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Collective spin-wave oscillations in finite-size ferromagnetic samples

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A theory of collective oscillations in a system of spin waves parametrically excited by homogeneous parallel pumping in a finite-size ferromagnetic sample is developed. This theory is an extension of the well-known S theory of spin-wave collective oscillations in an infinite ferromagnetic medium for the case when the boundary conditions in a magnetic sample of finite size D are taken into account. The unstable collective oscillations in the system of parametric spin waves manifest themselves as low-frequency autooscillations of magnetization and demonstrate a variety of bifurcations and transition to chaos. We show that by introducing boundary conditions and taking into account the finite size of the sample it is possible to explain the following experimentally observed properties of spin-wave autooscillations that were not explained by the existing models of this phenomenon: (i) the difference between the threshold of parametric excitation of spin waves $h_{\rm th}$ and the threshold of autooscillations $h_{\rm osc}$; (ii) the finite value of the autooscillation frequency $f_{\rm osc}$ at the threshold of autooscillations; (iii) the dependences of the threshold $h_{\rm osc}$ and the frequency $f_{\rm osc}$ of the spin-wave autooscillations on the size of the ferromagnetic sample. The results of the theory are in good qualitative agreement with the results of experiments in which the influence of sample size on the spin-wave autooscillations was studied in yttrium iron garnet spheres.

I. INTRODUCTION

Strongly nonlinear systems that are far from thermodynamic equilibrium often demonstrate complicated nonstationary behavior such as periodic and chaotic autooscillations. It is well known that boundary conditions can have a considerable influence on the development of nonstationary nonlinear behavior in such systems even in the case when the size of the system under consideration is rather large. Autooscillations of magnetization beyond the threshold of parametric instability of spin waves in ferromagnets[1] are a typical example of a nonstationary behavior in a strongly nonequilibrium nonlinear system where the influence of boundaries of the experimental sample can result in interesting qualitative effects. The parametric excitation of spin waves in magnetic dielectrics (ferrites and antiferromagnets) is usually performed by means of a high amplitude microwave electromagnetic field and has a threshold character (see, e.g., Refs. 2–5). Beyond the parametric instability threshold both stationary and nonstationary behavior can be observed in the system of parametrically excited magnons (or spin waves). The nonstationary behavior manifests itself in a form of periodic and/or chaotic autooscillations of magnetization having a characteristic frequency which is several orders of magnitude lower than the microwave frequency of parametric spin waves.

Theoretical and experimental investigations of the properties of autooscillations of magnetization above the spin-wave instability threshold have been carried out for more than thirty years and considerable experimental experience is gained in this field.[2,5–20] The physical mechanism of spin-wave autooscillations in the case of parallel pump parametric instability[1] was explained by L'vov, Musher, and Starobinets.[21] The authors of Ref. 21 demonstrated theoretically that the autooscillations of magnetization observed in spin-wave experiments are nothing else but the unstable collective oscillations in the system of parametrically excited spin waves analogous to the second sound in liquid helium. The idea of unstable collective oscillations was further developed in a series of subsequent theoretical papers.[22–24] The frequency spectra of both spatially homogeneous[21] and spatially inhomogeneous[23] spin-wave collective oscillations were calculated in the framework of this approach, and the conditions for stability of these oscillations were determined.

The spectrum of spatially homogeneous collective oscillations against the background of spatially homogeneous parametrically excited magnons has the form[21,25]

$$\Omega = i\gamma \pm \sqrt{\Delta^2 - \gamma^2}, \quad \Delta^2 = 4S(2T + S)N^2, \qquad (1)$$

where γ is the equilibrium parameter of magnon dissipation (or the magnon relaxation frequency), S and T are the amplitudes of nonlinear interaction between magnons and N is the number of parametrically excited magnons.

It is clear from the Eq. (1) that if the product $S(2T+S)$ is positive the collective oscillations dissipate and the stationary state of the system of parametric magnons is stable. In the opposite case when $S(2T+S) < 0$ the stationary state is unstable for arbitrary N and the amplitude of collective oscillations increases exponentially with time.

It is worth noting, that the amplitudes S, T of nonlinear interaction between magnons demonstrate a very strong dependence on the crystallographic orientation of the sample even in almost isotropic cubic magnetic crystals like yttrium iron garnet (YIG), where the magnon spectrum is practically not influenced by the crystallographic anisotropy.[6] This property of the coefficients S, T explains the giant crystallographic anisotropy of spin-wave autooscillations observed in Refs. 5, 6, and 26. The expression (1) also explains such characteristic features of spin-wave autooscillations as the resonant properties of radio-frequency susceptibility and the dependence of the autooscillation frequency on the input power.[5–8]

However, the theory of autooscillations[6] based on expression (1) does not explain the following important qualitative properties of autooscillations observed in experiments.[5,6,27,28]

(i) The difference between the thresholds of parametric excitation of spin waves h_{th} and the threshold of spin-wave autooscillations h_{osc}. In experiment this difference is about 0.1–2.0 dB and depends on the wave number of the parametrically excited spin waves, while in the theory[6] these two thresholds coincide.

(ii) The finite frequency of autooscillations at the threshold. In the experiment this frequency is about 10–1000 kHz and it depends on the spin-wave wave number, while in the theory[6] this frequency is exactly zero at the threshold of autooscillations.

(iii) The dependence of the autooscillation threshold and the autooscillation frequency at the threshold on the size of the experimental sample. Experiments[27,28] have demonstrated a sample size effect on both the autooscillation threshold and frequency in yttrium iron garnet (YIG) spheres of the orientation [111] while the theory[6] does not take into account the finite size of a sample.

An attempt to develop a general theory of spin-wave autooscillations (for the cases of both parallel and perpendicular pumping) was undertaken by Zhang and Suhl in their paper.[24] They considered a rough (compared to S theory[21,6]) model for the spin-wave nonlinear dynamics and obtained not only the autooscillation spectrum, but also the stability condition for the periodic autooscillations of finite amplitude. In the case of perpendicular pumping the theory[24] gives the different threshold values for the spin-wave excitation and spin-wave autooscillations ($h_{\mathrm{osc}}/h_{\mathrm{th}} = 10$ dB) as well as the finite frequency of autooscillations at the threshold ($\Omega = 2\gamma$, where γ is the spin-wave relaxation parameter). In the case of parallel pumping the theory[24] gives an estimate for the autooscillation frequency $|\Omega| \sim \gamma$ when $SN \sim \gamma$. This result agrees with formula (1). Once again, as in the model[21,6] the influence of the sample size was not taken into account in the theory.[24]

The pronounced size dependence of the autooscillation frequency observed in Refs. 27 and 28 suggests, however, that the above-mentioned differences between the experimental results for autooscillations and the results of the theories[6,24] can be explained by the influence of the sample boundaries.

The aim of our present paper is to develop a theory of spin-wave autooscillations which takes into account the influence of the sample boundaries and explains all the above-mentioned peculiarities of spin-wave autooscillations observed in the experiments.

The outline of the paper is as follows. In Sec. II we derive the system of equations for the amplitudes of small spatially inhomogeneous perturbations on the background of a stationary distribution of parametrically excited spin waves (or magnons) and obtain the boundary conditions for these perturbations. It is shown that when the characteristic size of the sample D is much greater than the magnon mean free path l the stationary distribution of parametric magnons is spatially homogeneous everywhere in the sample with the exception of a relatively small region (of the size l) near the sample boundaries. As a result, in this limiting case ($D >> l$) we obtain a closed set of equations for the amplitudes of collective oscillations in the system of parametrically excited magnons.

In Sec. III we obtain the frequency spectrum of collective oscillations in a sample of a finite size using the system of equations derived in Sec. II. We consider the case when the distribution of parametrically excited magnons is quasi-one-dimensional and the ferromagnetic sample has a finite size D in the direction of propagation of parametric magnons. We study in detail the particular case when the product $S(2T+S)$ is negative. This condition can be fulfilled, for example, in a cubic ferromagnet magnetized along the easy axis [111]. In an infinite ferromagnetic medium the condition $S(2T+S) < 0$ leads to the instability of spin-wave collective oscillations for any arbitrary small number N of parametric magnons.[6] We show that contrary to that in a sample of a finite size D the spin-wave collective oscillations become unstable only when the number N of parametric magnons is large enough to obey the condition

$$4|S(2T+S)|N^2 > 2\pi^2 v^2/D^2 , \tag{2}$$

where v is the group velocity of parametric magnons. In the same section we find the frequency of spin-wave collective oscillations at the threshold of their instability, which turned out to be finite and dependent on the size of the sample. In the samples of a relatively large size ($D \gg l$) this frequency is inversely proportional to the square of the sample size

$$\Omega \sim \frac{v^2}{\gamma D^2} , \tag{3}$$

while in a relatively small samples ($D \ll l$) the frequency of collective oscillations is inversely proportional to the sample size itself.

$$\Omega \sim \frac{v}{D} . \tag{4}$$

In Sec. IV we present the comparison of our theoretical results with the results of experimental study of the sample size influence on the spin-wave autooscillations in YIG spheres.[27,28]

Conclusions are given in Sec. V.

II. FORMULATION OF THE PROBLEM AND BASIC EQUATIONS

In a ferromagnetic sample of a finite size D the distribution of parametric magnons above the threshold of their excitation and collective oscillations in the system of these parametric magnons can be spatially inhomogeneous. The equations for the distribution function of parametric magnons in the spatially inhomogeneous case were obtained in Ref. 23. These equations can be rewritten in the form of equations for the envelope $A_{\mathbf{k}}$ of the packet of parametric spin waves in the form

$$\left[\frac{\partial}{\partial t} + \mathbf{v_k}\frac{\partial}{\partial \mathbf{r}} + \gamma_{\mathbf{k}} + i\left(\tilde{\omega}_{\mathbf{k}} - \frac{\omega_p}{2}\right)\right] A_{\mathbf{k}} + iP_{\mathbf{k}}A^*_{-\mathbf{k}} = 0, \tag{5}$$

$$P_{\mathbf{k}} = hV_{\mathbf{k}} + \sum_{\mathbf{k}'} S_{\mathbf{k},\mathbf{k}'}A_{\mathbf{k}'}A_{-\mathbf{k}'}, \tag{6}$$

$$\tilde{\omega}_{\mathbf{k}} = \omega_{\mathbf{k}} + 2\sum_{\mathbf{k}'} T_{\mathbf{k},\mathbf{k}'}|A_{\mathbf{k}'}|^2. \tag{7}$$

Here $\gamma_{\mathbf{k}}$ and $\mathbf{v_k}$ are the dissipation parameter and the group velocity of parametric spin waves, $\omega_{\mathbf{k}}$ is the spectrum of spin waves in the sample when the pumping field is absent, $h(t) = h\exp(\omega_p t)$ is the magnetic field of parametric pumping which is parallel to the bias magnetic field $\mathbf{H_0}$ ($\mathbf{h} \parallel \mathbf{H_0}$), $V_{\mathbf{k}}$ is the coupling coefficient of the parametric magnons with the pumping field, and $S_{\mathbf{k},\mathbf{k}'}$, $T_{\mathbf{k},\mathbf{k}'}$ are the amplitudes of four magnon interactions calculated in Ref. 6.

This system of equations is, in a certain sense, analogous to the BCS equations in the theory of superconductivity. In particular, in a spatially homogeneous case the threshold of parametric excitation of spin waves is reached when $|hV_{\mathbf{k}}| = \gamma_{\mathbf{k}}$, which is equivalent to the condition of the superconducting transition $T = T_c$. In the spatially homogeneous case the distribution of parametric magnons above the threshold of parametric excitation has a form such that the renormalized pumping $P_{\mathbf{k}}$ (6) is fixed at its threshold level $|P_{\mathbf{k}}| = \gamma_{\mathbf{k}}$ and the renormalized spin-wave frequency $\tilde{\omega}_{\mathbf{k}}$ is equal to one-half of the frequency of the pumping field (i.e., it stays in the exact parametric resonance).

The boundary conditions for the spin-wave envelopes $A_{\mathbf{k}}$, $A_{-\mathbf{k}}$ are, strictly speaking, dependent on the form and properties of the sample surface and can be derived rigorously by means of the diagrammatic technique for nonequilibrium processes.[29] However, in the case when the sample has a rough surface (so that the roughness is of the order of the wave length of a parametric spin wave) and a spectrally narrow packet of spin waves is excited in the sample the spin waves scattered from the rough surface do not, as a rule, return back to the packet and simply dissipate in the volume of the sample.[30] In such a case the boundary condition for the envelope of the spin-wave packet is that the envelope amplitude of the spin-wave packet reflected from the sample surface vanishes. We note, that the boundary conditions for the individual spin waves forming a narrow wave packet can be quite different from the above formulated boundary condition for the packet envelope $A_{\mathbf{k}}$.

We consider in our present paper the simplest (but important for experiments) case when the magnetic sample is a thin monocrystalline ferrite film of the width D. The constant bias magnetic field $\mathbf{H}_0$ and the ac pumping magnetic field $\mathbf{h}$ lie in the film plane, are parallel to each other, and are both oriented along the crystallographic axis [111] of the film. In this case the parametrically excited spin waves form a narrow packet with wave vectors oriented perpendicularly to the axis [111] (along which the pumping field $\mathbf{h}$ is aligned). This experimental configuration was used in the recent measurements.[26]

The simple model described above should also give a good qualitative description of the distribution of parametrically excited spin waves in spherical magnetic samples as in these samples above the threshold of parametric excitation this distribution is also highly anisotropic and consists of several narrow packets of parametric spin waves (see Refs. 31 and 32).

Our aim is to study the small spatially inhomogeneous perturbations of the stationary solution of the system of equations (5) for the envelope $A_{\mathbf{k}}$ of the spin-wave packet in the case when we take into account the boundary conditions of absence of reflections on the sample boundaries

$$A_{\mathbf{k}}|_{x=-D/2} = 0, \quad A_{-\mathbf{k}}|_{x=D/2} = 0. \tag{8}$$

We give a detailed consideration of the situation when the sample size D is much larger than the mean free path of a spin wave in the sample $l = v_k/\gamma_k$ ($D >> l$). This condition corresponds to a typical experimental situation.[10,33]

To solve this problem we need to know the stationary state of the system of parametric spin waves in the sample beyond the threshold of parametric excitation. This stationary state was studied in Ref. 23. It was shown in Ref. 23 that when the condition $D >> L$ takes place the stationary state in the volume of the sample is spatially homogeneous and the same as in the case of unbounded ferromagnetic medium. The deviations from the spatial homogeneity of the stationary state appear only in a layer of the thickness l near the boundaries of the sample.

At the same time the characteristic scale of spatial inhomogeneity of the perturbation of spin-wave envelopes in the sample is of the order of sample size D (the wave number of spatially inhomogeneous autooscillations is usually of the order of $1/D$). It means that for these perturbations the deviations of the stationary state from the spatially homogeneous "unbounded" case are negligible.

In the case when two narrow spin-wave packets with opposite directions ($\mathbf{k}$ and $-\mathbf{k}$) of the spin-wave wave vectors exist in the sample Eq. (5) and its conjugate can be reduced to a system of four equations for the values $A_{\mathbf{k}}, A_{-\mathbf{k}}, A^*_{\mathbf{k}}, A^*_{-\mathbf{k}}$. The homogeneous stationary solution of these equations has been studied in detail in Refs. 6 and 9 and it has the form

$$|A_{\mathbf{k}}^0| = |A_{-\mathbf{k}}^0|,$$
$$N = \sum_{\mathbf{k}} |A_{\mathbf{k}}^0|^2 = \frac{|V_{\mathbf{k}}|}{|S|}\sqrt{h^2 - h_{th}^2}, \tag{9}$$
$$\arg\left(hV_{\mathbf{k}}A_{\mathbf{k}}^{0\,*}A_{-\mathbf{k}}^{0\,*}\right) = \arcsin\left(\gamma_{\mathbf{k}}/|hV_{\mathbf{k}}|\right).$$

Here $S = S_{\mathbf{k},\mathbf{k}}$ is the amplitude of four magnon interaction, N is the number of parametrically excited magnons, h_{th} is the threshold pumping field, and $V_{\mathbf{k}}$ is the coupling coefficient of magnons with the pumping field. Linearizing the dynamic equations (5) for small spatially inhomogeneous perturbations of the homogeneous stationary solution

$$A_{\pm\mathbf{k}} = A_{\pm\mathbf{k}}^0 + \alpha_{\pm\mathbf{k}} \exp\left(-i\Omega t + i\kappa x\right),$$
$$A_{\pm\mathbf{k}}^* = A_{\pm\mathbf{k}}^{0\,*} + \alpha_{\pm\mathbf{k}}^\dagger \exp\left(i\Omega t + i\kappa x\right) \tag{10}$$

we get the following system of equations for the renormalized perturbations ϕ_n of the magnon distribution

$$\frac{\partial\phi_1}{\partial t} + v\frac{\partial\phi_2}{\partial x} + 2iSN\phi_3 = 0,$$
$$v\frac{\partial\phi_1}{\partial x} + \left(\frac{\partial}{\partial t} + 2\gamma\right)\phi_2 = 0,$$
$$2i(T^+ + T^- + S)N\phi_1 + \left(\frac{\partial}{\partial t} + 2\gamma\right)\phi_3 + v\frac{\partial\phi_4}{\partial x} = 0,$$
$$2i(T^+ - T^-)N\phi_2 + v\frac{\partial\phi_3}{\partial x} + \frac{\partial\phi_4}{\partial t} = 0, \tag{11}$$

where

$$\phi_1 = A_{\mathbf{k}}^{0*}\alpha_{\mathbf{k}} + A_{\mathbf{k}}^0\alpha_{\mathbf{k}}^\dagger + A_{-\mathbf{k}}^{0*}\alpha_{-\mathbf{k}} + A_{-\mathbf{k}}^0\alpha_{-\mathbf{k}}^\dagger,$$
$$\phi_2 = A_{\mathbf{k}}^{0*}\alpha_{\mathbf{k}} + A_{\mathbf{k}}^0\alpha_{\mathbf{k}}^\dagger - A_{-\mathbf{k}}^{0*}\alpha_{-\mathbf{k}} - A_{-\mathbf{k}}^0\alpha_{-\mathbf{k}}^\dagger,$$
$$\phi_3 = A_{\mathbf{k}}^{0*}\alpha_{\mathbf{k}} - A_{\mathbf{k}}^0\alpha_{\mathbf{k}}^\dagger + A_{-\mathbf{k}}^{0*}\alpha_{-\mathbf{k}} - A_{-\mathbf{k}}^0\alpha_{-\mathbf{k}}^\dagger, \tag{12}$$
$$\phi_4 = A_{\mathbf{k}}^{0*}\alpha_{\mathbf{k}} - A_{\mathbf{k}}^0\alpha_{\mathbf{k}}^\dagger - A_{-\mathbf{k}}^{0*}\alpha_{-\mathbf{k}} + A_{-\mathbf{k}}^0\alpha_{-\mathbf{k}}^\dagger.$$

Here we introduced the notations

$$T^+ = T_{\mathbf{k},\mathbf{k}}, \quad T^- = T_{\mathbf{k},-\mathbf{k}}, \quad S = S_{\mathbf{k},\mathbf{k}} = S_{\mathbf{k},-\mathbf{k}}. \tag{13}$$

for the spin-wave interaction coefficients. We note that in the chosen geometry of the problem $T^- = S$. The coefficients T^+, T^-, and S in Eq. (11) can be expressed in terms of Fourier harmonics S_0, T_0, $S_{\pm 2}, T_{\pm 2}$ of the coefficients of the four-magnon Hamiltonian obtained in Ref. 6 [see Eqs. (4.5)–(4.9) in Ref. 6]

$$T = \frac{1}{2}\left(T^+ + T^-\right) = T_0 + T_2 + T_{-2},$$
$$S = S_0 + S_2 + S_{-2}. \tag{14}$$

Calculation of the determinant of the above derived system of linear equations (11) leads to the following dispersion equation for the spatially inhomogeneous collective oscillations of the envelope of the spin-wave packet

$$\Omega\left(\Omega + 2i\gamma\right) = \left[\frac{\Omega_0^2}{2} + (v\kappa)^2\right] \pm \sqrt{\left[\frac{\Omega_0^2}{2} + (v\kappa)^2\right]^2 - (v\kappa)^2\left[\delta^2 - (v\kappa)^2\right]}, \tag{15}$$

where

$$\Omega_0^2 = 4S(2T + S)N^2, \quad \delta^2 = 4S(T - S)N^2. \tag{16}$$

In accordance with the boundary conditions for the spin-wave envelopes $A_{\pm\mathbf{k}}$, $A_{\pm\mathbf{k}}^*$ (8) the boundary conditions for the small perturbations of the envelope $\alpha_{\pm\mathbf{k}}$, $\alpha_{\pm\mathbf{k}}^\dagger$ have the following form

$$\alpha_{\mathbf{k}}|_{x=-D/2} = 0, \quad \alpha_{-\mathbf{k}}|_{x=D/2} = 0,$$
$$\alpha_{\mathbf{k}}^\dagger|_{x=-D/2} = 0, \quad \alpha_{-\mathbf{k}}^\dagger|_{x=D/2} = 0. \tag{17}$$

In terms of the renormalized perturbations ϕ_1, ϕ_2, ϕ_3, ϕ_4 (12) the boundary conditions (17) can be reduced to

$$(\phi_1 + \phi_2)|_{x=-D/2} = 0, \quad (\phi_1 - \phi_2)|_{x=D/2} = 0,$$
$$(\phi_3 + \phi_4)|_{x=-D/2} = 0, \quad (\phi_3 - \phi_4)|_{x=D/2} = 0. \tag{18}$$

The linear equations (11) in combination with the boundary conditions (18) form a closed system of equations from which the spectrum of the collective spin-wave oscillations in a finite-size magnetic sample can be determined.

III. SPECTRUM OF COLLECTIVE SPIN-WAVE OSCILLATIONS IN A FINITE-SIZE MAGNETIC SAMPLE

The general solution of the linear system of equations (11) for the amplitudes of spatially inhomogeneous perturbations in magnon distribution can be written as a sum of exponentials

$$\phi_i(x) = \Phi_i^+ e^{i\kappa_1 x} + \Phi^- e^{i\kappa_1 x} + \Psi_i^+ e^{i\kappa_2 x} + \Psi_i^- e^{i\kappa_2 x}, \tag{19}$$

where $i = 1,\ 2,\ 3,\ 4$ and κ_1 and κ_2 are the roots of Eq. (15) corresponding to the same frequency Ω

$$(v\kappa_{1,2})^2 = \Omega\left(\Omega + 2i\gamma\right) + \frac{1}{2}\delta^2 \pm \left[\frac{1}{4}\delta^4 + \Omega\left(\Omega + 2i\gamma\right)\left(\Omega_0^2 + \delta^2\right)\right]^{1/2}. \tag{20}$$

Substituting Eqs. (19) for the perturbation amplitudes ϕ_i in the boundary conditions (18) we obtain the expression for the spectrum of the spin-wave collective oscillations in a finite-size magnetic sample as a condition of existence of a nontrivial solution of the linear system (18). The (4×4) matrix of boundary conditions (18) splits into two (2×2) matrices in the variables $(\phi_1 + \phi_2,\ \phi_3 + \phi_4)$ and $(\phi_1 - \phi_2,\ \phi_3 - \phi_4)$. Introducing the dimensionless variables

$$q_{1,2} = \frac{1}{2}\kappa_{1,2}D, \quad \tilde{\Omega} = \frac{\Omega}{2\gamma},$$
$$\Lambda^2 = i\tilde{\Omega}(1 - i\tilde{\Omega}), \quad \Delta^2 = \delta^2/(2\gamma)^2, \tag{21}$$
$$\rho^2 = \frac{\Omega_0^2}{\delta^2} \equiv \frac{2T+S}{T-S}, \quad \xi = \frac{v}{\gamma D} \ll 1$$

we can rewrite the determinant of the first of these (2×2) matrices in the form

$$\det M_{n,m} = 0 \ \ (n,\ m = 1,\ 2),$$
$$M_{1,m} = (1 - i\tilde{\Omega})\cos q_m - \xi q_m \sin q_m, \tag{22}$$
$$M_{2,m} = (\Lambda^2 - \xi^2 q_m^2)\tilde{\Omega}\cos q_m - i(\Lambda^2 - \xi^2 q_m^2 + \Delta^2)\xi q_m \sin q_m.$$

The second matrix differs from the first one only by substitution

$$\cos q_{1,2} \to \sin q_{1,2}, \quad \sin q_{1,2} \to -\cos q_{1,2}. \tag{23}$$

The problem of calculation of the threshold of spin-wave autooscillations in a finite-size magnetic sample is thus reduced to finding the minimum value of $|\Delta|$ (which is proportional to the number N of parametrically excited magnons) for which the imaginary part of the frequency Ω of spin-wave collective oscillations is positive

$$\mathrm{Im}\,\Omega \geq 0. \tag{24}$$

The fulfillment of the inequality (24) corresponds to the appearance of the nondecaying spatially inhomogeneous perturbations of the stationary distribution of parametric magnons in the sample, i.e., to the appearance of the autooscillations.

The general analytical solution of the transcendental equation (22) is difficult to find, but in the case of relatively small value of the coefficient $|\rho|^2 < 8$ the solution of Eq. (22) can be obtained in the closed form and the maximum of $|\Delta|^2$ is reached when

$$\kappa_1 \simeq \frac{\pi}{D}, \quad \kappa_2 \simeq i\frac{\sqrt{2}\pi}{\rho^2 D} \tag{25}$$

We note, that in the experimental situation which is most interesting for us (YIG sample magnetized along the [111] axis and pumped at the frequency $\omega_p/2\pi = 9.4$ GHz) the value of $\rho^2 = 0.78$.

The threshold value of the number of parametric magnons corresponding to the appearance of unstable (undamped) perturbations is

$$4\,|S\,(2T+S)|\,N_{\rm osc}^2 = 2\pi^2\frac{v^2}{D^2}. \tag{26}$$

The condition (26) means that the spin-wave collective oscillations become unstable only when the characteristic energy of interaction between magnons δE exceeds the energy splitting in the magnon spectrum caused by the influence of the sample boundaries $\delta E > \hbar\pi v/D$. The condition (26) also demonstrates that the threshold of parametric excitation of spin waves $h_{\rm th}$ and the threshold of spin-wave autooscillations $h_{\rm osc}$ do not coincide in a finite sample and the difference between these two thresholds depends both on the sample size D and on the wave number of parametric magnons (through the magnon group velocity v).

Using Eqs. (9) we can get from Eq. (26) the expression for the threshold amplitude of the pumping field $h_{\rm osc}$ corresponding to the appearance of unstable spin-wave collective oscillations, i.e., the threshold pumping amplitude for the spin-wave autooscillations

$$\frac{h_{\rm osc}^2}{h_{\rm th}^2} - 1 = \frac{1}{2}\left|\frac{S}{2T+S}\right|\left(\frac{\pi v}{\gamma D}\right)^2. \tag{27}$$

The frequency of the collective oscillations at the threshold of their instability is equal to

$$\Omega = \frac{\pi^2 v^2}{2\gamma D^2}\sqrt{\frac{1+2\rho^2}{\rho^2}}. \tag{28}$$

The calculation of the determinant of the second (2×2) matrix [analogous to Eq. (22)] yields a higher value for the magnon number $N_{\rm osc}$ corresponding to the threshold of autooscillations. Thus, the most unstable are the collective oscillations having the eigenvectors

$$(\phi_1 + \phi_2), \quad (\phi_3 + \phi_4) \tag{29}$$

and the instability threshold determined by Eq. (27).

The influence of the sample boundaries is the strongest in the case when the size of a sample D is small compared to the magnon mean free path l, so that the dimensionless parameter ξ is large ($\xi = l/D \gg 1$). Unfortunately, in this case the stationary distribution of parametric magnons is not known. However, it is clear that the characteristic spatial scale of this stationary distribution of magnons is of the order of D and the dissipation of magnons is caused mainly by their scattering on the sample boundaries. This means that only one parameter having the dimension of frequency exists in the problem:

$$\Omega_D = \frac{v}{D}. \tag{30}$$

Thus, in the case $\xi \gg 1$ the threshold of instability of spin-wave collective oscillations and their frequency at the threshold can be estimated as follows:

$$4\,|S\,(2T+S)|\,N_{\rm osc}^2 \approx \Omega_D^2, \tag{31}$$
$$\Omega \approx \Omega_D. \tag{32}$$

It is easy to prove that in the intermediate case, when the size of the sample D is comparable to the mean free path of parametric magnons l ($D \sim l$), the results of the exact solution for the threshold and the frequency of spin-wave collective oscillations Eqs. (26) and (28) and the results of estimations Eqs. (31) and (32) are smoothly transformed one into another.

IV. DISCUSSION AND COMPARISON WITH EXPERIMENTS

To make a detailed comparison of the theory with the experiment it is necessary to carry out some additional measurements where the influence of the size of the sample on the spin-wave autooscillations is methodically studied in a wide range of sample sizes and relaxation parameters of parametric magnons.

However, the experimental data available in the literature (see, e.g., Refs. 5 and 6) and, especially, the results of the recent measurements[27,28] enable us to make some preliminary estimations. The most detailed study of the influence of the sample size on the spin-wave autooscillations was done by Rezende, de Aguiar, and Azevedo[27,28] in the case when the size of the experimental sample D is of the order of the mean free path of parametric magnons l. In this case our theoretical results obtained in the limiting cases $D >> l$ and $D << l$ can give only qualitative description of the experimental data. Nevertheless, the estimations made using expressions (27), (28) for the conditions of the experiments[27,28] (parallel pumping at the frequency $f_p = 9.4$ GHz in the YIG spheres of the diameter $D_1 = 1$ mm and $D_2 = 0.52$ mm magnetized by the bias magnetic field $\mathbf{H}_0$ oriented along the crystallographic axis [111]) demonstrate that the field dependence of the difference between the thresholds $h_{\rm osc}^2 - h_{\rm th}^2$ obtained from Eq. (27)

$$h_{\rm osc}^2 - h_{\rm th}^2 \propto \frac{v_k^2}{D^2} \propto H_c - H \tag{33}$$

is in a good agreement with the experimental results in the region of relatively large spin-wave wave vectors $k > 10^5$ cm^{-1} (see Fig. 1 in Ref. 27). It is in this region $H < H_c$ that the significant size effect was experimentally observed.[27,28]

The numerical estimate made using Eq. (27) for the sample size $D_1 = 1$ mm and $k = 2 \times 10^5$ cm^{-1} gives $h_{\rm osc}/h_{\rm th} = 1.06$ while the experimental value in this point obtained from Fig. 1 in Ref. 27 is $h_{\rm osc}/h_{\rm th} = 1.09$. The theoretical estimate of $h_{\rm osc}/h_{\rm th}$ was done for the values of the spin-wave interaction coefficients $T = T_0 = -0.62\pi g^2$ and $S = S_0 = 0.32\pi g^2$ (g is the gyromagnetic ratio) calculated for the YIG sphere magnetized along the [111] axis using the formulas presented in Ref. 6 [see Eqs. (4.5)–(4.9) in Ref. 6].

We believe that such agreement between theory and experiment should be considered reasonable especially in the case when D is of the order of l and Eq. (27) is suitable only for qualitative estimates.

The calculation of the frequency on autooscillations at the threshold of their appearance for the spheres of $D_1 = 1$ mm and $D_2 = 0.52$ mm at the point $k = 2 \cdot 10^5$ cm^{-1} was made using Eq. (28) and gave the respective values of autooscillation frequency $\omega_1/2\pi = 100$ kHz and $\omega_2/2\pi = 390$ kHz. The corresponding experimental values obtained from Fig. 2 in Ref. 27 are $f_1 = 95$ kHz and $f_2 = 290$ kHz. In the region of relatively large spin-wave wave vectors ($k > 10^5$ cm^{-1}) Eq. (28) also gives the dependencies of the autooscillation frequency on the wave vector k and the sample size D that are close to those experimentally obtained in Refs. 27 and 28 (see, e.g., Fig. 8 in Ref. 28).

The small difference between the threshold of spin-wave autooscillations and the threshold of parametric excitation of spin waves as well as the small (but finite) autooscillation frequency experimentally observed in the region of relatively small spin-wave wave vectors $k < 10^5$ cm^{-1} are caused, in our opinion, by the effects that were neglected in the S theory of spin-wave interaction.[6] Among these effects we can name the scattering of parametric magnons on one another[34] and scattering on magnetic impurities.[35] The finite autooscillation frequency caused by these effects is proportional to the negative power of the product (kv), and due to this dependence these small effects can play a dominant role in the region of relatively low spin-wave wave vectors. It was shown in Ref. 22 that the above-mentioned small effects of magnon scattering may result in appearance of new modes of spin-wave collective oscillations with very low eigenfrequencies.

V. CONCLUSION

It was pointed out in Refs. 6, 27, and 28 that one of the principal questions in the theory of autooscillations in a system of parametric spin waves was the question of a physical nature of a finite frequency of spin-wave autooscillations at the threshold of their appearance. This frequency was measured experimentally to be 1 to 2 orders of magnitude lower that the magnon relaxation frequency γ — the only characteristic frequency in the infinite system. The autooscillation frequency also demonstrated a pronounced dependence on the size of the experimental sample.

In the present paper we took into account the finite size of a sample in the framework of the approximations of the S theory of magnon interaction.[6,10] As a result we were able to eliminate the main contradictions between the existing theories of spin-wave autooscillations[6,24] and the experimental data on autooscillations obtained in finite-size magnetic samples.

(i) The threshold of spin-wave autooscillations in a finite system $h_{\rm osc}$ turned out to be higher than the threshold of excitation of parametric spin waves $h_{\rm th}$. The difference between these two thresholds depends on the ratio of the size of the experimental sample D to the mean free path of parametric magnons l.

(ii) The frequency of spin-wave autooscillations at the threshold of their appearance is finite and it is determined by the size of the experimental sample.

(iii) The above theory of spin-wave autooscillations in finite-size magnetic samples is a natural extension of the S theory of spin-wave autooscillations.[6] It preserves all the positive features of the S theory such as explanation of the giant crystallographic anisotropy of autooscillations and the correct description of the power dependence of the autooscillation frequency.

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