PHYSICAL REVIEW B VOLUME 47, NUMBER 9 1 MARCH 1993-I

Staircase dynamics of Josephson-junction arrays

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(Received 30 July 1992)

We present ansatz dynamical equations for current-driven two-dimensional square Josephson-junction arrays with magnetic flux per unit plaquette f such that the ground states form "staircase" configurations. For $f=p/q$, we need only $(q+1)$ variables to specify a dynamical staircase state; a large reduction is thus achieved in computing time. I-V curves for $f=\frac{1}{3}$ and $\frac{2}{5}$ obtained from these equations for dc plus ac driving are identical to those from simulations of $lq\times lq$ arrays. We see fractional Shapiro steps for dc plus ac current driving as well as much weaker subharmonic Shapiro steps in some circumstances. In the limit of low-frequency and low average voltage, one can further reduce the staircase equations to an approximate single-junction equation with an almost-sinusoidal supercurrent function, which is consistent with the appearance of fractional steps and weak subharmonic steps. We also find suppression of fractional harmonic stepwidths relative to integer harmonic stepwidths in the high-frequency limit.

I. INTRODUCTION

A Josephson junction driven by an external dc and ac current exhibits Shapiro steps in its time-averaged I-V curve; these steps consist of finite intervals of dc currents with the same time-averaged voltages of $n\hbar\omega/2e$, where n is an integer and ω is the frequency of the external ac driving current.[1–4] Recent experiments have demonstrated an interesting generalization of this single-junction phase locking to two-dimensional square arrays of such junctions of linear size $N\sim 100-1000$ junctions.[5–7] When the external magnetic field is zero, giant Shapiro steps appear in I-V curves of these $N\times N$ arrays at voltages satisfying

$$V_N=n\left[\frac{N\hbar\omega}{2e}\right],\quad n=0,1,2,\ldots, \tag{1.1}$$

corresponding to uniform phase locking of all junctions in the array to the driving ac current. On these Shapiro steps, the voltage per junction $V=n\hbar\omega/2e$. Further fine structures are found when a finite magnetic field is applied perpendicular to the array with flux per unit cell $\Phi=f\Phi_0$, where Φ_0 is the superconducting flux quantum $\Phi_0=hc/2e$ and f is a rational fraction $f=p/q$, with p and q relative primes.[6] These new step structures, called "fractional giant Shapiro steps," are observed at voltages of

$$V_N=\frac{n}{q}\left[\frac{N\hbar\omega}{2e}\right],\quad n=0,1,2,\ldots. \tag{1.2}$$

We can define a winding number ν by $\nu=2eV/\hbar\omega$. Integral values of ν correspond to integral steps; values of ν for which $q\nu$ is integral correspond to fractional steps. All other rational values of ν correspond to subharmonic steps.

Numerical simulations based on a coupled resistively shunted junction (RSJ) model (see Sec. II for description) have reproduced the main features of these experimental results; a proposed phase locking between the $q\times q$ periodic vortex configuration of the ground state and the external driving current explains, at least qualitatively, these results.[8,9] In general, however, analytical confirmation of these results is difficult due to the nonlinear nature of the coupled equations, especially for current-driven arrays. Although voltage driven arrays were analyzed by Halsey, it appears difficult to make contact with the experimental current-driven arrays using this approach.[10]

Recent Rzchowski, Sohn, and Tinkham considered the dynamics of a fully frustrated array ($f=\frac{1}{2}$) and obtained an effective single-junction equation in the low-frequency limit.[11] This equation explains the existence of fractional Shapiro steps, and they were also able to describe the behavior of the stepwidths as the external driving frequency is varied.[11,12]

In this paper, we study the dynamics of a Josephson-junction array in more general "rational" magnetic fields $f=p/q$ in which the static ground state has a quasi-one-dimensional form, possessing "staircase" symmetry. We assume that the dynamical state preserves the staircase symmetry at all times. We first construct a static state with nonvanishing overall current by twisting the phases of the ground-state configuration such that staircase symmetry is still preserved by this current-carrying state. See Ref. 13 for details. For $f=p/q$, $(q+1)$ independent variables are needed to specify a generalized current-carrying staircase state. In order to incorporate the time-dependent phase dynamics of Josephson-junction arrays, we simply convert these phase twist parameters into time-dependent dynamical variables and apply the current conservation condition at all nodes of the array. We thus obtain a system of *ansatz* dynamical equations for a Josephson array, which may be termed "staircase equations."

Since this set of reduced staircase equations is the result of imposing a symmetry constraint on the current distribution in the array, any solution of these ansatz equations will be a possible solution of the full array equations. Of course this does not resolve the question of the stability of these solutions.

In order to check whether these staircase ansatz solutions can actually be realized in the dynamics of the full two-dimensional array equations, we performed simulations of $q \times q$ Josephson-junction arrays. For systems with $f=\frac{1}{3}$ and $\frac{2}{5}$ under dc plus ac driving parallel to a square lattice direction, time-averaged I-V curves from these simulations were identical to those obtained by integrating the staircase equations. We also performed simulations on $2q \times 2q$ arrays in order to check the stability of the staircase solutions. We found that the staircase solutions were stable, at least for the parameters used in these simulations.

The natural frequency scale for this problem is $\omega_0 = 2eI_cR/\hbar$, where I_c is the junction critical current, and R is the normal shunt resistance of a junction. In the limit of low driving frequency $\omega \ll \omega_0$ and low voltage, we can further reduce the staircase equations into an effective single-junction equation for an overall phase difference ϕ (averaged in the direction transverse to the current flow) across a $q \times q$ array. In general, the effective supercurrent term appears only in implicit form, which must be obtained using numerical methods. We find that the effective supercurrent function in this adiabatic limit has an almost-sinusoidal form with small corrections from higher harmonics. Defining the average phase difference ψ per single junction by $\phi = q\psi$, the effective supercurrent function is

$$F(\psi) = \epsilon_1 \sin(q\psi) + \epsilon_2 \sin(2q\psi) + \cdots \quad . \tag{1.3}$$

If the effective normal current is not very different from its single-junction form $\hbar\dot{\phi}/2e$, we can think of a $q \times q$ cell as representing an effective single junction with ϕ as its phase variable. Then the above form of the effective supercurrent explains the appearance of fractional giant Shapiro steps. I-V curves obtained from integration of the staircase equations show weak subharmonic steps under favorable conditions, as originally predicted in Ref. 10. We can attribute this to the small higher harmonic terms in the effective supercurrent function.

These conclusions are based on the dynamics of two-dimensional square arrays with current driving in directions parallel to square lattice directions. The staircase equations also provide a framework for analyzing the array dynamics for driving currents in arbitrary directions. This general case is complicated by the appearance of two nonequivalent staircase dynamics that are compatible with a given external driving current, which arise from the fact that the original staircase states have a symmetry different from that of the underlying square lattice. The question of the global stability of these two different modes cannot be resolved using our methods.[14–16] In the case of diagonal current driving, we find that there exist mode-locked solutions corresponding to fractional Shapiro steps, but these seem to be only locally stable.

In Sec. II, we derive the staircase dynamical equations. In Sec. III, we discuss the effective supercurrent function for a $q \times q$ array; the results of this section are especially useful in the limit of low voltages and low driving frequencies. In Sec. IV, we discuss some peculiar features of diagonal current driving; in Sec. V, we present simulation results and compare them to the results of the integration of the staircase equations. In Sec. VI, we conclude. In Appendix A, we extend this method to a nonstaircase state, the $f=\frac{1}{5}$ state. In Appendix B, we discuss the response of these states to inhomogeneities in the array, which we expect to wash out the weaker steps.

II. STAIRCASE DYNAMICS

The ground-state configuration of a two-dimensional Josephson-junction array in a magnetic field is determined by minimizing the Hamiltonian

$$H = \sum_{\langle ij \rangle} \frac{\hbar I_{c,ij}}{2e} \cos(\theta_i - \theta_j - A_{ij}) \ , \tag{2.1}$$

with respect to the phases $\{\theta_i\}$. Here the sum is over nearest neighbors and $I_{c,ij}$ denotes the critical current of the junction between islands i and j. A_{ij} is the line integral of the vector potential

$$A_{ij} = \frac{2e}{\hbar c} \int_j^i \mathbf{A} \cdot d\mathbf{r} \ , \tag{2.2}$$

which satisfies $\sum_P A_{ij} = 2\pi f$ with the sum in the counter-clockwise direction around a plaquette. We deal with square arrays with uniform critical currents $I_{c,ij} = I_c$. The extremum condition for H is equivalent to the requirement that the supercurrent be conserved at every site in the array; the supercurrent in the ij bond is $I_c \sin(\theta_i - \theta_j - A_{ij})$.

For some values of f, the ground states are "staircase states" in which the supercurrent along the parallel diagonal staircases of the square array is constant. See Fig. 1. In these states, junctions belonging to a staircase have the same gauge-invariant phase differences. Suppose that we denote the invariant phase difference for the mth staircase as $\phi_m \equiv \theta_i - \theta_j - A_{ij}$. Then it can be shown that there are locally stable states with

$$\phi_m = \pi f m + \alpha - \pi[fm + \alpha/\pi]_n, \quad m = 1, \ldots, q \ , \tag{2.3}$$

where $[x]_n \equiv \mathrm{int}[x + \frac{1}{2}]$ is the nearest integer function. The ground-state value of α, which we denote by α_0, is determined by minimizing the total energy. It can be shown that $\alpha_0 = 0$ for odd q and $\alpha_0 = \pi/2q$ for even q. See Ref. 13 for details. As mentioned already, these staircase states are not the true ground states for all values of f. But they are known to be true ground states for simple fractional values $f = \frac{1}{2}, \frac{1}{3}, \frac{2}{5}, \frac{3}{7}$, and $\frac{3}{8}$. In this section we deal only with those systems with staircase ground states.

Now we want to generalize these static staircase ground states into dynamical states with finite net current, while still preserving the staircase symmetry. As a first step, we construct static states with nonzero net currents. For a state with net current along the direction of staircases, we can simply put an arbitrary value for α

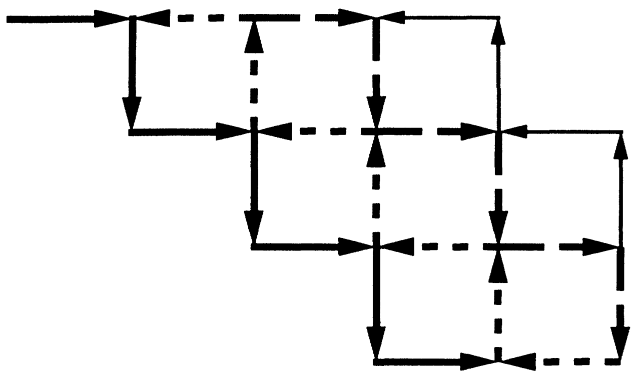

FIG. 1. Staircase form of the phase configuration. Horizontal and vertical junctions belonging to individual diagonal staircases have the same invariant phase differences and currents (directions of currents are denoted by arrows). Relative magnitudes of currents are determined by using current conservation for a given value of the frustration $f=p/q$.

in Eq. (2.3) instead of the ground-state value given above. We will call this direction, parallel to the staircases, the α direction. In order to generate a state with net current in a general direction, we have to form a state with a nonzero component of net current along the direction perpendicular to the staircase direction. This perpendicular direction we call the χ direction. To obtain states with finite current along the χ direction, one twists the phases on successive diagonal planes parallel to the α direction by constant angles χ_m, $m=1,\ldots,q$ so that the differences of phase shifts for neighboring diagonal planes are

$$\gamma_m \equiv \chi_m - \chi_{m-1}, \quad m=1,\ldots,q \ . \tag{2.4}$$

We require $\gamma_{m+q}=\gamma_m$ so that the current distribution retains its $q\times q$ periodicity. Again, see Ref. 13 for details. The following formulas for the net supercurrent along the α direction and the χ direction can be used to determine the parameters $\{\alpha,\gamma_m\}$ for static current-carrying states:

$$\begin{aligned} I_\alpha &= \frac{I_c}{2q}\sum_{m=1}^{q}[\sin(\phi_m+\gamma_m)+\sin(\phi_m-\gamma_m)] \\ &= \frac{I_c}{q}\sum_{m=1}^{q}\sin\phi_m\cos\gamma_m \ , \end{aligned} \tag{2.5a}$$

and

$$\begin{aligned} I_\chi &= \frac{I_c}{2}[\sin(\phi_m+\gamma_m)+\sin(-\phi_m+\gamma_m)] \\ &= I_c\cos\phi_m\sin\gamma_m \ . \end{aligned} \tag{2.5b}$$

I_α and I_χ denote currents per junction in the α direction and χ direction, respectively. Note that current conservation implies that $I_c\cos\phi_m\sin\gamma_m$ $(=I_\chi)$ cannot depend on m; this uniquely determines $\{\gamma_m\}$ as functions of I_χ.

The RSJ model (with no capacitance effect) is based on the following formula for the current between islands i and j:

$$I_{ij}=\frac{\hbar}{2eR}(\dot\theta_i-\dot\theta_j-\dot A_{ij})+I_c\sin(\theta_i-\theta_j-A_{ij}) \ , \tag{2.6}$$

where the first term on the right-hand side is the normal current through a shunt resistance R, which we assume to be uniform in the array, and the second term denotes the supercurrent contribution. From now on, for convenience, we will express time, current, and voltage in natural units of $\hbar/2eI_cR$, I_c, and I_cR, respectively.

We now regard α and the $\{\gamma_m\}$ as dynamical variables and include normal current terms by applying the RSJ model, thus obtaining

$$\begin{aligned} I_\alpha &= \frac{1}{q}\sum_{m=1}^{q}\left[\dot\phi_m+\sin\phi_m\cos\gamma_m\right] , \\ I_\chi &= \dot\gamma_m+\cos\phi_m\sin\gamma_m, \quad m=1,\ldots,q \ . \end{aligned} \tag{2.7}$$

Here,

$$\phi_m=\pi f m+\alpha_0-\pi[fm+\alpha_0/\pi]_n+\alpha(t) \ ,$$

which can take unrestricted values. Equations (2.7) we call the staircase equations. They are $(q+1)$ coupled equations for $(q+1)$ independent variables α and $\gamma_1,\ldots,\gamma_q$. We can use these equations to find the phase dynamics of an array with an arbitrary time-dependent driving current. These dynamics are constrained by the requirement that the state of the array retain the staircase symmetry at all times. When $f=\frac{1}{2}$, it can be shown that the staircase equations with current driving along a coordinate axis reduce to the results of Rzchowski, Sohn, and Tinkham.[11]

Most experiments and numerical simulations have examined the case of driving current $I(t)$ (per junction) injected parallel to a square lattice vector of the array. This corresponds to the case of $I_\alpha=I_\chi=I(t)/2$ in the staircase equations. In this case, all possible orientations of the current with respect to the staircase direction lead, by simple symmetry arguments, to the same behavior. For driving current $I(t)$ in an arbitrary direction, there are two possible inequivalent staircase dynamics due to the asymmetry arising from nonequivalence between the α direction and the χ direction. For example, if the net current is along a diagonal direction, we can have the staircase direction either parallel to the driving current or perpendicular to it. In these general cases, different orientations with respect to the staircase direction do not produce the same I-V curves. For the special case of $f=\frac{1}{2}$, the directions parallel and perpendicular to the staircase direction are equivalent.

III. LOW-FREQUENCY BEHAVIOR

We now consider in more detail the staircase dynamical equations in the limit of small frequency ω and low overall voltage. Our goal is to find a further reduction of the system of staircase equations into an effective single-junction equation. We put $I_\alpha=I_\chi=I(t)/2$, so that the current driving is parallel to one of the coordinate axes.

First we note that the average phase difference per junction in the direction of the driving current (the x axis) is

$$\psi \equiv \frac{1}{q} \sum_{m=1}^{q} (\phi_m + \gamma_m) . \tag{3.1}$$

In order to find an effective dynamical equation for ψ, we can combine staircase equations with suitable weights to obtain the following.

$$\begin{aligned} I(t) &= \frac{1}{q} \sum_{m=1}^{q} (\dot{\phi}_m + \dot{\gamma}_m) \\ &\quad + \frac{1}{q} \sum_{m=1}^{q} (\sin\phi_m \cos\gamma_m + \cos\phi_m \sin\gamma_m) \\ &\equiv \dot{\psi} + \frac{1}{q} \sum_{m=1}^{q} \sin(\phi_m + \gamma_m) . \end{aligned} \tag{3.2}$$

The second term on the right-hand side in the above equation, as it stands, depends on all the phase parameters $\{\alpha, \gamma_m\}$. In principle, this sum can be expressed in terms of ψ and its derivatives by using the remaining relations. These are

$$\dot{\gamma}_m + \cos\phi_m \sin\gamma_m = \dot{\alpha} + \frac{1}{q} \sum_{j=1}^{q} \sin\phi_j \cos\gamma_j , \quad m = 1, \ldots, q , \tag{3.3}$$

and

$$\begin{aligned} \psi &= \frac{1}{q} \sum_{m=1}^{q} (\phi_m + \gamma_m) \\ &= \alpha + \frac{1}{q} \sum_{m=1}^{q} \gamma_m . \end{aligned} \tag{3.4}$$

It is very difficult to solve these equations for $\{\alpha, \gamma_m\}$ in terms of a given ψ and its time derivatives. However, in the limit of low frequency ω and low overall voltage, we can ignore higher-order terms in time derivatives. After some algebra, we get the following effective equation.

$$\begin{aligned} I(t) &= \dot{\psi} + F(\psi) + G(\psi)\dot{\psi} + O(\omega^2) \\ &\simeq \dot{\psi}[1 + G(\psi)] + F(\psi) , \end{aligned} \tag{3.5}$$

where $F(\psi)$ is the sum of supercurrent terms in Eq. (3.2),

$$F(\psi) = \frac{1}{q} \sum_{m=1}^{q} \sin(\phi_m + \gamma_m) , \tag{3.6}$$

expressed in terms of the average phase variable ψ using the constraint equations in the adiabatic limit.

$F(\psi)$ is an effective supercurrent function because it represents the averaged sum of supercurrents of all junctions. The additional term $G(\psi)\dot{\psi}$ gives a first-order correction that modifies the normal current. Except when $f = \frac{1}{2}$, the constraint equations are quite complicated. Thus closed analytical expressions for $F(\psi)$ and $G(\psi)$ are impossible to obtain. However, numerical evaluation of $F(\psi)$ can be done with ease. Sine-series decomposition shows that $F(\psi)$ is close to a simple sinusoidal form of $\sin(q\psi)$ with small higher harmonic corrections. In particular, we find that if we write

$$F(\psi) = \epsilon_1 \sin(q\psi) + \epsilon_2 \sin(2q\psi) + \cdots , \tag{3.7}$$

then for $f = \frac{1}{3}$, $\epsilon_1 = 0.265$ and $\epsilon_2 = -0.036$; for $f = \frac{2}{5}$, $\epsilon_1 = 0.155$ and $\epsilon_2 = -0.030$. The fact that the lowest harmonic mode is $\sin(q\psi)$ is consistent with the fractional giant Shapiro steps found in the numerical work of Lee and Stroud[8] and Free *et al.*[9] In addition to fractional steps, we expect weaker subharmonic steps corresponding to the small higher harmonic components in the effective supercurrent function. However, as discussed in Appendix B, these small steps could easily be washed out by the effect of quenched disorder in real arrays.

IV. SHAPIRO STEPS AND DIAGONAL DRIVING

As mentioned earlier, there are two physically inequivalent modes of staircase dynamics for driving currents that are not parallel to a square lattice direction. We will now discuss the dynamics of arrays under current driving in diagonal directions. In this case, the driving current can be along the staircase direction (α direction) or perpendicular to it (χ direction).

First suppose that the driving current is in the α direction. Then from Eq. (2.6), $I_\chi = 0$ and the $\{\gamma_m\}$ are given by $\gamma_m = 0$. Thus we obtain single-junction dynamics with a purely sinusoidal effective supercurrent function for α. This will give only integer Shapiro steps. The situation is not so simple if the driving current is in the χ direction. We can numerically calculate the effective supercurrent function in a similar way as that shown above for the case of parallel current driving. Interestingly, we find that the fundamental sinusoidal mode of the effective supercurrent function in this case is $F(\psi) \approx \sin(q\psi/2)$, where ψ is defined to be the average phase difference per junction along the current driving direction. We see that the period of $F(\psi)$ is doubled when compared with parallel array dynamics. This is probably related to the fact that the vortices are moving along diagonal directions and a unit vortex shift in this mode involves crossing effectively two junctions at a time. When this staircase form of dynamics is realized, we expect to find fractional Shapiro steps at voltages satisfying the following relation

$$V = \frac{2n}{q} \left[\frac{\hbar\omega}{2e} \right], \quad n = 1, \ldots, q , \tag{4.1}$$

where V is again the voltage per junction. This means that we would observe $[q/2]_n - 1$ fractional steps between any two neighboring integer steps. Here, $[x]_n$, denotes the integer part of $x + \frac{1}{2}$. For example, if $f = \frac{1}{2}$ ($q = 2$), then $[2/2]_n - 1 = 0$, and there will be no fractional steps. If $f = \frac{1}{3}$ then we expect one fractional step in between integer steps. However, these steps will only be seen if these dynamical modes are stable.

It has been claimed that fractional steps do not exist for diagonal current driving.[14,15] In our simulations (see below), we found that it was possible to observe mode-locked solutions with the driving current perpendicular to the staircase direction; these solutions correspond to fractional Shapiro steps. However, these solutions turned out to be only *locally* stable. In order to develop such a mode-locked solution at fractional voltages, the initial configuration should be chosen to lie close to a staircase ground-state configuration with the correct orientation. Random initial configurations tended to develop into dynamical states that were not mode locked. This strongly hysteretic behavior seems to come from the two distinct vortex configurations having different staircase directions relative to the driving current. Thus in real arrays, we expect it to be very hard to observe these weakly stable fractional Shapiro steps.

V. INTEGRATION OF THE STAIRCASE EQUATIONS AND SIMULATION RESULTS

To see if staircase dynamics can be realized in the full array dynamics, we compared the I-V curves obtained from integration of the staircase equations with those from simulations of the full array equations for $q\times q$ arrays with $q\times q$ periodic distributions of boundary currents. The external driving current per junction is chosen to have the form $I(t)=I_{\mathrm{dc}}+I_{\mathrm{ac}}\sin(\omega t)$. We chose the current to be parallel to the square-lattice directions. We performed simulations on systems with $f=\frac{1}{3}$ and $f=\frac{2}{5}$ using simple Gauss-Jordan inversion of the conductivity matrix and the Bulirsch-Stoer method of time integration over a time of $T\geq 1000\tau_0$, where $\tau_0=\omega_0^{-1}=(\hbar/2eI_cR)$ is the natural time scale for the array dynamics. We used both the staircase ground state and random states as initial configurations for the full array simulations. We found no difference between these two cases. Simulations on $2q\times 2q$ arrays were also performed in the case of $f=\frac{1}{3}$ and the results were the same as those for $q\times q$ arrays.

Figures 2 and 3 show I-V curves for these systems for dc plus ac driving current for $f=\frac{1}{3}$ and $f=\frac{2}{5}$, respectively using both the staircase equations and the full array equations. We can see that I-V curves from the staircase equations are identical to those from the full $q\times q$ array dynamics. Fractional Shapiro steps are seen (for dc plus ac current driving) and there are also weak subharmonic Shapiro steps. See Fig. 4 for an example of this.

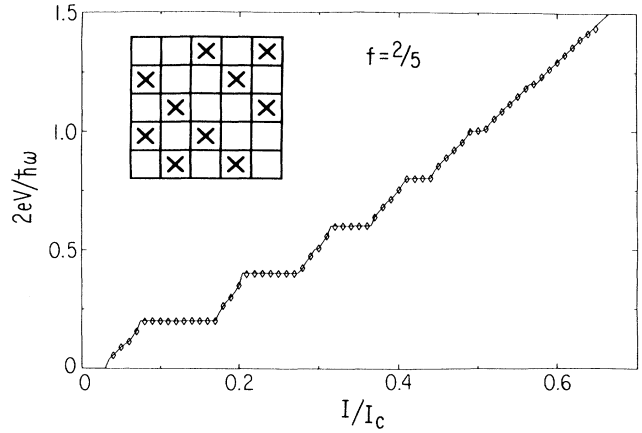

FIG. 3. I-V curve for $f=\frac{2}{5}$ with $\omega=0.3\omega_0$, $I_{\mathrm{ac}}=0.4I_c$. The solid line represents results from the staircase equations, while the diamonds represent simulation results; inset is the vortex configuration of the ground state for $f=\frac{2}{5}$ with crosses representing vortices. V denotes the average voltage per junction.

We investigated the ac-current dependence and frequency dependence of various Shapiro stepwidths using the staircase equations. Results for the case of $f=\frac{1}{3}$ are shown in Figs. 5 and 6. We can see that at a low frequency of $\omega=0.3\omega_0$ the stepwidths show "Bessel-law" behavior quite similar to that of a single junction. At higher frequencies, fractional stepwidths are considerably suppressed relative to integer stepwidths in agreement with the predictions of Rzchowski, Sohn, and Tinkham.[11,12]

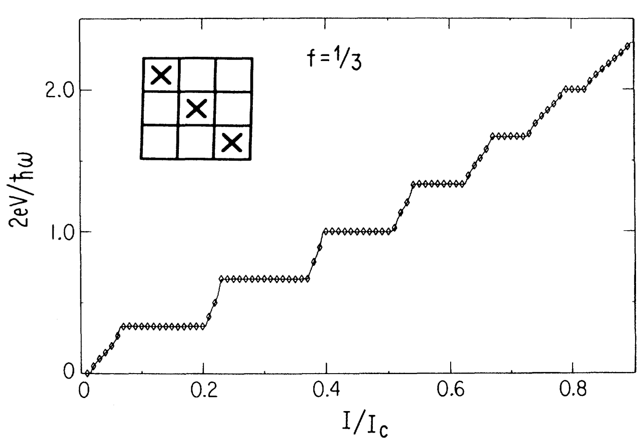

FIG. 2. I-V curve for $f=\frac{1}{3}$ with $\omega=0.3\omega_0$, $I_{\mathrm{ac}}=0.6I_c$. The solid line represents results from the staircase equations, while the diamonds represent simulation results; inset is the vortex configuration of the ground state for $f=\frac{1}{3}$ with crosses representing vortices. Note that V denotes the average voltage per junction.

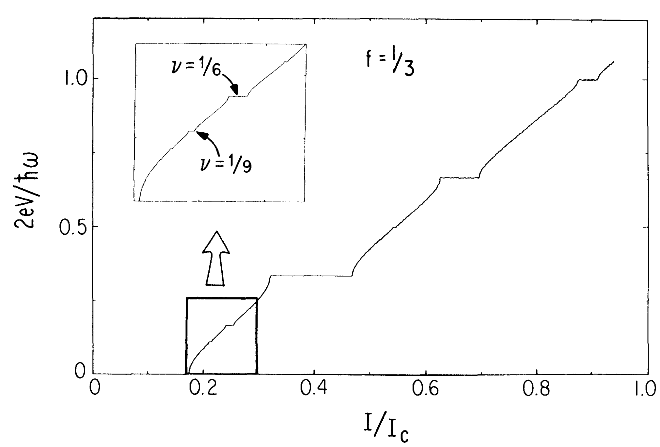

FIG. 4. I-V curve for $f=\frac{1}{3}$ obtained from the staircase equations when $\omega=0.7\omega_0$, $I_{\mathrm{ac}}=0.6I_c$. Weak subharmonic Shapiro steps appear at various voltages corresponding to winding number $\nu=\frac{1}{6},\ \frac{1}{9},\ldots$; the inset shows a detailed view of the subharmonic steps. V denotes the average voltage per junction.

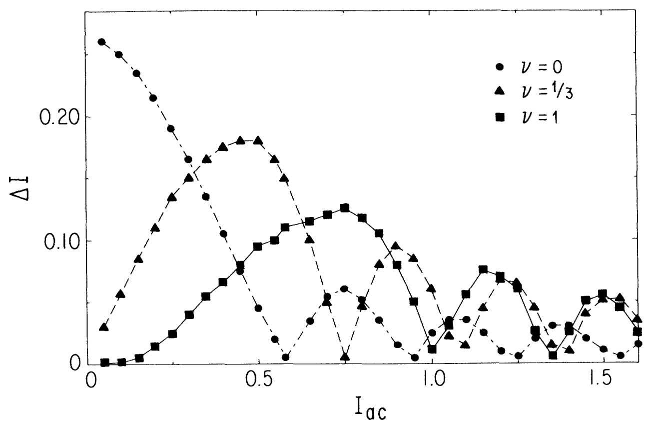

FIG. 5. I_{ac} dependence of stepwidths ΔI obtained from staircase equations for $f=\frac{1}{3}$ and $\omega=0.3\omega_0$. Shown are data for winding number $\nu=0$ (circle), $\frac{1}{3}$ (triangle), and 1 (square), respectively. The lines are only guides to the eye. The stepwidths obey a qualitative Bessel-law form.

VI. CONCLUSION

In this paper, a set of reduced dynamical equations was presented for the dynamics of Josephson arrays under rational magnetic fields such that the ground states have the staircase form. I-V curves from these reduced equations for dc plus ac driving were identical to those from simulations of $lq \times lq$ periodic array dynamics with external currents parallel to a square lattice direction. In the limit of low-frequency and low average voltage, we can further reduce this set of equations into an effective single-junction equation with a nearly sinusoidal effective supercurrent function. This effective supercurrent function is consistent with fractional Shapiro steps and much weaker subharmonic steps. In the high-frequency limit, the fractional stepwidths are suppressed relative to the integer stepwidths.

In contrast to the case of the external current parallel to a square lattice direction, staircase dynamics for diagonal current driving in the x direction turns out to be in general only locally stable, which will make it hard to realize fractional Shapiro steps in experimental situations.

It is possible to generalize the above approach to nonstaircase ground states. The case of $f=\frac{1}{5}$ is discussed in Appendix A. In principle, one might also analyze triangular arrays with transverse magnetic fields using similar methods.

ACKNOWLEDGMENTS

This work was supported by the Science and Technology Center for Superconductivity and by the Materials Research Laboratory at the University of Chicago. We are grateful to C. J. Lobb for discussing the results of Ref. 14 with us prior to publication.

APPENDIX A: $f=\frac{1}{5}$

In this appendix we describe a reduced set of dynamical equations for a square Josephson-junction array with $f=\frac{1}{5}$. In this case, the ground state is no longer a staircase state. Nevertheless, this state has a translational symmetry similar to that of the staircase states. Namely, if we translate the whole vortex lattice by two lattice units in the positive x direction (relative to the square lattice) together with one lattice unit in the negative y direction, we recover the initial vortex lattice configuration. See the inset in Fig. 7.

We assume that this translational symmetry holds in dynamical situations, in addition to assuming 5×5 periodicity. At the least, we expect this assumption to be valid for the mode-locked states.

Based on this assumption, we can consider, in analo-

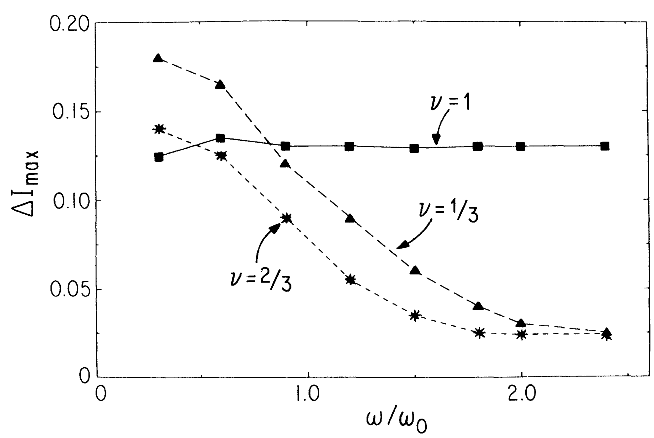

FIG. 6. Frequency dependence of the first maxima ΔI_{max} of stepwidths (as obtained from plotting the stepwidths in terms of I_{ac} for a given frequency) for winding number $\nu=\frac{1}{3}$ (triangle), $\frac{2}{3}$ (star), and 1(square). The lines are only guides to the eye. Fractional stepwidths are suppressed at higher frequencies relative to integer stepwidths.

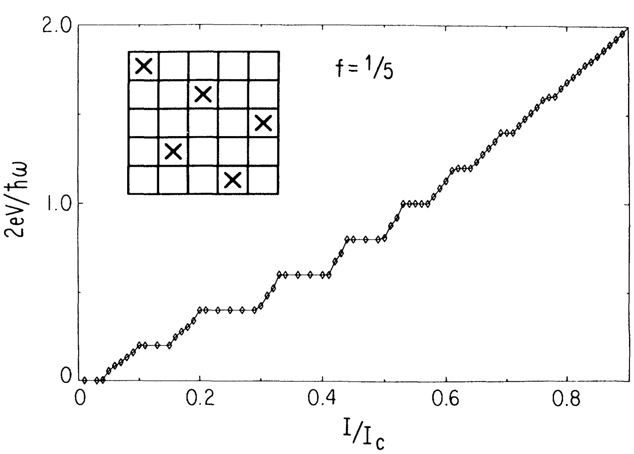

FIG. 7. I-V curve for $f=\frac{1}{5}$ when $\omega=0.3\omega_0$, $I_{ac}=0.6I_c$. The solid line represents results from the reduced equations, while the diamonds represent simulation results; inset is the vortex configuration of the ground state for $f=\frac{1}{5}$ with crosses representing vortices. V denotes the average voltage per junction.

gous fashion to the case of staircase states, the invariant phase-difference configuration of the array with external driving currents I_x and I_y. We can describe the dynamics of the array by using the invariant phase differences along horizontal and vertical junctions pertaining to only one row of plaquettes with its length equal to $5(=q)$. We denote these by ϕ_i and θ_i, $i=1,\ldots,5$, respectively. These ten variables cannot all be independent, since the magnetic flux constraint for each plaquette must be satisfied. We have four $(=q-1)$ independent constraints corresponding to four independent plaquettes; as a result we have six independent phase variables, which describe the dynamics of the array.

The four constraints are

$$\begin{aligned} &\theta_2+\phi_4-\theta_1-\phi_1=2\pi f \ , \\ &\theta_3+\phi_5-\theta_2-\phi_2=2\pi f \ , \\ &\theta_4+\phi_1-\theta_3-\phi_3=2\pi f \ , \\ &\theta_5+\phi_2-\theta_4-\phi_4=2\pi f \ . \end{aligned} \tag{A1}$$

Now, in order to write the time-dependent dynamics of the array, we use current conservation for each island in addition to requiring that the total current equal the external driving currents I_x, I_y. If we denote the current along horizontal junctions and vertical junctions by $J_{x,i}$ and $J_{y,i}$ $i=1,\ldots,5$, respectively, then, in reduced units, we have

$$\begin{aligned} &J_{x,i}=\dot{\phi}_i+\sin\phi_i \ , \\ &J_{y,i}=\dot{\theta}_i+\sin\theta_i \ . \end{aligned} \tag{A2}$$

Therefore, current conservation implies

$$\begin{aligned} &J_{y,1}+J_{x,4}=J_{x,3}+J_{y,4} \ , \\ &J_{y,2}+J_{x,5}=J_{x,4}+J_{y,5} \ , \\ &J_{y,3}+J_{x,1}=J_{x,5}+J_{y,1} \ , \\ &J_{y,4}+J_{x,2}=J_{x,1}+J_{y,2} \ , \end{aligned} \tag{A3}$$

And the constraints on the total current give

$$\begin{aligned} &I_x=\frac{1}{5}\sum_{i=1}^{5} J_{x,i} \ , \\ &I_y=\frac{1}{5}\sum_{i=1}^{5} J_{y,i} \ , \end{aligned} \tag{A4}$$

where I_x and I_y are the driving currents.

Eqs. (A3) and (A4) together with the four constraints [Eq. (A1)] constitute the complete set of equations for the reduced dynamics of the array with $f=\frac{1}{5}$. Figure 7 compares the I-V curve obtained from the reduced set of equations with that obtained from full array simulations for a 5×5 array with the boundary condition of periodic current distribution when dc plus ac current is applied parallel to a square lattice direction. We see good agreement between the two results.

APPENDIX B: INFLUENCE OF QUENCHED RANDOMNESS

In this appendix, we will analyze the effect of inhomogeneity of normal-state resistances R on the array. We will show that although low-order, wide steps should survive such inhomogeneities; high-order, narrow steps will be washed out by these effects.[16] Since the subharmonic steps predicted above are quite narrow, they will thus probably be unobservable in any real experiment.

Suppose that we assume that a junction array is in a locked state, with fixed voltage differences V corresponding to the pure array Shapiro steps on every junction of the array. Because the step corresponds to a phase locking of the oscillations corresponding to the voltages with the ac driving, we regard the time-averaged voltages as being fixed, and not responding to the quenched disorder. Because the resistances are no longer homogeneous, these fixed voltages will now lead to excess normal currents $i_n \sim V\delta R/R^2$ at the nodes of the array, where R is the average resistance, and δR is a typical variation in the resistances. Now consider a region of area L_xL_y, with L_x, L_y the characteristic dimensions of the region perpendicular and parallel to the net current flow. The total normal current generated in this region will be

$$I_n(L_x,L_y)\sim\frac{V\delta R}{R^2}\sqrt{L_xL_y} \ . \tag{B1}$$

This current must flow through the region boundary entirely as supercurrent, or else the original assumption that the voltages are locked is incorrect. In the direction parallel to the net current flow, an additional current per bond of order ΔI, where ΔI is the step width, can flow without moving the array locally off the current step. In the direction perpendicular to the net current, the maximum current per bond is of the order of I_0, where I_0 is the zero-temperature critical current of the array at that value of f for dc driving. Thus the total current that can flow through the boundary as supercurrent is

$$I_s(L_x,L_y)\sim L_x\Delta I+L_yI_0 \ . \tag{B2}$$

If we realize that we must have $I_n < I_s$ for domains of all shapes and sizes, we obtain the criterion for step stability,

$$\sqrt{I_0\Delta I} > V\frac{\delta R}{R^2} \ . \tag{B3}$$

This implies that the higher-order steps, for which ΔI is small, will be washed out in any real experiment.

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