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Gauge freedom, anholonomy, and Hopf index of a three-dimensional unit vector field

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By identifying a continuous, three-dimensional classical unit vector field with the tangent to a space curve, we show that the associated global anholonomy is related to certain topological quantities, which depend on the dimensionality d of the field. In three dimensions (3D) there exists a conserved topological current. The three components of the corresponding vector potential are the torsions associated with three space curves. Also, there exist two types of global anholonomy: one of them always vanishes identically while the other takes the form of the Hopf term. This is a consequence of the existence of an inherent gauge freedom in the space-curve formalism, which plays no role in 1D and 2D, but becomes important in 3D. An application to classical spin systems is briefly discussed.

There are many problems in physics that can be described in terms of a three-dimensional vector field $\mathbf{t}$ of unit magnitude, normalized such that $\mathbf{t}\cdot\mathbf{t}=1$. For instance: (i) the propagation of light in a twisted optical fiber is studied[1] in terms of $\mathbf{t}(x)$, a function of *one* spatial variable x, the distance along the fiber; (ii) the time evolution of the (normalized) classical spin vector at a site in the continuum version of a one-dimensional Heisenberg chain is described[2] by $\mathbf{t}(x,y)$, a function of *two* variables, one spatial and the other temporal; and (iii) the vector field in the $(2+1)$ dimensional $O(3)$ nonlinear sigma model[3] in field theory is described by $\mathbf{t}(x,y,z)$. There exists a connection between this last model and 2D antiferromagnets, which in turn is relevant in the study of high-T_c superconductors.

In recent years, the notion of anholonomy in physical problems has been receiving much attention.[4] Emphasized by Berry[5] in the context of an adiabatic, cyclic, and unitary evolution of a quantum state, the phenomenon has been extended to more general contexts[6] and finds applications in purely classical[7] problems as well. Anholonomy is a geometrical concept in which a quantity fails to recover its original value when the parameters on which it depends are varied round a closed circuit. In the case of a unit vector field, the tips of the vector lie on the surface S^2 of a unit sphere. Now, the closed circuit may be in the *target* space (i.e., on the unit sphere) or in the *configuration* space. For $\mathbf{t}(x)$ (in 1D) it is the former that plays a role (since there is no "closed" path in 1D configuration space) and it is by now well known that this anholonomy manifests itself as a "geometric phase" which is the solid angle subtended at the center by the area enclosed by the closed path on the sphere.

In this paper, we present a unified method of studying the anholonomy associated with a continuous differentiable unit vector field $\mathbf{t}$ by identifying it with the tangent to a space curve. First, given the tangent $\mathbf{t}(x)$ the space curve is specified uniquely[8,9] (except for its location in space) by its curvature

$$\kappa(x)=\sqrt{\mathbf{t}_x\cdot\mathbf{t}_x} \tag{1}$$

and its torsion

$$\tau(x)=\mathbf{t}\cdot(\mathbf{t}_x\times\mathbf{t}_{xx})/\kappa^2(x)\ . \tag{2}$$

The subscript x denotes d/dx, x being the length along the curve. Second, it is possible to form an orthogonal triad of unit vectors $\mathbf{t}$, $\mathbf{u}$, and $\mathbf{v}$ at any point x. Using vector identities $\mathbf{t}\cdot\mathbf{t}_x=\mathbf{u}\cdot\mathbf{u}_x=\mathbf{v}\cdot\mathbf{v}_x=0$, etc., it is easy to obtain the following Darboux-Ribaucour equations:[8]

$$\begin{aligned}\mathbf{t}_x&=\kappa_g^{(1)}\mathbf{u}-\kappa_n^{(1)}\mathbf{v}\ ,\\ \mathbf{u}_x&=-\kappa_g^{(1)}\mathbf{t}+\tau_g^{(1)}\mathbf{v}\ ,\\ \mathbf{v}_x&=\kappa_n^{(1)}\mathbf{t}-\tau_g^{(1)}\mathbf{u}\ ,\end{aligned} \tag{3}$$

where $\kappa_n^{(1)}$, $\kappa_g^{(1)}$, and $\tau_g^{(1)}$ are called the normal curvature, geodesic curvature, and geodesic torsion, respectively. From Eqs. (1) and (3), $\kappa^2=\kappa_g^{(1)2}+\kappa_n^{(1)2}$. In problems for which the direction of $\mathbf{t}_x$ (i.e., the orientation of the curve in space) is unimportant, one may choose $\kappa_n^{(1)}=0$ without loss of generality. Then Eqs. (1)–(3) show that $\kappa=\kappa_g^{(1)}$ and $\tau=\tau_g^{(1)}$, and Eqs. (3) reduce to the well-known Frenet-Serret equations,[9] $\mathbf{u}$ and $\mathbf{v}$ being the normal and binormal vectors. Equations (3) can be written in the more compact form:

$$\mathbf{F}_x=\xi\times\mathbf{F}\ ,$$

where $\mathbf{F}=\mathbf{t}$, $\mathbf{u}$, or $\mathbf{v}$, and ξ is the Darboux vector

$$\xi=\tau_g^{(1)}\mathbf{t}+\kappa_n^{(1)}\mathbf{u}+\kappa_g^{(1)}\mathbf{v}\ . \tag{4}$$

Here ξ denotes the angular velocity of rotation of the triad as one moves along the curve. In particular, $\tau_g^{(1)}$ represents the angular velocity of rotation of the $\mathbf{u}-\mathbf{v}$ plane around $\mathbf{t}$. We define a nonrotational frame[10] by using the usual Fermi-Walker parallel transport:

$$\frac{DG^i}{dx}=[(\kappa_n\mathbf{u}+\kappa_g\mathbf{v})\times\mathbf{G}]^i\ .$$

Then it is clear that as x is increased from x to $x+dx$, the corresponding infinitesimal angle of rotation of the $\mathbf{u}$ (or $\mathbf{v}$) axis around $\mathbf{t}$, with respect to the nonrotational frame, is given by

$$\delta\phi = \tau_g^{(1)} dx \ . \tag{5}$$

An expression for $\tau_g^{(1)}$ may be found as follows. We first reformulate our theory in terms of Euler angles. If we denote the column matrix consisting of the unit vectors **t**, **u**, and **v** by $\underline{T}$, then Eqs. (3) may be written as

$$\frac{D\underline{T}}{dx} = \underline{A}\,\underline{T} \ , \tag{6}$$

where $\underline{A}$ is the antisymmetric matrix with entries

$$a_{12} = \kappa_g^{(1)}, a_{13} = \kappa_n^{(1)}, a_{23} = \tau_g^{(1)}, \quad \text{and } a_{ij} \in \underline{A} \ .$$

If we introduce the rotational matrices $\underline{R}$, parametrized by the Euler angles, θ, φ, and ψ, Eq. (6) may be written as follows:[10,11]

$$\frac{d\underline{T}}{dx} = \frac{d\underline{R}}{dx}\underline{R}^{-1}\underline{T} \ . \tag{7}$$

Comparing Eqs. (3) and (7) allows us to express κ_g, κ_n, and τ_g as functions of the Euler angles:

$$\kappa_g^{(1)} = \sin\theta \sin\psi \frac{\partial\varphi}{\partial x} + \cos\psi \frac{\partial\theta}{\partial x} \ , \tag{8}$$

$$\kappa_n^{(1)} = \sin\theta \cos\psi \frac{\partial\varphi}{\partial x} - \sin\psi \frac{\partial\theta}{\partial x} \ , \tag{9}$$

$$\tau_g^{(1)} = \cos\theta \frac{\partial\varphi}{\partial x} + \frac{\partial\psi}{\partial x} \ . \tag{10}$$

An alternative interpretation of the angles θ, φ, and ψ is as follows. If θ and φ are identified as the polar and azimuthal angles of $\mathbf{t}(x)$, we may write

$$\mathbf{t} = (\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta)$$

in a cartesian coordinate system. Using Eqs. (8) and (9) in Eqs. (3), it is then readily verified that $\mathbf{t}_x = \sin\theta\,\varphi_x \mathbf{u}' + \theta_x \mathbf{v}'$. Hence $\mathbf{u}' = \mathbf{u}\cos\psi + \mathbf{v}\sin\psi$ and $\mathbf{v}' = -\mathbf{u}\sin\psi + \mathbf{v}\cos\psi$ may be immediately identified as the $(\mathbf{u},\mathbf{v})$ axes rotated through an arbitrary angle ψ about the **t** axis, with $0 \le \psi \le 2\pi$. It is this arbitrariness in defining the orientation of the $(\mathbf{u},\mathbf{v})$ axes in the plane perpendicular to **t** that represents the *gauge degree of freedom* inherent in the problem.

Equations (3) reduce to the Frenet-Serret equations when $\kappa_n^{(1)} = 0$. This yields $\kappa_g^{(1)} = \kappa$ and

$$\tan\psi = \sin\theta \frac{\partial\varphi/\partial x}{\partial\theta/\partial x} \ .$$

Note also that $\tau_g^{(1)}$ is given by the same expression as in Eq. (10), but ψ is no longer arbitrary. Thus, working in the Frenet-Serret frame amounts to choosing a special gauge function ψ. From Eq. (5) it is clear that the total anholonomy associated with going from $x=0$ to $x=x_0$, with the boundary condition $\mathbf{t}(x_0) = \mathbf{t}(0)$, will be given by[10]

$$\Phi = \int \tau_g^{(1)}(x)dx = \int \left[\cos\theta \frac{\partial\varphi}{\partial x} + \frac{\partial\psi}{\partial x} \right] dx \ ,$$

which becomes, on using Stokes theorem,

$$\Phi = 2\pi - \int\int_{\text{surface}} \sin\theta \, d\theta \, d\varphi \ .$$

This anholonomy is associated with a *closed path* in target space and represents the solid angle subtended at the center by the surface enclosing the circuit, which is just Berry's geometric phase, as mentioned earlier.

Next, consider $\mathbf{t}(x,y)$, a function of two variables. Clearly, y can be regarded as a continuously varying parameter for the curve in the sense that for every $y = y_0$, one can associate a space curve with the tangent vector $\mathbf{t}(x,y_0)$. Or alternatively, x plays the role of the parameter for the curve $\mathbf{t}(x_0,y)$. Let $\kappa_n^{(i)}$, $\kappa_g^{(i)}$, and $\tau_g^{(i)}$, where $i=1,2$, represent the parameters that occur in the Darboux-Ribaucour equation (3) corresponding to the x and y space curves, respectively.

If we consider an infinitesimal closed path in the configuration space, it is clear from Eq. (5) that the anholonomy associated with it is measured[12] by the net angle of rotation δQ of the **u** (or **v**) axis around **t**. It is given by[12]

$$\begin{aligned} \delta Q &= \tau_g^{(1)}(x,y)\Delta x + \tau_g^{(2)}(x+\Delta x,y)\Delta y \\ &\quad - [\tau_g^{(2)}(x,y)\Delta y + \tau_g^{(1)}(x,y+\Delta y)\Delta x] \\ &= \left[\frac{\partial\tau_g^{(2)}}{\partial x} - \frac{\partial\tau_g^{(1)}}{\partial y} \right] \Delta x \, \Delta y \ . \end{aligned} \tag{11}$$

Using Eq. (10) in Eq. (11), we find

$$\delta Q = \sin\theta \left[\frac{\partial\theta}{\partial y}\frac{\partial\varphi}{\partial x} - \frac{\partial\theta}{\partial x}\frac{\partial\varphi}{\partial y} \right] dx \, dy \ .$$

Note that δQ is independent of the gauge ψ, and is the determinant of the Jacobian of the transformation $(x,y) \rightarrow \mathbf{t}(x,y)$. Also,

$$\delta Q = \mathbf{t} \cdot (\mathbf{t}_x \times \mathbf{t}_y) dx \, dy \ , \tag{12}$$

as is seen by a direct substitution of $\mathbf{t} = \mathbf{t}(\theta,\varphi)$ in Eq. (12). Hence if $\mathbf{t}(x,y)$ is such that it takes on the same value at the boundaries at infinity, it is readily concluded that[12] the total anholonomy Q is 4π times the Pontryagin index, i.e.,

$$Q = 4\pi n \ , \tag{13}$$

n being the number of times S^2 wraps around S^2.

Finally, consider the case when **t** is a function of three variables, i.e., $\mathbf{t}(x,y,z)$. Construct an infinitesimal cube with edges Δx, Δy, and Δz along the respective axes. In this case, there are two distinct ways of computing anholonomy densities associated with this problem which, as we will show, will lead to two distinct topological terms H_1 and H_2.

To find H_1, we use Eq. (11), which gives the surface anholonomy densities associated with faces $dx\,dy$, $dy\,dz$, and $dz\,dx$ as, respectively,

$$\begin{aligned} J^{(3)} &= \frac{\partial\tau_g^{(1)}}{\partial y} - \frac{\partial\tau_g^{(2)}}{\partial x} \ , \\ J^{(2)} &= \frac{\partial\tau_g^{(3)}}{\partial x} - \frac{\partial\tau_g^{(1)}}{\partial z} \ , \\ J^{(1)} &= \frac{\partial\tau_g^{(2)}}{\partial z} - \frac{\partial\tau_g^{(3)}}{\partial y} \ . \end{aligned} \tag{14}$$

These may be regarded as components of a (topological) current **J**. Adding up the anholonomy densities corresponding to the faces, we get

$$\delta H_1 = [J^{(1)}(x+\Delta x,y,z) - J^{(1)}(x,y,z)]\Delta y\,\Delta z + [J^{(2)}(x,y+\Delta y,z) - J^{(2)}(x,y,z)]\Delta x\,\Delta z + [J^{(3)}(x,y,z+\Delta z) - J^{(3)}(x,y,z)]\Delta x\,\Delta y = \mathrm{div}\mathbf{J}\,\Delta x\,\Delta y\,\Delta z \ .$$

However, div**J** vanishes identically, essentially because

$$\mathbf{J} = \mathrm{curl}\,\mathbf{A}, \quad \text{with} \quad \mathbf{A} = (\tau_g^{(1)}, \tau_g^{(2)}, \tau_g^{(3)}) \ . \tag{15}$$

Thus

$$H_1 = \int\int\int \nabla\cdot\mathbf{J}dx\,dy\,dz = 0 \ . \tag{16}$$

Note that **A** is a local function whose components are given by the three torsions. This avoids the problem of having to solve for **A** from curl **A** = **J**, as is done in existing formalisms.[13] The geometric significance of **A** and **J** also become clear now.

The other anholonomy term H_2 is found as follows. Considering a point (x,y,z) in configuration space, anholonomy density may also be computed by finding the *product* of the anholonomy density associated with a closed circuit in a 2D *configuration space* corresponding to *any two* (x,y) of the variables, i.e., $(\partial\tau_g^{(1)}/\partial y - \partial\tau_g^{(2)}/\partial x)dx\,dy$, and the anholonomy density associated with a closed circuit in the *target space* corresponding to the *third* variable z, i.e., $\tau_g^{(3)}dz$. On adding up all the three contributions, we obtain

$$H_2 = \int\int\int \left[\tau_g^{(1)}\left[\frac{\partial\tau_g^{(2)}}{\partial z} - \frac{\partial\tau_g^{(3)}}{\partial y}\right] + \tau_g^{(2)}\left[\frac{\partial\tau_g^{(3)}}{\partial x} - \frac{\partial\tau_g^{(1)}}{\partial z}\right] + \tau_g^{(3)}\left[\frac{\partial\tau_g^{(1)}}{\partial y} - \frac{\partial\tau_g^{(2)}}{\partial x}\right]\right] dx\,dy\,dz \ ,$$

giving

$$H_2 = \int\int\int \mathbf{A}\cdot\mathbf{J}dx\,dy\,dz \ , \tag{17}$$

where we have used Eqs. (14) and (15). This is just the Hopf invariant[13] for appropriate boundary conditions. In fact, by using Eq. (10), we get

$$H_2 = \int\int\int dx\,dy\,dz\,\sin\theta[\psi_x(\varphi_y\theta_z - \varphi_z\theta_y) + \psi_y(\varphi_x\theta_z - \varphi_z\theta_x) + \psi_z(\varphi_x\theta_y - \varphi_y\theta_x)] \ . \tag{18}$$

H_2 is explicitly dependent on the gauge function ψ. We may also write

$$H_2 = \int\int\int dx\,dy\,dz\,\partial_\lambda\omega^\lambda, \quad \lambda = x,y,z \ ,$$

where $\omega^\lambda = -\epsilon^{\lambda\mu\nu}\cos\theta\,\partial_\mu\varphi\,\partial_\nu\psi$. Thus H_2 is a pure divergence term[14] and adding such a term to the Lagrangian of the system does not change the equations of motion of the field.

As an application of the above formalism, we now briefly consider a problem that has received considerable attention recently, namely, the role of topological terms in antiferromagnets.[15–18] In a (1+1)D problem, the existence of such a term is well established. However, all the methods that attempt to directly generalize the (1+1)D result to (2+1)D indicate that the corresponding topological term should be absent.[17] Thus, Fradkin and Stone[18] conjecture that "if there is a nontrivial Hopf term in the two-dimensional antiferromagnet, it must come from somewhere else, other than the spins themselves." Our analysis shows that this conjecture is indeed true and that a nonzero topological term H_2 arises essentially because of the angle ψ which signifies a gauge freedom inherent in the problem. The topological term H_1 (studied in Ref. 16), which depends *only* on the angles θ and φ describing the spins, vanishes identically.

Let us consider a specific configuration of the (2+1)D continuous classical antiferromagnet. One possible solution is of the form[19] $\theta=\theta(x,y)$ and $\varphi=\varphi(x,y)$. z now denotes the time variable t. Equation (18) yields [if ψ is a function of time alone and if (θ,φ) take on the same values as $x,y\to\pm\infty$]

$$H_2 = \int dt\frac{\partial\psi}{\partial t}\int\int dx\,dy\,\sin\theta(\varphi_x\theta_y - \varphi_y\theta_x) = 4\pi\times 4\pi n = 16\pi^2 n \ .$$

Geometrically, the Hopf index $H_2/16\pi^2$ represents the linking number of two curves in R^3. In this special case, curves in the z direction become the trajectories of **t** in time. H_2 is clearly nonzero for this case, and it should be possible to construct other such examples. This may suggest that there is indeed a Hopf term in the Lagrangian for the (2+1)D antiferromagnet, as originally suggested in Ref. 20.

In conclusion, we have developed a theory of space curves to understand anholonomy effects associated with continuous unit vector fields in one, two, and three dimensions. For the three-dimensional case, we have identified the components of the vector potential with torsions of the space curves with natural parameters x, y, and z, respectively. We have generalized the notion of Berry phase which, in the context of continuous systems, can be regarded as the manifestation of the anholonomy of a vector field in one dimension, to higher dimensions, using a unified approach. Our analysis shows that in two and three dimensions, the associated anholonomy manifests itself as Pontryagin- and Hopf-type terms, respectively. In 3D, the Hopf term depends crucially on the third Euler angle ψ, which shows that if one wishes to consider the global properties of the field, the local-order parameter space should be SO(3) rather than S^2. Finally, we emphasize that our formalism is general, independent of the form of the Hamiltonian or Lagrangian of the vector field, and therefore has many possible applications.

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