle\equiv V^{mn,kl}_{\mu\nu}=v([\delta^{m,k+1}e^{in(\varphi/2)}+\delta^{m+1,k}e^{-in(\varphi/2)}]\delta^{nl}+[\delta^{n,l+1}e^{-im(\varphi/2)}+\delta^{n+1,l}e^{im(\varphi/2)}]\delta^{mk})\delta_{\mu\nu} . \tag{2.5}$$

Here, $\varphi=(e/\hbar)Bl^2=2\pi\phi_{l^2}/\phi_0$, where $\phi_0=h/e$ is the elementary flux quantum and ϕ_{l^2} denotes the magnetic flux through a single box. The phase accumulated by an electron which follows a path enclosing flux ϕ is given by $2\pi\phi/\phi_0$. For the parameters λ and v appearing in Eqs. (2.4) and (2.5) we have the estimate[12] $v^2/\lambda^2\approx 1/(k_F l)$, where k_F denotes the Fermi momentum. Finally, we connect the ideal leads with the disordered region through

$$\langle E,a,\kappa|V|(m,n)\mu\rangle=W^{mn}_{a\mu}(E,\kappa)(\delta_{m1}+\delta_{mK_\parallel})\equiv W^m_{a\mu} . \tag{2.6}$$

We neglect the dependence of the quantities $W^{mn}_{a\mu}(E,\kappa)$ on E, κ, and n and assume (without loss of generality[12]) that they obey the orthogonality relation

$$\pi\sum_\mu W^m_{a\mu}W^m_{\mu b}=\delta_{ab}x^m_a \quad (m=1,K_\parallel) . \tag{2.7}$$

Here, $W^m_{a\mu}=W^m_{\mu a}$ for the GOE and $W^m_{a\mu}=W^{m*}_{\mu a}$ for the GUE. Our complete Hamiltonian now reads

$$H=\sum_\kappa\sum_a\int_{\epsilon_a}dE|E,a,\kappa\rangle E\langle E,a,\kappa+|\sum_{m,n,\mu,\nu}|(m,n)\mu\rangle(H_0)^{mn}_{\mu\nu}\langle(m,n)\nu|$$
$$+\sum_{m,n,k,l,\mu,\nu}|(m,n)\mu\rangle V^{mn,kl}_{\mu\nu}\langle(k,l)\nu|+\sum_{m,n,\mu}\sum_{a,\kappa}\int_{\epsilon_a}dE\{|E,a,\kappa\rangle W^{mn}_{a\mu}(E,\kappa)\langle(m,n)\mu|+\text{c.c.}\} . \tag{2.8}$$

The corresponding S matrix can be written as[18]

$$S^{\kappa\kappa'}_{ab}=\delta_{ab}\delta^{\kappa\kappa'}-2\pi i\sum_{m,n,k,l,\mu,\nu}W^m_{a\mu}(D^{-1})^{mn,kl}_{\mu\nu}W^k_{\nu b} , \tag{2.9}$$

with

$$D^{mn,kl}_{\mu\nu}=E\delta_{\mu\nu}\delta^{mk}\delta^{nl}-(H_0+V)^{mn,kl}_{\mu\nu}+i\pi\sum_a W^m_{\mu a}W^k_{a\nu} . \tag{2.10}$$

The conductance coefficient between lead κ and lead κ' is then given by the multichannel Landauer formula

$$g_{\kappa\kappa'}=\sum_{a,b}[|S^{\kappa\kappa'}_{ab}|^2+|S^{\kappa'\kappa}_{ba}|^2] . \tag{2.11}$$

The average magnetic-field correlation function, the quantity of central importance in this paper, is defined by

$$F(B,\Delta B)=\langle g(B)g(B+\Delta B)\rangle-\langle g(B)\rangle\langle g(B+\Delta B)\rangle . \tag{2.12}$$

In general it depends on both B and ΔB. As explained in the Introduction we will examine the dependence on these arguments separately. Assuming B to be large enough to completely break time-reversal symmetry we arrive at the case of unitary symmetry. We employ the GUE to model the disorder and investigate the ΔB dependence of Eq. (2.12). This is the main purpose of our paper. We find it instructive, however, to compare this case to the situation where time-reversal symmetry is gradually broken by an increasing magnetic field B, and $\Delta B=0$. Here, the disorder must be represented by GOE matrices. Of course, the effect on the correlation function in Eq. (2.12) is well known (the fluctuations are multiplied by one-half) so that this case does not require intensive discussion.

B. Supersymmetric functional

To represent the magnetic-field correlation function in Eq. (2.12) we define a supersymmetric generating functional ($c=1$ for GUE, and $c=2$ for GOE):

$$Z[\hat{J}]=\int d[\Psi]\exp\left\{\frac{i}{c}\Psi^\dagger\hat{L}^{1/2}(\hat{D}+\hat{J})\hat{L}^{1/2}\Psi\right\}$$
$$=\mathrm{detg}^{-1/c}(\hat{D}+\hat{J}) . \quad (2.13)$$

The definitions of the graded determinant "detg" and the graded trace "trg" (see below) can be found in Ref. 19. We have to explain various quantities appearing in Eq. (2.13) in detail and start with the case of unitary symmetry.

Let S_i and χ_i ($i=1,\ldots,4$) be vectors of commuting complex variables $(S_i)_\mu^{mn}$ and of anticommuting variables $(\chi_i)_\mu^{mn}$, respectively. The supervector Ψ is then given by

$$\Psi^T=(\varphi_1^T,\varphi_2^T,\varphi_3^T,\varphi_4^T) , \quad (2.14)$$

where $\varphi_i^T=(S_i^T,\chi_i^T)$. The integration measure is defined as

$$d[\Psi]=\prod_{m,n,\mu,i} d(S_i)_\mu^{mn}d(S_i^*)_\mu^{mn}d(\chi_i)_\mu^{mn}d(\chi_i^*)_\mu^{mn} . \quad (2.15)$$

Within the supervector Ψ, we have to distinguish between the space $\mathcal{M}_S$ of states at a site (indices $\mu,\nu,\ldots$), the space $\mathcal{M}_b$ of boxes (indices $m,n,k,l,\ldots$), and the remaining "graded" space $\mathcal{M}_g$ for which we introduce the indices $\alpha,\beta,\ldots$. Obviously, $\mathcal{M}_g$ is eight dimensional for the GUE. In graded space, the operator $\hat{D}$, the source field $\hat{J}$, and the metric tensor $\hat{L}$ are given by

$$\hat{D}=\mathrm{diag}[D_B,D_B,D_{B'},D_{B'},D_B^\dagger,D_B^\dagger,D_{B'}^\dagger,D_{B'}^\dagger] ,$$
$$\hat{J}=\mathrm{diag}[-J_1,J_1,-J_2,J_2,-J_3,J_3-J_4,J_4] , \quad (2.16)$$
$$\hat{L}=\mathrm{diag}[1,1,1,1,-1,-1,-1,-1] .$$

We have abbreviated $B'=B+\Delta B$. The structure of $D_B/D_{B'}$ in the spaces $\mathcal{M}_S$ and $\mathcal{M}_b$ has been given in Eq. (2.10). The source matrices J_i have the same symmetries as D, i.e., they are Hermitian (GUE). Looking at the first line in Eq. (2.16) we may view the space $\mathcal{M}_g$ as the direct product of "conductance space" $\mathcal{M}_g^c$ (where we distinguish between the arguments B and B', i.e., between conductances), "D space" $\mathcal{M}_g^{a/r}$ (where we distinguish between advanced and retarded inverse propagators D), and "supersymmetry space" $\mathcal{M}_g^{S\chi}$: $\mathcal{M}_g=\mathcal{M}_g^c\otimes\mathcal{M}_g^{a/r}\otimes\mathcal{M}_g^{S\chi}$.

In the case of GOE symmetry we have to combine the necessarily Hermitian coupling matrix V with the orthogonal rest of the inverse propagator D. This incompatibility of symmetries reflects the fact that the original invariance against time reversal (leading to an orthogonal representation of the Hamilton operator) is broken by the magnetic field (which has to be represented by a Hermitian matrix). Writing

$$\hat{D}+\hat{J}=\hat{M}-\hat{V}+\hat{J} ,$$
$$(\hat{M}=\hat{M}^T,\hat{V}=\hat{V}^\dagger,\hat{J}=\hat{J}^\dagger) , \quad (2.17)$$

we find that the GOE quantities can be defined in terms of the GUE quantities,

$$\Psi_{\mathrm{GOE}}=\begin{bmatrix}\Psi\\ \Psi^*\end{bmatrix},\quad \hat{M}_{\mathrm{GOE}}=\hat{M}\otimes 1_2 ,$$
$$\hat{V}_{\mathrm{GOE}}=\begin{bmatrix}\hat{V} & 0\\ 0 & \hat{V}^T\end{bmatrix},\quad \hat{J}_{\mathrm{GOE}}=\begin{bmatrix}\hat{J} & 0\\ 0 & \hat{J}^T\end{bmatrix} . \quad (2.18)$$

The additional structure in the GOE supervector relates to time-reversal symmetry and serves to define a corresponding operator.[20] For our purposes, it is enough to note that the graded space is now 16 dimensional, $\mathcal{M}_g=\mathcal{M}_g^c\otimes\mathcal{M}_g^{a/r}\otimes\mathcal{M}_g^{S\chi}\otimes\mathcal{M}_g^d$. The magnetic field explicitly breaks the symmetry in $\mathcal{M}_g^d$: While the original supervector Ψ is associated with $\hat{V}$, its "doubling image" Ψ^* is connected with the transpose of the coupling matrix, $\hat{V}^T$. In the following, we will suppress the explicit distinction between GOE and GUE quantities.

The propagator D^{-1} can be reexpressed as a derivative of the generating functional in Eq. (2.13) with respect to the source field:

$$\partial_{(J_i)_{\nu\mu}^{lk}}\langle Z[\hat{J}]\rangle|_{\hat{J}=0}=2\langle (D^{-1})_{\mu\nu}^{kl}\rangle . \quad (2.19)$$

This is an essential property since the generating functional—in contrast to the propagators D^{-1}—can readily be averaged. It follows from Eqs. (2.9), (2.11), and (2.12) that we need a product of four propagators (i.e., a four-point function) to express the correlation function $F(B,\Delta B)$. For this reason we have introduced four supervectors in Eq. (2.14) and four source fields in Eq. (2.16).

After averaging the functional we perform a Hubbard-Stratonovitch transformation and integrate over the variables in the supervector Ψ. For details, see Ref. 19. The result is

$$\langle Z[J]\rangle=\int d[Q]\exp\left\{-\frac{N}{2c\lambda^2}\sum_{kl}\mathrm{trg}[Q^{kl}Q^{kl}] -\frac{1}{c}\mathrm{trg}[\ln(\hat{E}+i\hat{W}+\hat{J} -\hat{V}-\hat{Q})]\right\} . \quad (2.20)$$

The relevant degrees of freedom are now represented by $(8c\times 8c)$-dimensional graded matrices Q^{kl}. In graded space they have the symmetry properties of the dyadic product $\Psi\Psi^\dagger$. If we denote by (a_{ik}) the matrix containing elements a_{ik} we have for the quantities appearing in Eq. (2.20)

$$Q^{kl}=(Q_{\alpha\beta}^{kl}),\quad \hat{Q}=(Q_{\alpha\beta}^{kl})\otimes 1_N ,$$
$$i\hat{W}=\left[i\pi\sum_a W_{\mu a}^m W_{a\nu}^k\right]\otimes\hat{L} , \quad (2.21)$$
$$\hat{E}=E\otimes 1_{NK_\perp K_\parallel 8c} .$$

Further progress relies on the saddle-point approximation. The general Q matrix can be decomposed[19] according to $Q=Q_G+\delta Q$ where the "Goldstone modes" explore the saddle-point manifold of the integrand in Eq. (2.20) while δQ represents so-called massive modes. The latter are integrated out in Gaussian approximation.[19] Putting $E=0$ [by ergodicity the averaged functional in Eq. (2.20) cannot depend on E] the Goldstone modes may be parametrized as[19]

$$Q_G=T_0^{-1}Q_D T_0 \ , \tag{2.22}$$

where

$$Q_D=-i\lambda\hat{L} \ . \tag{2.23}$$

The transformation matrices are given by

$$\begin{bmatrix} \sqrt{1+ab} & ia \\ -ib & \sqrt{1+ba} \end{bmatrix} . \tag{2.24}$$

The symmetry properties of the $(4c\times 4c)$-dimensional graded matrices a,b can be found in Ref. 19. The Goldstone modes obey the nonlinear constraint $Q_G^2=-\lambda^2$. Therefore $\mathrm{trg}[Q_G^2]=0$.

Having restricted the generating functional to the Goldstone modes we expand the exponent to lowest nonvanishing order in the coupling matrix V. Higher terms are suppressed by powers of $v^2/\lambda^2 \sim 1/(k_F l)$. Finally we arrive at

$$\langle Z[J]\rangle=\int d\mu(a,b)\exp\left\{+\frac{Nv^2}{2c\lambda^4}\mathrm{trg}[Q\tilde{V}Q\tilde{V}] -\frac{1}{c}\mathrm{trg}[\ln(\hat{E}+i\hat{W}+\hat{J}-\hat{Q})]\right\} . \tag{2.25}$$

Here, $d\mu(a,b)$ denotes the measure associated with the integration over the saddle-point manifold and $\tilde{V}$ is defined by $\hat{V}=v\tilde{V}$. Equation (2.25) constitutes a nonlinear σ model.

C. Continuum limit and perturbation theory

As in previous similar cases[12,15,16] our strategy will be to evaluate the integral on the right-hand side of Eq. (2.25) perturbatively. This treatment is known to give results equivalent to those derived by disorder perturbation theory, at least in the cases considered in Refs. 12 and 16. We express the exponent in Eq. (2.25) in terms of the independent (unconstrained) variables contained in the matrices a,b and expand the integrand in a Taylor series keeping only terms of second order in the exponent. The resulting integrals can be calculated by means of a generalized Wick theorem.[12] The result is a perturbation series proceeding essentially in inverse powers of Λ, the channel number. We write for the exponent in Eq. (2.25)

$$\begin{aligned}\mathcal{L}&=\frac{Nv^2}{2c\lambda^4}\mathrm{trg}[Q_G\tilde{V}Q_G\tilde{V}]-\frac{1}{c}\mathrm{trg}[\ln(i\hat{W}+\hat{J}-\hat{Q}_G)]\\ &=\frac{Nv^2}{2c\lambda^4}\mathrm{trg}[Q_G\tilde{V}Q_G\tilde{V}]-\frac{1}{c}\mathrm{trg}[\ln(1-i\hat{Q}_G^{-1}\hat{W}^{-1})]\\ &\quad -\frac{1}{c}\mathrm{trg}\{\ln[1-(1-i\hat{Q}_G^{-1}\hat{W})^{-1}\hat{Q}_G^{-1}\hat{J}]\}\\ &=\mathcal{L}_{\mathrm{kin}}+\mathcal{L}_{\mathrm{pot}}+\mathcal{L}_J \ .\end{aligned} \tag{2.26}$$

The terms $\mathcal{L}_{\mathrm{pot}}$ and $\mathcal{L}_J$ are identical to those appearing in previous models[15,16] and we will not discuss them in detail. The kinetic term $\mathcal{L}_{\mathrm{kin}}$, however, contains the magnetic field. Therefore we will explicitly display its expansion. At the same time we introduce a continuum limit which simplifies some formulas and their interpretation considerably. Inserting the explicit form of $\hat{V}$ [see Eq. (2.5)] into $\mathcal{L}_{\mathrm{kin}}$ we get (setting $Q\equiv Q_G$)

$$\mathcal{L}_{\mathrm{kin}}=\frac{Nv^2}{c\lambda^4}\sum_{k,l}\mathrm{trg}[P_lQ^{kl}P_l^\dagger Q^{k+1l}+P_kQ^{kl}P_k^\dagger Q^{kl-1}] \ , \tag{2.27}$$

where

$$P_l=\exp\left[il\sigma_3^c\frac{\Delta\varphi}{2}\right],\quad \Delta\varphi=\frac{e}{\hbar}\frac{\Delta B}{2}l^2 \quad \text{(GUE correlation function)} ,$$

$$P_l=\exp\left[il\sigma_3^d\frac{\varphi}{2}\right],\quad \varphi=\frac{e}{\hbar}Bl^2 \quad \text{(GOE/GUE crossover)} . \tag{2.28}$$

The matrices σ_3^c and σ_3^d are Pauli matrices having their nontrivial structure in $\mathcal{M}_g^c$ and $\mathcal{M}_g^d$, respectively. It is evident from the first line in Eq. (2.28) that the generating functional does not depend on the absolute value of the magnetic field in the case of the GUE correlation function: Having broken time-reversal symmetry completely by the use of a GUE, a symmetry-breaking field is without effect.

We take the continuum limit by considering the elastic mean free path l to be very small against all other lengths involved in our model. In this way, the box indices acquire a quasicontinuous character and we are led to the replacements

$$\begin{aligned}&\sum_{k,l}\to\frac{1}{l^2}\int dx\,dy \ ,\\ &Q^{kl}\to Q(x,y) \ ,\\ &Q^{k+1l}\to Q(x+l,y)\\ &\quad =Q(x,y)+\partial_x Q(x,y)l+\tfrac{1}{2}\partial_x^2Q(x,y)l^2+\cdots \ .\end{aligned} \tag{2.29}$$

Inserting the continuum version of the Q matrices into the graded trace in Eq. (2.27) we get

$$\begin{aligned}\mathrm{trg}[P_lQ^{kl}P_l^\dagger Q^{k+1l}]\to&\ \mathrm{trg}[P_lQ(x,y)P_l^\dagger Q(x,y)]\\ &+\mathrm{trg}[P_lQ(x,y)P_l^\dagger\partial_xQ(x,y)]l\\ &+\tfrac{1}{2}\mathrm{trg}[P_lQ(x,y)P_l^\dagger\partial_x^2Q(x,y)]l^2\\ &+\cdots \ .\end{aligned} \tag{2.30}$$

Expanding P_l to second order in l we find the following terms contribute to $\mathcal{L}_{\mathrm{kin}}$:

$$\begin{aligned}&\mathrm{trg}[P_lQP_l^\dagger Q]\to\mathrm{trg}[(-i\sigma_3\alpha_x)Q(i\sigma_3\alpha_x)Q]l^2 \ ,\\ &\mathrm{trg}[P_lQP_l^\dagger\partial_xQ]l\to\mathrm{trg}[(-i\sigma_3\alpha_x)Q\partial_xQ]l^2 +\mathrm{trg}[Q(i\sigma_3\alpha_x)\partial_xQ]l^2 \ ,\\ &\tfrac{1}{2}\mathrm{trg}[P_lQP_l^\dagger\partial_x^2Q]l^2\to\tfrac{1}{2}\mathrm{trg}[Q\partial_x^2Q]l^2 \ .\end{aligned} \tag{2.31}$$

We have defined

$$\alpha=\frac{e}{\hbar}\frac{\Delta\mathbf{A}}{2}\quad(\mathrm{GUE})\,,\qquad \alpha=\frac{e}{\hbar}\mathbf{A}\quad(\mathrm{GOE})\,, \tag{2.32}$$

where $\mathbf{A}=(B/2)(-y,x,0)$ is the vector potential in symmetric gauge for a static, homogeneous magnetic field in the z direction. Introducing a covariant derivative by

$$d_xQ\equiv\partial_xQ+i\alpha_x[\sigma_3,Q],\quad D_xQ\equiv d_x^2Q \tag{2.33}$$

we see that the continuum form of $\mathcal{L}_{\mathrm{kin}}$ reads

$$\mathcal{L}_{\mathrm{kin}}=\frac{Nv^2}{2c\lambda^4}\int dx\,dy\,\mathrm{trg}[QDQ],\quad D=D_x+D_y\,. \tag{2.34}$$

In Appendix A we prove the gauge invariance of our model. One can show that the covariant derivative d obeys the same rules for differentiation as the ordinary derivative ∂. Therefore we get the Taylor expansion of Eq. (2.34) by substituting d for ∂ in the corresponding field-free series. Up to sixth order in the matrices a,b we have

$$\mathcal{L}_{\mathrm{kin}}=\frac{\xi}{c}\int dx\,dy(X^{(2)}+X^{(4)}+X^{(6)})\,, \tag{2.35}$$

with

$$\begin{aligned}X^{(2)}&=-4\Delta^2\mathrm{trg}[\mathbf{d}a\cdot\mathbf{d}b]\,,\\ X^{(4)}&=+4\Delta^2\mathrm{trg}[a(\mathbf{d}b)a(\mathbf{d}b)+(\mathbf{d}a)b(\mathbf{d}a)b]\,,\\ X^{(6)}&=-2\Delta^2\mathrm{trg}[a\mathbf{d}(ba)b\mathbf{d}(ab)]\,.\end{aligned} \tag{2.36}$$

We have introduced $\xi=4Nv^2/\lambda^2\sim4N/(k_Fl)\sim\Lambda$ and the notation $\mathbf{d}=[d_x,d_y]$.

In the following we explicitly construct the term of second order in a and b which will be kept in the exponent. It defines the elementary propagators of our perturbation series (the diffusion propagators) and some insight can be gained by its inspection. In addition to $\mathcal{L}_{\mathrm{kin}}$ we have to consider contributions coming from the potential term $\mathcal{L}_{\mathrm{pot}}$. In discrete notation it reads

$$\mathcal{L}_{\mathrm{pot}}=-\frac{1}{c}\sum_{k,l}\sum_{a}\mathrm{trg}[\ln(1+T_a^k a^{kl}b^{kl})] \tag{2.37}$$

leading to a second-order term

$$\begin{aligned}\mathcal{L}_{\mathrm{pot}}^{(2)}&=-\frac{1}{c}\sum_{k,l}\left[\sum_a T_a^k\right]\mathrm{trg}[a^{kl}b^{kl}]\\ &=-\frac{\xi}{c}\sum_{k,l}\gamma^k\mathrm{trg}[a^{kl}b^{kl}]\\ &\to-\frac{\xi}{c}\int dx\,dy\frac{\gamma}{l}[\delta(x)+\delta(x-L_\parallel)]\mathrm{trg}[ab]\,.\end{aligned} \tag{2.38}$$

In the last line we have taken the continuum limit. The symbol T_a^k denotes the transmission coefficient

$$T_a^k=\frac{4\lambda x_a}{(\lambda+x_a)^2}(\delta_{k1}+\delta_{kK_\parallel})\,, \tag{2.39}$$

with x_a defined in Eq. (2.7). The coupling γ^k is defined by $\gamma^k=\xi^{-1}\sum_a(T_a^k)$. A nonvanishing γ indicates that the disordered probe is coupled to the external leads. The physics governed by this coupling coefficient has been discussed in detail in Ref. 12. Integrating by parts in $\mathcal{L}_{\mathrm{kin}}^{(2)}=(\xi/c)\int dx\,dy\,X^{(2)}$ and adding $\mathcal{L}_{\mathrm{pot}}^{(2)}$ we get for the quadratic action

$$\mathcal{L}^{(2)}=\mathcal{L}_{\mathrm{kin}}^{(2)}+\mathcal{L}_{\mathrm{pot}}^{(2)}=\frac{\xi}{c}\int dx\,dy\,\mathrm{trg}\left\{a\left[D+\delta(x)\left[d_x-\frac{\gamma}{l}\right]-\delta(x-L_\parallel)\left[d_x+\frac{\gamma}{l}\right]+[\delta(y)-\delta(y-L_\perp)]d_y\right]b\right\}\,. \tag{2.40}$$

The magnetic field defines a decomposition of the graded space $\mathcal{M}_g$ into two subspaces which we denote by $\mathcal{M}_g^{(1)}$ and $\mathcal{M}_g^{(2)}$. From our previous discussion it is clear that

$$\begin{aligned}&\mathcal{M}_g^{(1)}=\mathcal{M}_g^{(2)}=\mathcal{M}_g^{a/r}\otimes\mathcal{M}_g^{S\chi}\quad(\text{GUE correlation function})\,,\\ &\mathcal{M}_g^{(1)}=\mathcal{M}_g^{(2)}=\mathcal{M}_g^{c}\otimes\mathcal{M}_g^{a/r}\otimes\mathcal{M}_g^{S\chi}\quad(\text{GOE/GUE crossover})\,.\end{aligned} \tag{2.41}$$

In the case of the GUE the magnetic-field *difference* breaks the symmetry in $\mathcal{M}_g^c$. In the orthogonal case, it is the magnetic field itself which plays the analogous role in $\mathcal{M}_g^d$. We now decompose a and b accordingly:

$$a=\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix},\quad b=\begin{bmatrix}b_{11}&b_{12}\\b_{21}&b_{22}\end{bmatrix}\,. \tag{2.42}$$

The indices refer to the subspaces $\mathcal{M}_g^{(1)}$ and $\mathcal{M}_g^{(2)}$ and have therefore different meanings for the two cases given in Eq. (2.41). For the quadratic action we get

$$\begin{aligned}\mathcal{L}^{(2)}=\frac{\xi}{c}\int dx\,dy\,\mathrm{trg}[&a_{11}\hat{\Pi}_D^{-1}b_{11}+a_{22}\hat{\Pi}_D^{-1}b_{22}\\ &+a_{12}(\hat{\Pi}_C^{-1})^*b_{21}+a_{21}\hat{\Pi}_C^{-1}b_{12}]\end{aligned} \tag{2.43}$$

with the operators

$$\begin{aligned}\hat{\Pi}_D^{-1}&=\Delta+\delta(x)\left[\partial_x-\frac{\gamma}{l}\right]-\delta(x-L_\parallel)\left[\partial_x+\frac{\gamma}{l}\right]\\ &\quad+[\delta(y)-\delta(y-L_\perp)]\partial_y\,,\\ \hat{\Pi}_C^{-1}&=(\partial+2i\alpha)^2+\delta(x)\left[\partial_x+2i\alpha_x-\frac{\gamma}{l}\right]\\ &\quad-\delta(x-L_\parallel)\left[\partial_x+2i\alpha_x+\frac{\gamma}{l}\right]\\ &\quad+[\delta(y)-\delta(y-L_\perp)][\partial_y+2i\alpha_y]\,.\end{aligned} \tag{2.44}$$

Thus the diffusion propagators $\hat{\Pi}_D$ and $\hat{\Pi}_C$ are given by

$$\Delta\hat{\Pi}_D(\mathbf{r},\mathbf{r}')=-\delta(\mathbf{r}-\mathbf{r}')\ ,$$
$$(\boldsymbol{\partial}+2i\boldsymbol{\alpha})^2\hat{\Pi}_C(\mathbf{r},\mathbf{r}')=-\delta(\mathbf{r}-\mathbf{r}')\ , \tag{2.45}$$

together with the boundary conditions [determined by the δ-function terms in Eq. (2.44)]

$$\left[\partial_x-\frac{\gamma}{l}\right]\hat{\Pi}_D(\mathbf{r},\mathbf{r}')|_{x=0}=\left[\partial_x+\frac{\gamma}{l}\right]\hat{\Pi}_D(\mathbf{r},\mathbf{r}')|_{x=L_\parallel}=0\ ,$$
$$\left[\partial_x+2i\alpha_x-\frac{\gamma}{l}\right]\hat{\Pi}_C(\mathbf{r},\mathbf{r}')|_{x=0}=\left[\partial_x+2i\alpha_x+\frac{\gamma}{l}\right]\times\hat{\Pi}_C(\mathbf{r},\mathbf{r}')|_{x=L_\parallel}=0\ , \tag{2.46}$$
$$\partial_y\hat{\Pi}_D(\mathbf{r},\mathbf{r}')|_{y=0}=\partial_y\hat{\Pi}_D(\mathbf{r},\mathbf{r}')|_{y=L_\perp}=0\ ,$$
$$(\partial_y+2i\alpha_y)\hat{\Pi}_C(\mathbf{r},\mathbf{r}')|_{y=0}=(\partial_y+2i\alpha_y)\hat{\Pi}_C(\mathbf{r},\mathbf{r}')|_{y=L_\perp}=0\ .$$

In the following we will use the terms diffusion and cooperon for $\hat{\Pi}_D$ and $\hat{\Pi}_C$, respectively.[21]

Anticipating that cooperons are damped as the magnetic field increases (in the sense that $\int|\Pi_C|^2dV$ gradually vanishes) we can see important mechanisms already at this stage. It follows from Eq. (2.43) that connections between indices "1" and "2" are due to cooperons only. If the cooperon propagators vanish the spaces $\mathcal{M}_g^{(1)}$ and $\mathcal{M}_g^{(2)}$ will be decoupled. This has the following consequences for the two physical situations we have in mind.

(1) For the GUE correlation function it means decoupling of the conductances $g(B)$ and $g(B+\Delta B)$ in Eq. (2.12): $\mathcal{M}_g^{(1)}$ and $\mathcal{M}_g^{(2)}$ denote their respective "conductance spaces." Therefore we expect $\langle g(B)g(B+\Delta B)\rangle\rightarrow\langle g(B)\rangle\langle g(B+\Delta B)\rangle$ and consequently $F(B,\Delta B)\rightarrow 0$ as ΔB increases.

(2) In the case of the GOE/GUE crossover $\mathcal{M}_g^{(1)}$ and $\mathcal{M}_g^{(2)}$ are associated with the "doubling image" structure we introduced to account for time-reversal symmetry. It is precisely this symmetry which is broken by the magnetic field and decoupling of $\mathcal{M}_g^{(1)}$ and $\mathcal{M}_g^{(2)}$ reduces the GOE to a GUE representation.[22]

D. Asymptotic terms

The structure of the perturbation series originating from the functional in Eq. (2.25) is well known from the calculation of $\langle \mathrm{var}(g)\rangle$ for quasi-one-dimensional wires.[12] There, the first nonvanishing order (Λ^0) has been worked out completely, i.e., including also terms representing the influence of the coupling to the leads. Naturally, these terms are suppressed at the length of the system increases. Put differently, all terms of the series scale with $(l/L_\parallel)^n$ and as $L_\parallel\rightarrow\infty$ only those contributions with $n=0$, the *asymptotic terms,* survive. They no longer depend on the coupling parameter γ. The continuum limit defined in the preceding subsection $(l/L_\parallel\rightarrow 0)$ has precisely the effect of selecting the asymptotic terms. We will restrict ourselves to their discussion in all that follows.

The series contains two asymptotic contributions[12] which we denote by A_5 and A_6. They arise from the combination of certain source terms with higher-order terms of the kinetic action $\mathcal{L}_{\mathrm{kin}}$. We have

$$\exp(\mathcal{L}_{\mathrm{kin}}^{(4)}+\mathcal{L}_{\mathrm{kin}}^{(6)})=1+\mathcal{L}_{\mathrm{kin}}^{(4)}+\mathcal{L}_{\mathrm{kin}}^{(6)}+\tfrac{1}{2}(\mathcal{L}_{\mathrm{kin}}^{(4)})^2+\cdots\ . \tag{2.47}$$

Relevant source term contributions have the form

$$\frac{1}{l^4}\int dS\,d\hat{S}\ \mathrm{trg}[a_{11}(\mathbf{o})Ib_{11}(\mathbf{L})I]\mathrm{trg}[a_{22}(\hat{\mathbf{o}})Ib_{22}(\hat{\mathbf{L}})I]\ . \tag{2.48}$$

Here, the indices refer to the conductance space $\mathcal{M}_g^c$ in the GUE as well as in the GOE case. We had to take four derivatives with respect to the source field $\hat{J}$ to represent the four-point function $F(B,\Delta B)$. The matrices I in Eq. (2.48) reflect the source field structure in graded space after differentiation. For the GUE [see Eq. (2.16)] they are given by

$$I\varphi=\begin{bmatrix}-1 & \\ & 1\end{bmatrix}\begin{bmatrix}S\\ \chi\end{bmatrix}\ . \tag{2.49}$$

In the case of GOE symmetry we have to distinguish I_1 (for J) and I_2 (for J^T), see Eq. (2.18):

$$I_1\varphi=\begin{bmatrix}-1 & & & \\ & 0 & & \\ & & 1 & \\ & & & 0\end{bmatrix}\begin{bmatrix}S\\ S^*\\ \chi\\ \chi^*\end{bmatrix}\ ,\qquad I_2\varphi=\begin{bmatrix}0 & & & \\ & -1 & & \\ & & 0 & \\ & & & 1\end{bmatrix}\begin{bmatrix}S\\ S^*\\ \chi\\ \chi^*\end{bmatrix}\ . \tag{2.50}$$

By $\int dS\,d\hat{S}$ in Eq. (2.48) we denote two independent surface integrations over the vectors

$$\mathbf{o}=(0,y_0),\quad \hat{\mathbf{o}}=(0,\hat{y}_0)\ ,$$
$$\mathbf{L}=(L_\parallel,y_L),\quad \hat{\mathbf{L}}=(L_\parallel,\hat{y}_L)\ , \tag{2.51}$$

i.e., $dS=dy_0dy_L$ and $d\hat{S}=d\hat{y}_0d\hat{y}_L$. The asymptotic terms A_5 and A_6 are now given by combining $\mathcal{L}_{\mathrm{kin}}^{(6)}$ and $(\mathcal{L}_{\mathrm{kin}}^{(4)})^2/2$ with source terms of the form of Eq. (2.48). With the definition

$$\langle\langle f(a,b)\rangle\rangle=\int d\mu(t)f(a,b)e^{\mathcal{L}^{(2)}(a,b)} \tag{2.52}$$

we can write A_5 and A_6 as

$$A_5=\left\langle\!\!\left\langle -\frac{\xi^5}{32c^5}\frac{\gamma^4}{l^4}\int dS\, d\hat{S}\, dx\, dy \operatorname{trg}[a_{11}(\mathbf{o})Ib_{11}(\mathbf{L})I]\{\operatorname{trg}[a_{22}(\hat{\mathbf{o}})\hat{I}b_{22}(\hat{\mathbf{L}})\hat{I}]+(\hat{\mathbf{o}}\leftrightarrow\hat{\mathbf{L}})\}\operatorname{trg}[a\mathbf{d}(ab)b\mathbf{d}(ab)]_{xy}\right\rangle\!\!\right\rangle ,$$
$$A_6=\left\langle\!\!\left\langle \frac{\xi^6}{64c^6}\frac{\gamma^4}{l^4}\int dS\, d\hat{S}\, dx\, dy\, dx'dy' \operatorname{trg}[a_{11}(\mathbf{o})Ib_{11}(\mathbf{L})I]\{\operatorname{trg}[a_{22}(\hat{\mathbf{o}})\hat{I}b_{22}(\hat{\mathbf{L}})\hat{I}]+(\hat{\mathbf{o}}\leftrightarrow\hat{\mathbf{L}})\} \times \operatorname{trg}[a(\mathbf{d}b)a(\mathbf{d}b)+(\mathbf{d}a)b(\mathbf{d}a)b]_{xy}\operatorname{trg}[a(\mathbf{d}b)a(\mathbf{d}b)+(\mathbf{d}a)b(\mathbf{d}a)b]_{x'y'}\right\rangle\!\!\right\rangle . \tag{2.53}$$

In the case of GOE symmetry, I and $\hat{I}$ may be independently identified with I_1 and I_2, respectively, while for the GUE we have $I\equiv\hat{I}$. The asymptotic correlation function is now given by the sum of A_5 and A_6, $F(B,\Delta B)=A_5+A_6$.

III. DIFFUSION PROPAGATORS AND MAGNETIC-FIELD CORRELATION FUNCTION

In this section we focus attention on the case of the GUE correlation function. Results and derivations concerning the crossover behavior of $F(B,\Delta B)$ are sketched in Appendix D for completeness.

After expressing the correlation function [which we denote by $F(\Delta B)$ henceforth] in terms of diffusion propagators we introduce some modifications and approximations related to the ring geometry. Finally we solve the differential equations for diffuson and cooperon[21] in a certain limit and derive an analytical expression for $F(\Delta B)$.

A. Contraction rules

To evaluate the functional averages in Eq. (2.53) we employ the following (Wick-type) contraction rules, valid for the GUE:

$$\langle\langle \operatorname{trg}[Aa_{ii}(\mathbf{r})Bb_{ii}(\mathbf{r}')]\rangle\rangle=\hat{\Pi}_D(\mathbf{r},\mathbf{r}')\langle\langle\operatorname{trg}[A]\operatorname{trg}[B]\rangle\rangle/\xi ,$$
$$\langle\langle \operatorname{trg}[Aa_{ii}(\mathbf{r})]\operatorname{trg}[Bb_{ii}(\mathbf{r}')]\rangle\rangle=\hat{\Pi}_D(\mathbf{r},\mathbf{r}')\langle\langle\operatorname{trg}[AB]\rangle\rangle/\xi ,$$
$$\langle\langle \operatorname{trg}[Aa_{ij}(\mathbf{r})Bb_{ji}(\mathbf{r}')]\rangle\rangle=\hat{\Pi}_C(\mathbf{r},\mathbf{r}')\langle\langle\operatorname{trg}[A]\operatorname{trg}[B]\rangle\rangle/\xi ,$$
$$\langle\langle \operatorname{trg}[Aa_{ij}(\mathbf{r})]\operatorname{trg}[Bb_{ji}(\mathbf{r}')]\rangle\rangle=\hat{\Pi}_C(\mathbf{r},\mathbf{r}')\langle\langle\operatorname{trg}[AB]\rangle\rangle/\xi . \tag{3.1}$$

It is understood that $i\neq j$. Applying Eq. (3.1) repeatedly we can write A_5 and A_6 as integrals over products of diffusion propagators. With $\mathbf{r}=(r_1,r_2)=(x,y)$, $\mathbf{r}'=(r_1',r_2')=(x',y')$, $\hat{\gamma}=\gamma/l$, and the definitions

$$\nabla_{r_i}=\partial_{r_i}+i\alpha_{r_i} ,\qquad \nabla^*_{r_i}=\partial_{r_i}-i\alpha_{r_i} , \tag{3.2}$$

we get

$$A_5=-\hat{\gamma}^4\int dS\, d\hat{S}\, d\mathbf{r}\{\tfrac{1}{2}F_s^{\mathrm{oL}}(\mathbf{r},\mathbf{r})F_s^{\hat{\mathrm{o}}\hat{\mathrm{L}}}(\mathbf{r},\mathbf{r})[\nabla_{r_i}\nabla^*_{r_i'}\hat{\Pi}_C(\mathbf{r},\mathbf{r}')+\nabla^*_{r_i}\nabla_{r_i'}\hat{\Pi}^*_C(\mathbf{r},\mathbf{r}')]|_{r_i=r_i'} +\tfrac{1}{4}\partial_{r_i}[F_s^{\mathrm{oL}}(\mathbf{r},\mathbf{r})F_s^{\hat{\mathrm{o}}\hat{\mathrm{L}}}(\mathbf{r},\mathbf{r})][\partial_{r_i}\hat{\Pi}_C(\mathbf{r},\mathbf{r}')+\partial_{r_i}\hat{\Pi}^*_C(\mathbf{r},\mathbf{r}')]|_{r_i=r_i'}+\tfrac{3}{4}\partial_{r_i}F_s^{\mathrm{oL}}(\mathbf{r},\mathbf{r})\partial_{r_i}F_s^{\hat{\mathrm{o}}\hat{\mathrm{L}}}(\mathbf{r},\mathbf{r})\hat{\Pi}_C(\mathbf{r},\mathbf{r})\} ,$$
$$A_6=\hat{\gamma}^4\int dS\, d\hat{S}\, d\mathbf{r}\, d\mathbf{r}'\{F_s^{\mathrm{oL}}(\mathbf{r},\mathbf{r}')F_s^{\hat{\mathrm{o}}\hat{\mathrm{L}}}(\mathbf{r},\mathbf{r}')\nabla_{r_i}\nabla^*_{r_j'}\hat{\Pi}_C(\mathbf{r},\mathbf{r}')\nabla^*_{r_i}\nabla_{r_j'}\hat{\Pi}^*_C(\mathbf{r},\mathbf{r}')+\partial_{r_j'}F_s^{\mathrm{oL}}(\mathbf{r},\mathbf{r}')\partial_{r_j'}F_s^{\hat{\mathrm{o}}\hat{\mathrm{L}}}(\mathbf{r},\mathbf{r}')\nabla^*_{r_i}\hat{\Pi}^*_C(\mathbf{r},\mathbf{r}')\nabla_{r_i}\hat{\Pi}_C(\mathbf{r},\mathbf{r}')$$
$$+\partial_{r_i}F_s^{\mathrm{oL}}(\mathbf{r},\mathbf{r}')\partial_{r_i}F_s^{\hat{\mathrm{o}}\hat{\mathrm{L}}}(\mathbf{r},\mathbf{r}')\nabla^*_{r_j'}\hat{\Pi}_C(\mathbf{r},\mathbf{r}')\nabla_{r_j'}\hat{\Pi}^*_C(\mathbf{r},\mathbf{r}')$$
$$+\partial_{r_i}\partial_{r_j'}F_s^{\mathrm{oL}}(\mathbf{r},\mathbf{r}')\partial_{r_i}\partial_{r_j'}F_s^{\hat{\mathrm{o}}\hat{\mathrm{L}}}(\mathbf{r},\mathbf{r}')\hat{\Pi}_C(\mathbf{r},\mathbf{r}')\hat{\Pi}^*_C(\mathbf{r},\mathbf{r}')$$
$$+\tfrac{1}{4}[\partial_{r_i}F_s^{\mathrm{oL}}(\mathbf{r},\mathbf{r})\partial_{r_j'}F_s^{\hat{\mathrm{o}}\hat{\mathrm{L}}}(\mathbf{r}',\mathbf{r}')+(\mathbf{o}\leftrightarrow\hat{\mathbf{o}},\mathbf{L}\leftrightarrow\hat{\mathbf{L}})]\partial_{r_i}\partial_{r_j'}[\hat{\Pi}_C(\mathbf{r},\mathbf{r}')\hat{\Pi}^*_C(\mathbf{r},\mathbf{r}')]\} . \tag{3.3}$$

Here, a summation over repeated indices is implied and by $F_s^{\mathrm{oL}}(\mathbf{r},\mathbf{r}')$ we abbreviate the following symmetrized product of diffusons:

$$F_s^{\mathrm{oL}}(\mathbf{r},\mathbf{r}')=\hat{\Pi}_D(\mathbf{o},\mathbf{r})\hat{\Pi}_D(\mathbf{r}',\mathbf{L})+\hat{\Pi}_D(\mathbf{o},\mathbf{r}')\hat{\Pi}_D(\mathbf{r},\mathbf{L}) . \tag{3.4}$$

Using some technical manipulations detailed in Appendix B, A_5 and A_6 can be reduced to the rather simple form

$$A_5+A_6=4\hat{\gamma}^4\int dS\, d\hat{S}\, d\mathbf{r}\, d\mathbf{r}'\hat{\Pi}_C(\mathbf{r},\mathbf{r}')\hat{\Pi}^*_C(\mathbf{r},\mathbf{r}') \times\partial_{r_i}\partial_{r_j'}F_s^{\mathrm{oL}}(\mathbf{r},\mathbf{r}')\partial_{r_i}\partial_{r_j'}F_s^{\hat{\mathrm{o}}\hat{\mathrm{L}}}(\mathbf{r},\mathbf{r}')$$
$$+\tfrac{1}{2}\hat{\gamma}^4\int dS\, d\hat{S}\, d\mathbf{r}\, d\mathbf{r}'\,\hat{\Pi}_C(\mathbf{r},\mathbf{r}')\hat{\Pi}^*_C(\mathbf{r},\mathbf{r}')\partial_{r_i}\partial_{r_i} \times F_s^{\mathrm{oL}}(\mathbf{r},\mathbf{r})\partial_{r_j'}\partial_{r_j'}F_s^{\hat{\mathrm{o}}\hat{\mathrm{L}}}(\mathbf{r}',\mathbf{r}') . \tag{3.5}$$

This result is manifestly gauge invariant. A change of gauge in Eq. (2.45) leads to a corresponding phase factor multiplying $\hat{\Pi}_C(\mathbf{r},\mathbf{r}')$. This phase factor, however, cancels out in Eq. (3.5).

B. Ring geometry

Up to now we had a probe of rectangular shape in mind. But this assumption—although convenient—has not been essential for our derivations. The main geometry dependence of the formalism resides in the coupling matrix $\hat{V}$. This matrix determines which sites are connected via nearest-neighbor electron hopping. The second important aspect of the geometry is the position of the ideal leads, reflected in the site dependence of the potential term $\mathcal{L}_{\text{pot}}$. Given an arbitrary geometry, all we have to do is to specify the decomposition into sites and the interfaces to ideal leads. The formalism set up in Sec. II will then be well defined. On the level of the diffusion propagators this amounts to specifying (i) the region where the differential equations (2.45) have to be solved and (ii) the boundary conditions.

To deal with the ring geometry in Fig. 2(a) we introduce the following approximations. First, we neglect the curvature of the ring. We disregard the fact that the number of sites at the outer circumference should be larger than the one at the inner circumference. Consequently the structure of the coupling matrix $\hat{V}$ is very similar to the case of the rectangle. Second, we replace the disordered external leads by radial electron sources (or sinks). Essential properties of the correlation function should not depend on the details of the coupling. As a result of these simplifications we have to consider the model geometry shown in Fig. 2(b) which is easily described in terms of polar coordinates r and ϑ. We denote the inner and the outer radius by $r_<$ and $r_>$, respectively, and define the mean radius $\bar{r}=(r_< + r_>)/2$. The electron sources are located at $\vartheta=0$ and π. We directly turn to the continuum formulation for the diffusion propagators. Choosing the gauge

$$A_r=0, \quad A_\vartheta=\frac{\Delta B}{2}r, \quad A_z=0 \tag{3.6}$$

we write the differential operator for the cooperon as

$$\begin{aligned} D_\alpha &\equiv (\boldsymbol{\partial}+2i\boldsymbol{\alpha})^2 \\ &= (\boldsymbol{\partial}-i\frac{e}{\hbar}\Delta\,\mathbf{A})^2 \\ &= \left[\partial_r^2+\frac{1}{r}\partial_r+\frac{1}{r^2}\partial_\vartheta^2+i\frac{1}{l_{\Delta B}^2}\partial_\vartheta-\frac{1}{l_{\Delta B}^4}\frac{r^2}{4}\right], \end{aligned} \tag{3.7}$$

where we have introduced the magnetic length $l_{\Delta B}^2=\hbar/(e\Delta B)$. The electron sources are modeled in analogy to Eq. (2.44) by δ-function potentials,

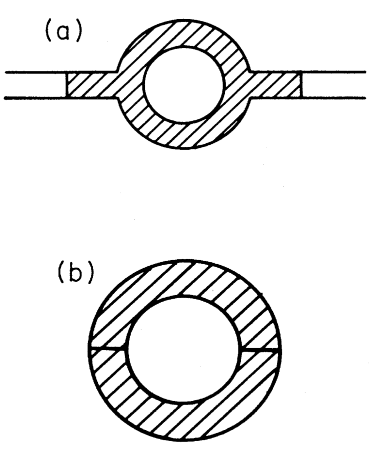

FIG. 2. (a) Ring geometry with two attached external leads. The disordered region is represented by shaded areas. (b) The ring geometry considered in this paper. The external leads are replaced by radial electron sources.

$$\left[\boldsymbol{\partial}+i\frac{e}{\hbar}\Delta\,\mathbf{A}\right]^2\hat{\Pi}_C(\mathbf{r},\mathbf{r}')-\frac{\mu}{l}[\delta(r\vartheta-r\pi)+\delta(r\vartheta)]\times\hat{\Pi}_C(\mathbf{r},\mathbf{r}')=-\delta(\mathbf{r}-\mathbf{r}') \tag{3.8}$$

leading to the boundary conditions ($x=0,\pi$)

$$[\partial_\vartheta\hat{\Pi}_C(\mathbf{r},\mathbf{r}')|_{\vartheta=x^+}-\partial_\vartheta\hat{\Pi}_C(\mathbf{r},\mathbf{r}')|_{\vartheta=x^-}] =\frac{\mu}{l}r\hat{\Pi}_C(\mathbf{r},\mathbf{r}')|_{\vartheta=x}=\tilde{c}\hat{\Pi}_C(\mathbf{r},\mathbf{r}')|_{\vartheta=x} . \tag{3.9}$$

In radial direction we have, due to the isolating walls at $r=r_<$ and $r=r_>$,

$$\partial_r\hat{\Pi}_C(\mathbf{r},\mathbf{r}')|_{r=r_<}=\partial_r\hat{\Pi}_C(\mathbf{r},\mathbf{r}')|_{r=r_>}=0 . \tag{3.10}$$

Therefore we have to solve the differential equation

$$\left[r\partial_r^2+\partial_r+\frac{1}{r}\partial_\vartheta^2+i\frac{r}{l_{\Delta B}^2}\partial_\vartheta-\frac{r^3}{4l_{\Delta B}^4}\right]\hat{\Pi}_C(\mathbf{r},\mathbf{r}') =-\delta(r-r')\delta(\vartheta-\vartheta') \tag{3.11}$$

subject to the boundary conditions Eqs. (3.9) and (3.10).

Experimentally relevant rings for the observation of the AB effect are characterized by ratios $\sigma \equiv(r_>-r_<)/\bar{r}\equiv L_\perp/\bar{r}\approx 0.1,\dots,0.2$.[1,2] This justifies setting $\tilde{c}=\mu r/l\equiv\hat{\mu}r\approx\hat{\mu}\bar{r}=$const. We show in Appendix C that the parameter $\tilde{c}$ may be interpreted as the ratio $\bar{r}/L$ of the mean ring radius and the length of the disordered leads in Fig. 2(a). This establishes a close connection between our model system in Fig. 2(b) and the realistic geometry in Fig. 2(a). The two limiting cases, $\tilde{c}=0$ and $\tilde{c}\to\infty$, are associated with an isolated ring and two disconnected half rings, respectively. In the first case, the angular derivative of $\hat{\Pi}_C(\mathbf{r},\mathbf{r}')$ becomes continuous everywhere while in the second case we must have $\hat{\Pi}_C(\mathbf{r},\mathbf{r}')|_{\vartheta=0,\pi}=0$ in order to fulfill Eq. (3.9).

Neglecting the ring's curvature as described above is equivalent to approximating the Laplace operator by its "Cartesian" form

$$\Delta\equiv\partial_r^2+\frac{1}{r}\partial_r+\frac{1}{r^2}\partial_\vartheta^2\approx\partial_r^2+\frac{1}{r^2}\partial_\vartheta^2 . \tag{3.12}$$

The Cartesian derivatives ∂_{r_i} in Eq. (3.5) are replaced by derivatives with respect to r and ϑ according to

$$\begin{aligned} \partial_{r_1}&\equiv\partial_x\to\frac{1}{r}\partial_\vartheta , \\ \partial_{r_2}&\equiv\partial_y\to\partial_r . \end{aligned} \tag{3.13}$$

Finally, we have to replace $\hat{\gamma}$ by $\hat{\mu}$ and to interpret the surface integrations in Eq. (3.5) as radial integrations ranging from $r=r_<$ to $r=r_>$ at $\vartheta=0$ and π. We are now in the position to evaluate Eq. (3.5) for the ring geometry.

C. Surface integrals

To calculate the surface integrals over dS and $d\hat{S}$ in Eq. (3.5) we have to solve the differential equation for the diffuson,

$$\Delta\hat{\Pi}_D(\mathbf{r},\mathbf{r}')=\left[\partial_r^2+\frac{1}{r^2}\partial_\vartheta^2\right]\hat{\Pi}_D(\mathbf{r},\mathbf{r}')=-\delta(\mathbf{r}-\mathbf{r}') , \quad (3.14)$$

with the boundary conditions stated for the cooperon in Eqs. (3.9) and (3.10). We expand the diffuson in a "radial" series

$$\hat{\Pi}_D(\mathbf{r},\mathbf{r}')=\sum_{m=0}^{\infty}\frac{\epsilon_m}{L_\perp}\cos[k_m(r-r_<)]\cos[k_m(r'-r_<)] \times f(\vartheta,\vartheta';m) , \quad (3.15)$$

where $k_m=m\pi/L_\perp$ $(m=0,1,2,\dots)$, $\epsilon_0=1$, and $\epsilon_{m>0}=2$. Typically we have to consider integrals of the form

$$O=\int dS\,\partial_{r_i}\hat{\Pi}_D(\mathbf{o},\mathbf{r})\partial_{r_j'}\hat{\Pi}_D(\mathbf{r}',\mathbf{L}) = \int dr_0 dr_\pi \partial_{r_i}\hat{\Pi}_D(\vartheta=0,r_0;\mathbf{r})\partial_{r_j'}\hat{\Pi}_D(\mathbf{r}',\vartheta=\pi,r_\pi) . \quad (3.16)$$

The integrations over r_0 and r_π project the associated series to their respective $m=0$ terms. This means that the derivatives ∂_{r_i} and $\partial_{r_j'}$ in Eq. (3.16) have to act on $f(\vartheta,\vartheta';m=0)$ and not on the remaining cosine. Otherwise, even the $m=0$ contributions vanish. Therefore the typical surface integral reduces to

$$O=\int dS\,\partial_{r_1}\hat{\Pi}_D(\mathbf{o},\mathbf{r})\partial_{r_1'}\hat{\Pi}_D(\mathbf{r}',\mathbf{L}) =\frac{1}{\bar r^2}\partial_\vartheta f(0,\vartheta;m=0)\partial_{\vartheta'}f(\vartheta',\pi;m=0) , \quad (3.17)$$

where we have replaced r^{-2} by $\bar r^{-2}$.

Inserting the series expansion Eq. (3.15) into the partial differential equation (3.14) yields an ordinary differential equation for f:

$$\frac{1}{\bar r}\partial_\vartheta^2 f(\vartheta,\vartheta';m=0)=-\delta(\vartheta-\vartheta') . \quad (3.18)$$

This equation can be solved by standard methods, see Appendix C. With $F(\vartheta,\vartheta')=f(\vartheta,\vartheta')/\bar r$, $\vartheta_<=\min(\vartheta,\vartheta')$, and $\vartheta_>=\max(\vartheta,\vartheta')$ the solution is given by

$$F(\vartheta,\vartheta')=\begin{cases}\dfrac{2+\tilde c(\pi-\vartheta_>)+\tilde c\vartheta_<+\tilde c(2+\tilde c\pi)\vartheta_<(\pi-\vartheta_>)/\pi}{\tilde c(4+\tilde c\pi)} & (\vartheta'\le\pi,\vartheta\le\pi)\\[2ex] \dfrac{2+\tilde c(2\pi-\vartheta_>)+\tilde c(\vartheta_<-\pi)+\tilde c(2+\tilde c\pi)(\vartheta_<-\pi)(2\pi-\vartheta_>)/\pi}{\tilde c(4+\tilde c\pi)} & (\vartheta'\ge\pi,\vartheta\ge\pi)\\[2ex] \dfrac{2-\tilde c(\pi-\vartheta)+\tilde c\vartheta'+\tilde c2\vartheta'(\pi-\vartheta)/\pi}{\tilde c(4+\tilde c\pi)} & (\vartheta'\le\pi,\vartheta\ge\pi)\\[2ex] \dfrac{2-\tilde c(\pi-\vartheta')+\tilde c\vartheta+\tilde c2\vartheta(\pi-\vartheta')/\pi}{\tilde c(4+\tilde c\pi)} & (\vartheta'\ge\pi,\vartheta\le\pi) .\end{cases} \quad (3.19)$$

Inserting this into Eq. (3.17) we arrive at

$$O=\int dS\,\partial_{r_i}\hat{\Pi}_D(\mathbf{o},\mathbf{r})\partial_{r_j'}\hat{\Pi}_D(\mathbf{r}',\mathbf{L})=-\frac{\tilde c^2}{\bar r^2(4+\tilde c\pi)^2}\delta_{i1}\delta_{j1} . \quad (3.20)$$

The typical surface integral is *constant.* This leads, of course, to a tremendous simplification of $F(\Delta B)$ in Eq. (3.5). With

$$S_1\equiv\hat\mu^4\int dS\,d\hat S\,\partial_{r_i}\partial_{r_j'}F_s^{\text{oL}}(\mathbf{r},\mathbf{r}')\partial_{r_i}\partial_{r_j'}F_s^{\hat{\text{o}}\hat{\text{L}}}(\mathbf{r},\mathbf{r}') =\frac{4\tilde c^4}{\bar r^4(4+\tilde c\pi)^4} ,$$

$$S_2\equiv\hat\mu^4\int dS\,d\hat S\,\partial_{r_i}\partial_{r_i}F_s^{\text{oL}}(\mathbf{r},\mathbf{r})\partial_{r_j'}\partial_{r_j'}F_s^{\hat{\text{o}}\hat{\text{L}}}(\mathbf{r}',\mathbf{r}') =\frac{16\tilde c^4}{\bar r^4(4+\tilde c\pi)^4} , \quad (3.21)$$

we write

$$F(\Delta B)=\frac{24\tilde c^4}{\bar r^4(4+\tilde c\pi)^4}\int d\mathbf{r}\,d\mathbf{r}'\hat{\Pi}_C(\mathbf{r},\mathbf{r}')\hat{\Pi}_C^*(\mathbf{r},\mathbf{r}') =\frac{24\tilde c^4}{\bar r^4(4+\tilde c\pi)^4}\text{tr}[\hat{\Pi}_C^2] . \quad (3.22)$$

In the following subsection we present the approximate calculation of the "volume integral" $I=\text{tr}[\hat{\Pi}_C^2]$.

D. Volume integral

We take advantage of the fact that the thickness $L_\perp$ of the mesoscopic ring is much smaller than its circumference $2\pi\bar r$: $L_\perp/(2\pi\bar r)=\sigma/(2\pi)$. We recall that $\sigma=0.1,\dots,0.2$ for relevant samples. Our strategy will be to expand the cooperon into radial modes keeping only the lowest (i.e., the constant) one. This will be an excellent approximation to the full result for all relevant values of $L_\perp$.

We employ a notation similar to the one used in quantum mechanics. Let $|mn\rangle$ be defined by $\langle r\vartheta|mn\rangle=f_m(r)g_n(\vartheta)$, where f_m and g_n are orthonormal (radial and angular) systems of functions obeying the boundary conditions Eqs. (3.9) and (3.10). We write the differential equation (3.11) in operator form and project it onto $f_0(r)=1/(L_\perp)^{1/2}$:

$$D_\alpha \hat{\Pi}_C = -1$$
$$\Longleftrightarrow \sum_j \langle 0n|D_\alpha|0j\rangle\langle 0j|\hat{\Pi}_C|0l\rangle = -\langle 0n|0l\rangle$$
$$\Longleftrightarrow \int d\vartheta' \langle 0\vartheta|d|0\vartheta'\rangle\langle 0\vartheta'|\hat{\Pi}_C|0\vartheta''\rangle = -\langle 0\vartheta|0\vartheta''\rangle \ . \qquad (3.23)$$

Defining

$$\langle 0\vartheta|\hat{\Pi}_C|0\vartheta'\rangle = F_C(\vartheta,\vartheta') \qquad (3.24)$$

and with

$$\langle 0\vartheta|D_\alpha|0\vartheta'\rangle = \delta(\vartheta-\vartheta')\frac{1}{L_\perp}\left[\ln\frac{r_>}{r_<}\partial_\vartheta^2 + \frac{i}{2l_{\Delta B}^2}(r_>^2 - r_<^2) - \frac{1}{16 l_{\Delta B}^4}(r_>^4 - r_<^4)\right] \qquad (3.25)$$

we arrive at the differential equation for $F_C(\vartheta,\vartheta')$:

$$\frac{1}{L_\perp}\left[\ln\frac{r_>}{r_<}\partial_\vartheta^2 + \frac{i}{2l_{\Delta B}^2}(r_>^2 - r_<^2) - \frac{1}{16 l_{\Delta B}^4}(r_>^4 - r_<^4)\right] \times F(\vartheta,\vartheta') = -\delta(\vartheta-\vartheta') \ . \qquad (3.26)$$

For sufficiently small $L_\perp / r_<$ we can write

$$\frac{1}{L_\perp}\ln\frac{r_>}{r_<} \approx \frac{1}{\bar{r}} = \frac{2}{r_> + r_<} \ , \qquad (3.27)$$

so that Eq. (3.26) simplifies further:

$$\left[\frac{1}{\bar{r}}\partial_\vartheta^2 + i\frac{\bar{r}}{l_{\Delta B}^2}\partial_\vartheta - \frac{\bar{r}}{8 l_{\Delta B}^4}(r_>^2 + r_<^2)\right]F(\vartheta,\vartheta') = -\delta(\vartheta-\vartheta') \ . \qquad (3.28)$$

Equation (3.28) can be solved as it stands and the essential steps of this rather involved (although in principle straightforward) calculation can be found in Appendix C. But before we present the result it is instructive to discuss Eq. (3.28) qualitatively.

The Green's function $F_C(\vartheta,\vartheta')$ can be constructed from the solutions of the homogeneous equation corresponding to Eq. (3.28). This homogeneous equation in turn can be reduced with an ansatz of the form $F = \exp(i\omega\vartheta)$ to an algebraic equation for ω. Its solutions are given by

$$\omega_{1/2} = \frac{(r_> + r_<)^2}{8 l_{\Delta B}^2} \pm i\frac{r_>^2 - r_<^2}{8 l_{\Delta B}^2} \ . \qquad (3.29)$$

With

$$\frac{r^2}{2 l_{\Delta B}^2} = \frac{2(\Delta B) r^2}{2\hbar} = \frac{(\Delta B)\pi r^2}{h/e} = \frac{\phi}{\phi_0} \qquad (3.30)$$

Eq. (3.29) may be rewritten as

$$\omega_{1/2} = \frac{1}{4}\left[\frac{\phi_>}{\phi_0} + \frac{\phi_<}{\phi_0} + 2\frac{(\phi_>\phi_<)^{1/2}}{\phi_0}\right] \pm \frac{i}{4}\left[\frac{\phi_>}{\phi_0} - \frac{\phi_<}{\phi_0}\right] . \qquad (3.31)$$

We see that the homogeneous solutions and consequently $F_C(\vartheta,\vartheta')$ contain oscillatory as well as exponentially damped contributions. The field scale for the former is a peculiar mean value of the flux threading the ring. The field scale for the latter is just the flux penetrating the body of the ring. Obviously, these are the basic manifestations of the AB effect and the decay of the correlation function due to aperiodic fluctuations, respectively. In the limit of a one-dimensional ring we have $\phi_< = \phi_>$ and only the oscillations survive: There is no damping.

The result for the volume integral $I = \mathrm{tr}[\hat{\Pi}_C^2]$ reads (see Appendix C)

$$I = \bar{r}^4 \frac{\sum_{i=0}^{4} S^{(i)}}{\sum_{i=0}^{4} X^{(i)}} \ , \qquad (3.32)$$

where (with $\omega_{1/2} = \alpha \pm i\beta$)

$$S^{(0)} = 16\beta^4\pi^2[\cos(\alpha 2\pi)\cosh(2\beta\pi) - 1] + 8\beta^3\pi[\cosh(2\beta\pi) - \cos(\alpha 2\pi)]\sinh(2\beta\pi) \ ,$$
$$S^{(1)} = 16\tilde{c}\beta^3\pi^2\cos(\alpha 2\pi)\sinh(2\beta\pi) - 8\tilde{c}\beta^2\pi[3\cos(\alpha 2\pi)\cosh(2\beta\pi) - 2 - \cosh(4\beta\pi)] + 6\tilde{c}\beta[2\cos(\alpha 2\pi)\sinh(2\beta\pi) - \sinh(4\beta\pi)] \ ,$$
$$S^{(2)} = 16\tilde{c}^{\,2}\{[\cos(\alpha 2\pi) + 1]\sinh^2(\beta\pi) - \sinh^2(2\beta\pi)\} - 2\tilde{c}^{\,2}\beta\pi[5\cos(\alpha 2\pi)\sinh(2\beta\pi) - 3\sinh(2\beta\pi) - 3\sinh(4\beta\pi)] + 4\tilde{c}^{\,2}\beta^2\pi^2\{[\cos(\alpha 2\pi) + 1]\cosh(2\beta\pi) + 2\} \ , \qquad (3.33)$$
$$S^{(3)} = 4\tilde{c}^{\,3}\beta\pi^2\sinh(2\beta\pi) + 4\tilde{c}^{\,3}\pi[\sinh^2(2\beta\pi) + \sinh^2(\beta\pi)] - 14\tilde{c}^{\,3}\frac{1}{\beta}\sinh(2\beta\pi)\sinh^2(\beta\pi) \ ,$$
$$S^{(4)} = 2\tilde{c}^{\,4}\pi^2\sinh^2(\beta\pi) + \tilde{c}^{\,4}\frac{\pi}{\beta}\sinh(2\beta\pi)\sinh^2(\beta\pi) - 4\tilde{c}^{\,4}\frac{1}{\beta^2}\sinh^4(\beta\pi) \ ,$$

and

$$X^{(0)}=16\beta^6[\cosh(2\beta\pi)-\cos(\alpha 2\pi)]^2 ,$$
$$X^{(1)}=32\tilde{c}\beta^5\sinh(2\beta\pi)[\cosh(2\beta\pi)-\cos(\alpha 2\pi)] ,$$
$$X^{(2)}=24\tilde{c}^{\,2}\beta^4\sinh^2(2\beta\pi)$$
$$-16\tilde{c}^{\,2}\beta^4\sinh^2(\beta\pi)[1+\cos(\alpha 2\pi)] , \quad (3.34)$$
$$X^{(3)}=16\tilde{c}^{\,3}\beta^3\sinh^2(\beta\pi)\sinh(2\beta\pi) ,$$
$$X^{(4)}=4\tilde{c}^{\,4}\beta^2\sinh^4(\beta\pi) .$$

The complete correlation function, including the surface integrations, is then given by

$$F(\Delta B)=24\frac{\tilde{c}^{\,4}}{(4+\tilde{c}\pi)^4}\frac{\sum_{i=0}^{4}S^{(i)}}{\sum_{i=0}^{4}X^{(i)}} . \quad (3.35)$$

Unfortunately, this is a rather complex expression. The main reason for this complexity is the detailed dependence of both numerator and denominator on the parameter $\tilde{c}$. We recall that $\tilde{c}$ may be interpreted as the ratio $\bar{r}/L$ between the ring radius and the length of one of the identical disordered external leads. This shows that it is the inevitable geometry dependence which brings about the complicated result Eq. (3.35).

Most of our discussion will be based on a graphical presentation of the properties of $F(\Delta B)$. There is, however, one comparatively simple case, namely, the limit $\tilde{c}\to\infty$ where only the fourth-order terms contribute in numerator and denominator. We get

$$\lim_{\tilde{c}\to\infty} F(\Delta B)=\frac{24}{\pi^4}\frac{S^{(4)}}{X^{(4)}}=\frac{12}{\beta^2\pi^2\sinh^2(\beta\pi)}+\frac{6\sinh(2\beta\pi)}{\pi^3\beta^3\sinh^2(\beta\pi)}-\frac{24}{\pi^4\beta^4} . \quad (3.36)$$

This is a very interesting result. It explicitly shows that the ring is effectively cut into two halves as $\tilde{c}\to\infty$. Equation (3.36) is just twice the correlation function of a (narrow) rectangle (with $\beta\pi$ slightly modified due to the geometry) in a magnetic field.[23] Furthermore, an expression of precisely the same form as Eq. (3.36) has been derived in Ref. 15. There, the decay of var(g) with increasing phase-breaking length L_ϕ was considered in the framework of a phenomenological model for dephasing. This demonstrates explicitly that the magnetic field penetrating the body of the ring induces a phase-breaking mechanism.

IV. RESULTS AND DISCUSSION

The properties of the magnetic-field correlation function are demonstrated in Figs. 3 and 4.

In Fig. 3 we display the correlation function of a ring with $\sigma=L_\perp/\bar{r}=0.3$ as a function of the flux difference $\Delta\phi/\phi_0$ and the coupling constant $\tilde{c}$. The dependence on $\Delta\phi$ is as expected from general considerations and the

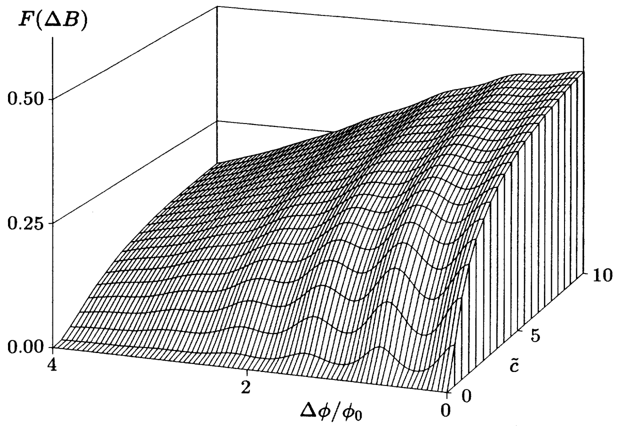

FIG. 3. The magnetic-field correlation function for a ring with $\sigma=L_\perp/\bar{r}=0.3$ as a function of $\Delta\phi/\phi_0$ and $\tilde{c}$.

qualitative discussion of Eq. (3.28): The mean value of $F(\Delta B)$ is damped as $\Delta\phi$ increases and there are superimposed AB oscillations of decreasing amplitude. As a function of $\tilde{c}$ there are two effects. First, $F(\Delta B)$ grows with increasing coupling constant $\tilde{c}$ until it reaches the limiting value in Eq. (3.36). In principle, one would expect some dependence on $\tilde{c}$ because the variation of $\tilde{c}$ can be interpreted as a variation of the ring geometry in Fig. 2(a). The particular behavior shown in Fig. 3 is, however, unrealistic. Small values of $\tilde{c}$ correspond to very long disordered external leads in Fig. 2(a). These leads—neglected in our present calculation—dominate the correlation function in the limit $\tilde{c}\leftrightarrow\bar{r}/L\to 0$ and the contribution from the ring itself becomes vanishingly small. Second, the amplitude of the AB oscillations gradually vanishes as $\tilde{c}\to\infty$. Clearly, this is a consequence of the decoupling of the upper and the lower halves of the mesoscopic ring. The potential barriers at $\vartheta=0$ and π become so high (or, put differently, the probability for electrons to be absorbed in the ideal leads which, in the limit $\tilde{c}\to\infty$, are directly attached to the disordered ring is so high) that the electrons cannot surround the ring

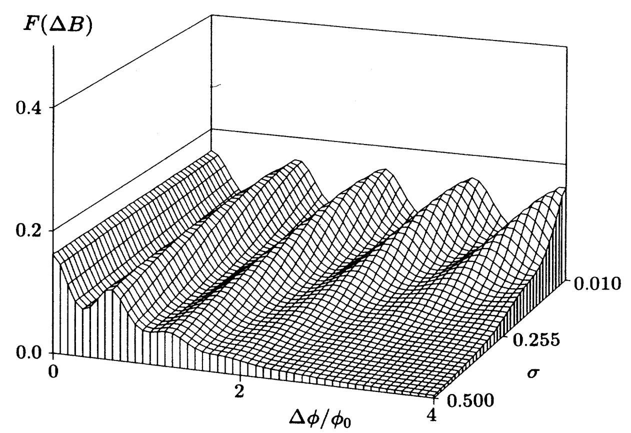

FIG. 4. The magnetic-field correlation function for a ring with $\tilde{c}=1$ as a function of $\Delta\phi/\phi_0$ and $\sigma=L_\perp/\bar{r}$.

coherently. Therefore the AB effect must vanish. These results complement and explain the outcome of the numerical simulations[9] mentioned in the Introduction where a similar dependence of the AB amplitude on the lengths of the disordered leads was found.

In Fig. 4 we present the correlation function for fixed $\tilde{c}=1$ as a function of $\Delta\phi/\phi_0$ and the ring thickness, characterized by $\sigma=L_\perp/\bar{r}$. We can see the crossover from the quasi-one-dimensional, periodic case to the regime of strong damping. At the same time, the period length of the oscillations decreases because the effective AB flux increases with the thickness of the ring.

We come to our most important result where we compare the relevant field scales for the damping of the aperiodic fluctuations and the AB oscillations, respectively. Our procedure is as follows: For different, but fixed values of $\tilde{c}$, and for a given value of $\sigma=I_\perp/\bar{r}$ we calculate $F_{\tilde{c}}(\Delta B)$ as a function of $\Delta\phi/\phi_0$. To separate the periodic and the aperiodic fluctuations, we perform a numerical Fourier transformation

$$G_{\tilde{c}}(y)=\int_0^\infty dx\, F_{\tilde{c}}(x)\cos(xy) \quad \left[x=\frac{\Delta\phi}{\phi_0}\right] . \qquad (4.1)$$

The structure of the Fourier transform $G_{\tilde{c}}(y)$ is quite simple. Apart from higher harmonics which determine the actual shape of the oscillations we have a peak at $y=0$ (aperiodic fluctuations) and another one at $y=2\pi$ (principal period of the AB oscillations). We take the widths of these peaks as a measure of the damping. Denoting these widths by $\Delta_{\rm ap}$ and $\Delta_{\rm AB}$, respectively, we then calculate the ratio $\lambda=\Delta_{\rm AB}/\Delta_{\rm ap}$. The result turns out to depend only on $\tilde{c}$, and not on σ. The curve $\lambda=\lambda(\tilde{c})$ is presented in Fig. 5. It rises from $\lambda\approx 1.4$ with increasing $\tilde{c}$ until $\lambda\approx 2.2$. Assuming $\tilde{c}\approx 1$ to be the experimentally relevant regime our calculation is in good agreement with the observed value $\lambda\approx 2$.

For all values of $\tilde{c}$ the AB oscillations decay considerably faster than the aperiodic fluctuations. This is to be expected from semiclassical arguments.[11] Furthermore, our model predicts a monotonic increase of λ with $\tilde{c}$. We offer the following qualitative explanation (based on the semiclassical picture given in Ref. 11) for this phenomenon: The magnetic correlation function of a rectangle is reduced to one-half of its value when the flux penetrating the probe is $\phi\approx\sqrt{3}\phi_0$.[4] This is the maximum

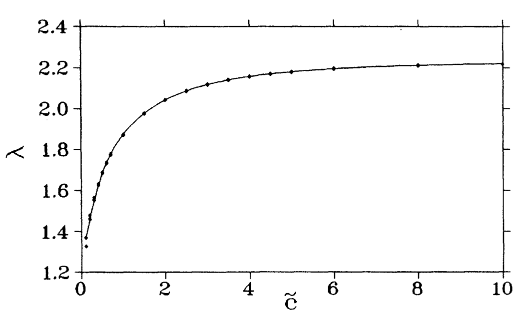

FIG. 5. The ratio λ of the decay widths of AB oscillations and aperiodic fluctuations, respectively, as a function of $\tilde{c}$.

flux two electron paths forming a simple loop can enclose. Let us consider such pairs of extremal paths in the ring. Electron paths contributing to the aperiodic fluctuations do not have to surround the ring. We assume for the moment that they stay in the upper half, say. On the contrary, AB paths have to go around the ring and, therefore, can enclose approximately twice the flux compared to extremal paths leading to aperiodic fluctuations. This is the root of the argument in Ref. 11. Now, it is of course not true that paths associated with aperiodic fluctuations are confined to one-half of the mescoscopic ring. They may coherently cross the coupling potentials at $\vartheta=0$ and π so that the enclosed flux increases. The same is true for the AB paths but their probability to increase the enclosed area by the same percentage is lower: They have to catch a larger *absolute* amount of flux. We conclude that the possibility of coherently crossing the coupling potentials reduces λ. But this mechanism is destroyed for large coupling parameters $\tilde{c}$. Crossing the potentials becomes extremely probable and coherent paths are indeed confined to one-half of the ring. This might explain the monotonic behavior of λ.

In summary, we have derived an analytical expression for the magnetic-field correlation function of a mesoscopic ring. We have taken into account the coupling to the external world, treated a ring with finite aspect ratio, and established a parameter ($\tilde{c}$) governing the geometry dependence of our results. In terms of this parameter the dependence of the AB amplitude on the length of the disordered leads could be fully understood. This dependence was observed previously in numerical simulations but remained unexplained up to now. Furthermore, it was possible to investigate the damping field scales for the aperiodic fluctuations and the AB oscillations, respectively. We found satisfactory agreement with observed experimental values and were able to predict the behavior of the ratio of both decay widths as a function of the geometry parameter $\tilde{c}$.

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APPENDIX A: GAUGE INVARIANCE OF THE GENERATING FUNCTIONAL

We introduce the notation

$$\mathcal{A}_x=i\alpha_x\sigma_3,\quad \mathcal{A}_y=i\alpha_y\sigma_3\ ,$$

$$\mathcal{A}=\begin{bmatrix}\mathcal{A}_x\\ \mathcal{A}_y\end{bmatrix}=i\boldsymbol{\alpha}\otimes\sigma_3\ , \qquad (\mathrm{A1})$$

$$\mathbf{d}=\boldsymbol{\partial}+[\mathcal{A},.]\ ,$$

where α_x and α_y have been defined in Eq. (2.32). We consider two different gauges for the vector potential, $\mathbf{A}$ and $\mathbf{A}'$, which are connected by the gradient of a scalar function:

$$\mathbf{A}+\partial g=\mathbf{A}' , \tag{A2}$$

The corresponding gauge transformation acting on $\mathcal{A}$ reads

$$U=\exp\left[-i\frac{e}{\hbar}\frac{c}{2}g\sigma_3\right] \tag{A3}$$

since $[\mathcal{A}_x,U]=[\mathcal{A}_y,U]=0$ and

$$\mathcal{A}\to\mathcal{A}'=U\mathcal{A}U^{-1}-(\partial U)U^{-1}=\mathcal{A}+i\frac{e}{\hbar}\frac{c}{2}\partial g\otimes\sigma_3 . \tag{A4}$$

Defining new Q fields

$$Q'=UQU^{-1} \tag{A5}$$

we have, due to the essential property

$$(DQ)'\equiv D'Q'=UDQU^{-1} , \tag{A6}$$

the identity

$$\mathrm{trg}[Q'D'Q']=\mathrm{trg}[QDQ] \tag{A7}$$

and gauge invariance seems to be trivially fulfilled. However, we still have to check that the reparametrization in Eq. (A5) does not contradict the original definition of the Q matrices as elements of a saddle-point manifold. In other words, we have to prove that $Q'=UQU^{-1}$ still belongs to the saddle-point manifold. Using Eq. (2.22) we may write

$$Q'=UT_0^{-1}Q_DT_0U^{-1}=(T_0U^{-1})^{-1}Q_DT_0U^{-1} . \tag{A8}$$

The transformations T_0 are determined by certain symmetry requirements.[19] In the case of the GUE they have to obey the pseudounitarity relation

$$T_0^\dagger\hat{L}^{1/2}K\hat{L}^{1/2}T_0=\hat{L}^{1/2}K\hat{L}^{1/2} . \tag{A9}$$

The transformation matrices for the GOE have to fulfill in addition the relation

$$T_0=(K\hat{L}M)^TT_0^*(K\hat{L}M) \tag{A10}$$

which determines the structure of T_0 in the subspace $\mathcal{M}_g^d$. The actual representation of K and the time-reversal matrix M (Ref. 20) are not important in the present context. It suffices to know certain commutation properties. Distinguishing explicitly between U_{GOE} and U_{GUE} we have

$$\begin{gathered}[U_{\mathrm{GUE}},K]=[U_{\mathrm{GOE}},K]=0 ,\\ [U_{\mathrm{GUE}},\hat{L}^{1/2}]=[U_{\mathrm{GOE}},\hat{L}^{1/2}]=0 ,\end{gathered} \tag{A11}$$

and

$$U_{\mathrm{GOE}}M=MU_{\mathrm{GOE}}^{-1} . \tag{A12}$$

Now, it is a simple matter to show that $T_0'=T_0U^{-1}$ indeed fulfills Eqs. (A9) and (A10):

$$T_0'^\dagger\hat{L}^{1/2}K\hat{L}^{1/2}T_0'=UT_0^\dagger\hat{L}^{1/2}K\hat{L}^{1/2}T_0U^{-1}=\hat{L}^{1/2}K\hat{L}^{1/2} , \tag{A13}$$

$$(K\hat{L}M)^TT_0'^*(K\hat{L}M)=(K\hat{L}M)^TT_0^*(K\hat{L}M)U_{\mathrm{GOE}}^{-1}=T_0' .$$

This completes the proof of gauge invariance.

APPENDIX B: SIMPLIFICATION PROCEDURE

We perform the calculation simplifying the expressions in Eq. (3.3) for a rectangle. We choose a gauge where $A_y=0$ so that

$$\begin{gathered}\nabla_{r_1}=\partial_{r_1}+i\alpha_{r_1} ,\\ \nabla_{r_2}=\partial_{r_2} .\end{gathered} \tag{B1}$$

The following treatment is in principle a generalization of Appendix A in Ref. 16. We are guided by the aim to remove all derivatives from the cooperon propagators. Let us denote the contributions to A_6 in Eq. (3.3) by a, b, c, d, and e, respectively. Inserting the identity

$$\begin{aligned}\nabla_{r_i}\nabla^*_{r_j'}\hat{\Pi}_C\nabla^*_{r_i}\nabla_{r_j'}\hat{\Pi}^*_C={}&\tfrac{1}{4}(\partial_{r_i}\partial_{r_i})(\partial_{r_j'}\partial_{r_j'})[\hat{\Pi}_C\hat{\Pi}^*_C]+\tfrac{1}{8}(\partial_{r_i}\partial_{r_i}+\partial_{r_j}\partial_{r_j})[\delta(\mathbf{r}-\mathbf{r}')(\hat{\Pi}_C+\hat{\Pi}^*_C)]\\ &-\tfrac{1}{4}\sum_{i,j}\partial_{r_i}[\delta(\mathbf{r}-\mathbf{r}')\partial_{r_j'}(\hat{\Pi}_C+\hat{\Pi}^*_C)]+\tfrac{1}{2}\delta(\mathbf{r}-\mathbf{r}')\partial_{r_i}\partial_{r_i'}(\hat{\Pi}_C+\hat{\Pi}^*_C)+\tfrac{1}{2}\alpha^2_{r_1}\delta(\mathbf{r}-\mathbf{r}')(\hat{\Pi}_C+\hat{\Pi}^*_C) ,\end{aligned} \tag{B2}$$

where a summation over repeated indices is implied into

$$a=\int dS\,d\hat{S}\,d\mathbf{r}\,d\mathbf{r}'F_S^{\mathrm{oL}}(\mathbf{r},\mathbf{r}')F_S^{\hat{\mathrm{o}}\hat{\mathrm{L}}}(\mathbf{r},\mathbf{r}')\nabla_{r_i}\nabla^*_{r_j'}\hat{\Pi}_C\nabla^*_{r_i}\nabla_{r_j'}\hat{\Pi}^*_C \tag{B3}$$

we get after some integrations by parts and with $f(\mathbf{r},\mathbf{r}')=F_s^{\mathrm{oL}}(\mathbf{r},\mathbf{r}')F_s^{\hat{\mathrm{o}},\hat{\mathrm{L}}}(\mathbf{r},\mathbf{r}')$

$$a=\int dS\,d\hat{S}\,d\mathbf{r}\,d\mathbf{r}'[\tfrac{1}{4}\hat{\Pi}_C\hat{\Pi}^*_C\partial_{r_i}\partial_{r_i}\partial_{r_j'}\partial_{r_j'}f(\mathbf{r},\mathbf{r}') \tag{B4}$$

$$\begin{aligned}&+\tfrac{1}{4}\partial_{r_i}(f(\mathbf{r},\mathbf{r}')\{\partial_{r_i}\partial_{r_j'}\partial_{r_j'}(\hat{\Pi}_C\hat{\Pi}^*_C)+\partial_{r_i}[\delta(\mathbf{r}-\mathbf{r}')(\hat{\Pi}_C+\hat{\Pi}^*_C)]-2\delta(\mathbf{r}-\mathbf{r}')\partial_{r_i'}(\hat{\Pi}_C+\hat{\Pi}^*_C)\}\\ &-\partial_{r_i}f(\mathbf{r},\mathbf{r}')\{\partial r_j'\partial r_j'(\hat{\Pi}_C\hat{\Pi}^*_C)+\delta(\mathbf{r}-\mathbf{r}')(\hat{\Pi}_C+\hat{\Pi}^*_C)\}\\ &+\partial_{r_i}(\hat{\Pi}_C\hat{\Pi}^*_C)\partial_{r_j'}\partial_{r_j'}f(\mathbf{r},\mathbf{r}')-\partial_{r_i}\partial_{r_j'}\partial_{r_j'}f(\mathbf{r},\mathbf{r}')\hat{\Pi}_C\hat{\Pi}^*_C)\end{aligned} \tag{B5}$$

$$\begin{aligned}&+[\tfrac{1}{4}\partial_{r_i}\partial_{r_i}f(\mathbf{r},\mathbf{r}')(\hat{\Pi}_C+\hat{\Pi}^*_C)+\tfrac{1}{2}\partial_{r_i}f(\mathbf{r},\mathbf{r}')\partial_{r_i'}(\hat{\Pi}_C+\hat{\Pi}^*_C)\\ &+\tfrac{1}{2}f(\mathbf{r},\mathbf{r}')\partial_{r_i}\partial_{r_i'}(\hat{\Pi}_C+\hat{\Pi}^*_C)+\tfrac{1}{2}f(\mathbf{r},\mathbf{r}')(\alpha_{r_1})^2(\hat{\Pi}_C+\hat{\Pi}^*_C)]\delta(\mathbf{r}-\mathbf{r}')] .\end{aligned} \tag{B6}$$

Repeating this procedure with

$$b=\int dS\,d\hat{S}\,d\mathbf{r}\,d\mathbf{r}'\partial_{r_j'}F_S^{\mathrm{oL}}(\mathbf{r},\mathbf{r}')\partial_{r_j'}F_S^{\hat{\mathrm{o}}\hat{\mathrm{L}}}(\mathbf{r},\mathbf{r}')\nabla_{r_i}^{*}\hat{\Pi}_C^{*}\nabla_{r_i}\hat{\Pi}_C \tag{B7}$$

and the identity

$$\nabla_{r_j}^{*}\hat{\Pi}_C^{*}\nabla_{r_j}\hat{\Pi}_C=\tfrac{1}{2}\partial_{r_j}\partial_{r_j}(\hat{\Pi}_C\hat{\Pi}_C^{*})+\delta(\mathbf{r}-\mathbf{r}')\tfrac{1}{2}(\hat{\Pi}_C+\hat{\Pi}_C^{*}) \tag{B8}$$

we arrive at

$$b=\int dS\,d\hat{S}\,d\mathbf{r}\,d\mathbf{r}'(\hat{\Pi}_C\hat{\Pi}_C^{*}\partial_{r_i}\partial_{r_j'}F_S^{\mathrm{oL}}(\mathbf{r},\mathbf{r}')\partial_{r_i}\partial_{r_j'}F_S^{\hat{\mathrm{o}}\hat{\mathrm{L}}}(\mathbf{r},\mathbf{r}') \tag{B9}$$

$$+\tfrac{1}{2}\partial_{r_i}[\partial_{r_i}(\hat{\Pi}_C\hat{\Pi}_C^{*})\partial_{r_j'}F_S^{\mathrm{oL}}(\mathbf{r},\mathbf{r}')\partial_{r_j'}F_S^{\hat{\mathrm{o}}\hat{\mathrm{L}}}(\mathbf{r},\mathbf{r}')]-\tfrac{1}{2}\partial_{r_i}\{\hat{\Pi}_C\hat{\Pi}_C^{*}\partial_{r_i}[\partial_{r_j'}F_S^{\mathrm{oL}}(\mathbf{r},\mathbf{r}')\partial_{r_j'}F_S^{\hat{\mathrm{o}}\hat{\mathrm{L}}}(\mathbf{r},\mathbf{r}')]\} \tag{B10}$$

$$+\tfrac{1}{2}(\hat{\Pi}_C+\hat{\Pi}_C^{*})\partial_{r_j'}F_S^{\mathrm{oL}}(\mathbf{r},\mathbf{r}')\partial_{r_j'}F_S^{\hat{\mathrm{o}}\hat{\mathrm{L}}}(\mathbf{r},\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}'))\ . \tag{B11}$$

Term c is analogous to b and leads to the same result. Term d is already of the desired form and the expression

$$e=\int dS\,d\hat{S}\,d\mathbf{r}\,d\mathbf{r}'\tfrac{1}{4}[\partial_{r_i}F_S^{\mathrm{oL}}(\mathbf{r},\mathbf{r})\partial_{r_j'}F_S^{\hat{\mathrm{o}}\hat{\mathrm{L}}}(\mathbf{r}',\mathbf{r}')+(\mathbf{o}\leftrightarrow\hat{\mathbf{o}},\mathbf{L}\leftrightarrow\hat{\mathbf{L}})]\partial_{r_i}\partial_{r_j'}(\hat{\Pi}_C\hat{\Pi}_C^{*}) \tag{B12}$$

can be directly transformed to give

$$e=\int dS\,d\hat{S}\,d\mathbf{r}\,d\mathbf{r}'(\tfrac{1}{2}\hat{\Pi}_C\hat{\Pi}_C^{*}\partial_{r_i}\partial_{r_i}F_S^{\mathrm{oL}}(\mathbf{r},\mathbf{r})\partial_{r_j'}\partial_{r_j'}F_S^{\hat{\mathrm{o}}\hat{\mathrm{L}}}(\mathbf{r}',\mathbf{r}') \tag{B13}$$

$$+\tfrac{1}{2}\partial_{r_i}\{\partial_{r_j'}(\hat{\Pi}_C\hat{\Pi}_C^{*})[\partial_{r_i}F_S^{\mathrm{oL}}(\mathbf{r},\mathbf{r})\partial_{r_j'}F_S^{\hat{\mathrm{o}}\hat{\mathrm{L}}}(\mathbf{r}',\mathbf{r}')+(\mathbf{o}\leftrightarrow\hat{\mathbf{o}},\mathbf{L}\leftrightarrow\hat{\mathbf{L}})]\}$$

$$-\tfrac{1}{2}\partial_{r_j'}\{\hat{\Pi}_C\hat{\Pi}_C^{*}[\partial_{r_i}\partial_{r_i}F_S^{\mathrm{oL}}(\mathbf{r},\mathbf{r})\partial_{r_j'}F_S^{\hat{\mathrm{o}}\hat{\mathrm{L}}}(\mathbf{r}',\mathbf{r}')+(\mathbf{o}\leftrightarrow\hat{\mathbf{o}},\mathbf{L}\leftrightarrow\hat{\mathbf{L}})]\})\ . \tag{B14}$$

At this stage we realize that (i) the contributions Eqs. (B4), (B9) (taken twice), (B13), and term d constitute the result Eq. (3.5) given in the text and (ii) the contributions Eqs. (B6) and (B11) (twice) just cancel the expression A_5 in Eq. (3.3).

We still have to deal with the surface terms in Eqs. (B5), (B10), and (B14). The first two lines of Eq. (B5) can be rewritten with the help of Eq. (B8):

$$\tfrac{1}{4}\partial_{r_i}\{f(\mathbf{r},\mathbf{r}')[2(\partial_{r_i}\nabla_{r_j'}^{*}\hat{\Pi}_C\nabla_{r_j'}\hat{\Pi}_C^{*}+\nabla_{r_j'}^{*}\hat{\Pi}_C\partial_{r_i}\nabla_{r_j'}\hat{\Pi}_C^{*})$$

$$-2\delta(\mathbf{r}-\mathbf{r}')\partial_{r_i'}(\hat{\Pi}_C+\hat{\Pi}_C^{*})]\} \tag{B15}$$

$$-\tfrac{1}{4}\partial_{r_i}[\partial_{r_i}f(\mathbf{r},\mathbf{r}')\{2\nabla_{r_j'}^{*}\hat{\Pi}_C\nabla_{r_j'}\hat{\Pi}_C^{*}\}]\ . \tag{B16}$$

For the expression in the innermost parentheses in Eq. (B15) we may write $\nabla_{r_i}\nabla_{r_j'}^{*}\hat{\Pi}_C\nabla_{r_j'}\hat{\Pi}_C^{*}+\nabla_{r_j'}^{*}\hat{\Pi}_C\nabla_{r_i}^{*}\nabla_{r_j'}\hat{\Pi}_C^{*}$. We decompose these terms into a regular and a singular part,

$$\begin{aligned}\nabla_{r_i}\nabla_{r_j'}^{*}\hat{\Pi}_C&=\delta_{ij}\delta(\mathbf{r}-\mathbf{r}')+\nabla_{r_i}\nabla_{r_j'}^{*}\hat{\Pi}_{C,\mathrm{reg}}\ ,\\ \nabla_{r_i}^{*}\nabla_{r_j'}\hat{\Pi}_C^{*}&=\delta_{ij}\delta(\mathbf{r}-\mathbf{r}')+\nabla_{r_i}^{*}\nabla_{r_j'}\hat{\Pi}_{C,\mathrm{reg}}^{*}\ ,\end{aligned} \tag{B17}$$

so that Eq. (B15) can be written as

$$\tfrac{1}{4}\partial_{r_i}[f(\mathbf{r},\mathbf{r}')2(\nabla_{r_i}\nabla_{r_j'}^{*}\hat{\Pi}_{C,\mathrm{reg}}\nabla_{r_j'}\hat{\Pi}_C^{*}+\nabla_{r_j'}^{*}\hat{\Pi}_C\nabla_{r_i}^{*}\nabla_{r_j'}\hat{\Pi}_{C,\mathrm{reg}}^{*})]\ . \tag{B18}$$

All surface terms are now expressed in the form of a divergence. This explicitly shows that these terms contribute at the boundary of the disordered region only. At the interfaces to the ideal leads they are suppressed by at least a factor $l/L_{\parallel}$ due to the boundary conditions for $\hat{\Pi}_D$ and $\hat{\Pi}_C$. At the isolating walls, the surface terms vanish identically: Each of the expressions contains at least one transverse derivative. Hence, we may neglect all surface contributions.

We have performed the calculation for a rectangle in explicit coordinates. A_6 was decomposed into a term compensating A_5, a volume term, and a contribution which could be expressed as a divergence. These statements do not depend on the coordinate system nor on the particular shape of the probe. They are therefore generally valid.

APPENDIX C: DIFFUSION PROPAGATORS AND THE VOLUME INTEGRAL

In this appendix, we solve the differential equations remaining after projecting to the lowest transverse mode for (i) the diffuson and (ii) the cooperon. In (ii) we then proceed to calculate the volume integral $I=\mathrm{tr}[\hat{\Pi}_C^2]$.

(i) The differential equation for the diffuson reads

$$\partial_{\vartheta}^2F(\vartheta,\vartheta')=-\delta(\vartheta-\vartheta')\ . \tag{C1}$$

The boundary conditions are given by

$$\partial_{\vartheta}F(\vartheta,\vartheta')|_{\vartheta=0^{+},\pi^{+}}-\partial_{\vartheta}F(\vartheta,\vartheta')|_{\partial=0^{-},\pi^{-}}=\tilde{c}F(\vartheta,\vartheta')|_{\vartheta=0,\pi}\ . \tag{C2}$$

For fixed ϑ', $F(\vartheta,\vartheta')$ is a linear function of ϑ and we may draw $F(\vartheta,\vartheta')$ schematically as shown in Fig. 6(a). The function $F(\vartheta,\vartheta')$ is completely determined by the parameters h_0, h_{π}, $m_{<}$, and $m_{>}$. These obey the following

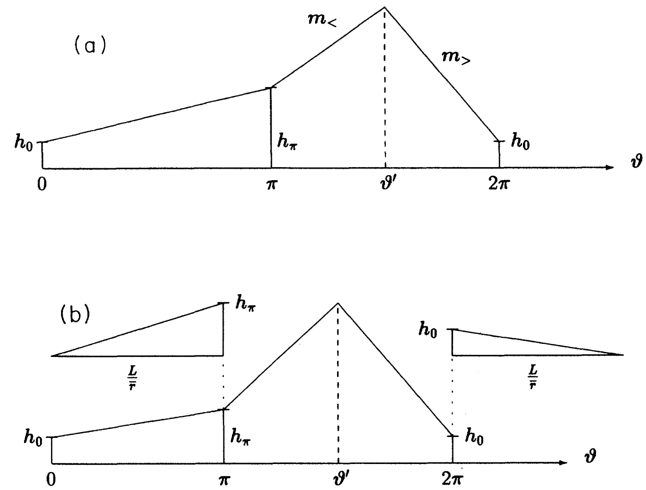

FIG. 6. (a) $F(\vartheta,\vartheta')$ for fixed ϑ' in a ring geometry without leads. (b) $F(\vartheta,\vartheta')$ for fixed ϑ' in a ring geometry including leads.

four conditions ($m_<, m_> > 0$):

$$\begin{aligned}
&\text{I} \quad (h_\pi - h_0)/\pi + m_> = \tilde c h_0 \ , \\
&\text{II} \quad m_< - (h_\pi - h_0)/\pi = \tilde c h_\pi \ , \\
&\text{III} \quad h_\pi + m_<(\vartheta' - \pi) = h_0 + m_>(2\pi - \vartheta') \ , \\
&\text{IV} \quad m_< + m_> = 1 \ .
\end{aligned} \tag{C3}$$

To interpret the parameter $\tilde c$ in Eq. (C2) we attach disordered leads of length $L/\bar r$ to the ring, see Fig. 6(b). At the boundary to the ideal leads the propagator has to vanish. Conditions I and II in Eq. (C3) have to be replaced by the following relations (current conservation at the junction points):

$$\begin{aligned}
&\text{I}' \quad (h_\pi - h_0)/\pi + m_> = \frac{\bar r}{L} h_0 \ , \\
&\text{II}' \quad m_< - (h_\pi - h_0)/\pi = \frac{\bar r}{L} h_\pi \ .
\end{aligned} \tag{C4}$$

Comparing this to the original equations I and II demonstrate the correspondence $\tilde c \leftrightarrow \bar r/L$ claimed in the text. The solution of Eq. (C3) is given by

$$\begin{aligned}
m_< &= \frac{1 + (2 + \tilde c\pi)(2 - \vartheta'/\pi)}{4 + \tilde c\pi} \ , \\
m_> &= \frac{3 + \tilde c\pi - (2 + \tilde c\pi)(2 - \vartheta'/\pi)}{4 + \tilde c\pi} \ , \\
h_0 &= \frac{2 + \tilde c\pi - \tilde c\pi(2 - \vartheta'/\pi)}{\tilde c(4 + \tilde c\pi)} \ , \\
h_\pi &= \frac{2 + \tilde c\pi(2 - \vartheta'/\pi)}{\tilde c(4 + \tilde c\pi)} \ .
\end{aligned} \tag{C5}$$

The expressions for $F(\vartheta,\vartheta')$ in the regions $\vartheta' \geq \pi$, $\vartheta \leq \pi$ and $\vartheta' \geq \pi$, $\vartheta \geq \pi$ follow immediately. Noting that

$$\begin{aligned}
&F(\vartheta,\vartheta') = F(\vartheta',\vartheta) \ , \\
&F(\vartheta \geq \pi, \vartheta' \geq \pi) = F(\vartheta - \pi, \vartheta' - \pi) \ ,
\end{aligned} \tag{C6}$$

we also get the remaining terms in Eq. (3.19).

(ii) To calculate the cooperon propagator we define $\hat F_C(\vartheta,\vartheta') = F_C(\vartheta,\vartheta')/\bar r$ and solve instead of Eq. (3.28)

$$\left[\partial_\vartheta^2 + \frac{i}{l_{\Delta B}^2}\bar r^2 \partial_\vartheta - \frac{1}{4 l_{\Delta B}^2}\bar r^2 \tfrac{1}{2}(r_>^2 + r_<^2)\right]\hat F_C(\vartheta,\vartheta') = -\delta(\vartheta - \vartheta') \ , \tag{C7}$$

with the boundary condition Eq. (C2). The typical frequencies ω_1 and ω_2 are still given by Eq. (3.31). In contrast to the case of the diffuson the solutions of the homogeneous equation are no longer linear in ϑ. Assuming for the moment that $\pi \leq \vartheta' \leq 2\pi$ we construct $\hat F_C(\vartheta,\vartheta')$ piecewise from the homogeneous solutions in the regions I ($0 \leq \vartheta \leq \pi$), II ($\pi \leq \vartheta \leq \vartheta'$), and III ($\vartheta' \leq \vartheta \leq 2\pi$). With the ansatz

$$\begin{aligned}
&\text{I} \quad f_{\text{I}}(\vartheta) = A e^{i\omega_1\vartheta} + B e^{i\omega_2\vartheta} \ , \\
&\text{II} \quad f_{\text{II}}(\vartheta) = C e^{i\omega_1\vartheta} + D e^{i\omega_2\vartheta} \ , \\
&\text{III} \quad f_{\text{III}}(\vartheta) = E e^{i\omega_1\vartheta} + F e^{i\omega_2\vartheta} \ ,
\end{aligned} \tag{C8}$$

we have six conditions for the six unknown quantities A, B, C, D, E, and F: (i) $f_{\text{I}}(\pi) = f_{\text{II}}(\pi)$, (ii) $f_{\text{II}}(\vartheta') = f_{\text{III}}(\vartheta')$, (iii) $f_{\text{III}}(2\pi) = f_{\text{I}}(0)$, (iv) $f'_{\text{I}}(0) - f'_{\text{III}}(2\pi) = \tilde c f_{\text{I}}(0)$, (v) $f'_{\text{II}}(\pi) - f'_{\text{I}}(\pi) = \tilde c f_{\text{I}}(\pi)$, (vi) $f'_{\text{III}}(\vartheta') - f'_{\text{II}}(\vartheta') = -1$. We denote with f' the derivative of f with respect to ϑ. We will only need the first four unknown terms A, B, C, and D. They can be written as

$$A = \frac{Z_A}{Y}, \quad B = \frac{Z_B}{Y}, \quad C = \frac{Z_C}{Y}, \quad \text{and } D = \frac{Z_D}{Y} \ , \tag{C9}$$

where ($\Delta\omega = \omega_2 - \omega_1$)

$$\begin{aligned}
Z_A &= -i\Delta\omega\{2\tilde c e^{i\omega_2(2\pi - \vartheta')} - [\tilde c - i\Delta\omega + (\tilde c + i\Delta\omega)e^{i\Delta\omega}]e^{i\omega_1(2\pi - \vartheta')}\} \ , \\
Z_B &= -i\Delta\omega\{2\tilde c e^{i\omega_1(2\pi - \vartheta')} - [\tilde c + i\Delta\omega + (\tilde c - i\Delta\omega)e^{-i\Delta\omega}]e^{i\omega_2(2\pi - \vartheta')}\} \ , \\
Z_C &= e^{i\omega_2(2\pi - \vartheta')}[-\tilde c(\tilde c + i\Delta\omega)e^{i\Delta\omega\pi} + \tilde c(\tilde c - i\Delta\omega)] + e^{i\omega_1(2\pi - \vartheta')}[\tilde c^{\,2} e^{i\Delta\omega\pi} - (\Delta\omega)^2 e^{i\omega_2 2\pi} - (\tilde c - i\Delta\omega)^2] \ , \\
Z_D &= e^{i\omega_1(2\pi - \vartheta')}[+\tilde c(\tilde c - i\Delta\omega)e^{-i\Delta\omega\pi} - \tilde c(\tilde c + i\Delta\omega)] + e^{i\omega_2(2\pi - \vartheta')}[-\tilde c^{\,2} e^{-i\Delta\omega\pi} + (\Delta\omega)^2 e^{i\omega_1 2\pi} + (\tilde c + i\Delta\omega)^2] \ ,
\end{aligned} \tag{C10}$$

and

$$
\begin{aligned}
Y=\Bigg\{&[i\Delta\omega(e^{i\omega_1\pi}-e^{-i\omega_1\pi})-\tilde{c}(e^{i\omega_1\pi}-e^{i\omega_2\pi})](e^{i\omega_1\pi}-e^{i\omega_2\pi})\frac{\tilde{c}}{i\Delta\omega}\\
&+[i\Delta\omega(e^{i\omega_1\pi}-e^{-i\omega_1\pi})-\tilde{c}(e^{i\omega_1\pi}+e^{-i\omega_1\pi})](e^{i\omega_2\pi}-e^{-i\omega_2\pi})e^{i(\omega_1+\omega_2)\pi}\\
&+\tilde{c}(e^{-i\omega_1\pi}+e^{i\omega_2\pi})(e^{i\omega_1\pi}-e^{-i\omega_1\pi})e^{i(\omega_1+\omega_2)\pi}\Bigg\}(\Delta\omega)^2 .
\end{aligned}
\tag{C11}
$$

The volume integral is given by

$$
\begin{aligned}
I&=\bar{r}^2\int_0^{2\pi}d\vartheta\, d\vartheta'|F(\vartheta,\vartheta')|^2\\
&=\bar{r}^4\int_0^{2\pi}d\vartheta\, d\vartheta'|\hat{F}(\vartheta,\vartheta')|^2\\
&=2\bar{r}^4\int_0^{2\pi}d\vartheta'\int_0^{\vartheta'}d\vartheta|\hat{F}(\vartheta,\vartheta')|^2\\
&=2\bar{r}^4\left\{\int_0^{\pi}d\vartheta'\int_0^{\vartheta'}d\vartheta|\hat{F}(\vartheta,\vartheta')|^2+\int_\pi^{2\pi}d\vartheta'\int_0^{\pi}d\vartheta|\hat{F}(\vartheta,\vartheta')|^2+\int_\pi^{2\pi}d\vartheta'\int_\pi^{\vartheta'}d\vartheta|\hat{F}(\vartheta,\vartheta')|^2\right\}\\
&\equiv 2\bar{r}^4\{I_1+I_2+I_3\} .
\end{aligned}
\tag{C12}
$$

Due to the symmetry of the ring we have $I_1=I_3$. We get

$$
\begin{aligned}
I_1=I_3&=\int_\pi^{2\pi}d\vartheta'\int_\pi^{\vartheta'}d\vartheta\, f_{\mathrm{II}}(\vartheta)f_{\mathrm{II}}^*(\vartheta)\\
&=\frac{1}{YY^*}\int_\pi^{2\pi}d\vartheta'\int_\pi^{\vartheta'}d\vartheta(Z_Ce^{i\omega_1\vartheta}+Z_De^{i\omega_2\vartheta})(Z_C^*e^{-i\omega_2\vartheta}+Z_D^*e^{-i\omega_1\vartheta}) ,\\
I_2&=\int_\pi^{2\pi}d\vartheta'\int_0^{\pi}d\vartheta\, f_{\mathrm{I}}(\vartheta)f_{\mathrm{I}}^*(\vartheta)\\
&=\frac{1}{YY^*}\int_\pi^{2\pi}d\vartheta'\int_0^{\pi}d\vartheta(Z_Ae^{i\omega_1\vartheta}+Z_Be^{i\omega_2\vartheta})(Z_A^*e^{-i\omega_2\vartheta}+Z_B^*e^{-i\omega_1\vartheta}) .
\end{aligned}
\tag{C13}
$$

After a lengthy calculation we arrive at Eqs. (3.32)–(3.34).

APPENDIX D: CROSSOVER FROM ORTHOGONAL TO UNITARY SYMMETRY

In the case of GOE symmetry we face an additional technical difficulty. The contraction rules Eq. (3.1) have a comparatively simple form because they were formulated with matrices a,b that were decomposed in conduction space $\mathcal{M}_g^c$. Such a formulation was useful because the symmetry breaking induced by the magnetic field acted in the same subspace. Therefore the decomposition in $\mathcal{M}_g^c$ naturally distinguished between diffusons and cooperons. In the present case, however, the magnetic field breaks the symmetry in "GOE-doubling" space $\mathcal{M}_g^d$. This forces us to perform an additional subdivision so that the contraction rules acquire a rather complicated structure. Suppressing indices referring to conductance space [which should be chosen according to Eq. (3.1)], but explicitly indicating the GOE indices we have

$$
\begin{aligned}
&\langle\langle \mathrm{trg}[Aa_{11}^{\mathbf{r}}Bb_{11}^{\mathbf{r}'}]\rangle\rangle=\langle\langle \mathrm{trg}[Aa_{22}^{\mathbf{r}}Bb_{22}^{\mathbf{r}'}]\rangle\rangle=\hat{\Pi}_D(\mathbf{r},\mathbf{r}')\langle\langle \mathrm{trg}[A]\mathrm{trg}[B]\rangle\rangle/\xi ,\\
&\langle\langle \mathrm{trg}[Aa_{11}^{\mathbf{r}}]\mathrm{trg}[Bb_{11}^{\mathbf{r}'}]\rangle\rangle=\langle\langle \mathrm{trg}[Aa_{22}^{\mathbf{r}}]\mathrm{trg}[Bb_{22}^{\mathbf{r}'}]\rangle\rangle=\hat{\Pi}_D(\mathbf{r},\mathbf{r}')\langle\langle \mathrm{trg}[AB]\rangle\rangle/\xi ,\\
&\langle\langle \mathrm{trg}[Aa_{12}^{\mathbf{r}}Bb_{21}^{\mathbf{r}'}]\rangle\rangle=\langle\langle \mathrm{trg}[Aa_{21}^{\mathbf{r}}Bb_{12}^{\mathbf{r}'}]\rangle\rangle=\hat{\Pi}_C(\mathbf{r},\mathbf{r}')\langle\langle \mathrm{trg}[A]\mathrm{trg}[B]\rangle\rangle/\xi ,\\
&\langle\langle \mathrm{trg}[Aa_{12}^{\mathbf{r}}]\mathrm{trg}[Bb_{21}^{\mathbf{r}'}]\rangle\rangle=\langle\langle \mathrm{trg}[Aa_{21}^{\mathbf{r}}]\mathrm{trg}[Bb_{12}^{\mathbf{r}'}]\rangle\rangle=\hat{\Pi}_C(\mathbf{r},\mathbf{r}')\langle\langle \mathrm{trg}[AB]\rangle\rangle/\xi ,\\
&\langle\langle \mathrm{trg}[Aa_{11}^{\mathbf{r}}Ba_{22}^{\mathbf{r}'}]\rangle\rangle=\langle\langle \mathrm{trg}[Ab_{11}^{\mathbf{r}}Bb_{22}^{\mathbf{r}'}]\rangle\rangle=\hat{\Pi}_D(\mathbf{r},\mathbf{r}')\langle\langle \mathrm{trg}[\sigma_3^{S\chi}AB^T]\rangle\rangle/\xi ,\\
&\langle\langle \mathrm{trg}[Aa_{11}^{\mathbf{r}}]\mathrm{trg}[Ba_{22}^{\mathbf{r}'}]\rangle\rangle=\langle\langle \mathrm{trg}[Ab_{11}^{\mathbf{r}}]\mathrm{trg}[Bb_{22}^{\mathbf{r}'}]\rangle\rangle=\hat{\Pi}_D(\mathbf{r},\mathbf{r}')\langle\langle \mathrm{trg}[AB^T]\rangle\rangle/\xi ,\\
&\langle\langle \mathrm{trg}[Aa_{12}^{\mathbf{r}}Ba_{21}^{\mathbf{r}'}]\rangle\rangle=\langle\langle \mathrm{trg}[Ab_{12}^{\mathbf{r}}Bb_{21}^{\mathbf{r}'}]\rangle\rangle=\hat{\Pi}_C(\mathbf{r},\mathbf{r}')\langle\langle \mathrm{trg}[AB^T]\rangle\rangle/\xi ,\\
&\langle\langle \mathrm{trg}[Aa_{12}^{\mathbf{r}}]\mathrm{trg}[Ba_{21}^{\mathbf{r}'}]\rangle\rangle=\langle\langle \mathrm{trg}[Ab_{12}^{\mathbf{r}}]\mathrm{trg}[Bb_{21}^{\mathbf{r}'}]\rangle\rangle=\hat{\Pi}_C(\mathbf{r},\mathbf{r}')\langle\langle \mathrm{trg}[\sigma_3^{S\chi}AB^T]\rangle\rangle/\xi .
\end{aligned}
\tag{D1}
$$

Here, $\sigma_3^{S\chi}$ has a nontrivial structure in $\mathcal{M}_g^{S\chi}$. The contraction rules Eq. (D1) have already been derived in Ref. 14. In view of the many different contraction schemes arising from Eq. (D1) it seems to be hopeless to calculate the crossover for the magnetic-field correlation function. The following construction, however, allows for a more compact formulation.

Let us define auxiliary matrices Ω^{r} and $\bar{\Omega}^{\mathrm{r}}$ having the following form in $\mathcal{M}_g^d$:

$$\Omega^{\mathrm{r}}=\begin{bmatrix}\alpha^{\mathrm{r}} & \\ & \beta^{\mathrm{r}}\end{bmatrix}, \quad \bar{\Omega}^{\mathrm{r}}=\begin{bmatrix}\beta^{\mathrm{r}} & \\ & \alpha^{\mathrm{r}}\end{bmatrix} . \tag{D2}$$

We associate diffusons and cooperons with products of the auxiliary variables α^{r} and β^{r} according to

$$\begin{aligned} &\alpha^{\mathrm{r}}\alpha^{\mathrm{r}'}=\beta^{\mathrm{r}}\beta^{\mathrm{r}'}\leftrightarrow\hat{\Pi}_D(\mathbf{r},\mathbf{r}') , \\ &\alpha^{\mathrm{r}}\beta^{\mathrm{r}'}\leftrightarrow\hat{\Pi}_C(\mathbf{r},\mathbf{r}') . \end{aligned} \tag{D3}$$

This gives, for example,

$$\begin{aligned} \langle\langle \operatorname{trg}[A\Omega^{\mathrm{r}}]\operatorname{trg}[B\Omega^{\mathrm{r}'}]\rangle\rangle &= \langle\langle \operatorname{trg}[\alpha^{\mathrm{r}}A_{11}+\beta^{\mathrm{r}}A_{22}]\operatorname{trg}[\alpha^{\mathrm{r}'}B_{11}+\beta^{\mathrm{r}'}B_{22}]\rangle\rangle \\ &\leftrightarrow \hat{\Pi}_D(\mathbf{r},\mathbf{r}')\langle\langle \operatorname{trg}[A_{11}]\operatorname{trg}[B_{11}]\rangle\rangle+\hat{\Pi}_D(\mathbf{r},\mathbf{r}')\langle\langle \operatorname{trg}[A_{22}]\operatorname{trg}[B_{22}]\rangle\rangle \\ &\quad +\hat{\Pi}_C(\mathbf{r},\mathbf{r}')\langle\langle \operatorname{trg}[A_{11}]\operatorname{trg}[B_{22}]\rangle\rangle+\hat{\Pi}_C^*(\mathbf{r},\mathbf{r}')\langle\langle \operatorname{trg}[A_{22}]\operatorname{trg}[B_{11}]\rangle\rangle . \end{aligned} \tag{D4}$$

With the help of Ω and $\bar{\Omega}$, Eq. (D1) can be reformulated and simplified to give

$$\begin{aligned} &\langle\langle \operatorname{trg}[Aa(\mathbf{r})Bb(\mathbf{r}')]\rangle\rangle=\langle\langle \operatorname{trg}[A\Omega^{\mathrm{r}}]\operatorname{trg}[B\Omega^{\mathrm{r}'}]\rangle\rangle , \\ &\langle\langle \operatorname{trg}[Aa(\mathbf{r})]\operatorname{trg}[Bb(\mathbf{r}')]\rangle\rangle=\langle\langle \operatorname{trg}[A\Omega^{\mathrm{r}}B\Omega^{\mathrm{r}'}]\rangle\rangle , \\ &\langle\langle \operatorname{trg}[Aa(\mathbf{r})Ba(\mathbf{r}')]\rangle\rangle=\langle\langle \operatorname{trg}[Ab(\mathbf{r})Bb(\mathbf{r}')]\rangle\rangle=\langle\langle \operatorname{trg}[A\Omega^{\mathrm{r}}MB^TM^T\bar{\Omega}^{\mathrm{r}'}]\rangle\rangle , \\ &\langle\langle \operatorname{trg}[Aa(\mathbf{r})]\operatorname{trg}[Ba(\mathbf{r}')]\rangle\rangle=\langle\langle \operatorname{trg}[Ab(\mathbf{r})]\operatorname{trg}[Bb(\mathbf{r}')]\rangle\rangle=\langle\langle \operatorname{trg}[A\Omega^{\mathrm{r}}MB^TM^T\Omega^{\mathrm{r}'}]\rangle\rangle . \end{aligned} \tag{D5}$$

The time-reversal matrix M has already been mentioned in Appendix A. The auxiliary matrices Ω, $\bar{\Omega}$ represent the symmetry breaking on the level of the contraction rules. In the limit of vanishing magnetic field we have $\alpha^{\mathrm{r}}=\beta^{\mathrm{r}}$ and Eq. (D5) reduces to the ordinary contraction rules for the GOE.[20] We can therefore start from the calculation of var(g) with GOE symmetry[12,15,16] [which gives the result Eq. (3.5) multiplied by 2 and with $\hat{\Pi}_C$ replaced by $\hat{\Pi}_D$] and look for the changes introduced by nontrivial auxiliary matrices. In a very abbreviated notation a typical term originating from A_6 in Eq. (2.53) reads

$$\operatorname{trg}[a_{11}^{\mathrm{o}}Ib_{11}^{\mathrm{L}}I]\operatorname{trg}[a_{22}^{\hat{\mathrm{o}}}\hat{I}b_{22}^{\hat{\mathrm{L}}}\hat{I}]\operatorname{trg}[a_{11}b_{11}a_{12}b_{21}]_{\mathrm{r}} \times\operatorname{trg}[a_{12}b_{22}a_{22}b_{21}]_{\mathrm{r}'} . \tag{D6}$$

The indices refer to conductance space $\mathcal{M}_g^c$ and we have left out all differential operators. Without specifying whether or not there is a magnetic field Eq. (D6) can be fully contracted in a series of steps identical to those performed in the calculation of var(g):

$$\operatorname{trg}[I\Omega_3^{\mathrm{r}}\Omega_4^{\mathrm{L}}]\operatorname{trg}[I\Omega_3^{\mathrm{o}}\Omega_4^{\mathrm{r}}\Omega_1^{\mathrm{r}}\Omega_2^{\mathrm{r}}]\operatorname{trg}[\Omega_1^{\mathrm{r}'}\Omega_2^{\mathrm{r}'}\Omega_5^{\mathrm{r}'}\Omega_6^{\hat{\mathrm{L}}}\hat{I}] \times\operatorname{trg}[\Omega_5^{\hat{\mathrm{o}}}\Omega_6^{\mathrm{r}'}\hat{I}] . \tag{D7}$$

The lower indices at the auxiliary matrices indicate the number of the contraction step where they were created (pairwise). As already stated at the end of Sec. II, the projector matrices I and $\hat{I}$ may be independently identified with either I_1 or I_2, see Eq. (2.50). Therefore the possible combinations are

$$\begin{aligned} II\hat{I}\hat{I}&\rightarrow(I_1I_1+I_2I_2)(I_1I_1+I_2I_2) \\ &=I_1I_1I_1I_1+I_2I_2I_2I_2+I_1I_1I_2I_2+I_2I_2I_1I_1 . \end{aligned} \tag{D8}$$

We come to the decisive step. The contractions 3, 4, 5, and 6 connect identical projectors and therefore lead to diffusons when applying the identification scheme Eq. (D3). But the first two contractions connect two traces containing I and $\hat{I}$, respectively. It follows from Eq. (D8) that *half* of the generated diffusion propagators are cooperons while the other half is formed by diffusons. The advantage of introducing auxiliary matrices is obvious: the difference between finite and vanishing magnetic field comes in in the very last step.

Our example Eq. (D6) turns out to be the generic case. Propagators depending on $\mathbf{r}$ and $\mathbf{r}'$ originate from contractions connecting I and $\hat{I}$. All we have to do is to replace in the result for var(g) the product of two diffusons $\hat{\Pi}_D(\mathbf{r},\mathbf{r}')\hat{\Pi}_D(\mathbf{r},\mathbf{r}')$ by $[\hat{\Pi}_C(\mathbf{r},\mathbf{r}')\hat{\Pi}_C^*(\mathbf{r},\mathbf{r}')+\hat{\Pi}_D(\mathbf{r},\mathbf{r}')\hat{\Pi}_D(\mathbf{r},\mathbf{r}')]/2$ leading to [cf. Eq. (3.5)]

$$\begin{aligned} F(B)=&4\hat{\gamma}^4\int dS\, d\hat{S}\, d\mathbf{r}\, d\mathbf{r}'(\hat{\Pi}_C\hat{\Pi}_C^*+\hat{\Pi}_D\hat{\Pi}_D)\partial_{r_i}\partial_{r_j'}F_s^{\mathrm{oL}}(\mathbf{r},\mathbf{r}')\partial_{r_i}\partial_{r_j'}F_s^{\hat{\mathrm{o}}\hat{\mathrm{L}}}(\mathbf{r},\mathbf{r}') \\ &+\tfrac{1}{2}\hat{\gamma}^4\int dS\, d\hat{S}\, d\mathbf{r}\, d\mathbf{r}'(\hat{\Pi}_C\hat{\Pi}_C^*+\hat{\Pi}_D\hat{\Pi}_D)\partial_{r_i}\partial_{r_i}F_s^{\mathrm{oL}}(\mathbf{r},\mathbf{r})\partial_{r_j'}\partial_{r_j'}F_s^{\hat{\mathrm{o}}\hat{\mathrm{L}}}(\mathbf{r}',\mathbf{r}') . \end{aligned} \tag{D9}$$

With growing magnetic-field strength the cooperons are gradually suppressed and we are left with the GUE result Eq. (3.5) (with $\Delta B=0$).

The result of this appendix is certainly not new but it has not been derived in the framework of a random matrix model. We feel that this treatment instructively complements the discussion of the GUE correlation function.

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[21]Strictly speaking this terminology is not quite correct. Cooperons are propagators describing pairs of time-reversed electron paths. In the GUE functional these are excluded from the beginning. In view of the formal analogy to the true cooperon we extend the meaning of the term to the ΔB-dependent propagators (which are actually diffusons).
[22]This reduction has been discussed in Ref. 12 where the conductance of a quasi-one-dimensional ring is considered.
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