PHYSICAL REVIEW B VOLUME 47, NUMBER 9 1 MARCH 1993-I

Coupling of microwaves to fluxon motion in Josephson junctions

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(Received 23 November 1992)

Phase locking of fluxon motion in a long Josephson junction is investigated theoretically for microwave coupling through the boundaries of the system. We demonstrate that the wave properties of the fluxon motion during reflection are crucial for predicting the size of the locking range. Analytical results are compared to direct numerical simulations of the sine-Gordon model and excellent agreement is found.

The effort to explain and predict the behavior of magnetic fluxon motion in long Josephson junctions (LJJ's) interacting with externally supplied microwave fields has been stimulated by several experiments showing phase-locking phenomena in both single junctions as well as phase locking of arrays of LJJ's.[1–4] Theoretically, experiments have been successfully explained by use of the sine-Gordon (SG) wave equation, modeling the LJJ.[5] Numerical calculations[6] have shown some characteristics of the phase-locking mechanism, but no direct comparisons have been made to analytical results. Due to the computing time of the problem, long time simulations and scans of the parameter space have typically been done by using the simple particle map approach suggested by Salerno *et al.*[7,8] and Malomed,[9] where a fluxon is treated as a particle with one degree of freedom and where the wave nature of the junction field is neglected. This formulation has proven to be a powerful tool for understanding the fluxon dynamics, as long as the interaction between the particle and the surroundings only takes place at the boundaries of the system. The reflection at a boundary is in the map approach assumed to be an instantaneous event, which means that the wave nature of the reflecting fluxon is neglected. In this paper we will demonstrate that this is not always a good approach if the time of reflection is not negligible compared to the characteristic time of the boundary effect. This has particular consequences for the study of phase locking of fluxon motion to subharmonics of the external drive.[10–12] In these cases the reflection time may be comparable to, or even larger than, the period of the external drive. For a well defined external frequency, we evaluate analytical expressions for the energy change of the system during a reflection and we give the analytical expression for the range in bias current for which the voltage is constant (the locking range).

We will study the SG equation in the form[5,8]

$$\phi_{xx} - \phi_{tt} - \sin\phi = \alpha\phi_t - \eta\,, \tag{1}$$

where ϕ is the phase difference between the quantum mechanical wave functions of the two superconductors defining the junction. The spatial dimension, x, is normalized to the Josephson penetration depth, λ_J, and the temporal dimension, t, is normalized to the inverse plasma frequency, ω_J, of the junction. The dissipative term, $\sim\alpha$, represents the tunneling of quasiparticles through the junction and the external bias current (overlap geometry) is represented by η (normalized to the critical current of the junction). The boundary conditions are given by

$$\phi_x(0) = \pm\phi_x(L) = \delta + \epsilon\sin\Omega t\,, \tag{2}$$

where "+" represents the magnetic coupling and "−" represents the electric coupling (see Ref. 8 and references therein). Here, ϵ and Ω are the normalized amplitude and frequency of the magnetic and electric field, respectively, and the normalized length of the terminated system is denoted by L. It should be noted that in case of the electric coupling, the above notation describes a system in the in-line geometry, where $\eta = 0$ and δ represents the bias current. The electric coupling in the overlap geometry has been treated in Ref. 13. For simplicity we will limit our analysis to the magnetic coupling to the overlap geometry. That is, in the following, we will assume $\delta = 0$ as well as only consider the "+" in Eq. (2).

Let us define the energy of the system in the usual form,

$$H = \int [\tfrac{1}{2}\phi_x^2 + \tfrac{1}{2}\phi_t^2 + 1 - \cos\phi]dx\,. \tag{3}$$

From this we find the power flow to the system as

$$\dot{H} = -\alpha\int\phi_t^2 dx + \eta\int\phi_t dx - \phi_t\phi_x|_{x=0} + \phi_t\phi_x|_{x=L}\,. \tag{4}$$

As in Ref. 13 we will consider the shuttling fluxon in the junction. In order to do this in a simple way, we will limit the analysis to a one-collision event by studying the analytical solution to the left-hand side of Eq. (1) for $L\to\infty$:

$$\phi = 4\tan^{-1}\left(\frac{1}{u}\frac{\sinh[u\gamma(u)(t-\tau)]}{\cosh[\gamma(u)x]}\right), \tag{5}$$

where $\gamma(u) = (1-u^2)^{-1/2}$ is the inverse Lorentz contraction of a traveling mode with the asymptotic velocity u and τ is the time of collision with the boundary at $x = 0$. Inserting the above solution into the power expression Eq. (4) we get

$$\dot{H} = -\alpha \int \phi_t^2 dx - \frac{2\pi u \eta}{\sqrt{1 - \gamma^{-1}(u)\mathrm{sech}^2[u\gamma(u)(t-\tau)]}} + 4\epsilon u^2 \gamma(u) \frac{\cosh[u\gamma(u)(t-\tau)]}{u^2 + \sinh^2[u\gamma(u)(t-\tau)]} \sin \Omega t. \quad (6)$$

Keeping the perturbation treatment correct up to first order in the perturbation parameters (α, η, and ϵ), we will regard the asymptotic velocity u as a constant. In a phase-locked state, where the fluxon performs one revolution of the junction in the time it takes the external signal to go through N periods, we can find the constant velocity u from the expression,[13]

$$\cosh[\gamma(u)L/2] = (1/u)\sinh[u\gamma(u)N\pi/2\Omega]\,. \quad (7)$$

This equation is derived by using Eq. (5) in half of the soliton period. In the phase-locked state, the energy of the system must remain unchanged after one period of motion, i.e., the following condition is fulfilled:

$$\Delta H = \int_0^{2\pi N/\Omega} \dot{H} dt = 0 \;\Rightarrow \quad (8)$$

$$0 = -2\alpha \int_0^{\pi N/\Omega} \int \phi_t^2 dx dt + 2I\eta + 2\Delta H_b\,, \quad (9)$$

where the I is given by

$$\sinh[I\gamma(u)/4\pi] = (1/u)\sinh[u\gamma(u)N\pi/2\Omega]\,, \quad (10)$$

and the energy change during reflection due to the external magnetic field is

$$\Delta H_b = 4\pi\epsilon \frac{\cosh[\frac{1}{2}\Omega u^{-1}\gamma^{-1}(u)\cos^{-1}(2u^2-1)]}{\cosh[\frac{1}{2}\Omega u^{-1}\gamma^{-1}(u)\pi]} \sin \Omega\tau\,, \quad (11)$$

where the time integral for this contribution is carried out for $-\infty < t < \infty$. The function $\cos^{-1}$ is defined in the interval $[0;\pi]$. This expression, Eq. (11), is the general perturbation result for a reflecting fluxon at a boundary, oscillating with the frequency, Ω. As is clear, this contribution has its maximum ($\Delta H_b = 4\pi\epsilon \cos \Omega\tau$) for $\Omega = 0$, $u = 1$, or $u = 0$. We note that the energy contribution is not well determined for low asymptotic velocities, since the velocity in this case cannot be treated as a constant during the collision.

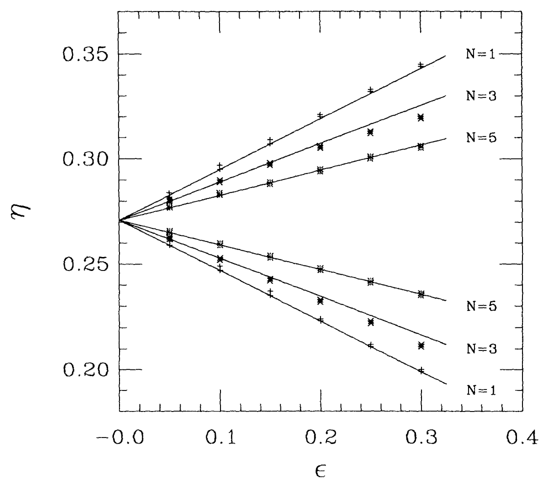

FIG. 1. The locking range in bias current as a function of the amplitude of the driving field. Markers show the results of direct numerical simulations of Eqs. (1) and (2) for the parameters $\alpha = 0.1$, $L = 8$, and $\Omega = N\omega_s$, where $\omega_s = 0.375$. The solid lines show the corresponding analytical results, given by Eqs. (12) and (14).

By adjusting the internal phase, $\Omega\tau$, the system may now exhibit a range in the bias current for which the voltage is constant. This range is given by Eq. (8),

$$\eta_0 - \tfrac{1}{2}\Delta\eta < \eta < \eta_0 + \tfrac{1}{2}\Delta\eta\,, \quad (12)$$

where the center, η_0, and the size of the locking range, $\Delta\eta$, are given by

$$\eta_0 = \frac{\alpha}{I}\int_0^{\pi N/\Omega}\int \phi_t^2 dx dt\,, \quad (13)$$

$$\Delta\eta = \frac{8\pi\epsilon}{I}\frac{\cosh[\frac{1}{2}\Omega u^{-1}\gamma^{-1}(u)\cos^{-1}(2u^2-1)]}{\cosh[\frac{1}{2}\Omega u^{-1}\gamma^{-1}(u)\pi]}\,, \quad (14)$$

and where u and I are given by Eqs. (7) and (10). For large velocities, $u \to 1$, we may write

$$u \approx \Omega L/N\pi \equiv \omega_s L/\pi\,, \quad (15)$$

$$I \approx 2\pi L\,, \quad (16)$$

and hereby get an approximative expression for the size of the locking range:

$$\Delta\eta \approx 4\frac{\epsilon}{L}\frac{\cosh\{\frac{N\pi}{2L}\sqrt{1-(L\omega_s/\pi)^2}\cos^{-1}[2(L\omega_s/\pi)^2-1]\}}{\cosh[\frac{N\pi^2}{2L}\sqrt{1-(L\omega_s/\pi)^2}]}\,. \quad (17)$$

Here, ω_s denotes the frequency of the fluxon motion. Under the approximation of constant asymptotic velocity expressions (11), (14), and (17) are valid for the electric coupling as well. These expressions are, however, only valid for odd N (even N) for the magnetic (electric) coupling.[11]

We have performed direct numerical simulations of the system described by Eqs. (1) and (2) for the overlap junc-

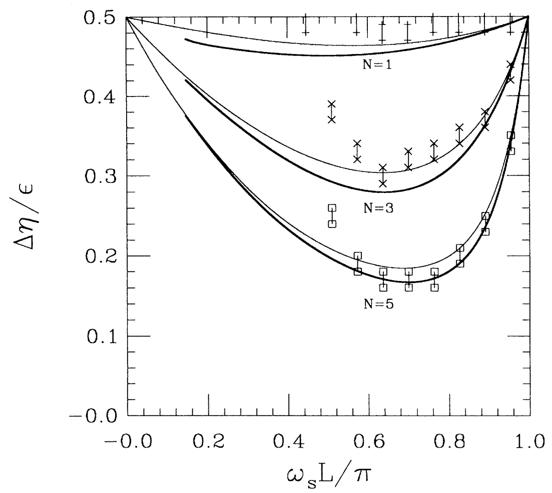

FIG. 2. The locking range in bias current as a function of the soliton frequency. Markers and error bars show the results of direct numerical simulations of Eqs. (1) and (2) for the parameters $\alpha = 0.1$, $L = 8$, $\Omega = N\omega_s$, and $\epsilon = 0.1$. $N = 1$: $+$; $N = 3$: $\times$; $N = 5$: $\Box$. The solid lines show the corresponding analytical results, given by Eqs. (12) and (14) (thick), and Eq. (17) (thin).

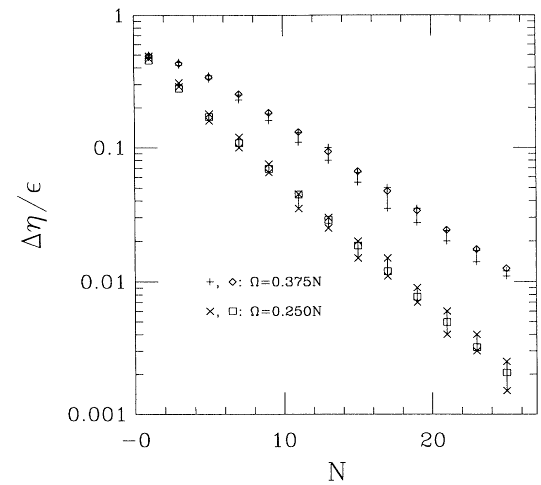

FIG. 3. The locking range in bias current as a function of the ratio N for two different soliton frequencies. The markers $\Diamond$ and $\Box$ show the analytical result Eq. (14) for $\omega_s = 0.375$ and $\omega_s = 0.25$, respectively, and the markers (with error bars) $+$ and $\times$ show the corresponding results of numerical simulations of Eqs. (1) and (2). Parameters are $\alpha = 0.1$, $L = 8$, $\Omega = N\omega_s$, and $\epsilon = 0.1$.

tion in an ac magnetic field. For all the numerical simulations we have chosen the damping and the length of the system to be $\alpha = 0.1$ and $L = 8$. The numerical method of integration is a finite difference scheme, second order in time and fourth order in space, with a spatial grid size of $dx = 0.025$ and a temporal grid size $dt \approx 0.8dx$. All simulations were carried out by allowing the system to relax over 100 soliton periods every time a change in a parameter was made. After this relaxation time, the system was allowed to run for an additional 100 periods to average the normalized voltage over the system.

In Fig. 1 we show the upper and lower limits of the locking range in bias current for different amplitudes and frequencies of the ac magnetic field. We have fixed the soliton frequency, $\omega_s = \Omega/N = 0.35$ for all the data in Fig. 1. The markers represent the results of direct numerical simulations of Eq. (1) and the solid lines are the analytical limits given by Eqs. (12) and (14). As is obvious from Fig. 1 we find excellent agreement between the analytically predicted range of phase locking and the dynamically evaluated locking ranges.

In Fig. 2 we have shown the frequency dependence of the locking range for different values of N. Here, we have fixed the amplitude of the driving field to $\epsilon = 0.1$ for all the data shown. The markers and error bars represent the results of numerical simulations and the solid lines represent the analytically predicted locking ranges. The thick lines represent the analytical expression Eq. (14), where u and I are given by Eqs. (7) and (10), and the thin lines represent the simplified expression Eq. (17), valid for $\omega_s \to \pi/L$. It is clear from Fig. 2 that there is a very close agreement between the results obtained numerically and the ones obtained from the perturbation analysis—even the simplified expression Eq. (17) seems to be very successful in predicting the locking range. We note, however, that for relatively low soliton frequencies there seem to be some deviations between the numerical and the analytical results. This is an artifact of the assumption of constant asymptotic velocity in the perturbation treatment, which is only a good approximation for a relatively high velocity (soliton frequency).

Finally, in Fig. 3 we have shown the locking range as a function of the driving frequency for two fixed values of the soliton frequency. As noted above, only the odd values of N will give rise to phase locking (to first order in the perturbation parameters) when the magnetic coupling is considered. The markers, $\Diamond$ and $\Box$, represent the analytical result Eq. (14) for $\omega_s = \Omega/N = 0.375$ and $\omega_s = \Omega/N = 0.25$, respectively. The markers (with error bars), $+$ and $\times$, show the corresponding results of numerical simulations. Again we find excellent agreement between the simulations and the analytical expressions. From Fig. 3 we see the exponential decay of the locking range as the ratio, N, between the driving frequency and the soliton frequency is increased [see Eq. (17) for large N].

We have evaluated an analytical expression for the energy that a fluxon absorbs from an external magnetic field (or electric field for an in-line geometry) during reflection at a boundary of the junction. We have demonstrated the correctness of the expression by comparing numerical simulations of phase-locking ranges in bias current to analytically evaluated expressions. Excellent agreement has been found. We note that this result is of particular importance for the numerical and analytical studies performed using the map approach.[7,8,10–12] These studies, performed for $\Delta H_b = 4\pi\epsilon \sin \Omega\tau$, have shown too large interaction between the fluxon and the external ac field. In particular, the studies of subharmonic steps must be corrected with the interaction energy proposed in this paper.

This work was performed under the auspices of the U.S. Department of Energy.

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