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Kosterlitz-Thouless transition in multipolar systems

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Using analytical and discretized Migdal-Kadanoff renormalization-group methods it is shown that similar to the standard XY model the two-component uniform multipolar systems undergo a Kosterlitz-Thouless transition in two dimensions. The critical behavior of the multipole-multipole correlation function is governed by a set of critical exponents that depend on the rank t of multipole moments. At the Kosterlitz-Thouless transition they become universal for all systems. An exactly renormalizable multipolar model is introduced. The behavior of this model on fractal and hierarchical lattices is studied. The $T=0$ scaling exponent of the two-component multipolar systems on a Sierpinski carpet is shown to be negative, so no Kosterlitz-Thouless transition can be expected to take place in this case.

I. INTRODUCTION

Multipolar glasses have become objects of recent experimental[1] and theoretical studies.[2] However, phase transitions in uniform systems consisting of multipole moments of a general rank t still remain to be understood. There are at least two reasons to study such systems. First of all, they exist in nature. For instance, systems made of the KCN molecules[3] and N_2 molecules[4] are examples of quadrupolar models [the dipole moment of $(CN)^-$ is small compared to the quadrupolar ones whereas the dipole moment of N_2 vanishes]. The compound CH_4, on the other hand, is an example of an octupolar system.[5] The second reason is that it is interesting to find out in what ways the multipolar systems may differ from their well-studied dipolar, or spin, counterparts. In this paper, we focus on one especially interesting problem: what would happen to the Kosterlitz-Thouless (KT) transition[6–8] in the two-dimensional (2D) classical XY model if the system consisted not of dipoles, but of multipoles of higher order.

Before we address the KT transition, it is useful to define Hamiltonians used in studies of multipolar systems. Interactions between moments of rank t and number of components m can be written as

$$H=-\sum_{ij}\sum_{\mu_1=1}^{m}\cdots\sum_{\mu_t=1}^{m} J_{ij} f_i^{\mu_1\cdots\mu_t} a^{\mu_1\cdots\mu_t} f_j^{\mu_1\cdots\mu_t} . \quad (1)$$

Here $f_i^{\mu_1\cdots\mu_t}$ is a tensor whereas $a^{\mu_1\cdots\mu_t}$ takes into account the symmetry of multipole-multipole interactions. These interactions can, in general, be long range but we shall be interested in couplings which are uniform, $J_{ij}=J$, and restricted to nearest neighbors. After Carmesin,[9,10] we consider the simplest case when the tensor $f_i^{\mu_1\cdots\mu_t}$ is uniaxial and can be represented in the following form:

$$f_i^{\mu_1\cdots\mu_t}=S_i^{\mu_1}S_i^{\mu_2}\cdots S_i^{\mu_t} , \quad (2)$$

where $\mathbf{S}_i$ is a unit vector with m components. We assume that the multipole-multipole interaction is isotropic. This allows us to bring the Hamiltonian to the form

$$H=-\sum_{ij} J_{ij}[\cos^t(\varphi_i-\varphi_j)-c(t,m)] , \quad (3)$$

where the angle φ_i describes direction at which the axis of a molecule is pointing. The constant $c(t,m)$ is subtracted for convenience and it is defined as the angular average of $\cos^t(\varphi_i-\varphi_j)$. It is equal to

$$c(t,m)=\begin{cases} 0 \quad \text{for odd } t , \\ \dfrac{3.5\cdots(t-1)}{m(m+2)\cdots(m+t-2)} \quad \text{for even } t . \end{cases} \quad (4)$$

It is well known that[11] the 2D uniform classical XY model ($t=1$) orders only at $T=0$ but it undergoes the KT transition at a finite temperature, T_{KT}. A similar behavior is believed to hold also in case of the quantum 2D XY model (see, e.g., Ref. 12 and references there). In this paper we focus on the behavior of XY multipole systems in 2D.

As pointed out by Carmesin,[9] the $m=2$ systems with an even t have an additional symmetry compared to systems with an odd t: the partition function for the Hamiltonian (3) is exactly equal to the one calculated with the Hamiltonian

$$H=-\sum_{ij} J_{ij}\{\cos^t[(\varphi_i-\varphi_j)/2]-c(t,2)\} . \quad (5)$$

The mapping of (3) onto (5) is obtained by changing variables and making other simple manipulations.[9] The immediate conclusion is that the quadrupolar ($t=2$) system is equivalent to the ordinary XY dipolar system.[9] It thus undergoes the KT transition in 2D.

What happens with the KT transition in systems with a higher t? In this paper, we show that for $T<T_{KT}$ the multipole-multipole correlation function is characterized by a set of critical exponents decreasing with the rank t of multipole moments. However, at high temperatures the decay of the correlation function is faster the larger t is. At $T=T_{KT}$ the exponents become independent of t and the critical behavior of all multipolar systems becomes similar to that of the ordinary XY model.

We have also applied the discretized Migdal-Kadanoff renormalization-group (MKRG) approach[13] to study scaling properties in the low-temperature phase of multipolar systems. Numerical calculations are carried out for the octupolar system ($t=4$). They point out that the exchange coupling scales similar to what was found for the dipolar XY model within the same approximation.

We also demonstrate that, in general, the harmonic approximation often used in the usual MKRG scheme[14] is valid for the multipolar systems except when couplings are "antiferromagnetic" ($J<0$) and t is even and greater than 2. The latter case was studied by the discretized scheme with the scaling factor $b=3$. Both the $T=0$ and discretized approximations show that the lower critical dimensionality of the multipolar systems of any t is equal to 2.

In Sec. IV, we turn our attention to multipolar systems on fractal lattices. The multipolar systems are shown not to have the KT transition on the 2D Sierpinski carpets.

Finally, in Sec. V, we consider an exactly renormalizable model which describes the XY-like multipolar systems at high temperatures. Renormalization-group equations are obtained for this model on hierarchical lattices and the role of the tensorial rank t is investigated.

II. CORRELATION FUNCTIONS

The general definition of the two-tensor multipole-multipole correlation function $g^t(\varphi_i-\varphi_j)$ is given by

$$g^t(\varphi_i-\varphi_j)=\left\langle \sum_{\mu_1=1}^{m} \cdots \sum_{\mu_t}^{m} [f_i^{\mu_1\cdots\mu_t} f_j^{\mu_1\cdots\mu_t} - \langle f_i^{\mu_1\cdots\mu_t} f_j^{\mu_1\cdots\mu_t}\rangle_{\text{angle}}]\right\rangle , \tag{6}$$

where $\langle\ \rangle_{\text{angle}}$ stands for the angular and $\langle\ \rangle$ for the thermal average. In our case this simplifies to

$$g^t(\varphi_i-\varphi_j)=\langle \cos^t(\varphi_i-\varphi_j)-c(t,m)\rangle . \tag{7}$$

In case of even t it is convenient to map the Hamiltonian into the form given by Eq. (5). The advantage of the mapped Hamiltonian is that, for example, if $J>0$, a bond energy has one minimum whereas the Hamiltonian (3) has two. This fact simplifies our analysis of the 2D octupolar system at low temperatures. So in what follows for even t we will deal with form (5). The corresponding correlation function takes the form

$$g^t(r_{ij})=\langle \cos^t[(\varphi_i-\varphi_j)/2]-c(t,2)\rangle . \tag{8}$$

In order to calculate the even- and odd-t correlation functions we generalize the approaches outlined in Refs. 7, 8, and 15, and start by representing the correlations in the form

$$g^t(r_{ij})=\sum_{p=1}^{t/2} a_p(t) g_p^t(r_{ij}) ,$$
$$g_p^t(r_{ij})=\langle e^{ip(\varphi_i-\varphi_j)}\rangle , \tag{9}$$
$$a_p(t)=\frac{t!}{(t/2+p)!(t/2-p)!2^{t-1}} ,$$

for even t and

$$g^t(r_{ij})=\sum_{p=1}^{(t+1)/2} a_{2p-1}(t) g_{2p-1}^t(r_{ij}) ,$$
$$g_{2p-1}^t(r_{ij})=\langle e^{i(2p-1)(\varphi_i-\varphi_j)}\rangle , \tag{10}$$
$$a_{2p-1}(t)=\frac{t!}{[(t-1+2p)/2]![(t-2p+1)/2]!2^{t-1}}$$

for odd t. The average is performed by using the Hamiltonian (3) for odd t and (5) for even t.

It should be noted that in the case of even t the correlation function $g_p^t(r_{ij})=\langle e^{ip(\varphi_i-\varphi_j)}\rangle$ calculated with respect to Hamiltonian (5) (after mapping) is equal to the function $g_{2p}^t(r_{ij})=\langle e^{i2p(\varphi_i-\varphi_j)}\rangle$ calculated with respect to the Hamiltonian (3) (before mapping). One can show that in this case all of the functions $g_{2p+1}^t(r_{ij}) =\langle e^{i(2p+1)(\varphi_i-\varphi_j)}\rangle$ ($p=0,1,\cdots$) calculated with respect to the Hamiltonian (3) are identically equal to zero at any temperature. For example, the dipole-dipole correlation function $g_1^t(r_{ij})$ of multipolar systems with even t is equal to zero.

High-temperature approximation. At high temperatures ($K=J/k_bT\ll 1$) the correlation function may be readily calculated since in this case one can use the following approximation:

$$\langle e^{ip(\varphi_i-\varphi_j)}\rangle \approx \langle e^{ip(\varphi_i-\varphi_{i+1})}\rangle\langle e^{ip(\varphi_{i+1}-\varphi_{i+2})}\rangle \cdots \langle e^{ip(\varphi_{j-1}-\varphi_j)}\rangle .$$

In this expression one should choose the nearest path between sites i and j. Using the expansion

$$\cos^t[(\varphi_i-\varphi_j)/2]-c(t,2)=\sum_{p=1}^{t/2} a_p(t)\cos p(\varphi_i-\varphi_j)$$

for even t and

$$\cos^t(\varphi_i-\varphi_j)=\sum_{p=1}^{(t+1)/2} a_{2p-1}(t)\cos[(2p-1)(\varphi_i-\varphi_j)]$$

for odd t, where $a_p(t)$ and $a_{2p-1}(t)$ are given by Eqs. (9) and (10). We can demonstrate that, to the first order in J/T, each nearest-neighbor term contributes in the amount

$$\langle e^{ip(\varphi_i-\varphi_{i+1})}\rangle = a_p(t)J/2k_BT\ . \quad (11)$$

Thus, in the high-T limit we obtain

$$g_p^t(r_{ij}) \sim [a_p(t)J/2k_BT]^{r_{ij}} \sim \exp\left[-r_{ij}\ln\frac{2k_BT}{a_p(t)J}\right]\ . \quad (12)$$

The correlation function decays exponentially with the distance between multipole moments. For a given t the coefficient $a_1(t)$ is the smallest, therefore at large separation the first function g_1^t dominates. Since $a_1(t)$ decreases with t the decay of the correlation function is faster for larger t. We shall see that at low temperatures it will be the other way around.

"Spin-wave" approximation. We consider now the low-temperature case ($K \gg 1$). The Hamiltonians (3) for odd t and (5) for even t have one minimum at $\varphi_{ij}=0$. So at low T's one can take into account only the quadratic term in the expansion of the Hamiltonians around this equilibrium angle. Apart from an irrelevant constant we have

$$H = \frac{\tilde{K}}{2}\sum_{\langle ij\rangle}(\varphi_i-\varphi_j)^2\ , \qquad \tilde{K} = \begin{cases} Jt/4 & \text{for even } t\ , \\ Jt & \text{for odd } t\ . \end{cases} \quad (13)$$

Using this Hamiltonian and following Refs. 8 and 15, we can obtain the algebraic decay of the correlation function at low temperatures

$$g_p^t(r_{ij}) \sim |r_{ij}|^{\eta_p(t)}\ , \quad (14)$$

where

$$\eta_p(t) = \begin{cases} 4p^2k_BT/2\pi tJ, & p=1,2,\ldots,t/2 \text{ for even } t\ , \\ p^2k_BT/2\pi tJ, & p=1,3,\ldots,(t-1)/2 \text{ for odd } t\ . \end{cases} \quad (15)$$

It is interesting to note that the exponent $\eta_p(t)$ depends on the rank t of the multipole moment. Contrary to the high-temperature case, the decay of correlation is slower the larger t is. At large distances between multipole moments the first term g_1^t in (9) and (10) dominates since $\eta_1(t)$ is the smallest. The algebraic behavior of the correlation indicates that similar to the XY model all the multipolar systems are topologically ordered at low T's.

Vortex perturbation correction. First we want to demonstrate how to construct the Villain potential[16] for the multipole systems. Such a potential allows one to consider the existence of vortices in a simple way.[8] To this end we introduce a function

$$V(\varphi_i-\varphi_j) = \begin{cases} -K[1-\cos^t(\varphi_i-\varphi_j)] & \text{for odd } t\ , \\ -K\{1-\cos^t[(\varphi_i-\varphi_j)/2]\} & \text{for even } t\ . \end{cases} \quad (16)$$

Using a Fourier series

$$e^{V(\varphi)} = \sum_{s=-\infty}^{s=\infty} e^{is\varphi}e^{\tilde{V}(s)}\ , \quad (17)$$

and the Poisson formula

$$\sum_{s=-\infty}^{\infty} g(s) = \sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty} d\Phi\, g(\Phi)e^{2\pi i m\Phi}\ , \quad (18)$$

one can rewrite the partition function as follows:

$$Z = \sum_{\{m(\mathbf{R})\}}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\prod d\Phi(\mathbf{R})\exp\left[\sum_{\langle\mathbf{RR}'\rangle}\tilde{V}[s(\mathbf{R})-s(\mathbf{R}')] + \sum_{\mathbf{R}} 2i\pi m(\mathbf{R})\Phi(\mathbf{R})\right]\ . \quad (19)$$

Here $\mathbf{R}$ and $\mathbf{R}'$ denote sites on the dual lattice.[8]

Taking into account (16) and (17) one has to calculate the Fourier coefficient

$$e^{\tilde{V}(s)} = \int_0^{2\pi}\frac{d\varphi}{2\pi}e^{-is\varphi+V(\varphi)}\ . \quad (20)$$

Note that for $t=1$ and 2 the last integral is calculated exactly. For $t>2$ this integral may be evaluated at high- and low-temperature limits. At high temperature ($K \ll 1$) we obtain

$$e^{\tilde{V}(s)} = \begin{cases} (K/2^t)^{(s+2m)/t}C_{s+2m}^m/[(s+2m)/t]! & \text{for odd } t\ , \\ (K/2^t)^{2(s+m)/t}C_{2(s+m)}^m/[2(s+m)/t]! & \text{for even } t\ . \end{cases} \quad (21)$$

The positive integer number m should be chosen so that $(2+2m)/t$ (for odd t) and $2(s+m)/t$ (for even t) will be the smallest integer number. For example, for $s=5$ and $t=3$ we have $m=2$. For low temperatures our calculations yield

$$e^{\tilde{V}(s)}=\frac{1}{\sqrt{2\pi\tilde{K}}}e^{-s^2/2\tilde{K}}\ , \tag{22}$$

where $\tilde{K}$ is given by Eq. (13). From (19) and (22) it is clear that the Villain potential has the same form as in the XY case[8] but with the modified interaction. The variable $\Phi(\mathbf{R})$ describes a "spin-wave" degree of freedom whereas the quantum number $m(\mathbf{R})$ describes a vortex excitation.

Similar to the familiar XY case,[8] taking into account only the "spin-wave"–vortex interaction we have the following expression for the KT transition temperature:

$$T_{\mathrm{KT}}=\pi\tilde{K}/2k_B=\begin{cases}\pi tJ/8k_B & \text{for even } t\ ,\\ \pi tJ/2k_B & \text{for odd } t\ .\end{cases} \tag{23}$$

The vortex-vortex interaction should change the critical temperature T_{KT}. In fact, the renormalization-group analysis[7,8] shows that in this case the actual value of T_{KT} may be defined from the following equation:

$$\pi\tilde{K}/k_BT_{\mathrm{KT}}-2=\exp(-\pi^2\tilde{K}/2K_BT_{\mathrm{KT}})\ . \tag{24}$$

Here $\tilde{K}$ is given by Eq. (13). The right part of Eq. (24) gives the correction to T_{KT} defined in (23). It is easy to show that the numerical differences between critical temperatures followed from Eqs. (23) and (24) are very small. These equations also suggest that $T_{\mathrm{KT}}\to\infty$ in the $t\to\infty$ limit. However, this is an artifact of the harmonic approximation which fails for large t. The reason is as follows. The Hamiltonians (3) and (5) may be expanded around $\varphi_{ij}=\varphi_i-\varphi_j=0$:

$$H=\sum_{ij}\sum_{n=0}^{\infty}J_{ij}b_{2n}(t)\varphi_{ij}^{2n}\ .$$

At large t the coefficients $b_{2n}(t)\sim t^n$ and the terms of order higher than 2 become important. We shall see in Sec. III B that a more accurate discretized MKRG eliminates this deficiency and yields T_{KT} which would saturate for large t. At $T=T_{\mathrm{KT}}$ all the critical exponents $\eta_p(t)$ become independent of t and they are equal to

$$\eta_p=\frac{p^2}{4},\quad p=1,2,\ldots,t/2 \tag{25}$$

for even t and

$$\eta_{2p+1}=\frac{(2p-1)^2}{4},\quad p=1,\ldots,(t+1)/2 \tag{26}$$

for odd t. Clearly at large separation the term characterized by the smallest exponent $\eta=\frac{1}{4}$ dominates. So at $T=T_{\mathrm{KT}}$ all the $m=2$ multipolar systems would behave similarly in $D=2$.

Following the renormalization-group analysis of Ref. 7 one can show that the critical behavior of the multipolar system is governed by the vortex-antivortex pairs correlation length $\xi(T)$. Below T_{KT} this length is infinite and diverges as

$$\xi(T)\sim\exp\{\mathrm{const}/(T-T_{\mathrm{KT}})^{1/2}\} \tag{27}$$

as $T\to T_{\mathrm{KT}}$ from above.

In the leading approximation the critical behavior of the susceptibility χ^t is the same for all multipolar systems. We have

$$\chi^t\sim\int g^t(r)d^2r\sim\begin{cases}\xi^{2-\eta}, & T>T_{\mathrm{KT}}\ ,\\ \infty, & T<T_{\mathrm{KT}}\ ,\end{cases} \tag{28}$$

where $\eta=\eta_1=\frac{1}{4}$.

External field. Fields applied to multipolar systems are, in general, of a tensorial character. Here we consider the simplest case assuming that the external field is parallel to x axis on the XY plane. Then one has to add energy

$$E_h=-h\sum_i[\cos^t\varphi_i-c(t,2)] \tag{29}$$

to the Hamiltonian (3). In the case of even t after mapping φ_i in the last equation should be replaced by $\varphi_i/2$. The field dependence of the multipole moment m which is a conjugate to h order parameter at T_c is characterized by a critical exponent δ (see, e.g., Ref. 7). At small fields

$$m\sim h^{1/\delta},\quad T=T_{\mathrm{KT}}\ . \tag{30}$$

Simple calculations show that δ is universal for all multipole systems and $\delta=15$. This value of δ was obtained by Kosterlitz[7] for the dipole system.

III. MIGDAL-KADANOFF ANALYSIS

A. Harmonic approximation

The MKRG scheme[17] based on the harmonic approximation gives correct results for the lower critical dimensionality for uniform XY and Heisenberg models.[14] Here we want to find out how useful is this approximation in reference to multipolar systems. For simplicity we consider the hierarchical lattice which is constructed by replacing a bond by b^{D-1} pieces each made of b bonds connected in series. The pieces are connected in parallel.

It is easy to show that in the odd-t case a $T=0$ harmonic approximation is appropriate for both "ferromagnetic" ($J>0$) and "antiferromagnetic" ($J<0$) interactions. For $m=2$ and 3 the recursion relation is as follows ($T=0$):

$$J'=b^{D-2}J\ . \tag{31}$$

The last equation is also valid for the even t with $J>0$ and for $t=2$ with $J<0$. From Eq. (30) it follows that the lower critical dimensionality of multipolar systems with odd t and of systems with even t and $J>0$, and with $t=2$ and $J<0$ is equal to 2. The harmonic approximation does not work in the case when $t=2n$ ($n=2,3,\ldots$) and the coupling $J<0$. The reason for this is as follows. The harmonic approximation requires expanding the Hamiltonian (5) about the equilibrium angles between neighboring multipole moments up to quadratic terms. In the

$m=2$ case, for example, the equilibrium angles corresponding to negative J are $\varphi_{ij}^{(\mathrm{eq})}=\pi$. However, all the derivatives of the Hamiltonian (5) with respect to an angle of order less than t disappear at $\varphi_{ij}^{(\mathrm{eq})}=\pi$. Taking these equilibrium states into account would require expanding the Hamiltonian up to φ_{ij}^t terms. Then, for $t=2n(n=2,3,\ldots)$, the effective partition function would cease to be Gaussian. This model can, however, be studied within the framework of the discretized scheme.[13]

B. The discretized MKRG scheme

Case $b=2$. To study the low-temperature properties of the 2D uniform multipolar system one can use the MKRG approach developed by José *et al.*[8] This scheme is based on expansion of $\exp(-H/k_BT)$ in a Fourier series and Migdal-Kadanoff recursion relations for the Fourier coefficients. However, we shall use our discretized MKRG approach[13] which has an advantage that it allows us to investigate not only uniform but also nonuniform systems at arbitrary temperature.

The idea of the discretized scheme is as follows: instead of allowing φ to be a continuous variable, we allow it take one of $q \gg 1$ discrete values which are uniformly distributed between 0 and 2π. The Hamiltonian is now defined for values of φ restricted to be $2\pi k/q$, where q is the number of clock states, and $k=0,1,\ldots,(q-1)$. Considering $m=2$ we define

$$J_{ij}(q,k)=J_{ij}[\cos^t(2\pi k/2q)-c(t,2)] \tag{32}$$

for even t and

$$J_{ij}(q,k)=J_{ij}\cos^t(2\pi k/q) \tag{33}$$

for odd t. Then we find

$$\sum_{k=0}^{q-1} J_{ij}(q,k)=0 , \tag{34}$$

where

$$q=\begin{cases} tn/2 & \text{for even } t, \\ tn & \text{for odd } t, \end{cases} \quad n=2,3,\ldots . \tag{35}$$

The restriction (35) imposed on q is needed to satisfy condition (34). Clearly, a minimal number of clock states q is equal to t and $2t$ for even and odd t, respectively. The recursion relations for the discretized clock models corresponding to multipolar systems can be derived straightforwardly and for the 1D decimation step they read[13] (the scaling factor $b=2$)

$$J'_{AB}(q,k,i)=k_BT\left[\ln F_{AiB}(q,k,T) - (1/q)\sum_{l=0}^{q-1}\ln F_{AiB}(q,l,T)\right] , \tag{36}$$

where

$$F_{AiB}(q,k,T)=\sum_{l=0}^{q-1}\exp\{J_{Ai}(q,l)+J_{iB}[q,\mathrm{mod}(q+k-l,q)]\}/k_BT . \tag{37}$$

Equation (36) is derived by noting that the renormalized Hamiltonian is characterized by renormalized exchange interactions and by a constant term. The latter was determined by imposing condition (34) on the rescaled couplings. The recursion scheme is completed by combining 2^{D-1} bonds decimated according to Eq. (36) into one rescaled bond $J'_{AB}(q,k)$ which is a q-valued variable

$$J'_{AB}(q,k)=\sum_{i=1}^{2^{D-1}} J'_{AB}(q,k,i) . \tag{38}$$

To study the influence of high polar moments on the critical behavior in 2D we take the octupolar model as an example. In this case the discretized coupling $J_{ij}(q,k)$ is as follows:

$$J_{ij}(q,k)=J_{ij}[\tfrac{1}{2}\cos(2\pi k/q)+\tfrac{1}{8}\cos(4\pi k/q)] . \tag{39}$$

The value of q depends on the number of the scaling iterations to be performed. As seen in Fig. 1 for the octupolar system, the first six iterations show essentially no crossover to the Ising-like behavior for $q=600$. Results presented in Fig. 1 also suggest that the $T=0$ scaling exponent y of the octupolar system is equal to $y=D-2$ (i.e., the lower critical dimensionality should be equal to 2). This happens for $q>2$ on length scales shorter than some critical value $L_c(q)$ which diverges as q tends to infinity. We found that $l_c(q)$ becomes very large when one calculates finite temperature couplings, even when T is infinitesimal. This is shown in Fig. 2 for the 2D $m=2$ octupolar model with $q=900$. No trace of a crossover is seen up to 20 iterations at $T=0.03J/k$ in this case: the effective coupling has reached a fixed point. For temper-

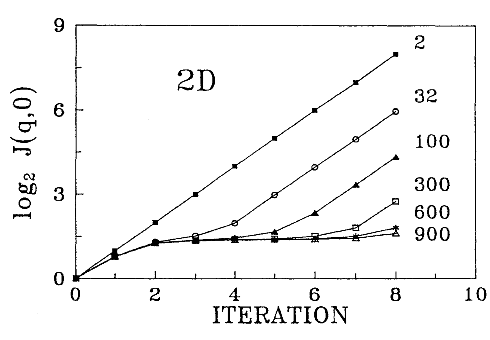

FIG. 1. Scaling of $J(q,0)$ for 2D at $T=0$ for various values of q for uniform octupolar model shown on the right-hand side of the figure. The result for the Ising model ($q=2$) is presented for comparison. For 3D the behavior is similar but the asymptotic law corresponds to a linear growth.

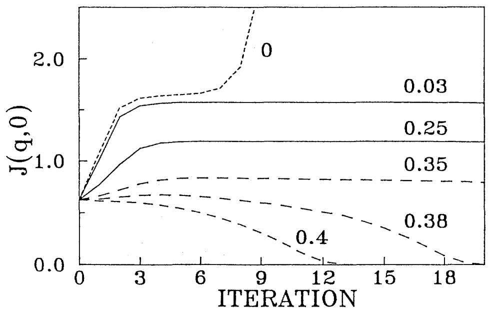

FIG. 2. Iterations of $J(q,0)$ for the uniform octupolar model with $q=900$ for the temperatures indicated in the plot (in units of J/k_B). The $T=0$ line was obtained from the recursion relations which were explicitly considered for this limiting T.

atures less than $0.26/Jk_BT$, $J(q,0)$ settles on a T-dependent fixed point value which is suggestive of a line of fixed points. This line seems to terminate at a $T_{\rm KT}$ of $0.26J/k_B$. At higher temperatures the coupling rescales to zero, indicating paramagnetic behavior. Figure 3 shows the behavior of the renormalized helicity modulus defined as the fixed point value of $J(q,0)$ normalized to its $T=0$ value. The plot clearly indicates a sharp transition to the paramagnetic phase.[18]

We have also calculated the critical temperatures for other t by the discretized scheme. In units of J/k_B we get $T_{\rm KT}=0.22$, 0.26, and 0.31 for $t=2$, 4, and 6, and $T_{\rm KT}=0.44$, 0.48, and 0.49 for $t=1$, 3, and 5, respectively. Clearly, this results supports the analytical one given by Eqs. (23) and (24) in the sense that $T_{\rm KT}$ should increase with t, and critical temperatures of systems with odd and even t belong to different branches. It seems, however, that $T_{\rm KT}$ should saturate with increasing t. It should be noted that it is not reasonable to compare numerical MKRG values of $T_{\rm KT}$ with those obtained by the analytical approach. The reason is that $T_{\rm KT}$ given by Eq. (24) depends on the chemical potential of a vortex which is not known exactly.[8] In the familiar XY case $T_{\rm KT}$ obtained by the Monte Carlo method[19] is equal to $0.89J/k_B$ (in Ref. 20 the same Monte Carlo method gives $T_{\rm KT}=0.88J/k_B$). This value is higher than our Migdal-Kadanoff value $0.44J/k_B$. Note that in the XY quantum case[12] the Monte Carlo value of $T_{\rm KT}$ is about $0.35J/k_B$.

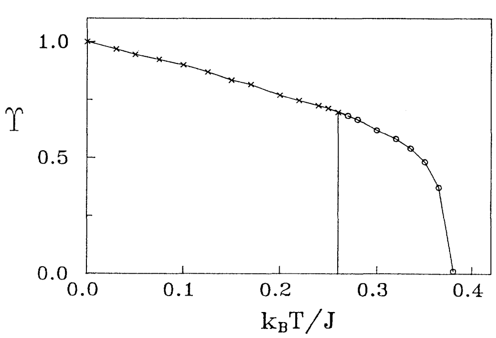

FIG. 3. Normalized helicity modulus for the 2D uniform octupolar model with $q=900$ as a function of temperature. The open circles indicate $J(q,0)$ after 20 iterations for those temperatures at which evidence for a gradual decrease in $J(q,0)$ is seen. Further iterations would result in the helicity modulus being zero. The data points denoted by the crosses are for temperatures at which no decrease in $J(q,0)$ is seen within 20 iterations.

We now focus on the angular (or k) dependence of $J(q,k)$. This is shown in Fig. 4. The microscopic coupling is given by the sum of two cosine functions (39). Below the Kosterlitz-Thouless transition the fixed point "potential" develops a minimum and a maximum slightly shifted from $\phi=\pi$ and 2π, respectively. For example, at $T=0.25J/k_B$ these minimum and maximum are at $\phi=174°$ and 356°. Such a departure is due to the second term in interaction (39). Similar to the study of the XY model by José *et al.*[8], our results suggest that the line of fixed points for the octupolar case is almost present in this approach. However, the behavior we observe in Figs. 2–4 is similar to that found by the same method for the XY model.[13] This means that similarity of properties of the XY and octupolar systems at low temperatures is predicted both by analytical and by discretized MKRG schemes. The systems with a higher t are also expected to behave similarly.

Case $b=3$. As mentioned above the case with even $t>2$ and "antiferromagnetic" couplings cannot be studied in the harmonic approximation. However, one can develop the discretized scheme for this case. To keep couplings antiferromagnetic after rescalings, one should take an odd b. For $b=3$ the recursion relations have been obtained by the discretized scheme. Using the corresponding recursion relations for the octupolar system with $J<0$, we can demonstrate that the lower critical dimensionality of this system is equal to 2. This result is probably valid for cases with higher t.

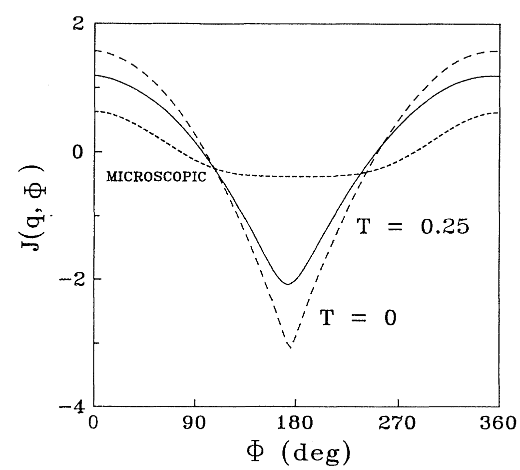

FIG. 4. Angular dependence of $J(q,k)$ in the 2D uniform octupolar model with $q=900$. The dotted line corresponds to the microscopic interactions. The remaining lines correspond to the fixed point situations at $T=0$ (dashed line) and $T=0.25$ (solid line).

IV. MULTIPOLAR MODELS ON FRACTAL LATTICES

A. Sierpinski gasket

Recently Vallat, Korshunov, and Beck[21] have shown that no KT transition takes place on the Sierpinski gasket (see Fig. 5). This is expected because the lattice is finitely ramified[22] and does not allow for any finite T transition even in the Ising case. Let us consider this problem for multipolar systems [the model (3) for odd t and (5) for even t] in the harmonic approximation. Then decimation may be carried out for the Sierpinski gasket. After s steps the coupling $\tilde{K}$ defined by Eq. (13) becomes

$$\tilde{K}^{(s)}=(\tfrac{3}{5})^s\tilde{K} \ . \qquad (40)$$

The coupling scales to zero at any T and there is no transition on the fractal lattice. The correlation function $g_p^t(r_{ij})$ defined by (9) and (10) may be calculated considering that r_{ij} is a distance between the sites on the corners of the same sth order plaquette ($r_{ij}=2^s$). Simple calculations give

$$g_p^t(r_{ij})\sim \exp\{-p^2(\tfrac{5}{3})^s/C\}\sim \exp[-p^2 r_{ij}^{\nu}/C] \ ,$$
$$\nu=\ln(\tfrac{5}{3})/\ln 2, \quad C\sim\tilde{K} \ . \qquad (41)$$

Thus, the exponential decay of the correlation function indicates that the long-range order is absent in the multipolar systems on the fractal lattice. This conclusion remains unchanged when vortex excitations are taken into account.

B. Sierpinski carpet

The Sierpinski carpets are constructed by subdividing a square into b^2 subsquares, then cutting out l^2 of these subsquares[22,23] (see Fig. 6). This yields a fractal of dimensionality $d_f=\ln(b^2-l^2)/\ln b$. It has been shown that, unlike the gaskets, the carpets generally are not finitely ramified and a nonzero temperature transition to the ordered phase takes place in the case of Ising spins.[22] The question we ask now is does the KT transition exist in multipolar systems on Sierpinski carpets? To seek the answer on this question we use the MKRG scheme based on harmonic approximation. Following Ref. 22 one should distinguish between a nearest-neighbor bond on the boundary of a cutout, $\tilde{K}_w$, and an internal bond, $\tilde{K}$. After decimation we find (the eliminated subsquares are taken in the center and all the bonds within a decorated square are moved to its perimeter)

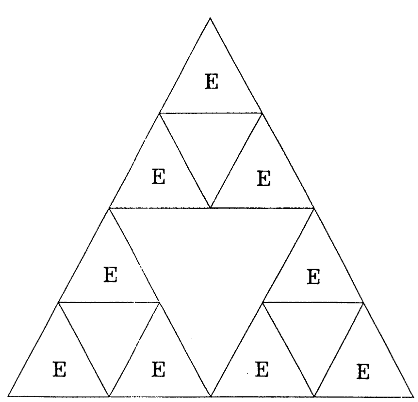

FIG. 5. Fragment of the Sierpinski gasket in two dimensions. The triangles labeled E denote elementary plaquettes.

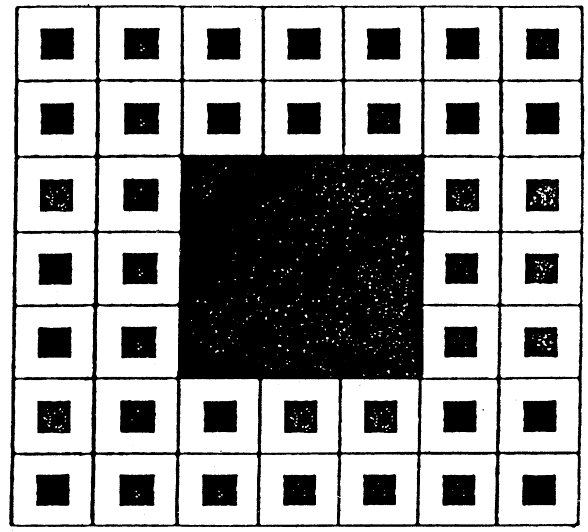

FIG. 6. Fragment of the Sierpinski carpet with $b=7$ and $l=3$.

$$\tilde{K}'=\frac{b\tilde{K}[(b-l-1)\tilde{K}+2\tilde{K}_w]}{[bl+(b-l)(b-l-1)]\tilde{K}+2(b-l)\tilde{K}_w} \ ,$$
$$\tilde{K}'_w=\frac{[(b-l-2)\tilde{K}+4\tilde{K}_w][(b-1)\tilde{K}+2\tilde{K}_w]}{2\{[l(b-1)+(b-l)(b-l-2)]\tilde{K}+2(2b-2l)\tilde{K}_w\}} \ . \qquad (42)$$

For $l=0$ we recover the $D=2$ result (31). For $l>0$ we have two fixed points. One of them is $\tilde{K}^*=\tilde{K}_w^*=0$, which describes the infinite-temperature paramagnetic phase. The second fixed point, which has coordinates $\tilde{K}^*=\infty$ and $\tilde{K}_w^*=\infty$, corresponds to zero temperature. From Eq. (42) it follows that $\tilde{K}^*/\tilde{K}_w^*=2/(l+1)$. Then one can demonstrate that this fixed point is governed by the $T=0$ scaling exponent

$$y=\ln[(b^2-l^2)/b^2]/\ln b=d_f-2 \ . \qquad (43)$$

Obviously, for $l>0$ the exponent y is negative and multipolar systems on Sierpinski carpets are below their lower critical dimensionalities. The absence of a fixed point with nonzero finite couplings suggests that within the MKRG scheme $m=2$ multipolar models would not undergo the KT transition on Sierpinski carpets. Again, this is not surprising since the fractal dimensionality of the carpets is smaller than 2.

V. EXACTLY RENORMALIZABLE MODEL ON FRACTAL AND HIERARCHICAL LATTICES

A. Sierpinski gasket

Vallat, Korshunov, and Beck[21] have introduced the exactly renormalizable XY-like model on a Sierpinski gasket described by the following Hamiltonian:

$$H=k_B T \sum \ln \left[1+K \sum \cos(\varphi_i-\varphi_j)\right] , \tag{44}$$

where the external sum is taken over all elementary plaquettes (see Fig. 5) and the internal one is taken over the perimeter of each such plaquette. This model coincides with the standard XY Hamiltonian only in the limit of high temperatures but it allows for exact renormalization. It does not give rise to the KT transition either.

Multipolar generalizations of (44) are certainly not expected to introduce the KT transition but it is interesting to see the impact of the multipolar rank t on the behavior of the correlation functions in exactly solvable toy models. Following Ref. 21 one can introduce the following Hamiltonian:

$$H=-k_B T \sum \ln \left[1+K \sum [\cos^t(\varphi_i-\varphi_j)-c(t,2)]\right] \tag{45}$$

for $m=2$ multipolar systems on the fractal lattice. It is easy show, however, that for $t>2$ the latter model is not exactly renormalizable. Instead of (45) we consider an even more general Hamiltonian described by

$$H=-k_B T \sum \ln \left[1+\sum \sum_{p=1} K_p \cos p(\varphi_i-\varphi_j)\right] . \tag{46}$$

In the $K_p \to 0$ limit the latter Hamiltonian describes t polar systems if we put $K_p=a_p K$ ($p=1,2,\ldots,t/2$ for even t and $p=1,3,\ldots,t$ for odd t), where coefficients a_p are given by (9) and (10). In what follows we will consider only the case when $K>0$ and the couplings K_p are therefore positive. From the condition that an argument of the logarithm should be positive, we have the following restriction on K_p:

$$K_p < a_p K_c(t) , \tag{47}$$

where $c(t,2)$ is defined in (4). Detailed analysis shows that

$$K_c(t)=\frac{1}{3[c(t,2)-(1/2)^t]} \tag{48}$$

for even t and

$$2/3 \le K_c(t) \le 1 \tag{49}$$

for odd t; $K_c(t=1)=\frac{2}{3}$, $K_c(t=\infty)=1$. It is easy to check that for both even and odd t all the couplings $K_p<1$. So model (46) corresponds to models (3) and (5) at high temperatures.

We now carry out the exact renormalization procedure for the extended model (46) on the Sierpinski gasket. After decimation of all sites on elementary plaquettes, the Hamiltonian has the same structure, but with rescaled couplings

$$K_p^{(1)}=(2+K_p)K_p^2 \Big/ \left[4+\sum_{p=1} K_p^3\right] , \tag{50}$$

where the upper index 1 denotes the coupling after the first renormalization step. For $K_p \ll 1$, Eq. (50) reduces to $K_p^{(1)} \simeq \frac{1}{2}K_p^2$ and after s steps we obtain

$$K_p^{(s)}=2(K_p/2)^{2^s} . \tag{51}$$

Clearly, similar to the harmonic approximation the exact renormalization leads to couplings which scale down to zero. The correlation function may be easy to determine:

$$g_p^t(r_{ij}) \sim K_p^{(s)} \sim \exp\{-[\ln(2/K_p)]r_{ij}\} . \tag{52}$$

From the Eq. (52) it follows that the exponential decay of the correlation function is characterized by a set of correlation lengths $\xi_p \sim 1/\ln(2/K_p)$. It should be noted that the correlation function obtained by the exact renormalization for high temperatures turned out to decay with distance faster than that obtained in the harmonic approximation [see Eq. (41) where $\nu<1$] for low temperatures.

B. Hierarchical lattice

Consider now model (46) on the hierarchical lattice described in Sec. III A. The internal sum over elementary plaquettes in (46) is absent for this geometry. The generalized XY-like Hamiltonian for hierarchical lattices then takes the form

$$H=-k_B T \sum_{\langle ij\rangle} \ln \left[1+\sum_{p=1} K_p \cos p(\varphi_i-\varphi_j)\right] . \tag{53}$$

Assuming that the latter Hamiltonian describes t polar systems on a microscopic level, one can obtain the same restrictions for K_p given by (47) but the critical value $K_c(t)$ is now equal to

$$K_c(t)=\begin{cases} 1/c(t,2) & \text{for even } t , \\ 1 & \text{for odd } t . \end{cases} \tag{54}$$

From Eqs. (4), (9), (10), (47), and (54) it follows that for odd t and $t=2$ all couplings K_p should be smaller than 1 whereas for even $t>2$ only K_1 may be greater than 1. For instance, for $t=4$ and 6, K_1 should be smaller than $\frac{4}{3}$ and $\frac{3}{2}$, respectively. However, model (53) may, in general, describe the multipolar systems on hierarchical lattices in the high-temperature region and one can consider all couplings K_p to be smaller than 1 for any t.

The model given by (53) is also exactly renormalizable on the hierarchical lattice. For 2D and $b=2$ we have the following exact recursion relation:

$$K_p^{(s)}=K_p^{2^s} . \tag{55}$$

Obviously, starting from $K_p<1$ all couplings scale down to zero. Using Eq. (55) one can determine the correlation function

$$g_p^t(r_{ij}) \sim \exp\{-[r_{ij}\ln(1/K_p)]\} . \tag{56}$$

From (52) and (56) it follows that the correlation between multipolar moments decays faster on the fractal than on the hierarchical lattice. The reason for this is that the effective dimension of the Sierpinski gasket is smaller than 2 and corresponding fluctuations should be stronger than in 2D.

VI. FINAL REMARKS

It should be interesting to verify experimentally the dependence of critical exponents on rank t of multipole moments. Our results suggest that the region where the KT transition takes place broadens up with t and low-temperature correlations decay slower. This means that the KT transition may be easier to observe in systems of high-ranking multipolar moments.

If is known that the Heisenberg ($m=3$) spin model does not order in 2D. However, spin orientations have a well-defined structure known as an instanton.[24] The interesting question is then what is the nature of excitation in the $m=3$ multipolar systems ($t>1$) at low temperature. This problem requires further investigation.

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