One-dimensional Hubbard model in a magnetic field and the multicomponent Tomonaga-Luttinger model

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(Received 29 June 1992)

The one-dimensional Hubbard model in a magnetic field is equivalent under renormalization-group transformation to a multicomponent Tomonaga-Luttinger model. The mapping between the Coulomb repulsion U of the Hubbard model and the couplings of the Tomonaga-Luttinger model is obtained for arbitrary magnetic field h and band filling n from a comparison of the correlation function exponents. These quantities are calculated for the Hubbard model from the finite-size corrections to the excitation energies, since the assumption of conformal invariance relates them to the critical exponents. On the other hand, the correlation functions of the multicomponent Tomonaga-Luttinger model are determined by solving the exact equation of motion derived by the use of generalized Ward identities.

I. INTRODUCTION

The properties of strongly correlated electron systems have been studied intensively recently, especially since the discovery of heavy-fermion compounds and high-temperature superconductors. Although it is not yet clear how well the simple one-parameter Hubbard Hamiltonian can describe these systems, it is important to get as much information as possible on this model.

In one spatial dimension the Hubbard model can be solved[1] exactly by Bethe's ansatz, leading to decoupled charge (holon) and spin (spinon) excitations. Although the spectrum has been known for many years, the calculation of the correlation functions proved to be a delicate problem. After early attempts[2] to determine numerically the correlation functions, an analytic approach has become possible only recently. In the large-U limit and for special filling of the band it is possible[3] to determine some correlation functions by analyzing the properties of the ground-state wave function.

Another exactly solvable one-dimensional fermion model is the Tomonaga-Luttinger[4] model, where the solvability is due to the linearity of the spectrum and to the neglect of the backward scattering and umklapp processes. It turns out that the Fermi-liquid theory fails in this case. The momentum distribution function n_k is not sharp and it has a power-law-like behavior. There are no fermionic quasiparticles; instead the low-energy physics is determined by the bosonic collective modes.

In the early 1980s, in a series of papers Haldane[5] pointed out that the non-Fermi-liquid-like behavior of the Tomonaga-Luttinger model is generic for a large class of one-dimensional quantum systems. In one dimension the statistics does not play a distinctive role; even in fermionic systems the low-lying excitations are bosons. The parameters in the excitation spectrum will determine all the exponents in the correlation functions. These systems are called Luttinger liquids.

The numerical evaluations of the correlation functions[2] and the analytic results[3] indicated clearly that the one-dimensional Hubbard model is a Luttinger liquid. The relationship between the Hubbard model and the Tomonaga-Luttinger model can be justified in another way too. The renormalization-group treatment of the so-called "g-ology" model[6] has shown that, at least for weak couplings, where perturbation theory works, the repulsive Hubbard model is in the same universality class as the exactly solvable Tomonaga-Luttinger model. There is indication from the above-mentioned works that this is valid even in the strong-coupling case of the one-dimensional Hubbard model.

Assuming that the one-dimensional Hubbard model is a Luttinger liquid, it then becomes possible to calculate the correlation functions from the knowledge of the energy spectrum. Using this procedure Schulz[7] studied the correlation function exponents for different U and band filling n. Ren and Anderson[8] gave a more general prescription to determine the correlation function exponents using the Luttinger-liquid picture. These calculations also have been extended to include the effect of external magnetic field.[9,10]

Recently the use of conformal field theories has turned out to be very fruitful in the description of the critical phenomena of two-dimensional classical or one-dimensional quantum systems. A large class of exactly solvable models can be uniquely characterized by a single-dimensionless number, the central charge c of the underlying Virasoro algebra. This central charge or conformal anomaly determines the critical exponents of the correlation functions of primary operators. It is closely related to the finite-size corrections of the ground-state energy and can be calculated from it. If $c = 1$, the critical exponents are not determined by the central charge alone; an extra anomalous dimension appears in the correlation functions. The anomalous dimension can be related to the parameters of the towers formed by the energies and momenta of the low-lying excited states.[11]

Although the Hubbard model is critical at zero temperature, it does not belong directly to the group of conformal invariant models. Conformal invariance requires that the group velocity be the same for all elementary excitations. In the Hubbard model, in general the holon and spinon excitations have different group velocities. Nevertheless, it has been suggested by Kawakami and Yang[12] and by Frahm and Korepin[13] that the concept of conformal invariance can be applied to the Hubbard model as well. They assumed that the spectrum of excitations, which in this case has a similar tower structure as in conformal theories, can be described in terms of a semidirect product of two Virasoro algebras. If there is an underlying conformal field theory in both the holon and spinon sectors, the correlation functions appear in product form. The correlation function exponents will be composed of two terms, each related to the parameters of the towers in the appropriate sector. For the Hubbard model this tower structure of the low-lying excitations has been calculated[14] by a careful analysis of the Bethe ansatz equations; hence the anomalous dimensions of the correlation functions could be determined.[13] Since the Bethe ansatz equations are valid even in an external magnetic field, this analysis has been extended to this situation as well.[15]

In an indirect way the correlation function exponents establish the relationship between the parameters of the Hubbard model and those of the Tomonaga-Luttinger model. The mapping between the two models is known explicitly in the weak-coupling case only, where the low-order renormalization-group treatment allows us to follow the scaling trajectories from the bare couplings to the fixed point. It is therefore of interest to see what happens for larger U. In this paper we study the Hubbard model in an arbitrary external magnetic field h and for general band filling n. Our aim is to determine explicitly the Fermi velocities and couplings of that generalized, multicomponent Tomonaga-Luttinger model, whose behavior is equivalent to that of the Hubbard model with Coulomb coupling U.

The paper is organized as follows. In Sec. II the Bethe ansatz solution of the Hubbard model is reviewed and the analytic form of the correlation functions predicted by the conformal field theory is given. The weak-coupling limit of the Hubbard model and its relationship to the g-ology model is discussed in Sec. III. A generalized Tomonaga-Luttinger model that could be equivalent to the Hubbard model in magnetic field is introduced in Sec. IV, and the tower structure of the excitation spectrum is determined. The correlation functions of this multicomponent Tomonaga-Luttinger model are calculated in Sec. V, using the equation of motion method and exact Ward identities. The mapping that relates the Hubbard U and the couplings g of the Tomonaga-Luttinger model is given Sec. VI. The special cases of zero magnetic field and the small- and large-U limits are considered separately. For small U a comparison to the results of the renormalization-group theory is given. Our results are summarized in Sec. VII.

II. HUBBARD MODEL IN MAGNETIC FIELD

The Hubbard model is the simplest nontrivial model of interacting spin-1/2 fermions on a lattice. It is defined by the Hamiltonian

$$H = -\sum_{j,\sigma}(c^\dagger_{j+1,\sigma}c_{j,\sigma} + \text{H.c.}) + U\sum_j n_{j,\uparrow}n_{j,\downarrow} -\mu\sum_j(n_{j,\uparrow}+n_{j,\downarrow}) - \frac{h}{2}\sum_j(n_{j,\uparrow}-n_{j,\downarrow}) \, . \tag{2.1}$$

Here $c^\dagger_{j,\sigma}$ ($c_{j,\sigma}$) is the creation (annihilation) operator of electrons of spin σ at site j and $n_{j,\sigma} = c^\dagger_{j,\sigma}c_{j,\sigma}$ is the number operator. U is the on-site repulsion, μ is the chemical potential, and h is the external magnetic field. The hopping integral (which determines the bandwidth of the free fermions) is taken to be unity. This sets the energy scale for the quantities U, μ, and h.

Lieb and Wu[1] have shown that this model can be solved by the Bethe ansatz. The wave function and the energy of $N_c = N_\uparrow + N_\downarrow$ electrons, where $N_\uparrow$ and $N_\downarrow$ denote the number of electrons with spin up and down, respectively, on a chain with N sites can be written in terms of N_c pseudomomentum variables k_j and $N_s = N_\downarrow$ rapidities λ_α. These quantities have to be determined from the Lieb-Wu equations:

$$Nk_j = 2\pi I_j - \sum_{\beta=1}^{N_s} 2\arctan\frac{\sin k_j - \lambda_\beta}{U/4} \, , \tag{2.2}$$

$$\sum_{j=1}^{N_c} 2\arctan\frac{\lambda_\alpha - \sin k_j}{U/4} = 2\pi J_\alpha + \sum_{\beta=1}^{N_s} 2\arctan\frac{\lambda_\alpha-\lambda_\beta}{U/2} \, , \tag{2.3}$$

where the parameters I_j and J_α are integers or half-integers, depending on the parity of N_c and N_s. The eigenstates of the model can be characterized by giving either the pseudomomenta k_j and rapidities λ_α, or alternatively the quantum numbers I_j and J_α.

The ground state is obtained by choosing the sets $\{I_j\}$ and $\{J_\alpha\}$ to contain consecutive integers (half-integers) centered around zero. For simplicity we assume that N_c is even and N_s is odd, i.e., I_j takes half-integer values with $-(N_c-1)/2 \le I_j \le (N_c-1)/2$ and J_α is integer with $-(N_s-1)/2 \le J_\alpha \le (N_s-1)/2$, so that the ground-state sets can be chosen symmetrically, otherwise the ground state would be degenerate. The corresponding pseudomomenta k_j and rapidities λ_α form a Fermi sea with maximal pseudomomentum k_0 for the charge variables and with maximal rapidity λ_0 for the spin variables, i.e., they are restricted to the ranges $-k_0 < k_j < k_0$ and $-\lambda_0 < \lambda_\alpha < \lambda_0$.

Contrary to the parameters I_j and J_α, the quantities k_j and λ_α are not distributed uniformly in the ground state. In the thermodynamic limit, where the momentum and rapidity variables are continuous, the Lieb-Wu equations become integral equations for the ground-state distribution functions of momenta $\rho_c(k)$ and of rapidities $\rho_s(\lambda)$, satisfying the integral equations

$$\rho_c(k) = \frac{1}{2\pi} + \cos k\int_{-\lambda_0}^{\lambda_0}\frac{d\lambda}{2\pi}K_1(\sin k - \lambda)\rho_s(\lambda) \, , \tag{2.4}$$

$$\rho_s(\lambda) = \int_{-k_0}^{k_0} \frac{dk}{2\pi} K_1(\lambda - \sin k)\rho_c(k) - \int_{-\lambda_0}^{\lambda_0} \frac{d\lambda'}{2\pi} K_2(\lambda - \lambda')\rho_s(\lambda') , \tag{2.5}$$

with the kernels

$$K_1(x) = \frac{U/2}{(U/4)^2 + x^2} , \quad K_2(x) = \frac{U}{(U/2)^2 + x^2} .$$

The values of k_0 and λ_0 are fixed by the constraints

$$\int_{-k_0}^{k_0} dk\ \rho_c(k) = n , \tag{2.6}$$

$$\int_{-\lambda_0}^{\lambda_0} d\lambda\ \rho_s(\lambda) = \frac{1}{2}(n - m), \tag{2.7}$$

where $n = N_c/N$ is the total charge density and $m = (N_\uparrow - N_\downarrow)/N$ is the density of magnetization. The charge density n is related to the band filling ν by $n = 2\nu$.

The quantities k_0 and λ_0 are, however, not the real Fermi momenta. These latters are related to the number of electrons by

$$k_{F\uparrow} = \pi \frac{N_\uparrow}{N} , \quad k_{F\downarrow} = \pi \frac{N_\downarrow}{N} , \tag{2.8}$$

or

$$k_{F\uparrow} + k_{F\downarrow} = \pi \frac{N_c}{N} , \quad k_{F\downarrow} = \pi \frac{N_s}{N} . \tag{2.9}$$

The low-lying excitations can be of several types. Changing the number of electrons of spin up and spin down by $\Delta N_\uparrow$ and $\Delta N_\downarrow$, respectively, i.e., changing the charge by $\Delta N_c = \Delta N_\uparrow + \Delta N_\downarrow$ and the magnetization by $\Delta M = \Delta N_\uparrow - \Delta N_\downarrow$, is equivalent to adding (removing) ΔN_c extra I_j and $\Delta N_s = \Delta N_\downarrow$ extra J_α values. It has to be taken into account that an odd number of $\Delta N_\uparrow$ particles will change the set of integers $\{J_\alpha\}$ to half-integers, and an odd number of $\Delta N_\downarrow$ particles will change the set of half-integers $\{I_j\}$ to integers. This may generate an asymmetry in the distribution of the new I_j and J_α values. This asymmetry will be characterized by the numbers D_c and D_s, where $2D_c$ denotes the difference in the number of positive and negative I_j values, and similarly $2D_s$ denotes the difference in the number of positive and negative J_α values.

Adding a spin-up particle to the system ($\Delta N_c = 1$, $\Delta N_s = 0$), the lowest-energy excited state is obtained by choosing the new I_j value to be $I_j = \pm(N_c + 1)/2$, i.e., it can be on the positive or negative side of the ground-state set, producing an asymmetry $D_c = \pm 1/2$. The change in the number of spin-up particles changes the possible values of J_α to half-integers and this leads to an asymmetry $D_s = \mp 1/2$. Since an extra I_j or J_α carries a momentum

$$k = \frac{2\pi}{N} I_j \quad \text{or} \quad k = \frac{2\pi}{N} J_\alpha , \tag{2.10}$$

it is easily seen, using Eq. (2.9), that the total momentum of the system will change by $\pm k_{F\uparrow}$. Similarly, adding a spin-down particle corresponds to $\Delta N_c = 1$, $\Delta N_s = 1$, $D_c = 0$, and $D_s = \pm 1/2$ and its momentum is $\pm k_{F\downarrow}$.

We can thus distinguish right- and left-moving particles that have positive or negative momenta. Let us denote by $\Delta N_\uparrow^+$, $\Delta N_\uparrow^-$, $\Delta N_\downarrow^+$, and $\Delta N_\downarrow^-$ the number of particles added to the system with spin up and momentum $k_{F\uparrow}$, spin up and momentum $-k_{F\uparrow}$, spin down and momentum $k_{F\downarrow}$, and spin down and momentum $-k_{F\downarrow}$, respectively.

It is easy to check, using the above considerations, that the numbers $\Delta N_\uparrow^+$, $\Delta N_\uparrow^-$, $\Delta N_\downarrow^+$, and $\Delta N_\downarrow^-$ are related to ΔN_c, ΔN_s, D_c, and D_s by

$$\Delta N_\uparrow^+ = \tfrac{1}{2}(\Delta N_c - \Delta N_s + 2D_c) , \tag{2.11}$$

$$\Delta N_\uparrow^- = \tfrac{1}{2}(\Delta N_c - \Delta N_s - 2D_c) , \tag{2.12}$$

$$\Delta N_\downarrow^+ = \tfrac{1}{2}(\Delta N_s + 2D_c + 2D_s) , \tag{2.13}$$

$$\Delta N_\downarrow^- = \tfrac{1}{2}(\Delta N_s - 2D_c - 2D_s) . \tag{2.14}$$

Naturally

$$\Delta N_c = \Delta N_\uparrow^+ + \Delta N_\uparrow^- + \Delta N_\downarrow^+ + \Delta N_\downarrow^- , \tag{2.15}$$

while the change in the magnetization is

$$\Delta M = \Delta N_\uparrow^+ + \Delta N_\uparrow^- - \Delta N_\downarrow^+ - \Delta N_\downarrow^- = \Delta N_c - 2\Delta N_s . \tag{2.16}$$

The asymmetries D_c and D_s are related to the charge and spin currents carried by the excitations. For a given number of extra particles or for a given change in the magnetization there is still some arbitrariness in the choice of the values of the sets $\{I_j\}$ and $\{J_\alpha\}$. The whole sets can be shifted to the left or right, increasing the asymmetries D_c and D_s.

It follows from the above considerations that these shifted states correspond to large momentum ($2k_F$) excitations. These excitations can carry charge and spin current J_c and J_s defined by

$$J_c = (\Delta N_\uparrow^+ - \Delta N_\uparrow^-)k_{F\uparrow} + (\Delta N_\downarrow^+ - \Delta N_\downarrow^-)k_{F\downarrow} , \tag{2.17}$$

$$J_s = (\Delta N_\uparrow^+ - \Delta N_\uparrow^-)k_{F\uparrow} - (\Delta N_\downarrow^+ - \Delta N_\downarrow^-)k_{F\downarrow} . \tag{2.18}$$

For later purposes we give here the relationship between these currents and the asymmetries D_c and D_s. From Eqs. (2.11)–(2.14)

$$J_c = 2D_c k_{F\uparrow} + (2D_c + 2D_s)k_{F\downarrow} , \tag{2.19}$$

$$J_s = 2D_c k_{F\uparrow} - (2D_c + 2D_s)k_{F\downarrow} . \tag{2.20}$$

If originally the system is unpolarized, i.e., $k_{F\uparrow} = k_{F\downarrow} = k_F$, then

$$J_c = 2(2D_c + D_s)k_F , \tag{2.21}$$

$$J_s = -2D_s k_F . \tag{2.22}$$

Other types of excitations can be created by removing some I_j values from the densely occupied shifted ground-state set, leaving behind holes, and choosing the same number of new I_j values. The removed and new I_j values can be considered as particle-hole pairs in the distribution of these parameters. If the I_j values of the pairs are close to the end points of the densely occupied intervals, $N_c/2$ or $-N_c/2$, the excitations will have small momenta. The $\{I_j\}$ sets obtained this way correspond in fact to small momentum particle-hole excitations. The number of such particle-hole charge excitations around $N_c/2$ ($-N_c/2$) will be denoted by n_c^+ (n_c^-). Similarly n_s^+ (n_s^-) denotes the number of small momentum spin excitations obtained by creating particle-hole pairs in the distribution of J_α near $N_s/2$ ($-N_s/2$). The excitations in the charge and spin sectors are called holons and spinons, respectively.

These small momentum excitations are superimposed on the large momentum ($2k_F$) excitations. For momenta in the neighborhood of multiples of $2k_F$ both the holon and spinon excitations have linear spectra with velocities u_c and u_s, respectively. These velocities can be expressed in terms of the derivatives of the dressed energies

$$u_c = \frac{\varepsilon_c'(k_0)}{2\pi\rho_c(k_0)}, \quad u_s = \frac{\varepsilon_s'(\lambda_0)}{2\pi\rho_s(\lambda_0)}, \tag{2.23}$$

where the denominators appear to relate the pseudomomentum and the rapidity to the real momentum of the excitations. $\varepsilon_c(k)$ and $\varepsilon_s(\lambda)$ satisfy the equations

$$\varepsilon_c(k) = \varepsilon_c^{(0)}(k) + \int_{-\lambda_0}^{\lambda_0} \frac{d\lambda}{2\pi} K_1(\sin k - \lambda)\varepsilon_s(\lambda), \tag{2.24}$$

$$\varepsilon_s(\lambda) = \varepsilon_s^{(0)}(\lambda) + \int_{-k_0}^{k_0} \frac{dk}{2\pi} \cos k\, K_1(\lambda - \sin k)\varepsilon_c(k) - \int_{-\lambda_0}^{\lambda_0} \frac{d\lambda'}{2\pi} K_2(\lambda - \lambda')\varepsilon_s(\lambda'), \tag{2.25}$$

where

$$\varepsilon_c^{(0)}(k) = -\mu - \frac{h}{2} - 2\cos k, \qquad \varepsilon_s^{(0)}(\lambda) = h.$$

In large but finite systems the energies and momenta of these excitations form towers. The position of a tower can be characterized by the number of extra particles ΔN_c and extra down spins ΔN_s and by the number of large momentum charge and spin excitations D_c, D_s, while the elements of the tower are characterized by the number of small momentum excitations in the holon and spinon sectors n_c^+, n_c^-, n_s^+, and n_s^-. According to Ref. 14, to leading order in $1/N$ the energy and momentum of the excitations can be written in the form

$$E(\Delta N_{c,s}, D_{c,s}, n_{c,s}^\pm) - E_0 = \frac{2\pi}{N}u_c(\Delta_c^+ + \Delta_c^- + n_c^+ + n_c^- - \tfrac{1}{12}) + \frac{2\pi}{N}u_s(\Delta_s^+ + \Delta_s^- + n_s^+ + n_s^- - \tfrac{1}{12}) + O\left(\frac{1}{N}\right) \tag{2.26}$$

and

$$P(\Delta N_{c,s}, D_{c,s}, n_{c,s}^\pm) = \frac{2\pi}{N}(\Delta_c^+ - \Delta_c^- + n_c^+ - n_c^-) + \frac{2\pi}{N}(\Delta_s^+ - \Delta_s^- + n_s^+ - n_s^-) + 2D_c(k_{F\uparrow} + k_{F\downarrow}) + 2D_s k_{F\downarrow} + O\left(\frac{1}{N}\right), \tag{2.27}$$

where the dependence of $\Delta_{c,s}^\pm$ on the numbers ΔN_c, ΔN_s, D_c, and D_s is given by

$$2\Delta_c^\pm = \left(\pm\frac{Z_{ss}\Delta N_c - Z_{cs}\Delta N_s}{2\det \underline{Z}} + Z_{cc}D_c + Z_{sc}D_s\right)^2, \tag{2.28}$$

$$2\Delta_s^\pm = \left(\mp\frac{Z_{sc}\Delta N_c - Z_{cc}\Delta N_s}{2\det \underline{Z}} + Z_{cs}D_c + Z_{ss}D_s\right)^2. \tag{2.29}$$

In this expression $\underline{Z}$ is the dressed charge matrix taken at the Fermi points

$$\underline{Z} = \begin{pmatrix} Z_{cc} & Z_{cs} \\ Z_{sc} & Z_{ss} \end{pmatrix} = \begin{pmatrix} \xi_{cc}(k_0) & \xi_{cs}(\lambda_0) \\ \xi_{sc}(k_0) & \xi_{ss}(\lambda_0) \end{pmatrix}, \tag{2.30}$$

and for general k and λ the dressed charge matrix itself

$$\begin{pmatrix} \xi_{cc}(k) & \xi_{cs}(\lambda) \\ \xi_{sc}(k) & \xi_{ss}(\lambda) \end{pmatrix} \tag{2.31}$$

is defined by the following integral equations:

$$\xi_{cc}(k) = 1 + \int_{-\lambda_0}^{\lambda_0} \frac{d\lambda}{2\pi} K_1(\sin k - \lambda)\xi_{cs}(\lambda), \tag{2.32}$$

$$\xi_{cs}(\lambda) = \int_{-k_0}^{k_0} \frac{dk}{2\pi} \cos k\, K_1(\lambda - \sin k)\xi_{cc}(k) - \int_{-\lambda_0}^{\lambda_0} \frac{d\lambda'}{2\pi} K_2(\lambda - \lambda')\xi_{cs}(\lambda'), \tag{2.33}$$

$$\xi_{sc}(k) = \int_{-\lambda_0}^{\lambda_0} \frac{d\lambda}{2\pi} K_1(\sin k - \lambda)\xi_{ss}(\lambda), \tag{2.34}$$

$$\xi_{ss}(\lambda) = 1 + \int_{-k_0}^{k_0} \frac{dk}{2\pi} \cos k\, K_1(\lambda - \sin k)\xi_{sc}(k) - \int_{-\lambda_0}^{\lambda_0} \frac{d\lambda'}{2\pi} K_2(\lambda - \lambda')\xi_{ss}(\lambda'). \tag{2.35}$$

As given in Eqs. (2.28) and (2.29) $\Delta_{c,s}^\pm$ are squares of bilinear expressions of the elements of the dressed charge matrix and of the parameters of the towers.

It is convenient to use a shorthand notation for such quantities introducing column vectors $|w_H^{(\pm u_{c,s})}\rangle$ constructed from the elements of the dressed charge matrix,

$$|w_H^{(+u_c)}\rangle = \frac{1}{2}\begin{bmatrix} \frac{Z_{ss}}{\det \underline{Z}} \\ -\frac{Z_{cs}}{\det \underline{Z}} \\ Z_{cc} \\ Z_{sc} \end{bmatrix}, \quad |w_H^{(-u_c)}\rangle = \frac{1}{2}\begin{bmatrix} -\frac{Z_{ss}}{\det \underline{Z}} \\ \frac{Z_{cs}}{\det \underline{Z}} \\ Z_{cc} \\ Z_{sc} \end{bmatrix}, \tag{2.36}$$

$$|w_H^{(+u_s)}\rangle = \frac{1}{2}\begin{bmatrix} -\frac{Z_{sc}}{\det \underline{Z}} \\ \frac{Z_{cc}}{\det \underline{Z}} \\ Z_{cs} \\ Z_{ss} \end{bmatrix}, \quad |w_H^{(-u_s)}\rangle = \frac{1}{2}\begin{bmatrix} \frac{Z_{sc}}{\det \underline{Z}} \\ -\frac{Z_{cc}}{\det \underline{Z}} \\ Z_{cs} \\ Z_{ss} \end{bmatrix},$$

the vector $|\Delta N_H\rangle$ that contains the parameters of the towers,

$$|\Delta N_H\rangle = \begin{bmatrix} \Delta N_c \\ \Delta N_s \\ 2D_c \\ 2D_s \end{bmatrix}, \tag{2.37}$$

and defining the scalar product of two vectors $|a\rangle$ and $|b\rangle$ as

$$\langle a|b\rangle = \sum_i a_i^* b_i \ . \tag{2.38}$$

In this notation $\Delta_{c,s}^{\pm}$ can be written in the form

$$2\Delta_{c,s}^{\pm} = \langle w_H^{(\pm u_{c,s})}|\Delta N_H\rangle^2 \ . \tag{2.39}$$

A primary operator ϕ may add (or remove) electrons or spins to (or from) the ground state, so the parameters $\Delta N_{c,s}$ are uniquely defined by the operator. It may also change the momentum by multiples of $2k_F$, the asymmetries $D_{c,s}$ can, however, take arbitrary integer (half-integer) values. The operator ϕ will have finite matrix elements between the ground state and states belonging to a series of towers in the excitation spectrum, where the tower is characterized by the four quantum numbers $\Delta N_{c,s}$ and $D_{c,s}$. These numbers and the dressed charge matrix define the parameters $\Delta_{c,s}^{\pm}$ given in Eqs. (2.28) and (2.29). According to Frahm and Korepin[13] the long distance asymptotics of the zero temperature correlation functions of the operator ϕ will have the form

$$\langle\phi(x,t)\phi(0,0)\rangle \sim \sum_{D_c,D_s} \frac{a(D_c,D_s)e^{-2iD_ck_{F\uparrow}x}e^{-2i(D_c+D_s)k_{F\downarrow}x}}{(x-u_ct)^{2\Delta_c^+}(x+u_ct)^{2\Delta_c^-}(x-u_st)^{2\Delta_s^+}(x+u_st)^{2\Delta_s^-}} \ . \tag{2.40}$$

Since the parameters $\Delta_{c,s}^{\pm}$ appear in the exponents, they are sometimes called anomalous dimensions.

It is important to emphasize that the four parameters $\Delta N_{c,s}$ and $D_{c,s}$ do not determine uniquely the four parameters $\Delta_{c,s}^{\pm}$. In fact, the finite-size corrections give three quantities: the coefficients of u_c and u_s in the expression for the energy corrections in Eq. (2.26) and the momentum of the state given by Eq. (2.27). The introduction of the four anomalous dimensions and the separation of the above-mentioned three parameters in the form given in Eqs. (2.28) and (2.29), although reasonable, is somewhat ambiguous. Instead of choosing them in the form given above, Eqs. (2.26) and (2.27) are equally well satisfied by

$$\tilde{\Delta}_c^{\pm} = \Delta_c^{\pm} \pm \Delta_0 \ , \quad \tilde{\Delta}_s^{\pm} = \Delta_s^{\pm} \mp \Delta_0 \ . \tag{2.41}$$

where Δ_0 can be arbitrarily chosen. As is known, without external magnetic field the charge and spin separation holds. It is therefore natural to assume that Δ_c depends on the number of extra charges ΔN_c and on the charge current J_c, while Δ_s depends on the extra magnetization ΔM and on the spin current J_s and there cannot exist a term Δ_0 that appears in both the charge and spin part of the correlation functions. This, however, is not the case when an external magnetic field is applied. As was shown by Frahm and Korepin[15] the charge-spin separation does not hold in general. The choice proposed by Frahm and Korepin[13] ($\Delta_0 = 0$) is motivated by the fact that in this case the anomalous dimensions $\Delta_{c,s}^{\pm}$ can be written in a simple quadratic form as shown in Eqs. (2.28) and (2.29).

III. WEAK-COUPLING LIMIT OF THE HUBBARD MODEL IN MAGNETIC FIELD

In the weak-coupling limit it is convenient to rewrite the Hamiltonian in momentum representation. The hopping term will give rise to the kinetic energy. The spectrum of noninteracting electrons has two branches, one for spin-up and one for spin-down electrons. This is described by the free part of the Hamiltonian,

$$H_0 = \sum_k [\varepsilon_\uparrow(k)c_{k,\uparrow}^\dagger c_{k,\uparrow} + \varepsilon_\downarrow(k)c_{k,\downarrow}^\dagger c_{k,\downarrow}] \ , \tag{3.1}$$

where

$$\varepsilon_\uparrow(k) = -2\cos k - \frac{h}{2} - \mu \ ,$$
$$\varepsilon_\downarrow(k) = -2\cos k + \frac{h}{2} - \mu \ .$$

This energy spectrum is shown in Fig. 1(a). The Fermi energy will cut both branches at two points and this will define four Fermi points at momenta $\pm k_{F\uparrow}$ and $\pm k_{F\downarrow}$. Since in the weak-coupling situation states near the Fermi energy only are important, the spectrum can be linearized around the Fermi points. This linearized spectrum is shown in Fig. 1(b). The slope will define the four Fermi velocities $\pm v_\uparrow$ and $\pm v_\downarrow$. The creation operators on the four branches will be denoted by $a_{k,R\uparrow}^\dagger$, $a_{k,L\uparrow}^\dagger$, $a_{k,R\downarrow}^\dagger$, and $a_{k,L\downarrow}^\dagger$, respectively, where R and L stand for right- and left-moving particles.

The interaction U will couple particles on the same branch as well as between different branches. Using the language of the g-ology model[6] the interaction between particles on the same branch is a g_4 process, while interaction between particles on different branches can give backward-scattering (g_1), forward-scattering (g_2), or umklapp processes (g_3). In the most general, spin-dependent case the coupling strength will depend on whether the incoming particles have identical or opposite spins. The corresponding coupling are denoted by $g_\parallel$ and $g_\perp$, respectively. However, since in the presence of a magnetic field the Fermi velocities, $v_\uparrow$ and $v_\downarrow$ and the Fermi momenta $k_{F\uparrow}$ and $k_{F\downarrow}$ are different, the couplings $g_\parallel$ will be further split, into $g_\uparrow$ and $g_\downarrow$, depending on whether both incoming particles have spin $\uparrow$ or $\downarrow$.

Fourier transformation of the on-site Coulomb coupling gives identical values for the backward- and forward-scattering processes. This coupling exists, however, between electrons of opposite spin only. Thus the Hubbard model, in its weak-coupling limit and in zero magnetic field is equivalent to a g-ology model in which $g_{1\perp} = g_{2\perp} = g_{4\perp} = U$ and $g_{1\parallel} - g_{2\parallel} = g_{4\parallel} = 0$. In the general, non-half-filled case $g_3 = 0$.

The g-ology model has been studied in detail by a renormalization-group treatment.[6] Under a scaling transformation of the bandwidth the bare couplings get renormalized. It has been found that in this way the backward scattering processes can be eliminated since they scale to zero if originally they were repulsive. Assuming that the band is not half-filled, the umklapp scattering processes can also be neglected. It is therefore expected that the fixed-point Hamiltonian of the repulsive Hubbard model is a model in which g_2- and g_4-type processes only are present. This is the Tomonaga-Luttinger model.

In the most general case the interaction between the fermions in the Tomonaga-Luttinger model can be written in the form

$$\begin{aligned}
H_{\rm int} = {} & \frac{g_{2\uparrow}}{N} \sum_{k,k',q} a^\dagger_{k,R\uparrow} a^\dagger_{k',L\uparrow} a_{k'+q,L\uparrow} a_{k-q,R\uparrow} + \frac{g_{2\downarrow}}{N} \sum_{k,k',q} a^\dagger_{k,R\downarrow} a^\dagger_{k',L\downarrow} a_{k'+q,L\downarrow} a_{k-q,R\downarrow} \\
& + \frac{g_{2\perp}}{N} \sum_{k,k',q} \left(a^\dagger_{k,R\uparrow} a^\dagger_{k',L\downarrow} a_{k'+q,L\downarrow} a_{k-q,R\uparrow} + a^\dagger_{k,R\downarrow} a^\dagger_{k',L\uparrow} a_{k'+q,L\uparrow} a_{k-q,R\downarrow} \right) \\
& + \frac{g_{4\uparrow}}{2N} \sum_{k,k',q} \left(a^\dagger_{k,R\uparrow} a^\dagger_{k',R\uparrow} a_{k'+q,R\uparrow} a_{k-q,R\uparrow} + a^\dagger_{k,L\uparrow} a^\dagger_{k',L\uparrow} a_{k'+q,L\uparrow} a_{k-q,L\uparrow} \right) \\
& + \frac{g_{4\downarrow}}{2N} \sum_{k,k',q} \left(a^\dagger_{k,R\downarrow} a^\dagger_{k',R\downarrow} a_{k'+q,R\downarrow} a_{k-q,R\downarrow} + a^\dagger_{k,L\downarrow} a^\dagger_{k',L\downarrow} a_{k'+q,L\downarrow} a_{k-q,L\downarrow} \right) \\
& + \frac{g_{4\perp}}{N} \sum_{k,k',q} \left(a^\dagger_{k,R\uparrow} a^\dagger_{k',R\downarrow} a_{k'+q,R\downarrow} a_{k-q,R\uparrow} + a^\dagger_{k,L\uparrow} a^\dagger_{k',L\downarrow} a_{k'+q,L\downarrow} a_{k-q,L\uparrow} \right) .
\end{aligned} \tag{3.2}$$

It is easily seen that in the form given above the interaction is in fact between the densities on the different branches. Introducing density operators

$$\rho_\lambda(q) = \sum_k a^\dagger_{k+q,\lambda} a_{k,\lambda} , \tag{3.3a}$$

$$\rho_\lambda(-q) = \sum_k a^\dagger_{k,\lambda} a_{k+q,\lambda} , \tag{3.3b}$$

where λ stands for $R\uparrow$, $L\uparrow$, $R\downarrow$, or $L\downarrow$, the Hamiltonian describing the interaction is

$$\begin{aligned}
H_{\rm int} = {} & \frac{g_{2\uparrow}}{N} \sum_q \rho_{R\uparrow}(q) \rho_{L\uparrow}(-q) + \frac{g_{2\downarrow}}{N} \sum_q \rho_{R\downarrow}(q) \rho_{L\downarrow}(-q) \\
& + \frac{g_{2\perp}}{N} \sum_q [\rho_{R\uparrow}(q)\rho_{L\downarrow}(-q) + \rho_{R\downarrow}(q)\rho_{L\uparrow}(-q)] + \frac{g_{4\uparrow}}{2N} \sum_q [\rho_{R\uparrow}(q)\rho_{R\uparrow}(-q) + \rho_{L\uparrow}(q)\rho_{L\uparrow}(-q)] \\
& + \frac{g_{4\downarrow}}{2N} \sum_q [\rho_{R\downarrow}(q)\rho_{R\downarrow}(-q) + \rho_{L\downarrow}(q)\rho_{L\downarrow}(-q)] + \frac{g_{4\perp}}{N} \sum_q [\rho_{R\uparrow}(q)\rho_{R\downarrow}(-q) + \rho_{L\uparrow}(q)\rho_{L\downarrow}(-q)] .
\end{aligned} \tag{3.4}$$

In the case when no external magnetic field is applied, the scaling theory can also predict the values of these fixed-point couplings. The combination $g_{1\parallel} - g_{2\parallel} - g_{2\perp}$, which in our case is equal to $-U$, is an exact invariant of the g-ology model. On the other hand, the scaling equations show that the coupling constant combination $g_{1\parallel} - g_{2\parallel} + g_{2\perp}$ and $g_{1\perp}$ scale together to zero if originally they are equal and positive, as is the case for the repulsive Hubbard model. The fixed-point couplings satisfy

$$g^*_{1\perp} = 0 , \tag{3.5}$$

$$g^*_{1\parallel} - g^*_{2\parallel} + g^*_{2\perp} = 0 , \tag{3.6}$$

$$g^*_{1\parallel} - g^*_{2\parallel} - g^*_{2\perp} = -U . \tag{3.7}$$

Since $g_{1\parallel}$ and $g_{2\parallel}$ are physically indistinguishable, we are free to choose the backscattering as $g^*_{1\parallel} = 0$ and so

$$g^*_{1\parallel} = g^*_{1\perp} = 0 , \quad g^*_{2\parallel} = g^*_{2\perp} = U/2 . \tag{3.8}$$

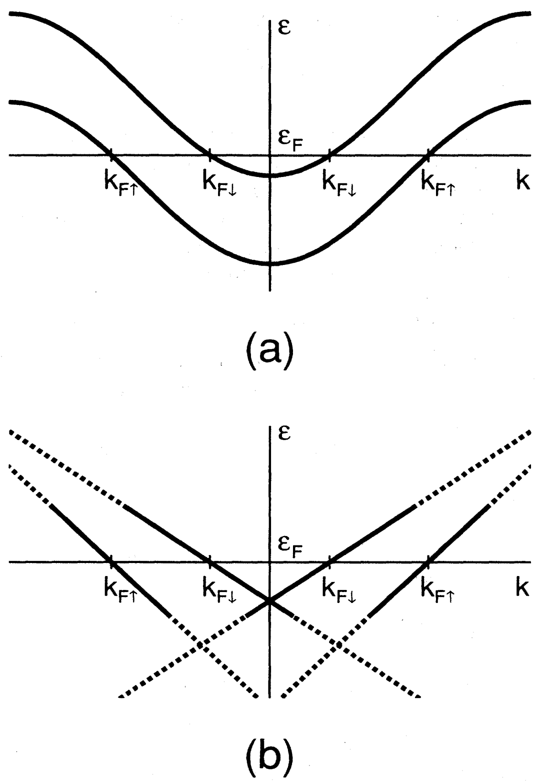

FIG. 1. (a) The energy spectrum of free fermions on a lattice in an external magnetic field. (b) The linearized spectrum of noninteracting fermions with velocities $\pm v_\uparrow$ and $\pm v_\downarrow$ of the corresponding Tomonaga-Luttinger model. The Fermi momenta $\pm k_{F\uparrow}$ and $\pm k_{F\downarrow}$ are also shown.

The couplings g_4 do not get renormalized to lowest order and they can be incorporated into a Fermi velocity renormalization giving

$$v_c = v_F + \frac{1}{2\pi}(g_{4\parallel} + g_{4\perp}) , \quad v_s = v_F + \frac{1}{2\pi}(g_{4\parallel} - g_{4\perp}) . \tag{3.9}$$

In the present case, where $g_{4\parallel} = 0$ and $g_{4\perp} = U$, the renormalized velocities will be

$$v_c = v_F + \frac{U}{2\pi} , \quad v_s = v_F - \frac{U}{2\pi} . \tag{3.10}$$

The above considerations are valid for $h = 0$. The scaling theory can be generalized to the g-ology model in magnetic field. This cannot, however, be equivalent to the Hubbard model in magnetic field, since in the latter model the Fermi velocities are different for the two spin orientations. In the next section we will introduce a generalized Tomonaga-Luttinger model with several kinds of particles, each having different velocities, and will calculate exactly its properties.

IV. ENERGY SPECTRUM OF THE MULTICOMPONENT TOMONAGA-LUTTINGER MODEL

As we have seen in the preceding section the Hubbard model in magnetic field is expected to be equivalent to a Tomonaga-Luttinger-like model. The four branches of the bare spectrum, labeled by $R\uparrow$, $L\uparrow$, $R\downarrow$, and $L\downarrow$ have different but pairwise opposite Fermi velocities and Fermi momenta. Here we will define a generalized, multicomponent Tomonaga-Luttinger model and will show how its spectrum and correlation functions can be determined exactly.

Instead of the spin index a "color" index λ will be introduced, where the number of colors can be arbitrary. For each color λ there will be right- and left-moving fermions with opposite Fermi velocities and opposite Fermi momenta,

$$v_{L,\lambda} = -v_{R,\lambda} , \quad k_{F,L,\lambda} = -k_{F,R,\lambda} . \tag{4.1}$$

In what follow λ will stand both for the color and the R, L index.

The Hamiltonian will be assumed in the form

$$H = \sum_\lambda H_\lambda + \sum_{\lambda\lambda'} H_{\lambda\lambda'} , \tag{4.2}$$

where H_λ is the kinetic energy of free particles of color λ and $H_{\lambda\lambda'}$ describes the interaction between particles of color λ and λ'. Assuming a linear dispersion relation with Fermi velocity v_λ and Fermi momentum $k_{F\lambda}$, H_λ can be written as

$$H_\lambda = \sum_k v_\lambda (k - k_{F\lambda}) a^\dagger_{k,\lambda} a_{k,\lambda} , \tag{4.3}$$

where $a^\dagger_{k,\lambda}$ and $a_{k,\lambda}$ are the creation and annihilation operators, respectively of fermions of color λ.

As a generalization of Eqs. (3.2) and (3.4) the interaction $H_{\lambda\lambda'}$ is supposed to exist between the densities of fermions of color λ and λ',

$$\begin{aligned} H_{\lambda\lambda'} &= \frac{1}{2N} \sum_{k,k',q} g_{\lambda\lambda'}(q) a^\dagger_{k,\lambda} a^\dagger_{k',\lambda'} a_{k'+q,\lambda'} a_{k-q,\lambda} \\ &= \frac{1}{2N} \sum_q g_{\lambda\lambda'}(q) \rho_\lambda(q) \rho_{\lambda'}(-q) , \end{aligned} \tag{4.4}$$

where N is the number of sites in the chain and the coupling constants $g_{\lambda\lambda'}$ are symmetric in the color indices $g_{\lambda\lambda'} = g_{\lambda'\lambda}$. The momentum transfer in the scattering processes is limited by choosing $g(k) = g e^{-|k|/\Lambda}$, with the momentum transfer cutoff $\Lambda \ll k_{F\lambda}$, i.e., only small-momentum-transfer processes are allowed.

Following Mattis and Lieb,[4] let us introduce the density operators defined in Eqs. (3.3a) and (3.3b). They obey the commutation relations

$$\begin{aligned} [\rho_\lambda(-q), \rho_{\lambda'}(q')] &= \frac{qN}{2\pi} \delta_{\lambda,\lambda'} \delta_{q,q'} \operatorname{sgn} v_\lambda , \\ [\rho_\lambda(-q), \rho_{\lambda'}(-q')] &= [\rho_\lambda(q), \rho_{\lambda'}(q')] = 0 . \end{aligned} \tag{4.5}$$

Mattis and Lieb have shown that not only the interaction is bilinear in the densities, but the kinetic energy part of the Hamiltonian Eq. (4.3) can also be written in terms of the density operators

$$H'_\lambda = \frac{2\pi}{N}\sum_{q>0}|v_\lambda| \times \begin{cases}\rho_\lambda(q)\rho_\lambda(-q) & \text{if } v_\lambda > 0\\ \rho_\lambda(-q)\rho_\lambda(q) & \text{if } v_\lambda < 0\,.\end{cases} \qquad (4.6)$$

In what follows we will use the notation H_λ for the kinetic energy when it is written in its original form Eq. (4.3) and the notation H'_λ when written in terms of the densities. Both H_λ and H'_λ obey the same commutation relation with the density operators

$$[H_\lambda, \rho_\lambda(\pm q)] = [H'_\lambda, \rho_\lambda(\pm q)] = \pm q v_\lambda \rho_\lambda(\pm q)\,, \qquad (4.7)$$

and with the interaction, so they can only differ in a shift of the ground-state energy.

Haldane[5] has pointed out that the expression (4.6) is valid in the Hilbert space of a fixed number of particles. An extra term should appear in the energy if the number of particles is changed. If ΔN_λ particles of type λ are added to the system, the additional energy is

$$\Delta E = (\pi/N)|v_\lambda|(\Delta N_\lambda)^2\,. \qquad (4.8)$$

In an operator form

$$\Delta N_\lambda = \sum_k [a^\dagger_{k,\lambda} a_{k,\lambda} - \Theta(k_{F\lambda} - k\,\mathrm{sgn}\,v_\lambda)]\,, \qquad (4.9)$$

where $\Theta(x)$ is the Heaviside step function. Adding this term to Eq. (4.6) and neglecting a constant term coming from commuting the density operators, the kinetic energy is equivalent to

$$H'_\lambda = \frac{\pi}{N}|v_\lambda|(\Delta N_\lambda)^2 + \frac{2\pi}{N}\sum_{q>0}|v_\lambda|\rho_\lambda(q)\rho_\lambda(-q)\,. \qquad (4.10)$$

As mentioned before, the interaction between the fermions is in fact an interaction between the densities, so it is straightforward to express the interacting part of the Hamiltonian in a bilinear form of the densities as given in Eq. (4.4). Analogous to Eq. (4.10), the $q > 0$, $q = 0$, and $q < 0$ parts will be treated separately. Regularizing the $q = 0$ part in the interaction by subtracting the infinite ground-state density [using ΔN_λ of Eq. (4.9) for $\rho_\lambda(q = 0)$], the full Hamiltonian of the model is

$$H = \frac{2\pi}{N}\sum_{\lambda,\lambda',q>0} A_{\lambda\lambda'}\rho_\lambda(q)\rho_{\lambda'}(-q) + \frac{\pi}{N}\sum_{\lambda,\lambda'} A_{\lambda\lambda'}\Delta N_\lambda \Delta N_{\lambda'}\,, \qquad (4.11)$$

where

$$\mathrm{A}_{\lambda\lambda'} = |v_\lambda|\delta_{\lambda\lambda'} + \frac{g_{\lambda\lambda'}}{2\pi}\,. \qquad (4.12)$$

Introducing the vectors $|\rho(q)\rangle$ and $|\Delta N\rangle$ with components $\rho_\lambda(q)$ and ΔN_λ, respectively, and the matrix $\underline{A}$ defined by the matrix elements $A_{\lambda\lambda'}$, the Hamiltonian can be written in a shorthand notation as

$$H = \frac{2\pi}{N}\sum_{q>0}\langle\rho(q)|\underline{A}|\rho(q)\rangle + \frac{\pi}{N}\langle\Delta N|\underline{A}|\Delta N\rangle\,. \qquad (4.13)$$

The usual procedure is to diagonalize this Hamiltonian by a canonical transformation. Instead we will use a different procedure. Let us suppose that the new densities $\tilde{\rho}_j(\pm q)$ that diagonalize the first part of the Hamiltonian can be obtained from the densities $\rho_\lambda(\pm q)$ by multiplying the column vector $|\rho(q)\rangle$ by the vectors $|w^{(j)}\rangle$, i.e.,

$$\tilde{\rho}_j(\pm q) = \langle w^{(j)}|\rho(\pm q)\rangle\,. \qquad (4.14)$$

We require that the first part of the Hamiltonian (4.13) be the sum of the contribution of these decoupled modes,

$$H = \frac{2\pi}{N}\sum_{q>0}\sum_j |u_j|\tilde{\rho}_j(q)\tilde{\rho}_j(-q) + \frac{\pi}{N}\langle\Delta N|\underline{A}|\Delta N\rangle\,, \qquad (4.15)$$

where u_j is the velocity of the jth mode. The matrix $\underline{A}$ can then be written as

$$\underline{A} = \sum_j |u_j||w^{(j)}\rangle\langle w^{(j)}|\,. \qquad (4.16)$$

Since the matrix $\underline{A}$ is symmetric and real, this equation can be satisfied with real vectors $|w^{(j)}\rangle$ and real velocities u_j unless the couplings $g_{\lambda\lambda'}/2\pi$ are of the order of or larger than $|v_\lambda|$. We will not consider this strong-coupling situation, where different kinds of instabilities may occur, since, as we will see, even in the strong-coupling limit of the Hubbard model the couplings of the equivalent Tomonaga-Luttinger model are less than the velocities.

Using the commutation relations for ρ_λ given in Eq. (4.5), the new density operators $\tilde{\rho}_j$ satisfy the relations

$$[\tilde{\rho}_j(-q), \tilde{\rho}_{j'}(q')] = \frac{qN}{2\pi}\delta_{q,q'}\langle w^{(j)}|\underline{B}|w^{(j')}\rangle\,,$$
$$[\tilde{\rho}_j(-q), \tilde{\rho}_{j'}(-q')] = [\tilde{\rho}_j(q), \tilde{\rho}_{j'}(q')] = 0\,, \qquad (4.17)$$

where the elements of the matrix $\underline{B}$ are

$$B_{\lambda\lambda'} = \delta_{\lambda\lambda'}\,\mathrm{sgn}\,v_\lambda\,. \qquad (4.18)$$

The matrices $\underline{A}$ and $\underline{B}$ do not commute; futhermore $\underline{B}^2 = \underline{1}$, where $\underline{1}$ denotes the unit matrix. Since we want the modes $\tilde{\rho}_j$ to satisfy the usual commutation relations, we could try to require

$$\langle w^{(j)}|\underline{B}|w^{(j')}\rangle = \delta_{jj'}\,. \qquad (4.19)$$

It turns out that this is not possible because $\underline{B}$ is not positive definite. Some of the diagonal elements have to be -1. Analogous to the first equation of (4.5), we will assign these negative signs to the respective velocities u_j, writing the orthogonality relations of the $|w^{(j)}\rangle$ vectors in the form

$$\langle w^{(j)}|\underline{B}|w^{(j')}\rangle = \delta_{jj'}\,\mathrm{sgn}\,u_j\,. \qquad (4.20)$$

This shows that the diagonalization of the Hamiltonian with this orthogonality constraint is equivalent to the generalized eigenvalue problem

$$\underline{A}|\tilde{w}^{(j)}\rangle = u_j\underline{B}|\tilde{w}^{(j)}\rangle \ , \tag{4.21}$$

where

$$|\tilde{w}^{(j)}\rangle = \underline{B}|w^{(j)}\rangle \ . \tag{4.22}$$

It follows from the commutation relations (4.17) that the properly normalized density operators

$$b^\dagger_{q,j} = \sqrt{\frac{2\pi}{qN}}\tilde{\rho}_j(q), \quad b_{q,j} = \sqrt{\frac{2\pi}{qN}}\tilde{\rho}_j(-q) \ , \tag{4.23}$$

have the properties of boson creation and annihilation operators if $u_j > 0$ and $q > 0$, while for $u_j < 0$ these operators have to be defined by

$$b^\dagger_{-q,j} = \sqrt{\frac{2\pi}{qN}}\tilde{\rho}_j(-q) \quad \text{and} \quad b_{-q,j} = \sqrt{\frac{2\pi}{qN}}\tilde{\rho}_j(q) \ , \tag{4.24}$$

again for $q > 0$. For the symmetric model [see Eq. (4.1)] the velocities u_j are also symmetric, that is, there are pairs j and j' for which $u_j = -u_{j'}$. The corresponding operators $b^\dagger_{q,j}$ and $b^\dagger_{-q,j'}$ can then be considered as the creation operators of a single bosonic field defined for all values of q.

Introducing the diadic matrices

$$\underline{\alpha}^{(j)} = |w^{(j)}\rangle\langle w^{(j)}| \ , \tag{4.25}$$

with matrix elements

$$\alpha^{(j)}_{\lambda\lambda'} = w^{(j)}_\lambda w^{(j)}_{\lambda'} = \tilde{w}^{(j)}_\lambda \tilde{w}^{(j)}_{\lambda'} \operatorname{sgn}(v_\lambda v_{\lambda'}) \ , \tag{4.26}$$

the spectral decomposition of $\underline{A}$ in Eq. (4.16) takes the simple form

$$\underline{A} = \sum_j |u_j|\underline{\alpha}^{(j)} \ , \tag{4.27}$$

while the eigenvalue equation (4.21) gives

$$\underline{B} = \sum_j |w^{(j)}\rangle\langle w^{(j)}| \operatorname{sgn} u_j = \sum_j \underline{\alpha}^{(j)} \operatorname{sgn} u_j \ . \tag{4.28}$$

Using the spectral decomposition of $\underline{A}$ in Eq. (4.27), the second term in Eq. (4.15) can also be diagonalized. We get

$$H = \sum_{q>0,j} |u_j| q \times \begin{cases} b^\dagger_{q,j} b_{q,j} & \text{if } u_j > 0 \\ b^\dagger_{-q,j} b_{-q,j} & \text{if } u_j < 0 \end{cases}$$
$$+\frac{2\pi}{N}\sum_j |u_j| \frac{\langle \Delta N|\underline{\alpha}^{(j)}|\Delta N\rangle}{2} \ . \tag{4.29}$$

The momentum of the system can also be calculated easily. The addition of ΔN_λ fermions changes the Fermi momentum $k_{F\lambda}$ to $k_{F\lambda}+2\pi(\Delta N_\lambda/N)\operatorname{sgn} v_\lambda$. The change in the momentum is obtained by multiplying the average Fermi momentum by the number of added fermions ΔN_λ. The momentum of the bosonic excitations can be written in the usual form, and the total momentum is

$$P = \sum_\lambda \left(k_{F\lambda} + \pi\frac{\Delta N_\lambda}{N}\operatorname{sgn} v_\lambda\right)\Delta N_\lambda$$
$$+ \sum_{q>0,j} \times \begin{cases} q b^\dagger_{q,j} b_{q,j} & \text{if } u_j > 0 \\ -q b^\dagger_{-q,j} b_{-q,j} & \text{if } u_j < 0 \ . \end{cases} \tag{4.30}$$

The term $\sum_\lambda (\Delta N)^2_\lambda \operatorname{sgn} v_\lambda$ can be written in terms of the matrix $\underline{B}$ as $\langle \Delta N|\underline{B}|\Delta N\rangle$. Using the spectral decomposition of $\underline{B}$ in Eq. (4.28), the momentum can be written as

$$P = \sum_\lambda k_{F\lambda}\Delta N_\lambda + \frac{2\pi}{N}\sum_j \frac{\langle \Delta N|\underline{\alpha}^{(j)}|\Delta N\rangle}{2}\operatorname{sgn} u_j$$
$$+ \sum_{q>0,j} \times \begin{cases} q b^\dagger_{q,j} b_{q,j} & \text{if } u_j > 0 \\ -q b^\dagger_{-q,j} b_{-q,j} & \text{if } u_j < 0 \ . \end{cases} \tag{4.31}$$

Since the momentum of the bosonic excitations is quantized in units of $2\pi/N$, the quantity $\sum_q q b^\dagger_{q,j} b_{q,j}$ can take values that are integer multiples of $2\pi/N$. Denoting by n_j this integer number for the bosons with velocity u_j, the energy and momentum of the system take the form

$$E(\Delta N_\lambda, n_j) - E_0 = \frac{2\pi}{N}\sum_j |u_j|(\Delta^{(j)} + n_j) + O\left(\frac{1}{N}\right) \tag{4.32}$$

and

$$P(\Delta N_\lambda, n_j) = \frac{2\pi}{N}\sum_j (\Delta^{(j)} + n_j)\operatorname{sgn} u_j$$
$$+\sum_\lambda \Delta N_\lambda k_{F\lambda} + O\left(\frac{1}{N}\right), \tag{4.33}$$

where

$$\Delta^{(j)} = \langle \Delta N|\underline{\alpha}^{(j)}|\Delta N\rangle/2$$
$$= \langle w^{(j)}|\Delta N\rangle^2/2 \ . \tag{4.34}$$

Thus the excitation spectrum has the usual tower structure and the towers are characterized by the parameters ΔN_λ forming the vector $|\Delta N\rangle$.

These expressions for the energy and momentum Eqs. (4.32) and (4.33) should be compared to those expected in conformal field theories or to those obtained for the Hubbard model given in Eqs. (2.26) and (2.27), taking into account that if the noninteracting system was symmetric ($v_{R,\lambda} = -v_{L,\lambda}$), then the collective mode velocities are also such that each mode has its counterpart with opposite velocity.

Using the language of the conformal field theory, the quantities $\Delta^{(j)}$ can be associated with the anomalous dimensions of the primary fields ϕ. The operator ϕ determines the parameters ΔN_λ and thereby selects the towers that contribute to the correlation functions. The contribution of a given tower to the correlation function is

$$\langle\phi(x,t)\phi(0,0)\rangle \sim \frac{\exp\left(-ix\sum_\lambda k_{F\lambda}\Delta N_\lambda\right)}{\prod_j (x-u_j t)^{2\Delta^{(j)}}} . \qquad (4.35)$$

This assignment of $\Delta^{(j)}$ to be the critical exponent, although straightforward, is not rigorous. Since the velocity u_j appears in Eq. (4.32) in absolute value, the contribution of the modes with opposite velocities appear in the expression for the energy as a sum. Their separation into positive and negative components could be attempted using the expression for the momentum where, due to the factor $\operatorname{sign} u_j$, their difference appears. As we will see, and was emphasized by Frahm and Korepin[15] for the Hubbard model, this separation is not *a priori* possible, since in general the contribution of all modes are mixed in the expression for $\Delta^{(j)}$. The critical exponents cannot be determined unambiguously. It is therefore of interest to calculate these correlation functions directly, using an independent method. It will be shown that, indeed, the critical exponents are $\langle\Delta N|\underline{\alpha}^{(j)}|\Delta N\rangle$.

V. GENERALIZED CORRELATION FUNCTIONS OF THE MULTICOMPONENT TOMONAGA-LUTTINGER MODEL

The Green's function of the fermions has been calculated exactly by Dzyaloshinsky and Larkin.[16] They have shown that due to the linearity of the spectrum and due to the absence of large-momentum-transfer processes, all diagrams containing internal fermion loops with more than two interaction legs cancel. As a consequence, the diagrams that give nonvanishing contribution are very simple: the incoming fermion lines pass through the diagram conserving their color. These lines are coupled in all possible ways by effective interactions which are obtained by dressing the bare interactions with polarization bubbles.

Another important ingredient in the calculation of the Green's function was the proof by Dzyaloshinsky and Larkin[16] of an exact Ward identity for general momenta. For an alternative proof of the theorem using more elementary arguments see Ref. 17. This Ward identity is a consequence of a special conservation law of the Tomonaga-Luttinger model. In the absence of backward scattering and umklapp processes the particle number is conserved on each branch of the spectrum for each spin. In the case of a linear energy-momentum relation the particle-number conservation goes together with the current conservation for each branch. Two-particle Green's functions or correlation functions can also be calculated exactly using generalized Ward identities as shown by Sólyom.[6,18]

The same arguments are also valid in the multicomponent case as well, since the color current is conserved in the scattering processes described by the interaction Hamiltonian (4.4). Following Refs. 6, 16, and 17 it is possible to introduce effective interactions and to prove the generalized Ward identities by considering the diagrammatic expansion of the quantities. This allows us to write the Dyson equation for the correlation functions in a closed form and to calculate the exact form of the most general correlation function of fermions in the multicomponent Tomonaga-Luttinger model.

Alternatively, as was shown by Everts and Schulz,[19] the equation of motion method can be used, since a closed set of equations can be obtained in two steps. The same procedure has been applied by Di Castro and Metzner[20] in their study of the properties of Luttinger liquids.

Let us consider the most general correlation function

$$G_{\lambda_1\lambda_2\cdots\lambda_m}(\mathbf{x}'_1,\mathbf{x}'_2,\ldots,\mathbf{x}'_m;\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_m) = (-i)^m\langle \mathcal{T}\Psi_{\lambda_1}(\mathbf{x}'_1)\Psi_{\lambda_2}(\mathbf{x}'_2)\cdots\Psi_{\lambda_m}(\mathbf{x}'_m)\Psi^\dagger_{\lambda_m}(\mathbf{x}_m)\cdots\Psi^\dagger_{\lambda_2}(\mathbf{x}_2)\Psi^\dagger_{\lambda_1}(\mathbf{x}_1)\rangle , \qquad (5.1)$$

where $\Psi_\lambda(\mathbf{x})$ is the field operator of fermions of color λ, $\mathbf{x}$ is a shorthand notation for the space-time point (x,t), and $\mathcal{T}$ is the time ordering operator. For simplicity we will assume that the color indices $\lambda_1,\lambda_2,\ldots,\lambda_m$ are all different, although the results can be easily generalized by proper antisymmetrization to the case, when several particles of the same color are present.

The equation of motion is derived by taking the derivative of G with respect to the time variable t_m of the operator $\Psi^\dagger_{\lambda_m}(\mathbf{x}_m)$. For this we write first the equation of motion for $\Psi^\dagger_\lambda(x,t)$, $\partial\Psi^\dagger_\lambda(x,t)/\partial t = i[H,\Psi^\dagger_\lambda(x,t)]$. Instead of using the fermionic representation both for the kinetic energy and the interaction Eqs. (4.3) and (4.4), respectively, it is convenient rewrite the Hamiltonian as

$$H = \sum_\lambda H_\lambda + \left(H - \sum_\lambda H'_\lambda\right) , \qquad (5.2)$$

where for H_λ we use the fermionic form [Eq. (4.3)], while for H and H'_λ the diagonalized forms in terms of bosonic operators $\tilde{\rho}_j$ is applied. Then

$$\begin{aligned}\frac{\partial}{\partial t}\Psi^\dagger_\lambda(x,t) &= i[H_\lambda,\Psi^\dagger_\lambda(x,t)] + i[H-H'_\lambda,\Psi^\dagger_\lambda(x,t)]\\ &= -v_\lambda\frac{\partial}{\partial x}\Psi^\dagger_\lambda(x,t) + i[H-H'_\lambda,\Psi^\dagger_\lambda(x,t)] .\end{aligned} \qquad (5.3)$$

Since the Hamiltonian is written in terms of $\tilde{\rho}_j$ we need the commutators

$$\begin{aligned}[\tilde{\rho}_j(\pm q,t'),\Psi_\lambda(x,t)] &= -\delta(t-t')e^{\pm iqx}w^{(j)}_\lambda\Psi_\lambda(x,t) ,\\ [\tilde{\rho}_j(\pm q,t'),\Psi^\dagger_\lambda(x,t)] &= \delta(t-t')e^{\pm iqx}w^{(j)}_\lambda\Psi^\dagger_\lambda(x,t) ,\end{aligned} \qquad (5.4)$$

which are obtained from the relation between $\tilde{\rho}_j$ and ρ_λ in Eq. (4.14) and from the usual canonical relations between the fermion creation and annihilation operators and densities.

Using Eq. (4.15) for the full Hamiltonian, the commutator gives

$$[H,\Psi_\lambda^\dagger(x,t)]=\frac{2\pi}{N}\sum_{q>0,j}|u_j|w_\lambda^{(j)}[e^{iqx}\Psi_\lambda^\dagger(x,t)\tilde\rho_j(-q,t)+e^{-iqx}\tilde\rho_j(q,t)\Psi_\lambda^\dagger(x,t)]$$
$$=\frac{2\pi}{N}\sum_{q,j}|u_j|e^{-iqx}w_\lambda^{(j)}\Psi_\lambda^\dagger(x,t)\tilde\rho_j(q,t)\ , \qquad (5.5)$$

where we have neglected the commutator of $\Psi_\lambda^\dagger(x,t)$ and $\tilde\rho_j(q,t)$ that yields a shift of the ground-state energy.

In order to calculate the commutator with H'_λ given in Eq. (4.10), we express ρ_λ in terms of $\tilde\rho_j$. Using the spectral decomposition of $\underline{B}$ [Eqs. (4.18) and (4.20)],

$$\rho_\lambda(\pm q,t)\,\mathrm{sgn}\,v_\lambda=\sum_j w_\lambda^{(j)}\tilde\rho_j(\pm q,t)\,\mathrm{sgn}\,u_j\ . \qquad (5.6)$$

With this form we find

$$[H'_{\lambda'},\Psi_\lambda^\dagger(x,t)]=\frac{2\pi}{N}\sum_{q>0}|v_\lambda|\delta_{\lambda\lambda'}w_\lambda^{(j)}\,\mathrm{sgn}\,(u_jv_\lambda)[e^{iqx}\Psi_\lambda^\dagger(x,t)\tilde\rho_j(-q,t)+e^{-iqx}\tilde\rho_j(q,t)\Psi_\lambda^\dagger(x,t)]$$
$$=\frac{2\pi}{N}\sum_{q}|v_\lambda|\delta_{\lambda\lambda'}e^{-iqx}w_\lambda^{(j)}\tilde\rho_j(q,t)\Psi_\lambda^\dagger(x,t)\,\mathrm{sgn}\,(u_jv_\lambda)\ . \qquad (5.7)$$

Inserting these expressions into Eq. (5.3) we get

$$\left(\frac{\partial}{\partial t}+v_\lambda\frac{\partial}{\partial x}\right)\Psi_\lambda^\dagger(x,t)=i\frac{2\pi}{N}\sum_{q,j}(u_j-v_\lambda)e^{-iqx}w_\lambda^{(j)}\tilde\rho_j(q,t)\Psi_\lambda^\dagger(x,t)\,\mathrm{sgn}\,u_j\ . \qquad (5.8)$$

It is straightforward now to write the equation of motion for the correlation functions

$$\left(\frac{\partial}{\partial t_m}+v_{\lambda_m}\frac{\partial}{\partial x_m}\right)G_{\lambda_1\cdots\lambda_m}(\mathbf{x}'_1,\ldots,\mathbf{x}'_m;\mathbf{x}_1,\ldots,\mathbf{x}_m)$$
$$=i\delta(x'_m-x_m)\delta(t'_m-t_m)G_{\lambda_1\cdots\lambda_{m-1}}(\mathbf{x}'_1,\ldots,\mathbf{x}'_{m-1};\mathbf{x}_1,\ldots,\mathbf{x}_{m-1})$$
$$+i\frac{2\pi}{N}\sum_{q,j}\mathrm{sgn}\,u_j(u_j-v_{\lambda_m})e^{-iqx_m}w_{\lambda_m}^{(j)}F_{\lambda_1\cdots\lambda_m}^{(j)}(\mathbf{x}'_1,\ldots,\mathbf{x}'_m;\mathbf{x}_1,\ldots,\mathbf{x}_m;q,t_m)\ , \qquad (5.9)$$

where $F_{\lambda_1\cdots\lambda_m}^{(j)}$ is defined by

$$F_{\lambda_1\cdots\lambda_m}^{(j)}(\mathbf{x}'_1,\ldots,\mathbf{x}'_m;\mathbf{x}_1,\ldots,\mathbf{x}_m;q,t)=(-i)^{m+1}\langle T\Psi_{\lambda_1}(\mathbf{x}'_1)\cdots\Psi_{\lambda_m}(\mathbf{x}'_m)\tilde\rho_j(q,t)\Psi_{\lambda_m}^\dagger(\mathbf{x}_m)\cdots\Psi_{\lambda_1}^\dagger(\mathbf{x}_1)\rangle. \qquad (5.10)$$

As a next step we calculate the equation of motion for the generated new quantity. H is now used in its bosonic diagonal form (4.15). The singularities due to the time ordering produce the original Green's function; thus

$$\frac{\partial}{\partial t}F_{\lambda_1\cdots\lambda_m}^{(j)}(\mathbf{x}'_1,\ldots,\mathbf{x}'_m;\mathbf{x}_1,\ldots,\mathbf{x}_m;q,t)$$
$$=iqu_jF_{\lambda_1\cdots\lambda_m}^{(j)}(\mathbf{x}'_1,\ldots,\mathbf{x}'_m;\mathbf{x}_1,\ldots,\mathbf{x}_m;q,t)$$
$$+i\sum_{l=1}^{m}[e^{iqx'_l}\delta(t-t'_l)-e^{iqx_l}\delta(t-t_l)]w_{\lambda_l}^{(j)}G_{\lambda_1\cdots\lambda_m}(\mathbf{x}'_1,\ldots,\mathbf{x}'_m;\mathbf{x}_1,\ldots,\mathbf{x}_m). \qquad (5.11)$$

The hierarchy of equations terminates in this step, no further quantities are generated. Fourier transformation to frequency ω and inverse Fourier transforming back to the time variable t gives

$$F_{\lambda_1\cdots\lambda_m}^{(j)}(\mathbf{x}'_1,\ldots,\mathbf{x}'_m;\mathbf{x}_1,\ldots,\mathbf{x}_m;q,t)=\sum_{l=1}^{m}\int\frac{d\omega}{2\pi}[e^{i[qx'_l-\omega(t'_l-t)]}-e^{i[qx_l-\omega(t_l-t)]}]\frac{w_{\lambda_l}^{(j)}}{\omega-u_jq}$$
$$\times G_{\lambda_1\cdots\lambda_m}(\mathbf{x}'_1,\ldots,\mathbf{x}'_m;\mathbf{x}_1,\ldots,\mathbf{x}_m)\ . \qquad (5.12)$$

Finally, inserting this into Eq. (5.9), performing the integration with respect to q and ω, introducing the kernels

$$K_{\lambda\lambda'}(x,t) = \sum_j (u_j - v_\lambda)\frac{\alpha^{(j)}_{\lambda\lambda'}}{x - u_j t + i/\Lambda\,\mathrm{sgn}\,u_j t}\,, \tag{5.13}$$

where $\alpha^{(j)}_{\lambda\lambda'}$ is defined in Eq. (4.26), we get

$$\begin{aligned}\left(\frac{\partial}{\partial t_m} + v_{\lambda_m}\frac{\partial}{\partial x_m}\right) & G_{\lambda_1\lambda_2\cdots\lambda_m}(\mathbf{x}'_1,\mathbf{x}'_2,\ldots,\mathbf{x}'_m;\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_m)\\ &= i\delta(\mathbf{x}_m-\mathbf{x}'_m)G_{\lambda_1\cdots\lambda_{m-1}}(\mathbf{x}'_1,\ldots,\mathbf{x}'_{m-1};\mathbf{x}_1,\ldots,\mathbf{x}_{m-1})\\ &\quad+\sum_{l=1}^{m}\left[K_{\lambda_m\lambda_l}(\mathbf{x}_m-\mathbf{x}'_l) - K_{\lambda_m\lambda_l}(\mathbf{x}_m-\mathbf{x}_l)\right]G_{\lambda_1\lambda_2\cdots\lambda_m}(\mathbf{x}'_1,\mathbf{x}'_2,\ldots,\mathbf{x}'_m;\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_m)\,. \end{aligned}\tag{5.14}$$

Before turning to the solution of this equation for $G_{\lambda_1\lambda_2\cdots\lambda_m}$, let us consider an auxiliary problem, the differential equation

$$(\partial_t + v_\lambda\partial_x)f_{\lambda\lambda'}(x,t) = K_{\lambda\lambda'}(x,t)f_{\lambda\lambda'}(x,t)\,. \tag{5.15}$$

Using Eq. (5.13) its solution is found in the form

$$f_{\lambda\lambda'}(x,t) = f_0(x-v_\lambda t)\prod_j\left(\frac{x-u_jt+i/\Lambda\,\mathrm{sgn}\,u_jt}{x-v_\lambda t+i/\Lambda\,\mathrm{sgn}\,u_jt}\right)^{-\alpha^{(j)}_{\lambda\lambda'}}\,, \tag{5.16}$$

where $f_0(x - v_\lambda t)$ is an as yet undetermined function with boundary condition $f_0(0) = 1$. The proper analytic properties of f can be ensured if we choose f_0 in the following form:[16]

$$f_0(x - v_\lambda t) = \prod_{\substack{j\\(u_jv_\lambda)<0}}\left[1 + (x - v_\lambda t)^2\Lambda^2\right]^{-\alpha^{(j)}_{\lambda\lambda'}}\,. \tag{5.17}$$

Inserting this into Eq. (5.16) and using Eqs. (4.18) and (4.28), we find

$$\begin{aligned}f_{\lambda\lambda'}(x,t) = \Lambda^{-2\alpha'}&\left(x - v_\lambda t + \frac{i}{\Lambda}\,\mathrm{sgn}\,v_\lambda t\right)^{\delta_{\lambda\lambda'}}\\ &\times\prod_j\left(x-u_jt+\frac{i}{\Lambda}\,\mathrm{sgn}\,u_jt\right)^{-\alpha^{(j)}_{\lambda\lambda'}}\,,\end{aligned}\tag{5.18}$$

where

$$\alpha' = \sum_{\substack{j\\(u_jv_\lambda)<0}}\alpha^{(j)}_{\lambda\lambda'}\,. \tag{5.19}$$

We use this result first to calculate the one-particle Green's function $G_\lambda(\mathbf{x}',\mathbf{x})$. It satisfies the equation

$$\begin{aligned}\left(\frac{\partial}{\partial t}+v_\lambda\frac{\partial}{\partial x}\right)&G_\lambda(\mathbf{x}',\mathbf{x})\\ &= i\delta(\mathbf{x}'-\mathbf{x}) + [K_{\lambda\lambda}(\mathbf{x}-\mathbf{x}') - K_{\lambda\lambda}(\mathbf{x}-\mathbf{x})]G_\lambda(\mathbf{x}',\mathbf{x})\,.\end{aligned}\tag{5.20}$$

The term with $K_{\lambda\lambda}(\mathbf{x}-\mathbf{x})$ gives a shift in the ground-state energy and will be neglected. The differential equation is satisfied if the dressed Green's function has the form

$$G_\lambda(\mathbf{x}',\mathbf{x}) = G^{(0)}_\lambda(\mathbf{x}',\mathbf{x})f_{\lambda\lambda}(\mathbf{x}-\mathbf{x}')\,. \tag{5.21}$$

In the same way, using Eq. (5.15) it is straightforward to see that the solution of the differential equation (5.14) can be expressed as a product of functions f,

$$\begin{aligned}G_{\lambda_1\lambda_2\cdots\lambda_m}&(\mathbf{x}'_1,\mathbf{x}'_2,\ldots,\mathbf{x}'_m;\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_m)\\ &= G_{\lambda_m}(\mathbf{x}'_m,\mathbf{x}_m)\prod_{l<m}\frac{f_{\lambda_m\lambda_l}(\mathbf{x}_m-\mathbf{x}'_l)f_{\lambda_m\lambda_l}(\mathbf{x}'_m-\mathbf{x}_l)}{f_{\lambda_m\lambda_l}(\mathbf{x}_m-\mathbf{x}_l)f_{\lambda_m\lambda_l}(\mathbf{x}'_m-\mathbf{x}'_l)}G_{\lambda_1\cdots\lambda_{m-1}}(\mathbf{x}'_1,\ldots,\mathbf{x}'_{m-1};\mathbf{x}_1,\ldots,\mathbf{x}_{m-1})\,.\end{aligned}\tag{5.22}$$

Since Eq. (5.22) is a recursion relation for the correlation functions, the full solution is

$$\begin{aligned}G_{\lambda_1\lambda_2\cdots\lambda_m}&(\mathbf{x}'_1,\mathbf{x}'_2,\ldots,\mathbf{x}'_m;\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_m)\\ &= \prod_l G_{\lambda_l}(\mathbf{x}'_l,\mathbf{x}_l)\prod_{l'<l}\frac{f_{\lambda_l\lambda_{l'}}(\mathbf{x}_l-\mathbf{x}'_{l'})f_{\lambda_l\lambda_{l'}}(\mathbf{x}'_l-\mathbf{x}_{l'})}{f_{\lambda_l\lambda_{l'}}(\mathbf{x}_l-\mathbf{x}_{l'})f_{\lambda_l\lambda_{l'}}(\mathbf{x}'_l-\mathbf{x}'_{l'})}\,.\end{aligned}\tag{5.23}$$

The correlation functions appear in a fully factorized form. This is a typical feature of integrable systems and is due to the factorizability of the scattering matrix.

We will apply now this general formula to obtain the explicit form of the time-dependent response functions for which the conformal invariance gives the prediction (2.40). Let us consider the response function $\chi(\mathbf{x}) = \langle TO^\dagger(\mathbf{x})O(\mathbf{0})\rangle$, where $O(\mathbf{x})$ is a product of fermion field operators $\Psi_\lambda(\mathbf{x})$ and $\Psi^\dagger_\lambda(\mathbf{x})$. The particles propagate from the origin $\mathbf{0}$ to $\mathbf{x}$ or in the opposite direction. If two particles of color λ_l and $\lambda_{l'}$ propagate in the same direction the product of the f functions in Eq. (5.23) gives

$$\frac{f_{\lambda_l\lambda_{l'}}(\mathbf{x})f_{\lambda_l\lambda_{l'}}(-\mathbf{x})}{f_{\lambda_l\lambda_{l'}}(\mathbf{0})f_{\lambda_l\lambda_{l'}}(\mathbf{0})}\,, \tag{5.24}$$

while if they propagate in opposite directions, we get

$$\frac{f_{\lambda_l\lambda_{l'}}(\mathbf{0})f_{\lambda_l\lambda_{l'}}(\mathbf{0})}{f_{\lambda_l\lambda_{l'}}(\mathbf{x})f_{\lambda_l\lambda_{l'}}(-\mathbf{x})}\,. \tag{5.25}$$

Using Eq. (5.21) for the propagator of the particles and the explicit form (5.18) for the f functions, after some bookkeeping and elementary considerations, we arrive at

$$\chi(\mathbf{x}) \sim \prod_l e^{-ik_{\lambda_l}x}\frac{x - v_{\lambda_l}t + i/\Lambda\,\mathrm{sgn}\,v_{\lambda_l}t}{x - v_{\lambda_l}t + i\delta\,\mathrm{sgn}\,v_{\lambda_l}t} \times \prod_{j,l'}\left(x - u_jt + \frac{i}{\Lambda}\,\mathrm{sgn}\,u_jt\right)^{-\alpha^{(j)}_{\lambda_l\lambda_{l'}}C_{ll'}}\,, \tag{5.26}$$

where $C_{ll'}$ is either 1 or -1, depending on whether the fermions of color λ_l and $\lambda_{l'}$ propagate in the same or opposite direction. Denoting by ΔN_λ the number of fermions of color λ propagating from the origin to the point $\mathbf{x}$ (particles propagating backwards appear with negative sign), Eq. (5.26) can be written as

$$\chi(\mathbf{x},\{\Delta N_\lambda\}) \sim \exp\left(-ix\sum_\lambda \Delta N_\lambda k_\lambda\right) \times \prod_j (x - u_jt)^{-\langle\Delta N|\alpha^{(j)}|\Delta N\rangle}\,. \tag{5.27}$$

Comparing this formula with Eq. (4.35), the critical exponents are indeed

$$2\Delta^{(j)} = \langle\Delta N|\underline{\alpha}^{(j)}|\Delta N\rangle\,, \tag{5.28}$$

as conjectured from the conformal field theory. Using the definition (4.25) of the matrix $\underline{\alpha}^{(j)}$,

$$2\Delta^{(j)} = \langle w^{(j)}|\Delta N\rangle^2\,. \tag{5.29}$$

Strictly speaking this formula is valid for $\Delta N_\lambda = 0, \pm 1$. However, since $\chi(\mathbf{x})$ is an asymptotic correlation function, this formula could still be used if two or more fermions of the same color are created at very small distances compared to the propagation path $\mathbf{x}$.

VI. MAPPING OF THE HUBBARD MODEL TO THE TOMONAGA-LUTTINGER MODEL

Our aim is to find the mapping that, for a given value of the Coulomb repulsion U of the Hubbard model, determines the couplings of the equivalent Tomonaga-Luttinger model. This is achieved by requiring that the critical exponents, the anomalous dimensions be the same for the two models.

As mentioned in Sec. III the Tomonaga-Luttinger model, which could be equivalent to the Hubbard model in a magnetic field, has to have four components with Fermi velocities $\pm v_\uparrow$ and $\pm v_\downarrow$ and Fermi momenta $\pm k_{F\uparrow}$ and $\pm k_{F\downarrow}$, and it might have six different couplings $g_{2\uparrow}$, $g_{2\downarrow}$, $g_{2\perp}$, $g_{4\uparrow}$, $g_{4\downarrow}$, and $g_{4\perp}$.

The matrices $\underline{A}$ and $\underline{B}$ defined by Eqs. (4.12) and (4.18) will be denoted in this special Tomonaga-Luttinger model by $\underline{A}_{TL}$ and $\underline{B}_{TL}$. Taking the color indices in the order $R\uparrow$, $L\uparrow$, $R\downarrow$, and $L\downarrow$ they have the form

$$\underline{A}_{TL} = \begin{pmatrix} v_\uparrow + \tilde{g}_{4\uparrow} & \tilde{g}_{2\uparrow} & \tilde{g}_{4\perp} & \tilde{g}_{2\perp} \\ \tilde{g}_{2\uparrow} & v_\uparrow + \tilde{g}_{4\uparrow} & \tilde{g}_{2\perp} & \tilde{g}_{4\perp} \\ \tilde{g}_{4\perp} & \tilde{g}_{2\perp} & v_\downarrow + \tilde{g}_{4\downarrow} & \tilde{g}_{2\downarrow} \\ \tilde{g}_{2\perp} & \tilde{g}_{4\perp} & \tilde{g}_{2\downarrow} & v_\downarrow + \tilde{g}_{4\downarrow} \end{pmatrix}\,, \tag{6.1}$$

where $\tilde{g} = g/2\pi$ and

$$\underline{B}_{TL} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\,. \tag{6.2}$$

The eigenvalue problem Eq. (4.21) with $\underline{A} = \underline{A}_{TL}$ and $\underline{B} = \underline{B}_{TL}$ will produce the velocities $\pm u_c$ and $\pm u_s$ of the charge and spin collective modes, and the eigenvectors $|w_{TL}^{(u_c)}\rangle$, $|w_{TL}^{(-u_c)}\rangle$, $|w_{TL}^{(u_s)}\rangle$, and $|w_{TL}^{(-u_s)}\rangle$.

Using Eq. (5.29), it is straigthforward to give the exponents of the correlation functions in terms of the eigenvectors $|w_{TL}^{(\pm u_{c,s})}\rangle$,

$$2\Delta^{(j)} = \langle w_{TL}^{(j)}|\Delta N_{TL}\rangle^2\,, \tag{6.3}$$

where the elements of the vector

$$|\Delta N_{TL}\rangle = \begin{bmatrix} \Delta N_{R\uparrow} \\ \Delta N_{L\uparrow} \\ \Delta N_{R\downarrow} \\ \Delta N_{L\downarrow} \end{bmatrix} \tag{6.4}$$

count the number of different fermions in a given correlation function, the number of extra particles added to the system.

The requirement that this system should have the same behavior as the Hubbard model leads to the relation

$$\Delta^{(j)} = \Delta^\pm_{c,s}\,, \tag{6.5}$$

where the anomalous dimensions of the Hubbard model, $\Delta^\pm_{c,s}$ are defined in Eqs. (2.28) and (2.29). From the comparison of Eqs. (6.3) and (2.39) we find

$$\langle w_{TL}^{(j)}|\Delta N_{TL}\rangle^2 = \langle w_H^{(j)}|\Delta N_H\rangle^2\,. \tag{6.6}$$

It is natural to identify the numbers $\Delta N_{R\uparrow}$, $\Delta N_{L\uparrow}$, $\Delta N_{R\downarrow}$, and $\Delta N_{L\downarrow}$ with the number of right- and left-moving particles introduced in Sec. II for the Hubbard model. Equations (2.11)–(2.14) can then be used to relate the vectors $|\Delta N_{TL}\rangle$ and $|\Delta N_H\rangle$ defined in Eq. (2.37). These relations can be written in a matrix form,

$$|\Delta N_{TL}\rangle = \underline{V}|\Delta N_H\rangle, \tag{6.7}$$

where

$$\underline{V} = \frac{1}{2}\begin{pmatrix} 1 & -1 & 1 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & -1 & -1 \end{pmatrix} . \tag{6.8}$$

Using Eq. (6.7) that relates the vectors $|\Delta N_{TL}\rangle$ and $|\Delta N_H\rangle$, Eq. (6.6) is satisfied if the eigenvectors $|w_{TL}\rangle$ and $|w_H\rangle$ are related by

$$|w_{TL}^{(j)}\rangle = (\underline{V}^{-1})^T |w_H^{(j)}\rangle . \tag{6.9}$$

Making use of this relationship we are able to relate the matrix elements of $\underline{A}_{TL}$, which contain the couplings and velocities of the Tomonaga-Luttinger model, to the known elements of the dressed charge matrix of the Hubbard model. According to Eq. (4.27) the spectral decomposition of $\underline{A}_{TL}$ is

$$\underline{A}_{TL} = u_c(|w_{TL}^{(u_c)}\rangle\langle w_{TL}^{(u_c)}| + |w_{TL}^{(-u_c)}\rangle\langle w_{TL}^{(-u_c)}|) + u_s(|w_{TL}^{(u_s)}\rangle\langle w_{TL}^{(u_s)}| + |w_{TL}^{(-u_s)}\rangle\langle w_{TL}^{(-u_s)}|) . \tag{6.10}$$

Inserting Eq. (6.9) into this expression we get

$$\underline{A}_{TL} = u_c(\underline{V}^{-1})^T(|w_H^{(u_c)}\rangle\langle w_H^{(u_c)}| + |w_H^{(-u_c)}\rangle\langle w_H^{(-u_c)}|)\underline{V}^{-1} + u_s(\underline{V}^{-1})^T(|w_H^{(u_s)}\rangle\langle w_H^{(u_s)}| + |w_H^{(-u_s)}\rangle\langle w_H^{(-u_s)}|)\underline{V}^{-1} . \tag{6.11}$$

The vectors $|w_H^{(\pm u_{c,s})}\rangle$ are given in Eq. (2.36), while the matrix $\underline{A}_{TL}$ is defined in Eq. (6.1). It is straightforward now to find the couplings g of that Tomonaga-Luttinger model, which has the same critical behavior as the Hubbard model with coupling U in a magnetic field h,

$$v_\uparrow + \tilde{g}_{4\uparrow} = \frac{u_c}{2}\left[\frac{Z_{ss}^2}{(\det \underline{Z})^2} + (Z_{cc} - Z_{sc})^2\right] + \frac{u_s}{2}\left[\frac{Z_{sc}^2}{(\det \underline{Z})^2} + (Z_{cs} - Z_{ss})^2\right] , \tag{6.12}$$

$$\tilde{g}_{2\uparrow} = \frac{u_c}{2}\left[\frac{Z_{ss}^2}{(\det \underline{Z})^2} - (Z_{cc} - Z_{sc})^2\right] + \frac{u_s}{2}\left[\frac{Z_{sc}^2}{(\det \underline{Z})^2} - (Z_{cs} - Z_{ss})^2\right] , \tag{6.13}$$

$$v_\downarrow + \tilde{g}_{4\downarrow} = \frac{u_c}{2}\left[\frac{(Z_{ss} - Z_{cs})^2}{(\det \underline{Z})^2} + Z_{sc}^2\right] + \frac{u_s}{2}\left[\frac{(Z_{sc} - Z_{cc})^2}{(\det \underline{Z})^2} + Z_{ss}^2\right] , \tag{6.14}$$

$$\tilde{g}_{2\downarrow} = \frac{u_c}{2}\left[\frac{(Z_{ss} - Z_{cs})^2}{(\det \underline{Z})^2} - Z_{sc}^2\right] + \frac{u_s}{2}\left[\frac{(Z_{sc} - Z_{cc})^2}{(\det \underline{Z})^2} - Z_{ss}^2\right] , \tag{6.15}$$

$$\tilde{g}_{4\perp} = \frac{u_c}{2}\left[\frac{(Z_{ss} - Z_{cs})Z_{ss}}{(\det \underline{Z})^2} + Z_{sc}(Z_{cc} - Z_{sc})\right] + \frac{u_s}{2}\left[\frac{(Z_{sc} - Z_{cc})Z_{sc}}{(\det \underline{Z})^2} + Z_{ss}(Z_{cs} - Z_{ss})\right] , \tag{6.16}$$

$$\tilde{g}_{2\perp} = \frac{u_c}{2}\left[\frac{(Z_{ss} - Z_{cs})Z_{ss}}{(\det \underline{Z})^2} - Z_{sc}(Z_{cc} - Z_{sc})\right] + \frac{u_s}{2}\left[\frac{(Z_{sc} - Z_{cc})Z_{sc}}{(\det \underline{Z})^2} - Z_{ss}(Z_{cs} - Z_{ss})\right] , \tag{6.17}$$

where Z_{cc}, Z_{cs}, Z_{sc}, and Z_{ss} are defined in Eqs. (2.30)–(2.35).

A. The zero magnetic-field case

As a special case let us consider the model first in zero magnetic field where the magnetization vanishes. This case corresponds to $\lambda_0 \to \infty$. The Bethe ansatz equations have been analyzed in detail by several authors. Equations (2.4) and (2.24) for the charge degrees of freedom simplify to[14]

$$\rho_c(k) = \frac{1}{2\pi} + \cos k \int_{-k_0}^{k_0} \frac{dk'}{2\pi} \bar{K}(\sin k - \sin k')\rho_c(k') \tag{6.18}$$

and

$$\varepsilon_c'(k) = 2\sin k + \cos k \int_{-k_0}^{k_0} \frac{dk'}{2\pi} \bar{K}(\sin k - \sin k')\varepsilon_c'(k') , \tag{6.19}$$

where the kernel $\bar{K}(z)$ is given by

$$\bar{K}(z) = \int_0^\infty d\omega \frac{e^{-\omega U/4}}{\cosh(\omega U/4)} \cos \omega z . \tag{6.20}$$

The velocity of the charge modes, u_c is given by Eq. (2.23).

In the expression (2.23) for the velocity of the spin modes, both the numerator and the denominator vanish in the limit, when $\lambda_0 \to \infty$. This velocity can be calculated using the form[14]

$$u_s = \frac{1}{2\pi}\left[\int_{-k_0}^{k_0} dk\, e^{\frac{\pi}{2u}\sin k}\varepsilon_c'(k)\right] \times \left[\int_{-k_0}^{k_0} dk\, e^{\frac{\pi}{2u}\sin k}\rho_c(k)\right]^{-1} . \tag{6.21}$$

Using Eqs. (2.32)–(2.35), the dressed charge matrix defined in Eq. (2.30) takes now the form

$$\underline{Z} = \begin{pmatrix} \xi(k_0) & 0 \\ \frac{\xi(k_0)}{2} & \frac{1}{\sqrt{2}} \end{pmatrix} , \tag{6.22}$$

where

$$\xi(k) = 1 + \int_{-k_0}^{k_0} \frac{dk'}{2\pi} \cos k' \bar{K}(\sin k - \sin k')\xi(k') . \tag{6.23}$$

In this special case the charge and spin degrees of freedom are independent. To show this we use (6.22) in the expressions (2.28) and (2.29) for the anomalous dimensions $\Delta^{\pm}_{c,s}$

$$2\Delta^{\pm}_{c} = \frac{1}{4}\left(\xi(k_0)(2D_c + D_s) \mp \frac{\Delta N_c}{\xi(k_0)}\right)^2 , \tag{6.24}$$

$$2\Delta^{\pm}_{s} = \frac{1}{4}\left(\sqrt{2}D_s \mp \frac{\Delta N_c - 2\Delta N_s}{\sqrt{2}}\right)^2 . \tag{6.25}$$

The anomalous dimensions can be expressed in terms of the extra charge ΔN_c, the change in the magnetization ΔM, and the spin and charge current J_s and J_c defined in Eqs. (2.15), (2.16) and (2.21), (2.22). We find

$$2\Delta^{\pm}_{c} = \frac{1}{4}\left(\xi(k_0)\frac{J_c}{2k_F} \pm \frac{\Delta N_c}{\xi(k_0)}\right)^2 , \tag{6.26}$$

$$2\Delta^{\pm}_{s} = \frac{1}{4}\left(-\sqrt{2}\frac{J_s}{2k_F} \mp \frac{\Delta M}{\sqrt{2}}\right)^2 , \tag{6.27}$$

which clearly demonstrates the charge-spin separation. In the general, field-dependent case, however, both ΔN_c and ΔN_s as well as J_c and J_s will contribute to all the exponents.

Evidently in zero magnetic field the Fermi velocities and momenta of spin $\uparrow$ and spin $\downarrow$ fermions are equal. Furthermore, the couplings $g_{4\uparrow}$ and $g_{4\downarrow}$ are also equal and will be denoted by $g_{4\parallel}$. Similarly $g_{2\uparrow} = g_{2\downarrow} = g_{2\parallel}$. In this case Eqs. (6.12)–(6.17) reduce to

$$v_F + \tilde{g}_{4\parallel} = \frac{u_c}{2}\left(\frac{1}{\xi^2(k_0)} + \frac{\xi^2(k_0)}{4}\right) + \frac{u_s}{2} , \tag{6.28}$$

$$\tilde{g}_{4\perp} = \frac{u_c}{2}\left(\frac{1}{\xi^2(k_0)} + \frac{\xi^2(k_0)}{4}\right) - \frac{u_s}{2} , \tag{6.29}$$

$$\tilde{g}_{2\perp} = \tilde{g}_{2\parallel} = \frac{u_c}{2}\left(\frac{1}{\xi^2(k_0)} - \frac{\xi^2(k_0)}{4}\right) . \tag{6.30}$$

Equations (6.18)–(6.21) and (6.23) can be solved numerically to evaluate the velocities and $\xi(k_0)$. Performing this calculation for several densities n as a function of the Coulomb coupling U, we plot our results in Fig. 2 for $v_F + \tilde{g}_{4\parallel}$, in Fig. 3 for $\tilde{g}_{4\perp}$, and in Fig. 4 for $\tilde{g}_{2\perp} = \tilde{g}_{2\parallel}$. As we can see, the couplings $\tilde{g}_{4\perp}$, $\tilde{g}_{2\perp}$, and $\tilde{g}_{2\parallel}$ are linear in U for small U and they saturate for large U. Except for very low densities or for densities near half-filling, the linear region extends to U values of the order of the hopping integral, which is our energy unit. At small densities the bottom of the band is very close to the Fermi energy and a linear approximation of the dispersion relation is valid in a narrow energy range only. Near half filling the umklapp scattering g_3 becomes important, and this leads to the breakdown of linearity. We will analyze the small and large-U limits separately.

In Figs. 5–7 we present the velocity and the couplings for several values of U as a function of the band filling or density n. At low densities all three quantities go linearly with n. Close to half-filling $\tilde{g}_2$ decreases and goes to zero for any U. In the same region $\tilde{g}_{4\perp}$ becomes negative,

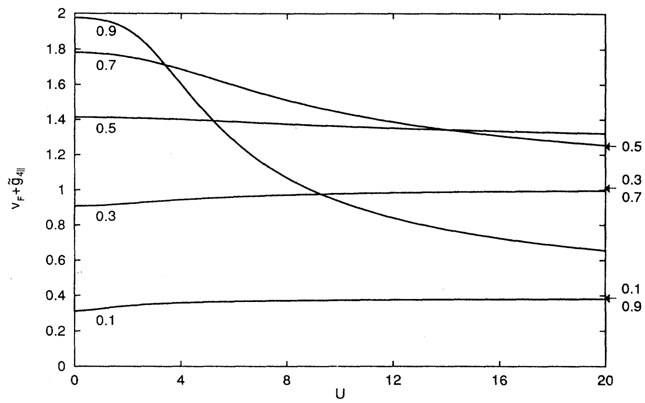

FIG. 2. The velocity with the Hartree-Fock correction $v_F + \tilde{g}_{4\parallel}$ of the Tomonaga-Luttinger model that shows the same critical behavior as the Hubbard model with the Coulomb repulsion U in zero magnetic field for density $n =$ 0.1, 0.3, 0.5, 0.7, and 0.9. The arrows on the right-hand side show the asymptotic value of the velocity in the large-U limit, $v_F + \tilde{g}_{4\parallel} \rightarrow (5/4)\sin \pi n$.

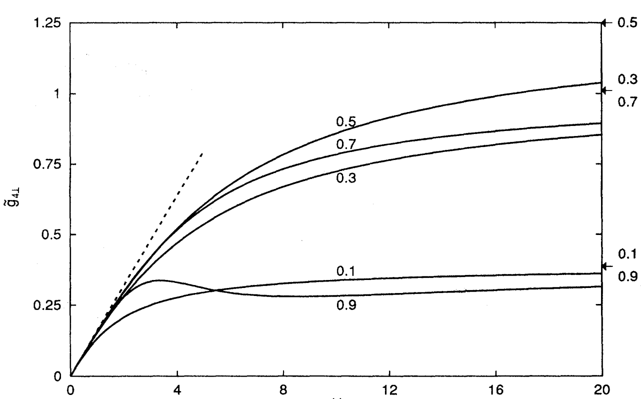

FIG. 3. The couplings $\tilde{g}_{4\perp}$ of the Tomonaga-Luttinger model that shows the same critical behavior as the Hubbard model with the Coulomb repulsion U in zero magnetic field for density n = 0.1, 0.3, 0.5, 0.7, and 0.9. In the small-U region the dashed line shows the predictions of the scaling theory, $g_{4\perp} = U$. The arrows on the right-hand side show the asymptotic value of the coupling in the large-U limit, $\tilde{g}_{4\perp} \rightarrow (5/4)\sin \pi n$.

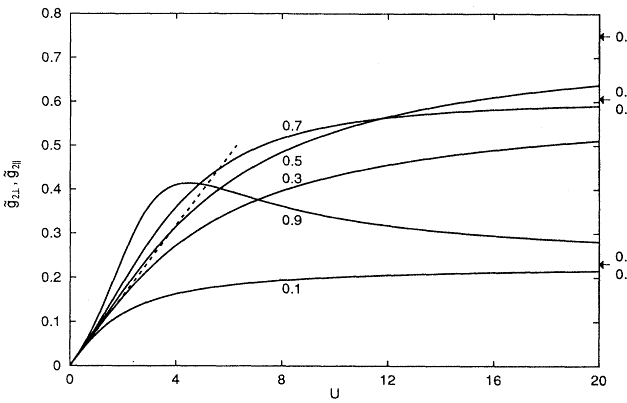

FIG. 4. The couplings $\tilde{g}_2 = \tilde{g}_{2\parallel} = \tilde{g}_{2\perp}$ of the Tomonaga-Luttinger model that shows the same critical behavior as the Hubbard model with the Coulomb repulsion U in zero magnetic field for density $n = 0.1$, 0.3, 0.5, 0.7, and 0.9. In the small-U region the dashed line shows the predictions of the scaling theory, $g_2 = U/2$. The arrows on the right-hand side show the asymptotic value of the coupling in the large-U limit, $\tilde{g}_2 \to (3/4)\sin \pi n$.

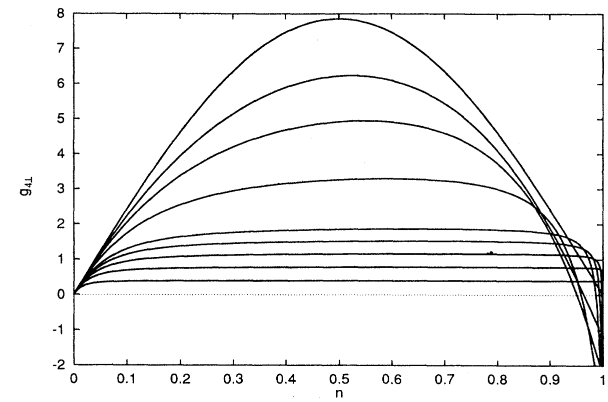

FIG. 6. The same as Fig. 5, for the coupling $g_{4\perp}$.

while $v_F + \tilde{g}_{4\parallel}$ remains positive such that $v_F + \tilde{g}_{4\parallel} + \tilde{g}_{4\perp} = 0$. Except for these extreme cases, very low density or nearly half-filled band, large plateaus appear for small U, indicating that the band filling plays little role. For $U < 1$ this plato extends from $n \approx 0.1$ to 0.9. This is the region where the low-order scaling theory and the g-ology model can be reasonably applied.

1. *Small-U limit ($U \ll \sin k_0$)*

As a check let us look at the results in the weak-coupling limit, where the results obtained in the g-ology model should be valid. In the noninteracting ($U = 0$) case the hopping on the lattice gives a dispersion $\varepsilon(k) = -2\cos k$. Since the spin-up and -down bands are equally occupied, and k_0 is equal to the common Fermi momentum $k_F = \pi n/2$, the charge and spin excitations have identical velocities $u_c = u_s = 2\sin k_F$. For small U Eqs. (6.18)–(6.19) and (6.23) can be solved in powers of U using the Wiener-Hopf technique (see Ref. 21). The leading terms in the interesting quantities are

$$\xi(k_0) = \sqrt{2}\left(1 - \frac{U}{8\pi \sin k_F}\right) + O(U^2) , \qquad (6.31)$$

$$u_c = 2\sin k_F + \frac{U}{2\pi} + O(U^2) , \qquad (6.32)$$

$$u_s = 2\sin k_F - \frac{U}{2\pi} + O(U^2) . \qquad (6.33)$$

The velocity $v_F + \tilde{g}_{4\parallel}$ is not renormalized to linear order in U, while for the couplings we get

$$g_{2\perp} = g_{2\parallel} = \frac{U}{2} + O(U^2) , \qquad (6.34)$$

$$g_{4\perp} = U + O(U^2) . \qquad (6.35)$$

This result is in complete agreement with the prediction of the scaling theory of the g-ology model,[6] as dis-

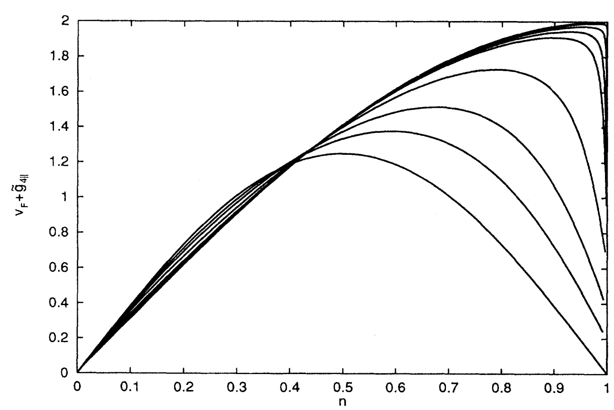

FIG. 5. The velocity with the Hartree-Fock correction $v_F + \tilde{g}_{4\parallel}$ of the Tomonaga-Luttinger model that shows the same critical behavior as the Hubbard model with different values of the Coulomb repulsion U ($U =$ 0.4, 0.8, 1.2, 1.6, 2, 4, 8, 16, ∞ from top to bottom on the right-hand side) in zero magnetic field as a function of the density n.

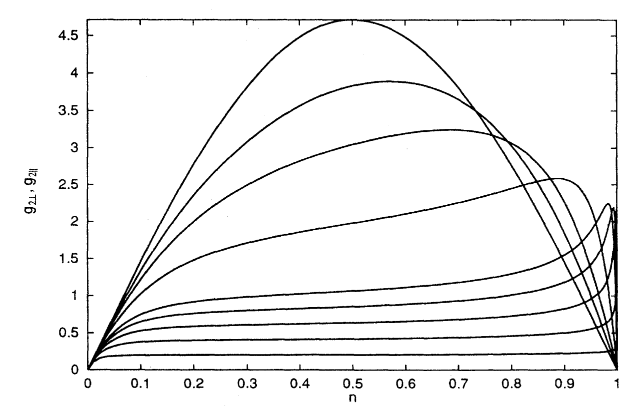

FIG. 7. The same as Fig. 5, for the coupling $g_{2\parallel}$, $g_{2\perp}$.

cussed in Sec. III. As mentioned before, the numerical results show that this linear approximation is valid until U becomes of the order of unity.

2. *Large-U limit* ($U \gg \sin k_0$)

The strong Coulomb coupling splits the bands. For infinitely strong repulsion only the lower band will be filled until $k_0 = \pi n = 2k_F$. The velocity of the charge excitations will be $u_c = 2\sin 2k_F$, while spin waves cannot propagate, since double occupancy of sites is forbidden. For finite but large U the equations for the dressed charge and the velocities can be calculated in inverse powers of U,

$$\xi(k_0) = 1 + \frac{4}{\pi}\frac{\ln 2}{U}\sin k_0 + O(1/U^2)\ , \tag{6.36}$$

$$u_c = 2\sin k_0 \left(1 - \frac{4k_0}{\pi}\frac{\ln 2}{U}\cos k_0\right) + O(1/U^2)\ , \tag{6.37}$$

$$u_s = \frac{2\pi}{U}\left(1 - \frac{\sin 2k_0}{2k_0}\right) + O(1/U^2)\ . \tag{6.38}$$

Both the charge and spin excitations remain gapless. The velocity of the spin excitations is of the order of $1/U$, as expected. To leading order in $1/U$ the couplings are

$$v_F + \tilde{g}_{4\parallel} = \tfrac{5}{4}\sin\pi n + O(1/U)\ , \tag{6.39}$$

$$\tilde{g}_{2\perp} = \tilde{g}_{2\parallel} = \tfrac{3}{4}\sin\pi n + O(1/U)\ , \tag{6.40}$$

$$\tilde{g}_{4\perp} = \tfrac{5}{4}\sin\pi n + O(1/U)\ . \tag{6.41}$$

The coupling saturate to a finite value. The asymptotic values are shown by arrows in Figs. 2–4. Looking at those figures we should also notice that the couplings remain always smaller than the velocity, thus no strong-coupling instabilities can occur in the model. The fractions 5/4 and 3/4 give the simple exponents found in this limit.[2,9]

B. The model in a magnetic field

In order to do calculations in a finite magnetic field we need to establish the connection between the parameters λ_0 and h. In the ground state those states of the two Hubbard bands are occupied for which $\varepsilon_c(k) < 0$ and $\varepsilon_s(\lambda) < 0$, i.e., k_0 and λ_0 satisfy the equations

$$\varepsilon_c(k_0) = 0\ , \quad \varepsilon_s(\lambda_0) = 0\ . \tag{6.42}$$

Following Woynarovich[14] this equation can be rewritten using the definition of the dressed charge in Eqs. (2.32)–(2.35) as

$$\varepsilon_c(k_0) = \bar{\varepsilon}_c(k_0) - \left(\mu + \frac{h}{2}\right)\xi_{cc}(k_0) + h\xi_{sc}(\lambda_0) = 0, \tag{6.43}$$

$$\varepsilon_s(\lambda_0) = \bar{\varepsilon}_s(\lambda_0) - \left(\mu + \frac{h}{2}\right)\xi_{cs}(k_0) + h\xi_{ss}(\lambda_0) = 0, \tag{6.44}$$

where $\bar{\varepsilon}_c(k)$ and $\bar{\varepsilon}_s(\lambda)$ are the band energies for $h = 0$ and $\mu = 0$.

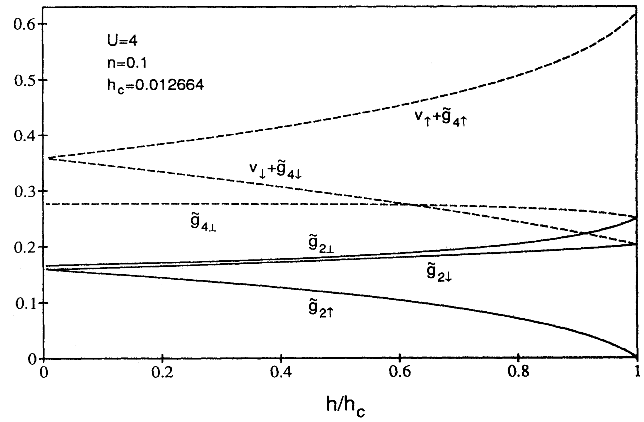

FIG. 8. The velocity and couplings of the Tomonaga-Luttinger model that has the same critical behavior as the Hubbard model with $U = 4$ and density $n = 0.1$ plotted against the magnetic field h/h_c (h_c is the magnetic field of saturation). As $h \to 0$ the couplings $\tilde{g}_{2\uparrow}$, $\tilde{g}_{2\downarrow}$, and $\tilde{g}_{2\perp}$ (solid lines) approach the common value in a nonanalytic way.

Solving this set of linear equations for the magnetic field we get

$$h = \frac{\bar{\varepsilon}_c(k_0)\xi_{cs}(\lambda_0) - \bar{\varepsilon}_s(\lambda_0)\xi_{cc}(k_0)}{\xi_{cc}(k_0)\xi_{ss}(\lambda_0) - \xi_{sc}(k_0)\xi_{cs}(\lambda_0)}\ . \tag{6.45}$$

As the strength of the magnetic field increases, in the ground state the number of spin down particles decreases and correspondingly λ_0 decreases. At a finite value h_c of the magnetic field all spins point upwards, and λ_0 vanishes. For larger fields the spin excitations will have a finite gap, the system ceases to be critical and the correlation functions decay exponentially. Our considerations based on mapping to the critical Tomonaga-Luttinger model are not valid any more for $h > h_c$.

In Figs. 8–10 we show the couplings for three densities for fixed $U = 4$ as a function of the magnetic field h. It can be seen that for $h \to 0$ the couplings $g_{2\uparrow}$, $g_{2\downarrow}$, and $g_{2\perp}$ approach the common value given in Eq. (6.30)

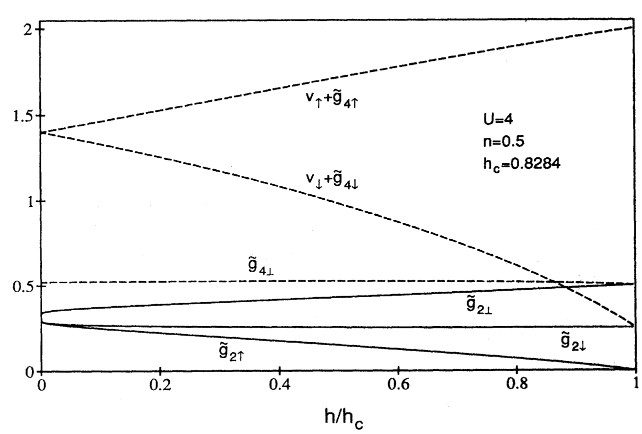

FIG. 9. The same as Fig. 8 at the density $n = 0.5$.

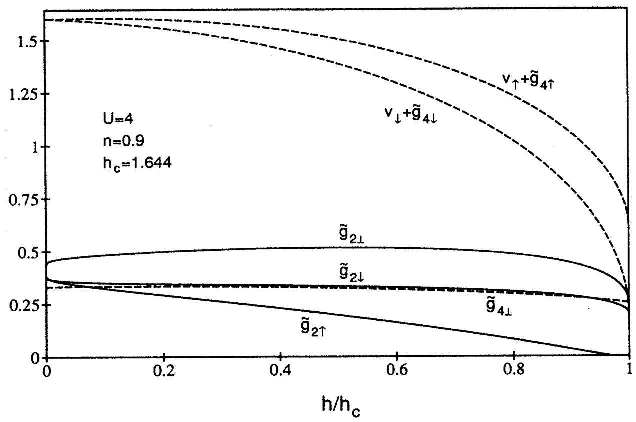

FIG. 10. The same as Fig. 8 at the density $n = 0.9$.

in a nonanalytic way. This splitting of $g_{2\uparrow}$ and $g_{2\downarrow}$ from $g_{2\perp}$ becomes more enhanced as the band filling increases. Frahm and Korepin[15] studied the behavior for weak fields and near the critical field in the strong-coupling limit. We will analyze the behavior for general U in the following subsections.

1. *Behavior in a weak magnetic field*

For weak fields, i.e., in the limit $\lambda_0 \gg U$, the dependence of λ_0 on the magnetic field is determined by evaluating the quantities appearing in Eq. (6.45) using the Wiener-Hopf method. By inverting the relation we find

$$\lambda_0 = \frac{U}{2\pi} \ln \frac{h_0}{h} , \tag{6.46}$$

with

$$h_0 = \sqrt{\frac{2\pi}{e}} \frac{2}{U\xi(k_0)} \times \int_{-k_0}^{k_0} dk \cos k \, [\bar{\varepsilon}_c(k_0)\xi(k) - \bar{\varepsilon}_c(k)\xi(k_0)] e^{\frac{2\pi \sin k}{U}} , \tag{6.47}$$

where $\xi(k)$ is the solution of Eq. (6.23) and $\bar{\varepsilon}_c(k)$ is obtained from

$$\bar{\varepsilon}_c(k) = -2\cos k + \int_{-k_0}^{k_0} \frac{dk'}{2\pi} \cos k' \bar{K}(\sin k - \sin k')\bar{\varepsilon}_c(k') . \tag{6.48}$$

The charge and spin velocities behave smoothly, as $h \to 0$.[22] However, this is not the case for the dressed charge matrix, since a logarithmic field dependence appears in Z_{ss}. Neglecting the linear corrections compared to this logarithm we find

$$\underline{Z} = \begin{pmatrix} \xi(k_0) + O(h^2) & O(h) \\ \frac{\xi(k_0)}{2} + O(h) & \frac{1}{\sqrt{2}}\left(1 + \frac{1}{4\ln(h_0/h)}\right) + O(h) \end{pmatrix} , \tag{6.49}$$

where $\xi(k_0)$ is given in Eq. (6.22). This logarithmic field dependence modifies the expression for the couplings as

$$\tilde{g}_{2\uparrow} = \tilde{g}_{2\downarrow} = \frac{u_c}{2}\left(\frac{1}{\xi^2(k_0)} - \frac{\xi^2(k_0)}{4}\right) - \frac{u_s}{2}\frac{1}{2\ln(h_0/h)} + O(h) , \tag{6.50}$$

$$\tilde{g}_{2\perp} = \frac{u_c}{2}\left(\frac{1}{\xi^2(k_0)} - \frac{\xi^2(k_0)}{4}\right) + \frac{u_s}{2}\frac{1}{2\ln(h_0/h)} + O(h) . \tag{6.51}$$

The corrections to the couplings $\tilde{g}_4$ are $O(h)$. A common value $g_{2\parallel} = g_{2\uparrow} = g_{2\downarrow}$ can be used if corrections of $O(h)$ are neglected.

This logarithmic correction is not surprising since the magnetic field couples directly to the spin degrees of freedom and it is known that such a logarithmic dependence appears in the magnetic susceptibility of the spin-1/2 Heisenberg chain.[23] The magnetic origin of this correction is even better seen if we consider the linear combinations $\tilde{g}_c = \tilde{g}_{2\parallel} + \tilde{g}_{2\perp}$ and $\tilde{g}_s = \tilde{g}_{2\parallel} - \tilde{g}_{2\perp}$, which couple directly to the charge and spin modes,[6] respectively:

$$\tilde{g}_c = u_c\left(\frac{1}{\xi^2(k_0)} - \frac{\xi^2(k_0)}{4}\right) + O(h) , \tag{6.52}$$

$$\tilde{g}_s = -u_s\frac{1}{2\ln(h_0/h)} + O(h) , \tag{6.53}$$

i.e., the logarithmic term appears in g_s only.

2. *Behavior at the critical field*

At $h = h_c$ it is straightforward to solve the Lieb-Wu equations. Since in this field all spins point upwards, $\lambda_0 = 0$ and therefore from Eqs. (2.6) and (2.4) $\rho_c(k) = 1/2\pi$ and $k_0 = \pi n$. Holon excitations only are present with velocity $u_c = 2\sin \pi n = 2\sin k_{F\uparrow}$, the spinon band is empty, and the velocity vanishes, $u_s = 0$. The magnetic field of saturation can be given[15] in the form

$$h_c = \frac{U}{2\pi}\int_{-\pi n}^{\pi n} dk \cos k \frac{\cos k - \cos \pi n}{(U/4)^2 + \sin^2 k} , \tag{6.54}$$

which for large U is

$$h_c = \frac{8}{U}\left(n - \frac{\sin 2\pi n}{2\pi}\right) + O(1/U^3) \tag{6.55}$$

and for small U

$$h_c = 2(1 - \cos \pi n) - Un + O(U^2) . \tag{6.56}$$

The dressed charge matrix defined in Eqs. (2.30)–(2.35) becomes

$$\underline{Z} = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} , \tag{6.57}$$

where

$$\alpha = \frac{2}{\pi}\arctan\frac{4\sin k_0}{U} . \tag{6.58}$$

Inserting this into the expressions for the couplings we get

$$v_\uparrow + \tilde{g}_{4\uparrow} = u_c \,, \tag{6.59}$$

$$\tilde{g}_{2\uparrow} = 0 \,, \tag{6.60}$$

$$v_\downarrow + \tilde{g}_{4\downarrow} = \tilde{g}_{2\downarrow} = \frac{u_c}{2}(1-\alpha)^2 \,, \tag{6.61}$$

$$\tilde{g}_{4\perp} = \tilde{g}_{2\perp} = \frac{u_c}{2}(1-\alpha) \,. \tag{6.62}$$

We can choose $v_\uparrow = u_c$, $v_\downarrow = 0$ yielding

$$\tilde{g}_{4\uparrow} = \tilde{g}_{2\uparrow} = 0 \,, \tag{6.63}$$

$$\tilde{g}_{4\downarrow} = \tilde{g}_{2\downarrow} = \frac{u_c}{2}(1-\alpha)^2 \,, \tag{6.64}$$

$$\tilde{g}_{4\perp} = \tilde{g}_{2\perp} = \frac{u_c}{2}(1-\alpha) \,. \tag{6.65}$$

This is easily understood, since if $k_{F\downarrow} = 0$ and $u_s = 0$, the distinction between right- and left-moving particles disappears for that branch. These degeneracies are clearly seen in Figs. 8–10.

In the small-U limit

$$\alpha = 1 - \frac{U}{2\pi \sin \pi n} \,, \tag{6.66}$$

so the leading corrections to the couplings are

$$g_{4\downarrow} = g_{2\downarrow} = \frac{U^2}{\pi u_c} \,, \tag{6.67}$$

$$g_{4\perp} = g_{2\perp} = U \,, \tag{6.68}$$

$$g_{4\uparrow} = g_{2\uparrow} = 0 \,. \tag{6.69}$$

The coupling between particles of opposite spin is simply the original Hubbard U, while particles of the same spin exercice a weaker mediated repulsion on each other.

VII. SUMMARY

It was known for some time already that the one-dimensional Hubbard model is in the universality class of Luttinger liquids. The explicit relationship between the parameters of the models were known, however, in the weak-coupling limit only, where the Hubbard model can be replaced by a g-ology model and the fixed point couplings can be determined using the scaling theory. The aim of the present paper was to find the mapping between the Tomonaga-Luttinger model and the Hubbard model in magnetic field h for arbitrary Coulomb coupling U and for general band filling n.

Since the Fermi velocities of free fermions on a lattice in an external magnetic field are different for the two spin orientations, first we generalized the Tomonaga-Luttinger model to the case when the fermions of different color have different Fermi velocities and Fermi momenta. Using the method of Mattis and Lieb it is possible to diagonalize the corresponding Hamiltonian exactly and the excitation spectrum could be determined. Assuming that this model can be considered as a direct product of Virasoro algebras, the finite-size corrections to the energy and momentum allowed us to obtain the anomalous dimensions of the primary fields that appear in the exponents of the correlation functions. Since there is some ambiguity in the choice of the anomalous dimensions, an alternative approach was used to determine the correlation functions exactly. The equation of motion method applied to the correlation functions gave a closed set of equations which are equivalent to generalized Ward identities between higher-order vertices. The correlation functions could be calculated in a closed form in real space and time representation.

Since the same correlation functions are known for the Hubbard model from the works of Frahm and Korepin,[13,15] it is possible to identify the equivalent models by putting equal the anomalous dimensions of the Hubbard model and the generalized Tomonaga-Luttinger model. We have found that in zero magnetic field in the weak-coupling limit the couplings of the Tomonaga-Luttinger model increase linearly with U in agreement with renormalization-group theory. Except for low density or near half-filling the couplings of the equivalent Tomonaga-Luttinger model are not very sensitive to the band filling, again in agreement with the g-ology model. For strong-coupling U the couplings of the Tomonaga-Luttinger model saturate at finite values. This allowed us to conclude that except for very low densities or near half-filling, the g-ology model with bare couplings U is a very good approximation for the Hubbard model if U is not larger than the hopping integral.

In an external magnetic field the equivalent Tomonaga-Luttinger model has two different velocities and the couplings become spin dependent. In small magnetic field we have found an interesting nonanalytic (logarithmic) behavior of the couplings as $h \to 0$ in much the same way as is found in the susceptibility of the one-dimensional Heisenberg chain. At a finite critical field h_c the system becomes fully polarized. The velocity of the spinon excitations vanishes and several couplings become degenerate. For larger fields the spinon excitations are massive and the system loses its criticality, the spin part of the correlation functions decays exponentially. The model cannot therefore be mapped to an equivalent Tomonaga-Luttinger model.

ACKNOWLEDGMENTS

The authors are grateful to F. Woynarovich for valuable discussions on this topic. One of the authors (J.S.) thanks Ph. Nozières and the other (K.P.) wishes to thank D. Haldane, A. Patkós, and A. Zawadowski for useful communications. This work was partially supported by the Hungarian Research Fund OTKA Grant No. 2979.

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