Efficiency of determinantal transport theory in the integer quantum Hall effect

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A steady-state solution of the density matrix, capable of properly handling a quasi-infinite density of states, is applied to the integer quantum Hall effect in a noninteracting two-dimensional electron gas, on the basis of unperturbed Landau's states. The width of the Landau subbands due to the crystal periodic potential is taken as an infinitely small quantity tending to zero, and the initial many-body density matrix at equilibrium is calculated up to first order of this bandwidth. This allows us to extract the finite principal part of the conductivity-tensor components from the statistical average of the current density. The quantized plateaus of the Hall conductivity are shown to result from both the very large relaxation parameters due to the quasi-infinite density of states in the Landau subbands and the effect of disorder on the center-of-orbit motion. These are responsible for the vanishing of the diagonal conductivity, except in the intermediate regions between the plateaus, where the density of states is strongly reduced and additional interband transitions take place. The present treatment formally amounts to a change of basis and explains that the underlying physical content is found to be in agreement with prevailing interpretations. The salient experimental features are obtained without an explicit knowledge of localized-extended states, nor with the use of gauge or topological invariants. Owing to its efficiency and simplicity, such an approach offers attractive promise in further quantitative developments.

I. INTRODUCTION

So far, the widespread interpretation of the quantum Hall effect (QHE) in two-dimensional electron systems rests on the singular properties of the Landau states once perturbed by a random scattering potential.[1] Each Landau level is broadened into a bell-shaped spectrum, involving both a narrow band of extended states in the central part, and localized states in the wings. Every time the Fermi level falls into localized states, only the full bands of extended states below contribute to current transport, leading to the well-known plateaus of the Hall current and the vanishing of the diagonal conductivity. The intriguing question of how all electrons can contribute to the Hall current even though a large fraction of them is trapped in localized states also seems answered in various ways by the theory. A great advantage in working on a collision-broadened basis is that the strong divergences resulting from the quasi-infinite degeneracy of the unperturbed Landau levels are removed. However, this requires a detailed knowledge of localized and extended wave functions. Unfortunately, despite a lot of effort devoted to that question in recent years, particularly to the crucial problem of the magnetic-field-induced localization, a reliable quantitative treatment based on the localized-extended state concept is still missing, and major questions such as the exact nature of extended states or the location of mobility edges are not properly answered yet. In fact, available materials are mainly restricted to qualitative discussions and numerical simulations, in particular, model systems.[2–7] This fact may explain why certain basic concepts sustaining present ideas are not free from criticism[5,6–10] and why several theories have been proposed to avoid them.[8–13]

This paper is not intended to present an alternative interpretation to the localization theory, but rather a parallel investigation which allows us to encompass the seemingly unattainable goal of constructing localized and extended states, with the major advantage that disorder effects will arise in a form amenable to calculation in a much more elementary way. Our starting remark is that it should obviously be possible to develop the QHE theory in the unperturbed Landau-state basis, since the quantum statistical average of the current density, as given by a trace, does not depend on the choice of basis. This, however, demands a mathematical method capable of dealing with strong density-of-states divergences. In recent years, the author has proposed a systematic way to overcome such difficulties in solving the time-dependent Schrödinger equation of the density matrix. This so-called "determinantal" method mainly rests on using linear system theory and elementary properties of determinants in the Liouville-Laplace space.[14(a),14(b)] Its applicability to quantum transport was recently studied from a formal point of view, leading to tractable expressions of the long-term solution,[14(c)] including predominant collisions and slow relaxation processes (mainly the coupling with the heat bath).

The QHE constitutes a particularly sensitive test of the relevance of the determinantal theory. This paper aims at giving a detailed account of its potentialities in that important problem. The essential basis of this approach consists in working out the finite limits of the physical quantities of interest when the Landau-level width (which, strictly speaking, is finite but very small) is regarded as tending to zero.[15–17] But, since the method formally amounts to a change of basis, the physical content is expected to be consistent with previous ideas. However, it will be shown to lead to a drastically simplified treatment of the salient experimental features, without explicit reference to the structure of localized and extended states, mobility gap, particular geometries

or boundary conditions, gauge or topological invariants.

For the sake of brevity we shall restrict this first application to the demonstration of the most important topics of the integer QHE, say the occurrence of the plateaus and the correlated absence of diagonal conductivity. The intermediate behavior of the Hall and diagonal conductivities in the transition regions between the plateaus will be introduced in a qualitative way only. Spinless independent electrons will be assumed.

In Sec. II the determinantal derivation of the steady-state density matrix is briefly recalled for notational convenience, with some further refinements required by the singularities of the Landau spectrum. In Sec. III the calculation of the equilibrium density matrix in the presence of collisions is carried out in the unperturbed Landau-state basis. In Sec. IV explicit determinantal expressions of the conductivity-tensor components are derived. These will be used to find out the quantization of the Hall conductivity in Sec. V and the vanishing of the diagonal conductivity in Sec. VI. The results will be discussed in Sec. VII with regard to existing interpretations and some conclusions will be drawn about the capabilities of the method in the present problem.

II. DENSITY MATRIX SOLUTION ON THE LANDAU-STATE BASIS

We shall recall the key issues of the determinantal method with particular emphasis on the extra difficulties resulting from the singularities of the Landau spectrum. The many-body Hamiltonian of the two-dimensional electron gas will be written as

$$H=H_0+V+F, \quad \text{with } H_0=\frac{1}{2m^*}\sum_i(\mathbf{p}_i+e\,\mathbf{A}_i)^2 . \tag{1}$$

H_0 is the unperturbed Hamiltonian of the system involving n_s free electrons per unit area, with an effective mass m^* and a charge $-e$. The constant magnetic field $\mathbf{B}$, parallel to the z axis, is described by the vector potential $\mathbf{A}(0,Bx,0)$ in the Landau gauge. V represents a collision potential and $F=eEx$ the coupling of electrons with the constant electric field E, assumed to be applied at $t=0$, along the x axis.

The unperturbed many-body eigenstates of H_0 will be denoted by $b,c,\ldots$, the related energy being $\varepsilon_b=\hbar\omega_b$, $\varepsilon_c=\hbar\omega_c,\ldots$. They are constructed out of the well-known one-electron Landau states, specified by the quantum numbers N and k_y, where $N=0,1,2,\ldots$ is the band index, and k_y the y component of the electron momentum. For each Landau level of energy $(N+\frac{1}{2})\hbar\Omega$, with $\Omega=eB/m^*$ denoting the cyclotron frequency, there are eB/h values of k_y per unit area. As no confusion can arise, k_y will be simply written as k in what follows.

We shall assume one-particle operators which can be written in the many-body ket space of a second quantization scheme, in terms of one-electron Landau states. For example, the collision operator is written as

$$V=\sum_{N'k'Nk} V_{Nk}^{N'k'} c_{N'k'}^{\dagger} c_{Nk} . \tag{2}$$

c_{Nk} and $c_{Nk}^{\dagger}$ denote the customary fermion annihilation and creation operators satisfying anticommutation rules. In the relevant temperature range ($\lesssim 1$ K), the predominant contribution to the collision potential V is that of impurities and crystal defects. One-particle matrix elements $V_{Nk}^{N'k'}$ in the case of random scatterers are briefly recalled in Appendix A, for later use. Diagonal elements can be ignored without loss of generality, for, as was pointed out by Kohn and Luttinger,[18] this simply amounts to a change of the energy-state origin in the whole system.

On the other hand, the operators a and $a^{\dagger}$ of the harmonic oscillator theory leading to Landau's states are also written by means of the c_{Nk}'s,

$$a=\sum_{Nk}\sqrt{N}\,c_{N-1k}^{\dagger}c_{Nk}, \quad a^{\dagger}=\sum_{Nk}\sqrt{N+1}\,c_{N+1k}^{\dagger}c_{Nk} . \tag{3}$$

The electron x coordinate is given in terms of a and $a^{\dagger}$ by

$$x=l(a+a^{\dagger})/\sqrt{2}-l^2k . \tag{4}$$

$l=\sqrt{\hbar/eB}$ is the cyclotron radius. As is well known, the first term in the right-hand side above refers to relative orbital coordinates of electrons, and the second one to the center-of-orbit coordinates. The former has only matrix elements between adjacent Landau levels, deduced from Eq. (3), while the many-body momentum component is diagonal in the Landau representation and, therefore, easily expressed in terms of single-particle k's as

$$\sum_{Nk} k c_{Nk}^{\dagger} c_{Nk} .$$

The same notation k will also be used for the many-body momentum without confusion, the related eigenvalues being specified by a suitable subscript.

At this point, it is worth recalling for the sake of further discussion that the single-level k degeneracy of Landau's states is only due to the somewhat simplified description of the crystal Hamiltonian (1), in which the Peierls substitution, $p_y\rightarrow p_y+eA_y$, is combined with the effective mass approximation, including thereby the periodic crystal potential. In fact, the problem of the broadening of free-electron Landau levels in a periodic potential has been carefully investigated in the past, mainly by Wannier and co-workers,[15–17,19–22] and found to be rather complex. The major conclusion which can be drawn from these works, at experimentally attainable values of magnetic field we are interested in, is that the ideal Landau levels are rather the limit of very narrow bands in which energy is still degenerate with respect to the k's. The essential argument[20–22] is given in Appendix B, with mathematics used in this paper. It follows that intra-N-band energy differences, $\varepsilon_{Nk'Nk}=\varepsilon_{Nk'}-\varepsilon_{Nk}$, are of very small but finite magnitude, the initial and final states requiring in fact some additional quantum numbers, besides k and k', to be completely specified. To reconcile our simplified description with the physical reality and for the purpose of the present work, it will be sufficient to bear in mind the finite but very small bandwidth, independent of the k's, additional quantum num-

bers being implied only when needed, in some places. In comparison with other significant quantities such as the cyclotron energy $\hbar\Omega$, the bandwidth is completely negligible. But insofar as it is also the key of the divergent density of states, it must be taken as tending mathematically to zero, if we want to extract properly the finite limit of relevant physical quantities.

Now, the Schrödinger equation of the many-body density matrix $\rho(t)$ is written as

$$d\rho(t)/dt=-i\hbar^{-1}[H_0+V+F,\rho(t)]-[\rho(t)-\rho_0]\epsilon_r \ , \tag{5}$$

where ρ_0 stands for the initial value of $\rho(t)$ at $t=0$, and ϵ_r a phenomenological relaxation rate taking into account, in a simplified way, the coupling of the system with the heat bath together with very slow relaxation processes. As explained in Ref. 14(c) this parameter improves the overall coherence of the theory and allows the equilibrium to be recovered if the external field is switched off. It will be regarded as a vanishingly small quantity in the present linear approach.

Since the external field is applied at $t=0$, it will be most convenient to seek the Laplace transform $R(\nu)$ of $\rho(t)$, satisfying the following transformed equation:

$$(\nu+\epsilon_r)R(\nu)+i\hbar^{-1}[H_0+V+F,R(\nu)]=\rho_0(1+\epsilon_r/\nu) \ , \tag{6}$$

or, equivalently, by taking the matrix elements of both sides between b and c,

$$(\nu+\epsilon_r+i\omega_{cb})R^c_b+i\hbar^{-1}[V+F,R\,]^c_b=\rho_{0b}^c(1+\epsilon_r/\nu) \ . \tag{7}$$

This equation is then looked upon as a linear system in the Liouville space[23] constructed by tensorial multiplication of the unperturbed Hilbert space $\mathcal{E}_{H_0}$ and its dual space $\mathcal{E}^*_{H_0}$, say $\mathcal{E}_L=\mathcal{E}_{H_0}\otimes\mathcal{E}^*_{H_0}$. In that space, the density matrix denoted by $\mathbb{R}$ becomes a vector with components $\langle cb|\mathbb{R}\rangle=\mathbb{R}^c{}_b$ on the diads cb defining the basis set. The Liouville superoperators describing the commutation kernels involving V and F become the operators $\mathbb{K}_V$ and $\mathbb{K}_F$, respectively, expressed by

$$\mathbb{K}_V=i\hbar^{-1}(V\delta-\delta V), \quad \mathbb{K}_F=i\hbar^{-1}(F\delta-\delta F) \ . \tag{8}$$

This means that the $c'b'$-cb matrix element of $\mathbb{K}_V$ will be written as

$$\mathbb{K}_{V\,cb'}^{\,c'b}=i\hbar^{-1}(V^{c'}_c\delta^b_{b'}-\delta^{c'}_c V^b_{b'}) \ .$$

We shall also introduce the diagonal superoperator

$$\mathbb{d}=(\nu+\epsilon_r)\mathbb{I}+\mathbb{K}_{H_0}=(\nu+\epsilon_r)\mathbb{I}+i\hbar^{-1}(H_0\delta-\delta H_0) \ , \tag{9}$$

$\mathbb{I}$ being the unity superoperator. The eigenvalues of $\mathbb{d}$ are simply related to the difference of unperturbed energies

$$\mathbb{d}_{cb}=\nu+i\omega_{cb}+\epsilon_r \quad (\omega_{cb}=\omega_c-\omega_b) \ .$$

With the help of these definitions, Eq. (6) can be formally rewritten as

$$(\mathbb{d}+\mathbb{K}_V+\mathbb{K}_F)|\mathbb{R}\rangle=|\rho_0\rangle(1+\epsilon_r/\nu) \ , \tag{10}$$

equivalent to the following system satisfied by the $\mathbb{R}$ components in $\mathcal{E}_L$:

$$\mathbb{R}^c{}_b+\mathbb{d}^{-1}_{cb}(\mathbb{K}_{V\,c_1b}^{\,cb_1}+\mathbb{K}_{F\,c_1b}^{\,cb_1})\mathbb{R}^{c_1}{}_{b_1}=\mathbb{d}^{-1}_{cb}\rho^c_{0b}(1+\epsilon_r/\nu) \ . \tag{11}$$

Summation over repeated indices will be implicit throughout.

In the spirit of the determinantal method, one starts with the primary Cramer solution of the system (11),

$$\mathbb{R}^m{}_l=\mathbb{D}^{mb}_{cl}\rho^c_{0b}/\mathbb{d}_{cb}\mathbb{D} \ ,$$

where $\mathbb{D}$ and $\mathbb{D}^{mb}_{cl}$, respectively, denote the determinant and the ml (row)-cb (column) minor of the matrix $\mathbb{I}+\mathbb{d}^{-1}(\mathbb{K}_V+\mathbb{K}_F)$. The Cramer solution is free from secular terms but the upper and lower determinants contain unlinked sets of transitions which must be removed so as to obtain much more tractable expansions of perturbation type.[14(b)] This is achieved through division by the diagonal minor $\mathbb{D}^{cb}_{cb}$, in each cb contribution, as

$$\mathbb{R}^m{}_l=\mathbb{D}^{mb}_{cl}(\mathbb{D}^{cb}_{cb})^{-1}\rho^c_{0b}/\nu\mathbb{D}(\mathbb{D}^{cb}_{cb})^{-1} \ . \tag{12}$$

This expression can formally be derived from Eq. (10). Let us take into account the contribution of V collisions to the initial equilibrium, giving rise to the commutation of ρ_0 with H_0+V, say $(\mathbb{K}_{H_0}+\mathbb{K}_V)|\rho_0\rangle=0$. Hence the definition (9) allows us to write the following identity:

$$\begin{aligned}(\mathbb{d}+\mathbb{K}_V)|\rho_0\rangle&=[(\nu+\epsilon_r)\mathbb{I}+\mathbb{K}_{H_0}+\mathbb{K}_V]|\rho_0\rangle\\&=(\nu+\epsilon_r)|\rho_0\rangle \ .\end{aligned}$$

Equation (10) can thus be rewritten, without change, as

$$\begin{aligned}(\mathbb{d}+\mathbb{K}_V+\mathbb{K}_F)|\mathbb{R}\rangle=(\mathbb{d}+\mathbb{K}_V+\mathbb{K}_F\mathbb{Q}_{cb})|cb\rangle\\ \times\rho^c_{0b}(1+\epsilon_r/\nu)/(\nu+\epsilon_r) \ .\end{aligned}$$

Each cb contribution has been detailed on the right; $\mathbb{Q}_{cb}$ denotes the projector on the cb complementary subspace in $\mathcal{E}_L$. We also have, after some minor arrangements,

$$\begin{aligned}[\mathbb{I}+\mathbb{d}^{-1}(\mathbb{K}_V+\mathbb{K}_F)]|\mathbb{R}\rangle=[\mathbb{I}+\mathbb{d}^{-1}(\mathbb{K}_V+\mathbb{K}_F\mathbb{Q}_{cb})]|cb\rangle\\ \times\rho^c_{0b}/\nu \ .\end{aligned} \tag{13}$$

Equation (12) is just recovered if one defines $\mathbb{D}^{cb}_{cb}=\det\mathbb{S}_{cb}$, with

$$\mathbb{S}_{cb}=\mathbb{I}+\mathbb{d}^{-1}(\mathbb{K}_V+\mathbb{K}_F\mathbb{Q}_{cb}) \ . \tag{14}$$

The slight change of $\mathbb{D}^{cb}_{cb}$ with respect to the cb-cb minor [determinant of the matrix $\mathbb{I}+\mathbb{d}^{-1}(\mathbb{K}_V+\mathbb{K}_F)\mathbb{Q}_{cb}$] is due to the commutation of ρ_0 with H_0+V.

The interest of the form (12) can now be well understood. Let us make use of the splitting $\mathbb{K}_F=\mathbb{K}_F\mathbb{Q}_{cb}+\mathbb{K}_F\mathbb{P}_{cb}$ in the left-hand side of Eq. (13), and multiply both sides by $\mathbb{S}^{-1}_{cb}$ from the left. This gives

$$(\mathbb{I}+\mathbb{S}^{-1}_{cb}\mathbb{d}^{-1}\mathbb{K}_F\mathbb{P}_{cb})|\nu\mathbb{R}\rangle=|cb\rangle\rho^c_{0b} \ . \tag{15}$$

The matrix of the operator in the left-hand member has nonzero elements along the first diagonal and the cb column only. The solution of the system is therefore straightforward. Taking the long-time limit by means of the elementary rule $\lim_{\nu\to+0}[\nu\mathbb{R}(\nu)]$, we obtain

$$\rho_l^m=\frac{\delta_{cl}^{mb}-\langle ml|\mathbb{S}_{cb}^{-1}\mathbb{d}^{-1}\mathbb{K}_F|cb\rangle}{1+\langle cb|\mathbb{S}_{cb}^{-1}\mathbb{d}^{-1}\mathbb{K}_F|cb\rangle}\rho_{0b}^c \ . \tag{16}$$

Of course, this result expressed in terms of $\mathbb{S}_{cb}^{-1}$ matrix elements could be obtained on carrying out the upper and lower determinant divisions in Eq. (12), as well.[14(b)] Note that the zeroth-order component, $ml=cb$, is singled out in Eq. (16) and so, *the ml-cb matrix elements in the numerator must be understood with the index restriction* $ml\neq cb$. In the linear theory we are restricted to, $\mathbb{K}_F$ can be removed from the definition of $\mathbb{S}_{cb}$ which therefore will be simply written as $\mathbb{S}=\mathbb{I}+\mathbb{d}^{-1}\mathbb{K}$, dropping the V subscript in $\mathbb{K}_V$, which becomes useless.

Observable quantum statistical averages could be derived from expression (16), but at this stage, the *cb-cb* matrix elements involving the applied field F, in the denominator, are connected with the natural broadening of the *ml-cb* transitions, which is quite negligible in most applications. In the present case of the Landau spectrum this also holds, even if divergences may occur when b and c pertain to the same unfilled N band. To see this, however, a more detailed investigation is required, which will be achieved self-consistently, below.

Ignoring, therefore, the *cb-cb* bracket in the denominator of Eq. (16) for the time being, ρ_l^m then reduces to the numerator which is, in fact, nearly identical to the perturbation series of the density matrix. To get a nontrivial result the determinantal procedure must be pushed one step further.[14(c)] To this end, the matrix elements

$$\mathbb{X}^m{}_l(cb)=\langle ml|\mathbb{S}_{cb}^{-1}\mathbb{d}^{-1}\mathbb{K}_F|cb\rangle$$

in Eq. (16) will be calculated, in turn, by solving the system

$$\begin{aligned}(\mathbb{I}+\mathbb{d}^{-1}\mathbb{K})|\mathbb{X}\rangle&=\mathbb{d}^{-1}\mathbb{K}_F|cb\rangle\\&=\mathbb{d}_{zb}^{-1}K_{Fc}^z|zb\rangle-\mathbb{d}_{cz}^{-1}K_{Fz}^b|cz\rangle \ ,\end{aligned} \tag{17}$$

where $\mathbb{S}_{cb}$ is taken in the simplified form $\mathbb{I}+\mathbb{d}^{-1}\mathbb{K}$, and the $\mathbb{K}_F$ commutation kernel has been detailed, with $K_F=i\hbar^{-1}F$. Proceeding strictly as above, we obtain for the cb contribution

$$\begin{aligned}\mathbb{X}^m{}_l(cb)&=\frac{\mathbb{D}_{zl}^{mb}(\mathbb{D}_{zb}^{zb})^{-1}}{\mathbb{d}_{zb}\mathbb{D}(\mathbb{D}_{zb}^{zb})^{-1}}K_{Fc}^z-\frac{\mathbb{D}_{cl}^{mz}(\mathbb{D}_{cz}^{cz})^{-1}}{\mathbb{d}_{cz}\mathbb{D}(\mathbb{D}_{cz}^{cz})^{-1}}K_{Fz}^b\\&=\frac{\delta_{zl}^{mb}-\mathbb{d}_{ml}^{-1}\langle ml|\mathbb{K}(\mathbb{I}+\mathbb{Q}_{zb}\mathbb{d}^{-1}\mathbb{K})^{-1}|zb\rangle}{\mathbb{d}_{zb}+\langle zb|\mathbb{K}(\mathbb{I}+\mathbb{Q}_{zb}\mathbb{d}^{-1}\mathbb{K})^{-1}|zb\rangle}K_{Fc}^z\\&\quad+p^*(m,l;c,b) \ ,\end{aligned} \tag{18}$$

with $ml\neq zb$ in the upper brackets; the determinants now refer to the matrix $\mathbb{I}+\mathbb{d}^{-1}\mathbb{K}$ and $p^*(m,l;c,b)$ denotes the conjugate quantity of the first one on the right, after exchanging b and c, l and m.

In particular, expression (18) can be used to evaluate, in a more precise way, the natural broadening bracket in the denominator of Eq. (16), previously neglected. This gives

$$\begin{aligned}\mathbb{X}^c{}_b(cb)=&-\frac{\mathbb{d}_{cb}^{-1}\langle cb|\mathbb{K}(\mathbb{I}+\mathbb{Q}_{zb}\mathbb{d}^{-1}\mathbb{K})^{-1}|zb\rangle}{\mathbb{d}_{zb}+\langle zb|\mathbb{K}(\mathbb{I}+\mathbb{Q}_{zb}\mathbb{d}^{-1}\mathbb{K})^{-1}|zb\rangle}\\&\times K_{Fc}^z+p^*(c,b) \ .\end{aligned} \tag{19}$$

If in the final state b and c belong to the same band, as will be shown below, this expression exhibits a divergence in $\mathbb{d}_{cb}^{-1}$. But, seemingly, such a divergence has no reason to occur in the starting Eq. (17) [which also reads $(\mathbb{d}+\mathbb{K})|\mathbb{X}\rangle=\mathbb{K}_F|cb\rangle$]. In fact, this behavior is analogous to the zero-frequency divergence of the electrical conductivity handled elsewhere,[14(d)] and can be lifted the same way. The argument is briefly recalled in Appendix C and leads indeed to a vanishing value of $\mathbb{X}^c{}_b(cb)$. The physical reason for it is that (19) represents the ratio of a field-induced first-order transition rate (in $\hbar^{-1}F_c^z$) to the very large intraband collision rate in Landau's subbands of quasi-infinite density of states. So, our initial assumption of neglecting the term in brackets in the denominator of Eq. (16) is well validated, now.

Expression (18) can be used as such, to calculate the response $\rho_l^m=-\mathbb{X}^m{}_l(cb)\rho_{0b}^c$ to the interband part of $K_F=i\hbar^{-1}eEx$, with x given in Eq. (4). Substituting from (18) into Eq. (16), we obtain, for $ml\neq cb$,

$$\begin{aligned}\rho_l^m=&-\frac{\delta_{zl}^{mb}-\mathbb{d}_{ml}^{-1}\langle ml|\mathbb{K}(\mathbb{I}+\mathbb{Q}_{zb}\mathbb{d}^{-1}\mathbb{K})^{-1}|zb\rangle}{\mathbb{d}_{zb}+\langle zb|\mathbb{K}(\mathbb{I}+\mathbb{Q}_{zb}\mathbb{d}^{-1}\mathbb{K})^{-1}|zb\rangle}\\&\times K_{Fc}^z\rho_{0b}^c+p^*(m,l) \ ,\end{aligned} \tag{20}$$

where c and z relate to distinct bands and $K_{Fc}^z=i\hbar^{-1}eE(l/\sqrt{2})(a+a^\dagger)_c^z$.

Instead, to get the response to the diagonal intraband elements of x in Eq. (4), the form (18) cannot be used. Coming back to Eq. (17), we then have

$$(\mathbb{I}+\mathbb{d}^{-1}\mathbb{K})|\mathbb{X}\rangle=\mathbb{d}_{cb}^{-1}K_{Fcb}|cb\rangle \ ,$$

with $K_{Fcb}=K_{Fc}^c-K_{Fb}^b=-i\hbar^{-1}eEl^2k_{cb}$. The subsequent step (18) gives

$$\begin{aligned}\mathbb{X}^m{}_l(cb)&=\frac{\mathbb{D}_{cl}^{mb}(\mathbb{D}_{cb}^{cb})^{-1}}{\mathbb{d}_{cb}\mathbb{D}(\mathbb{D}_{cb}^{cb})^{-1}}K_{Fcb}\\&=\frac{\delta_{cl}^{mb}-\langle ml|(\mathbb{I}+\mathbb{d}^{-1}\mathbb{K}\,\mathbb{Q}_{cb})^{-1}\mathbb{d}^{-1}\mathbb{K}|cb\rangle}{1+\langle cb|(\mathbb{I}+\mathbb{d}^{-1}\mathbb{K}\,\mathbb{Q}_{cb})^{-1}\mathbb{d}^{-1}\mathbb{K}|cb\rangle}\frac{K_{Fcb}}{\mathbb{d}_{cb}} \ ,\end{aligned} \tag{21}$$

which again leads to a vanishing value of the natural broadening parameter in Eq. (16), for the same reason as in Eq. (19):

$$\begin{aligned}\mathbb{X}^c{}_b(cb)&=K_{Fcb}/[\mathbb{d}_{cb}+\langle cb|\mathbb{K}(\mathbb{I}+\mathbb{d}^{-1}\mathbb{K}\,\mathbb{Q}_{cb})^{-1}|cb\rangle]\\&\to 0 \ .\end{aligned}$$

For $ml\neq cb$, Eq. (21) sends us back to Eq. (16) and can be handled the same way. The seemingly divergent bracket in $\mathbb{d}_{cb}$, in the denominator, will be again shown to vanish, and we are reduced to seeking

$$\mathbb{Y}^m{}_l(cb)=\langle ml|(\mathbb{I}+\mathbb{d}^{-1}\mathbb{K}\,\mathbb{Q}_{cb})^{-1}\mathbb{d}^{-1}\mathbb{K}|cb\rangle\ ,$$

as the solution of the system

$$(\mathbb{I}+\mathbb{d}^{-1}\mathbb{K}\,\mathbb{Q}_{cb})|\mathbb{Y}\rangle=\mathbb{d}^{-1}\mathbb{K}|cb\rangle$$
$$=\mathbb{d}_{zb}^{-1}K_c^z|zb\rangle-\mathbb{d}_{cz}^{-1}K_z^b|cz\rangle\ ,$$

similar to (17) with K in place of K_F. Proceeding as in (18), this yields

$$\mathbb{Y}^m{}_l(cb)=\frac{\mathbb{D}_{zl}^{mb}(\mathbb{D}_{zb}^{zb})^{-1}}{\mathbb{d}_{zb}\mathbb{D}(\mathbb{D}_{zb}^{zb})^{-1}}K_c^z-\frac{\mathbb{D}_{cl}^{mz}(\mathbb{D}_{cz}^{cz})^{-1}}{\mathbb{d}_{cz}\mathbb{D}(\mathbb{D}_{cz}^{cz})^{-1}}K_z^b$$
$$=\frac{\delta_{zl}^{mb}-\mathbb{d}_{ml}^{-1}\langle ml|\mathbb{K}(\mathbb{I}+\mathbb{Q}_{cb}\mathbb{Q}_{zb}\mathbb{d}^{-1}\mathbb{K})^{-1}|zb\rangle}{\mathbb{d}_{zb}+\langle zb|\mathbb{K}(\mathbb{I}+\mathbb{Q}_{cb}\mathbb{Q}_{zb}\mathbb{d}^{-1}\mathbb{K})^{-1}|zb\rangle}$$
$$\times K_c^z+p^*(m,l;c,b)\ . \tag{22}$$

The determinants now refer to the matrix $\mathbb{I}+\mathbb{Q}_{cb}\mathbb{d}^{-1}\mathbb{K}$. In particular this can be used to verify that the term in brackets in the denominator of Eq. (21), equal to $\mathbb{Y}^c{}_b(cb)$, still vanishes, as in (19). For $ml\neq cb$, $\mathbb{X}^m{}_l$ as given by Eq. (21) then identifies with $-\mathbb{Y}^m{}_l\mathbb{d}_{cb}^{-1}K_{Fcb}$, and from Eq. (16) (in which the denominator reduces to 1), the density matrix elements become

$$\rho_l^m=\frac{\delta_{zl}^{mb}-\mathbb{d}_{ml}^{-1}\langle ml|\mathbb{K}(\mathbb{I}+\mathbb{Q}_{zb}\mathbb{Q}_{cb}\mathbb{d}^{-1}\mathbb{K})^{-1}|zb\rangle}{\mathbb{d}_{zb}+\langle zb|\mathbb{K}(\mathbb{I}+\mathbb{Q}_{zb}\mathbb{Q}_{cb}\mathbb{d}^{-1}\mathbb{K})^{-1}|zb\rangle}$$
$$\times K_c^z\frac{K_{Fcb}}{\mathbb{d}_{cb}}\rho_{0b}^c+p^*(m,l)\ . \tag{23}$$

This somewhat new form of the ρ_l^m's as compared with (20) is due to the need of collision assistance ($K_c^z=i\hbar^{-1}V_c^z$) for interband transitions to be induced by the purely intraband excitation K_{Fcb} (compare with the occurrence of indirect transitions in semiconductors). Unlike in Eq. (19), the remaining divergence in $\mathbb{d}_{cb}^{-1}$ no longer comes from final states but, instead, from the same intraband feature of K_{Fcb}. It will be removed by working out the particular behavior of the ρ_{0b}^c's, in the next section.

III. EQUILIBRIUM DENSITY MATRIX

Before making use of our basic results (20) and (23) in the derivation of the conductivity, we have to calculate the ρ_{0b}^c's on the Landau-state basis. We start with the well-known expression of the equilibrium density matrix in the grand canonical scheme

$$\rho_0=\frac{\exp[-\beta(H_0+V-\mu n)]}{\mathrm{Tr}\exp[-\beta(H_0+V-\mu n)]}\ . \tag{24}$$

$\beta=1/k_BT$, μ is the Fermi energy, and n the number-of-fermions operator. In the occupation number representation

$$H_0-\mu n=\sum_{Nk}(\varepsilon_{Nk}-\mu n)c_{Nk}^\dagger c_{Nk}\ . \tag{25}$$

In the absence of scattering ($V=0$) ρ_0 takes a simple ideal form, say $\rho_0^{(i)}$, on the basis constructed out of the one-particle eigenstates (N,k). In the grand ensemble where the Fermi level is fixed but the number of particles is not, the trace appearing in Eq. (24) is then given by $\prod_{Nk}\{1+\exp[-\beta(\varepsilon_{Nk}-\mu)]\}$, leading to[24]

$$\rho_0^{(i)}=\prod_{Nk}[f_{Nk}c_{Nk}^\dagger c_{Nk}+(1-f_{Nk})c_{Nk}c_{Nk}^\dagger]\ , \tag{26}$$

where f_{Nk} is the Fermi occupation function of the (N,k) state.

In the quantum limit ($\hbar\Omega\gg k_BT$) the Fermi level is slowly varying during the filling (or emptying) of the Landau levels, as the field is varied. The filling of the N level, for example, occurs in the range

$$n_sh/(N+1)e\leq B\leq n_sh/Ne\ . \tag{27}$$

In what follows, the quantum limit will be assumed to be reached. Near the lower end of the range where $B\lesssim n_sh/(N+1)e$, we have an intermediate region in which the Fermi level jumps from N to $N+1$, so that

$$f_{Nk}\lesssim 1,\quad f_{N+1k}\gtrsim 0\ . \tag{28}$$

In contrast, in most of the range (27), the Landau levels are full up to $N-1$, N is on filling, and $N+1$ is empty,

$$f_{0k}=f_{1k}=\cdots=f_{N-1k}=1,\quad f_{Nk}<1\ ,$$
$$f_{N+1k}=\cdots=0\ .$$

Let us now recall the matrix elements of the unperturbed equilibrium density matrix $\rho_0^{(i)}$. Collective fermion states $|b\rangle,\ldots|s\rangle,\ldots$ are constructed, as usual, through creation of n_s fermions in the "vacuum" state $|0\rangle$,

$$|b\rangle=\cdots c_{N_2k_2}^\dagger c_{N_1k_1}^\dagger|0\rangle\ .$$

Hence, from Eq. (26)

$$\rho_{0b}^{(i)b}=\langle b|\rho_0^{(i)}|b\rangle=\prod_{N_1k_1N_2k_2}f_{N_1k_1}(1-f_{N_2k_2})\ , \tag{29}$$

where N_1k_1 refers to occupied states in $|b\rangle$ and N_2k_2 to empty states. In the quantum limit, the product over the N's can be dropped because of the filling of the Landau levels up to $N-1$ and, therefore,

$$\rho_{0b}^{(i)b}=\prod_{kk'}f_{Nk}(1-f_{Nk'})\ . \tag{30}$$

Near the lower edge of the range (27) [$B\sim n_sh/(N+1)e$] the population of the $N+1$ level is no longer negligible so that we must write

$$\rho_{0b}^{(i)b}=\prod_{k_2k_1k'k}f_{N+1k_2}f_{Nk'}(1-f_{N+1k_1})(1-f_{Nk})\ . \tag{31}$$

We now come to the derivation of tractable expressions of the collision-perturbed density matrix ρ_0, as given by Eq. (24), in the Landau-state basis. To this end we first notice, from Eqs. (2) and (25), the following commutation relations:

$$[H_0,V]=\sum_{N'k'Nk}\varepsilon_{N'k'Nk}V_{Nk}^{N'k'}c_{N'k'}^\dagger c_{Nk}\ , \tag{32a}$$

$$[H_0,[H_0,V]]=\sum_{N'k'Nk}\varepsilon_{N'k'Nk}^2V_{Nk}^{N'k'}c_{N'k'}^\dagger c_{Nk}\ . \tag{32b}$$

Expressions (32) suggest splitting the collision potential V into intraband (V^{NN}) and interband ($V^{N'N}, N' \neq N$) contributions as

$$V = \sum_{N,N'} (V^{NN} + V^{N'N}) , \tag{33}$$

with the evident following definitions:

$$V^{NN} = \sum_{k'k} V_{Nk}^{Nk'} c_{Nk'}^{\dagger} c_{Nk} , \quad V^{N'N} = \sum_{k'k} V_{Nk}^{N'k'} c_{N'k'}^{\dagger} c_{Nk} . \tag{34}$$

Between distinct Landau levels, e.g., nearest levels, we have $[H_0, V^{N'N}] = \hbar\Omega V^{N'N}$, $[H_0,[H_0,V^{N'N}]] = (\hbar\Omega)^2 V^{N'N}, \ldots$ while inside a given level, $[H_0, V^{NN}] = 0$ to zeroth order of the bandwidth.

The density matrix (24) is most conveniently handled with the help of the following Bourbaki expression[25] for the product of exponentials of noncommuting operators, say A and B,

$$e^{A} e^{B} e^{-A} = \exp\left[\sum_{n=0}^{\infty} \frac{1}{n!} L_A^n B \right] = \exp\{[\exp(L_A)]B\} , \tag{35}$$

where $L_A^n B$ represents the n-fold repeated commutation of A and B:

$$L_A^1 B = [A,B], \quad L_A^2 B = [A,[A,B]], \ldots , \quad \sum_{n=0}^{\infty} \frac{1}{n!} L_A^n B = (\exp L_A) B , \tag{36}$$

with the convention that $L_A^0 B = B$.

Taking $A = -\beta(H_0 - \mu n)$ and $B = -\beta(H_0 + V - \mu n)$, Eq. (35) then gives

$$\begin{aligned} \exp[-\beta(H_0 + V - \mu n)] = \exp[\beta(H_0 - \mu n)] & \\ \times \exp(-\beta\{H_0 - \mu n & \\ + [\exp(L_{-\beta H_0})]V\}) & \\ \times \exp[-\beta(H_0 - \mu n)] & \end{aligned} \tag{37}$$

(n can be ignored in $L_{-\beta(H_0-\mu n)}$ in the case of conservative particles). From the definition (36) and Eqs. (32), we also have

$$[\exp(L_{-\beta H_0})]V = \sum_{N'k'Nk} \exp(-\beta\varepsilon_{N'k'Nk}) V_{Nk}^{N'k'} c_{N'k'}^{\dagger} c_{Nk} .$$

In the quantum limit where $\beta\Omega \gg 1$, $|\varepsilon_{N'k'Nk}|$ is at least equal to Ω if $N' \neq N$, and therefore $\exp(-\beta\varepsilon_{N'k'Nk})$ is either a very large or a very small quantity. It follows that the second exponential factor in Eq. (37) is either zero or extremely close to $\exp[-\beta(H_0 - \mu n)]$, which amounts to saying that interband transitions do not contribute to within a very large accuracy. Thus the analysis can be restricted to the intraband transitions described by V^{NN} in the unfilled N band. Further, if this band is taken to be zero width, Eq. (32a) shows that the commutator $C = [H_0, V^{NN}]$ vanishes. Thus, if a response sensitive to intraband collisions is expected, it can only be obtained by extracting with some care the proper finite limit when the bandwidth, or equivalently the commutator C, tends to zero. For this essential reason we shall have to seek the relevant quantities to at least first order of the bandwidth, which is well typified by the C-matrix elements

$$\langle c|C|b\rangle = \varepsilon_{cb} (V^{NN})_b^c . \tag{38}$$

From Eq. (32b) we obviously have, to first order in C,

$$[C, H_0] = 0 .$$

But, in addition, the commutator $[C, V^{NN}]$, of which (intraband) matrix elements can be written as

$$\langle c|[C,V^{NN}]|b\rangle = \varepsilon_{ci} (V^{NN})_i^c (V^{NN})_b^i - \varepsilon_{jb} (V^{NN})_j^c (V^{NN})_b^j , \tag{39}$$

turns out to vanish, as well. The reason is that, in the light of the above discussion, the energy spacings inside the band are independent of the k's, while the matrix elements of V^{NN} are dependent on the k's only, at least to zeroth order or, in a more precise way, on the differences k_{ci}, k_{ib}. It follows that given k_c, k_b, either intermediate summations over i,j in (39), or distinct substates c,b of equal ω_{cb}, or even both of them, can be freely associated so as to lead to complete cancellation.

Finally, C commutes with both H_0 and V^{NN}, in which case the identity (37) is satisfied by the well-known solution

$$\begin{aligned} \exp[-\beta(H_0 + V^{NN} - \mu n)] &= \exp(-\beta V^{NN}) \exp(\tfrac{1}{2}\beta^2 [H_0, V^{NN}]) \exp[-\beta(H_0 - \mu n)] \\ &= \exp(-\beta V^{NN})(1 + \tfrac{1}{2}\beta^2 C) \exp[-\beta(H_0 - \mu n)] , \end{aligned} \tag{40}$$

where the last member is limited to first order. This solution, which can most easily be derived from the Baker-Campbell-Hausdorff expansion,[26] will enable us to calculate the Boltzmann operator matrix elements. Since C and V^{NN} commute with each other, we first have

$$\exp(-\beta V^{NN}) C = \sum_{n=0}^{\infty} \frac{(-\beta V^{NN})^n}{n!} C = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{q+p=n} \frac{1}{n+1} (-\beta V^{NN})^q C (-\beta V^{NN})^p .$$

Then, by taking into account the form (38) of C-matrix elements, we obtain by some simple algebra

$$\langle c|\exp(-\beta V^{NN}) C|b\rangle = \beta^{-1} \varepsilon_{cb} \langle c|[\exp(-\beta V^{NN}) - 1]|b\rangle .$$

Hence, for $c \neq b$,

$$\langle c|\exp[-\beta(H_0+V^{NN}-\mu n)]|b\rangle=\langle c|\exp(-\beta V^{NN})|b\rangle[1+\tfrac{1}{2}\beta\varepsilon_{cb}+\cdots]\exp[-\beta(\varepsilon_b-\mu)]\ . \tag{41}$$

Slow relaxation processes (mainly the coupling with the heat bath) which have been accounted for in the simplified form of the phenomenological parameter ϵ_r, should be coherently included in Eq. (41). This can be achieved most simply by means of a coupling potential V_B operating between diad basis states of both the system and the heat bath, which approximately amounts to replacing V^{NN} by $V^{NN}+V_B$ in Eq. (40), i.e., C by $C+[H_0,V_B]$ in the first-order correction of Eq. (41). $[H_0,V_B]$ can be connected, in turn, with the relaxation rate ϵ_r by[14(c)]

$$-i\hbar^{-1}[\rho,V_B]\cong(\rho-\rho_0)\epsilon_r\ , \tag{42}$$

where ρ_0 is taken in $\exp[-\beta(H_0+V^{NN}-\mu n)]$, and ρ in $\exp[-\beta(H_0+V^{NN}+V_B-\mu n)]$. Then, from the commutation relation $[H_0,[H_0,V_B]]=0$, valid to first order, we have

$$[\exp[-\beta(H_0+V^{NN}+V_B-\mu n)],V_B]=\exp[-\beta(H_0+V^{NN}-\mu n)][-\beta H_0,V_B]\ ,$$

which yields, on substituting into Eq. (42),

$$\epsilon_{rcb}\cong i\beta\hbar^{-1}\overline{\langle c|[H_0,V_B]|b\rangle}\ .$$

The overbar implies averaging over the related initial and final states of the heat bath which have not been detailed. If in addition ϵ_{rcb} is averaged into ϵ_r over all couples of states c-b (within the range of the collision potential), the lowest-order contribution of the heat bath amounts to replacing $\overline{\langle c|[H_0,V_B]|b\rangle}$ by $-i\hbar\epsilon_r/\beta$, or ε_{cb} by $\varepsilon_{cb}-i\hbar\epsilon_r$ in Eq. (41), as expected.

Finally, remembering that inside the range defined in (27) the only N band contributes to statistical average calculations, the matrix elements of $\exp[-\beta(H_0-\mu n)]$ are to be taken in the limit of vanishing bandwidth and between basic many-body states $b,c,\ldots,s,\ldots$, which differ from one another by the electron distribution in the unfilled N band. Thus the matrix elements of expression (24) will be explicitly rewritten to first order, as

$$\rho_{0b}^{c}=\frac{\langle c|\exp(-\beta V^{NN})|b\rangle[1+\frac{1}{2}\beta(\varepsilon_{cb}-i\hbar\epsilon_r)]\langle b|\prod_k\{c_{Nk}c_{Nk}^{\dagger}+\exp[-\beta(\varepsilon_{Nk}-\mu)]c_{Nk}^{\dagger}c_{Nk}\}|b\rangle}{\sum_s\langle s|\exp(-\beta V^{NN})|s\rangle\left\langle s\left|\prod_k\{c_{Nk}c_{Nk}^{\dagger}+\exp[-\beta(\varepsilon_{Nk}-\mu)]c_{Nk}^{\dagger}c_{Nk}\}\right|s\right\rangle}\ . \tag{43}$$

It is easily seen that the diagonal elements of the products over k in the above expression can be simplified out. The reason is that they take the same value in all states $b,c,\ldots,s,\ldots$ because they are only dependent on the electron population in the N band, which is imposed by the chemical potential μ to within very small fluctuations. We thus arrive at

$$\rho_{0b}^{c}=\frac{\langle c|\exp(-\beta V^{NN})|b\rangle[1+\frac{1}{2}\beta(\varepsilon_{cb}-i\hbar\epsilon_r)]}{\sum_s\langle s|\exp(-\beta V^{NN})|s\rangle}\ . \tag{44}$$

IV. CONDUCTIVITY-TENSOR COMPONENTS

The preceding results will enable us to set up detailed expressions of the conductivity of the two-dimensional electron gas. The statistical ensemble average of the current density is given by

$$\mathcal{J}=\mathrm{Tr}(\rho\mathbf{J})=\mathbf{J}_m^l\rho_l^m\ , \tag{45}$$

with $\mathbf{J}$ being the current density operator which will be written in terms of the rotating components

$$J_{\pm}=\mp\frac{ie}{L_xL_y}\left(\frac{2\hbar\omega_c}{m^*}\right)^{1/2}\begin{pmatrix}a^{\dagger}\\ a\end{pmatrix}\ .$$

L_x and L_y are the transverse dimensions of the sample. We thus have

$$\begin{aligned}J_x&=\tfrac{1}{2}(J_++J_-)=i(e\hbar/m^*l\sqrt{2})(a-a^{\dagger})\ ,\\ J_y&=\tfrac{1}{2}(J_+-J_-)=-(e\hbar/m^*l\sqrt{2})(a+a^{\dagger})\ .\end{aligned} \tag{46}$$

Now, substituting ρ_l^m from Eq. (20) or (23) into Eq. (45), we straightforwardly obtain the current density components and, thereby, the conductivity-tensor components in the following form:

$$\begin{pmatrix}\sigma_{xx}\\ \sigma_{yx}\end{pmatrix}=\begin{pmatrix}i\\ -1\end{pmatrix}\frac{e^2A_{\mp}{}_m^l}{2m^*\omega_{ml}}\times\frac{\langle ml|\mathbb{K}[\mathbb{I}+(\mathbb{Q}_{cb})\mathbb{Q}_{zb}\mathbb{d}^{-1}\mathbb{K}]^{-1}|zb\rangle}{\mathbb{d}_{zb}+\langle zb|\mathbb{K}[\mathbb{I}+(\mathbb{Q}_{cb})\mathbb{Q}_{zb}\mathbb{d}^{-1}\mathbb{K}]^{-1}|zb\rangle}\times X_c^z\rho_{0b}^{c}+\text{c.c.}\ , \tag{47}$$

valid for $ml\neq zb$. For brevity we have set $A_{\mp}=a\mp a^{\dagger}$, and defined the dimensionless position coordinate

$$X_c^z=A_{+}{}_c^z+i\hbar^{-1}l\sqrt{2}V_c^z\mathbb{d}_{cb}^{-1}k_{cb}\ . \tag{48}$$

In $\mathbb{d}_{ml}=i\omega_{ml}+\epsilon_r$, ϵ_r is dropped in comparison with ω_{ml} of magnitude Ω. We notice that the expansions implied

in the terms in brackets of Eqs. (20) and (23) do not involve the same index restrictions, as recalled by the parenthetic term ($\mathbb{Q}_{cb}$) in Eq. (47). This, however, will have no bearing on the result in the thermodynamic limit where all intraband indices become continuous variables.

Calculations should be performed in a many-body scheme, using, for instance, expressions such as (2) and (3). However, in our assumption of independent electrons, this can be simplified to a great extent. As far as electron motions can be regarded as uncorrelated, the initial Cramer solution (12) tends to split into one-particle terms to each of which the determinantal method can be applied separately. This amounts to assuming the electron under consideration to be moving in states which remain statistically free, all other electrons being frozen in their equilibrium initial state. If instead correlations are required in the relevant processes, for example, those resulting from the exclusion principle, we have to deal with the evolution of at once two electrons or more, all others remaining frozen, and so on. In practice, such points of view will often be sufficient, and these remarks will be borne in mind in the analysis below, to understand how the theory can be properly reduced to one-electron processes.

It will be convenient to begin with the "ideal case" in which collisions are ignored, as a reference. In the collision case which will be considered next, the respective contributions of the relative orbital motion and that of the center-of-orbit migration leading to completely distinct results will be separately treated.

A. Ideal behavior

Collisions are ignored. We have to return to Eq. (20) which reduces to its zeroth-order contribution, with $ml=zb$, giving

$$\rho_b^z=-i\hbar^{-1}(eEl/\sqrt{2})\mathbb{d}_{zb}^{-1}A_{+\,b}^{\,z}\rho_{0b}^{(i)b}\ ,$$

where the unperturbed density matrix $\rho_0^{(i)}$ is diagonal. Substituting from this equation into Eq. (45), and using expressions (46), yields the ideal conductivity

$$\begin{bmatrix}\sigma_{xx}^{(i)}\\ \sigma_{yx}^{(i)}\end{bmatrix}=\begin{bmatrix}-i\\ 1\end{bmatrix}\frac{e^2}{2m^*\omega_{zb}}A_{\mp\,z}^{\,b}A_{+\,b}^{\,z}\rho_{0b}^{(i)b}+\text{c.c.} \tag{49}$$

In the quantum limit, assuming filled levels up to $(N-1)(f_{N-1}=1)$, $\rho_{0b}^{(i)b}$ is given by Eq. (30). Since $\epsilon_r\to 0$ we have $\mathbb{d}_{zb}=i\omega_{zb}=i\Omega$. In Eq. (49) the two conjugate quantities add up in $\sigma_{yx}^{(i)}$ and cancel in $\sigma_{xx}^{(i)}$. Since the only possible transitions take place either between $N-1$ and N, or N and $N+1$, the result is as follows:

$$\sigma_{xx}^{(i)}=0\ ,$$
$$\sigma_{yx}^{(i)}=(e^2/m^*\Omega)\sum_k[N(1-f_{Nk})+(N+1)f_{Nk}]\ .$$

On summing over k, the degeneracy eB/h of the N band is factored out. Hence, dropping the subscript k which f_{Nk} does not depend on

$$\sigma_{yx}^{(i)}=(N+f_N)e^2/h=n_se/B\ , \tag{50}$$

because $n_s=(N+f_N)eB/h$.

In the intermediate region $B\sim n_sh/(N+1)e$ [see Eq. (28)] we have additional transitions from $N+1$ towards N and $N+2$, leading to

$$\sigma_{yx}^{(i)}=(e^2/m^*\Omega)\sum_k[N(1-f_N)+(N+1)f_N(1-f_{N+1})+(N+2)f_{N+1}-(N+1)f_{N+1}(1-f_N)]\ .$$

After arrangement, the term in square brackets reduces to $N+f_N+f_{N+1}$, which again leads to n_se/B, since we now have $n_s=(N+f_N+f_{N+1})eB/h$. The diagonal conductivity is still zero because of the absence of scattering.

As expected, we find the customary expression of the ideal Hall conductivity which is known to be associated with the relative motion of electrons in cyclotron orbits. The center-of-orbit contribution, indeed, falls in the absence of collision, as can be seen in Eq. (47).

B. Effect of collisions

Collision effects cause two major changes in the formalism. First, the equilibrium density matrix is perturbed by the collision potential arising in the Boltzmann function [Eq. (40)], which can be regarded as a broadening of Landau levels, leading to expression (44) of ρ_0. Second, collisions drastically influence electron motion after the field is switched on, as expressed by the terms in angular brackets in Eq. (47). The major task then consists of calculating these terms upon expansion.

To this end, we shall make use of definitions (8) and (9) of the $\mathbb{K}$'s and $\mathbb{d}$'s, along with the rules given elsewhere[14(a)] for the calculation of matrix elements in the Liouville space. For the sake of precision and clarity, let us write out the term in the upper brackets in Eq. (47), in second order,

$$\begin{aligned}\omega_{ml}^{-1}A_{\mp\,m}^{\,l}\langle ml|\mathbb{K}(\mathbb{I}+\mathbb{Q}_{zb}\mathbb{d}^{-1}\mathbb{K})^{-1}|zb\rangle X_c^z\rho_{0b}^c&=-\omega_{ml}^{-1}A_{\mp\,m}^{\,l}\langle ml|\mathbb{K}\mathbb{Q}_{zb}\mathbb{d}^{-1}\mathbb{K}|zb\rangle X_c^z\rho_{0b}^c+\cdots\\&=-\omega_{ml}^{-1}A_{\mp\,m}^{\,l}\hbar^{-2}(V_{c_1}^m\delta_l^{b_1}-\delta_{c_1}^mV_l^{b_1})\mathbb{d}_{c_1b_1}^{-1}(V_z^{c_1}\delta_{b_1}^b-\delta_z^{c_1}V_{b_1}^b)X_c^z\rho_{0b}^c+\cdots\\&=-\hbar^{-2}\left[\frac{A_{\mp\,m}^{\,b}V_{c_1}^mV_z^{c_1}X_c^z\rho_{0b}^c}{\omega_{mb}\mathbb{d}_{c_1b}}-\frac{A_{\mp\,m}^{\,l}V_z^mX_c^z\rho_{0b}^cV_l^b}{\omega_{ml}\mathbb{d}_{zl}}\right.\\&\qquad\left.-\frac{A_{\mp\,m}^{\,l}V_z^mX_c^z\rho_{0b}^cV_l^b}{\omega_{ml}\mathbb{d}_{mb}}+\frac{A_{\mp\,z}^{\,l}X_c^z\rho_{0b}^cV_{b_1}^bV_l^{b_1}}{\omega_{zl}\mathbb{d}_{zb_1}}\right]+\cdots\end{aligned} \tag{51}$$

($ml \neq zb$). Similarly, from the denominator of Eq. (47),

$$\begin{aligned}\mathbb{d}_{zb}+\langle zb|\mathbb{K}(\mathbb{I}+\mathbb{Q}_{zb}\mathbb{d}^{-1}\mathbb{K})^{-1}|zb\rangle &= \mathbb{d}_{zb}-\langle zb|\mathbb{K}\mathbb{Q}_{zb}\mathbb{d}^{-1}\mathbb{K}|zb\rangle+\cdots \\ &= \mathbb{d}_{zb}+\hbar^{-2}(V^z_{c_1}\delta^{b_1}_b-\delta^z_{c_1}V^{b_1}_b)\mathbb{d}^{-1}_{c_1b_1}(V^{c_1}_z\delta^b_{b_1}-\delta^{c_1}_zV^b_{b_1})+\cdots \\ &= \mathbb{d}_{zb}+\hbar^{-2}(\mathbb{d}^{-1}_{c_1b}V^z_{c_1}V^{c_1}_z+\mathbb{d}^{-1}_{zb_1}V^b_{b_1}V^{b_1}_b)+\cdots\ . \end{aligned} \tag{52}$$

The point is that the energy denominators $\mathbb{d}_{c_1b_1}$ become very small every time the energy of c_1 and b_1 fall inside the same Landau subband. The related transition sequences largely dominate the above expansions more strongly, the higher the order. This is the effect of the quasi-infinite density of states and is certainly true in a magnetic-field range such as that defined in (27), far enough from the ends, but not necessarily in the intermediate range where the ends are approached and new transitions take place towards almost filled bands. We shall come back qualitatively to this point in Sec. VI. For brevity, such complications will be excluded from the present quantitative analysis, and postponed to subsequent publications. Then, all expansions reduce to their divergent terms in comparison to which the cyclotron frequency can be disregarded. In fact this corresponds to the quantized plateaus, as we are going to see in the next section.

V. HALL CONDUCTIVITY

On detailing the transition sequences arising in the brackets of Eqs. (47) with the choice of A_+ referring to the Hall conductivity in the current operator, one finds a net trend to complete cancellation as long as the index restriction $ml \neq zb$ is ignored. Moreover, as this property seems to persist at higher orders, a simple underlying reason is thought to exist. To find it let us single out the final transitions in the numerator of Eq. (47):

$$-A_+{}^l_m\omega^{-1}_{ml}\langle ml|\mathbb{K}(\mathbb{I}+\mathbb{Q}_{zb}\mathbb{d}^{-1}\mathbb{K})^{-1}|zb\rangle X^z_c\rho^c_{0b} = A_+{}^l_m\omega^{-1}_{ml}\langle ml|\mathbb{K}|c_1b_1\rangle\mathbb{d}^{-1}_{c_1b_1}\langle c_1b_1|\mathbb{K}(\mathbb{I}+\mathbb{Q}_{zb}\mathbb{d}^{-1}\mathbb{K})^{-1}|zb\rangle X^z_c\rho^c_{0b}\ .$$

The term in the second set of brackets in the right-hand side can be taken from second order since $c_1b_1 \neq zb$. It represents c_1b_1 matrix elements of an expansion of the type

$$W^{c_1}_{b_1} = -\sum_{n=2}^{\infty}(-i\hbar^{-1})^nV^{c_1}\mathbb{d}^{-1}V\cdots V_zX^z_c\rho^c_{0b}\mathbb{d}^{-1}V^b\cdots\mathbb{d}^{-1}V_{b_1}\ .$$

The numerator of Eq. (47) can thus be written as

$$\frac{iA_{\mp}{}^l_m}{\hbar\omega_{ml}}\left[\frac{V^m_{c_1}W^{c_1}_l}{\mathbb{d}_{c_1l}}-\frac{W^m_{b_1}V^{b_1}_l}{\mathbb{d}_{mb_1}}\right]=\frac{i}{\hbar}\left[\frac{A_+{}^l_mV^m_n}{\omega_{ml}}-\frac{V^l_pA_+{}^p_n}{\omega_{np}}\right]\frac{W^n_l}{\mathbb{d}_{nl}} \tag{53}$$

($ml, np, nl \neq zb$), after some evident circular permutations and arrangements of dummy indices, in the right-hand side.

Again predominant contributions correspond to intraband elements of $\mathbb{d}_{nl}$, and calculations can be restricted to those elements. Then, within our assumption of single-particle collision operators, the $n \to l$ transitions inside the parentheses in the right-hand side (rhs) of (53) can be split into one-electron processes between the N and $N+1$ bands, and two-electron processes between the $N-1$ and N bands, as illustrated in Fig. 1. The related single-particle matrix elements, given in Eqs. (A6) and (A7) of Appendix A, involve two distinct contributions through the factors $(q_x \mp iu_y)$. It is easy to see that, in the $N \to N+1$ case [Fig. 1(a)], for example, in which $\omega_{ml}=\Omega$ and $\omega_{np}=-\Omega$, the terms in $u_y=k_l-k_n$ cancel in the rhs parentheses of (53), if the index restriction $ml(np) \neq zb$ is ignored. In a sense, these terms very nearly behave like the trace of a commutator. This is due to the selection rule of the A_+'s, coupling interband states with the *same* k. On the other hand, terms in q_x add up and give

$$-\frac{W^n_l}{\mathbb{d}_{nl}}\frac{lL_x}{\pi\hbar\Omega\sqrt{2}}\sum_I\int_{-\infty}^{+\infty}v(q_x,u_y)\exp\{-\tfrac{1}{2}\chi^2+iq_x[\tfrac{1}{2}l^2(k_l+k_n)-x_I]-iu_yy_I\}L_N^{(1)}(\tfrac{1}{2}\chi^2)q_xdq_x\ . \tag{54}$$

As $v(q_x,u_y)$ is currently an even function of q_x the integration yields an odd function of $k_l+k_n-2x_I/l^2$ which will be eliminated in subsequent summations over the k's and the coordinates x_I of the random scattering centers. The simultaneous vanishing of $\mathbb{d}_{nl}$ in the denominator of (54) has no effect if one remembers that the $\mathbb{d}_{nl}$'s are not sensitive to the k's and, therefore, can be associated in opposite values of ω_{nl} for the same k_n, k_l.

As a result, the only nonzero contribution in (53) stems from the index restriction $ml, np \neq zb$, leaving either term uncompensated in the parentheses on the rhs. The conclusion would be the same for the $N \to N-1$ transitions [Fig. 1(b)]. The summation in the numerator of (47) thus reduces to the opposite of the $ml=zb$ term, i.e.,

$$\begin{aligned}-\sum_{ml\neq zb}A_+{}^l_m\omega^{-1}_{ml}\langle ml|\mathbb{K}(\mathbb{I}+\mathbb{Q}_{zb}\mathbb{d}^{-1}\mathbb{K})^{-1}|zb\rangle X^z_c \\ = A_+{}^b_z\omega^{-1}_{zb}\langle zb|\mathbb{K}(\mathbb{I}+\mathbb{Q}_{zb}\mathbb{d}^{-1}\mathbb{K})^{-1}|zb\rangle X^z_c\ ,\end{aligned}$$

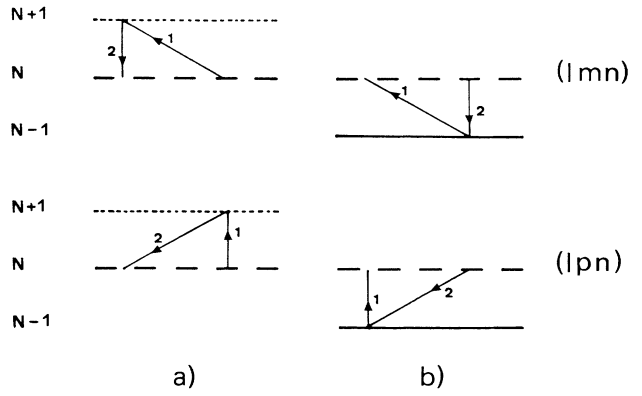

FIG. 1. Schematic representation of elementary *nml* (upper) and *npl* (lower) terminal processes arising in the terms in the rhs parentheses of (53), in the limit $\omega_{nl} \cong 0$. They involve either (a) one electron in $N \leftrightarrows N+1$ transitions, or (b) two electrons in $N-1 \leftrightarrows N$ transitions. The $N-1$ band is full (full line), the N one is partly occupied (broken line), and the $N+1$ one is empty (dotted line). Vertical lines refer to k-conserving $A_{\mp}$-induced processes, and inclined lines to V-induced processes. The numbers relate to the chronological order.

leading to the following expression of the Hall conductivity:

$$\sigma_{yx} = \frac{e^2 A_{+z}^{\ b}}{2m^* \omega_{zb}} \frac{\langle zb | \mathbb{K}(\mathbb{I}+\mathbb{Q}_{zb}\mathbb{d}^{-1}\mathbb{K})^{-1} | zb \rangle}{\mathbb{d}_{zb} + \langle zb | \mathbb{K}(\mathbb{I}+\mathbb{Q}_{zb}\mathbb{d}^{-1}\mathbb{K})^{-1} | zb \rangle} \times X_c^z \rho_{0b}^c + \text{c.c.} \tag{55}$$

z is now a unique state connected with b by the selection rule of A_+, and $\mathbb{d}_{zb}$ is necessarily of magnitude Ω. But since the zb-zb angular bracket represents a very large (divergent) collision frequency, the main fraction in (55) reduces in fact to 1. On substituting X_c^z from its definition (48), we thus arrive at a rather simple expression involving two distinct contributions,

$$\sigma_{yx} = \sigma_{yx}^{(0)} + \sigma_{yx}^{(0c)} , \tag{56}$$

with

$$\sigma_{yx}^{(0)} = (e^2/2m^*) A_{+z}^{\ b} A_{+\bar{b}}^{\ z} \rho_{0b}^{\bar{b}} / \omega_{zb} + \text{c.c.} \tag{57}$$

and

$$\sigma_{yx}^{(0c)} = \frac{ie^2 l}{\sqrt{2}m^*} \frac{A_{+z}^{\ b}}{\hbar\omega_{zb}} \frac{V_c^z k_{cb}}{\mathbb{d}_{cb}} \rho_{0b}^c + \text{c.c.} \tag{58}$$

A. Orbital motion

In the orbital component (57), the selection rules of the A_+'s impose diagonal density matrix elements, $\rho_{0b}^{\bar{b}}$. As in the ideal case [Eq. (49)], the product $A_{+z}^{\ b} A_{+\bar{b}}^{\ z}$ involves $(1-f_N)eB/h$ transitions from $N-1$ to N, and $f_N eB/h$ transitions from N to $N+1$, again leading to

$$A_{+z}^{\ b} A_{+\bar{b}}^{\ z} = [N(1-f_N)+(N+1)f_N]eB/h = (N+f_N)eB/h ,$$

independent of b. It follows that on summing over $c=b$, the upper and lower summations are simplified out in expression (44) of ρ_{0b}^c, and the ideal result (50) is just recovered,

$$\sigma_{yx}^{(0)} = (N+f_N)e^2/h = n_s e/B . \tag{59}$$

As expected, collisions do not influence the orbital Hall conductivity.

B. Center-of-orbit motion

We now come to the center-of-orbit contribution (58). We first stress noticing that because of the momentum transfer k_{cb}, and the presence of V_c^z associated with the applied field, it only exists insofar as ρ_0 has nondiagonal matrix elements, and by means of collision-assisted transitions. So, the *collisions now play a determinant role.* Consider in some detail the transitions arising in Eq. (58). Again the $A_{+z}^{\ b} V_c^z$ product splits up into single processes, schematically pictured in Fig. 2, for each of which b and c differ from each other by the only transition $Nk \rightarrow Nk'$, so that $k_{cb} = k' - k$, and

$$|c\rangle = c_{Nk'}^{\dagger} c_{Nk} |b\rangle . \tag{60}$$

Hence, from Eq. (3), the summation in the definition of A_+ runs over all occupied states of the N band in b, equivalent to multiplying by eBf_N/h. Further, the two electrons involved in the $(N-1 \rightarrow N)$ processes are interchanged in the final state, giving rise to a change of sign, according to Wick's theorem. The two sets can be combined term by term, giving

$$A_{+z}^{\ b} V_c^z \mathbb{d}_{cb}^{-1} k_{cb} \rho_{0b}^c = \frac{eBf_N}{h} \sum_{k'} \frac{\sqrt{N+1}\, V_{Nk'}^{N+1k} - \sqrt{N}\, V_{N-1k'}^{Nk}}{i\omega_{cb} + \epsilon_r} \times (k'-k)\rho_{0b}^c , \tag{61}$$

where Nk denotes any occupied state of the N band in b. This combination of interband matrix elements can be transformed into intraband one-electron processes, as shown in Appendix A. Using the result (A8) we find

$$A_{+z}^{\ b} V_c^z \mathbb{d}_{cb}^{-1} k_{cb} \rho_{0b}^c = -\frac{eBf_N}{h} \frac{l}{\sqrt{2}} \sum_{k'} \frac{V_{Nk'}^{Nk} (k'-k)^2}{i\omega_{cb} + \epsilon_r} \rho_{0b}^c . \tag{62}$$

The second term in the right-hand side of (A8) is omitted,

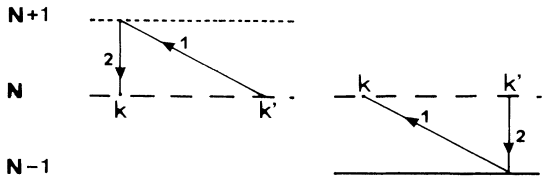

FIG. 2. Sketch of associated elementary transitions involved in Eq. (61). Conventions are those of Fig. 1.

because it yields a contribution of first order in $k'-k$ which will be removed in subsequent summations over k', k, and scatterer coordinates. Then, substituting from Eq. (62) into Eq. (58) with $\omega_{zb}=\Omega=eB/m^*$ gives

$$\sigma_{yx}^{(0)}=-\frac{e^2 f_N}{2h}\sum_{k'}\frac{V_{Nk'}^{Nk}l^2(k'-k)^2\rho_{0b}^c}{\hbar(\omega_{cb}-i\epsilon_r)}+\text{c.c.} \tag{63}$$

From Eq. (60), $\rho_{0b}^c=\langle b|c_{Nk}^\dagger c_{Nk'}\rho_0|b\rangle$, where ρ_0 is given by Eq. (44) up to first order in bandwidth. Once rewritten in its general form, expression (63) is easily seen to vanish, in zeroth order of the numerator

$$\sum_{cb}k_{cb}^2(V^{NN})_c^b\rho_{0b}^c/(\omega_{cb}-i\epsilon_r)+\text{c.c.}=0\ .$$

This is not surprising since intraband contributions brought about via (62) are expected to have no effect at zeroth order. The first-order term in Eq. (44) is then to be taken, thereby lifting the remaining bandwidth divergence in (63). The resulting value of $\sigma_{yx}^{(0c)}$ is as follows:

$$\sigma_{yx}^{(0c)}=-(f_N e^2/h)\sum_{bk'}\tfrac{1}{2}l^2(k'-k)^2\beta V_{Nk'}^{Nk}\langle b|c_{Nk}^\dagger c_{Nk'}\exp(-\beta V^{NN})|b\rangle\Big/\sum_s\langle s|\exp(-\beta V^{NN})|s\rangle\ . \tag{64}$$

Expression (64) will be further simplified in our assumption of independent electrons. The contribution of the electron under consideration, starting from the (N,k) state, can be separated out in both the upper and lower summations of the fraction, all others remaining in their initial configuration, say b', statistically unique. The related contribution in $\langle b'|\exp(-\beta V^{NN})|b'\rangle$ will be factored out and eliminated through simplification. We are thus left with one-electron terms, the nth order of which in the exponential expansion of the numerator can be more explicitly written as

$$\tfrac{1}{2}l^2(k-k')^2\beta V_{Nk'}^{Nk}\langle Nk'|(\beta V^{NN})^{n-1}|Nk\rangle/(n-1)!\ .$$

The subsequent summation over b in (64) now amounts to summing over all possibilities for k to be an occupied state, i.e., all states of the N band. It will be convenient to rewrite the denominator accordingly. In fact, given N,k, there are n identical terms of nth order, in $\langle Nk|(\beta V^{NN})^n|Nk\rangle/n!$ because any intermediate state in the product can be N,k, as well. Collecting them together we obtain a term-by-term correspondence between the numerator and the denominator which permits us to rewrite expression (64) in the following more symmetrical form:

$$\sigma_{yx}^{(0c)}=-(f_N e^2/h)\sum_{kk'}\tfrac{1}{2}l^2(k-k')^2V_{Nk'}^{Nk}\langle Nk'|\exp(-\beta V^{NN})|Nk\rangle\Big/\sum_k\langle Nk|V^{NN}\exp(-\beta V^{NN})|Nk\rangle\ . \tag{65}$$

The neglect of the zeroth-order-in-V terms which is implied in the denominator has no bearing on the result, because only large orders in the exponential expansions will be significant.

C. Effect of disorder and quantization

From Eq. (65) the center-of-orbit Hall conductivity turns out to be given by a Boltzmann-like average value of the squared momentum transfer over all collision events. The result is, therefore, mainly determined by the configurational disorder of scatterers. Even though more or less involved statistical methods could be implemented to determine the result, a derivation based on simple arguments is given in Appendix D. The average value of $\frac{1}{2}l^2(k-k')^2$ is shown to reduce to 1, owing to the following relationship between terms of the same order, say, $q+1$, in the upper and lower expansions:

$$\frac{\sum_{k'}\tfrac{1}{2}l^2(k-k')^2V_{Nk'}^{Nk}\langle Nk'|(V^{NN})^q|Nk\rangle}{\langle Nk|(V^{NN})^{q+1}|Nk\rangle}=1\ .$$

The physics underlying this property is simple. Due to the fact that $\beta|V|$ may take values much larger than 1 at low temperature, the expansion of $\exp(-\beta V^{NN})$ is dominated by terms of very high order, involving an extremely large number of collisions. This causes the effect of scattering to remove any correlation between k and k' in very large sequences such as $V_{Nk_{q-1}}^{Nk'}\cdots V_{Nk}^{Nk_1}$, if $|k'-k|$ is much smaller than the sum of all (reciprocal) scattering ranges therein. The average of $(k-k')^2$ is therefore strongly dependent on the terminal transition $V_{Nk'}^{Nk}$. Shape peculiarities of the V's are smoothed out by the configurational disorder so that the result is, in fact, mainly determined by the Gaussian localization over the length $l/\sqrt{2}$ specific to Landau's states, thus leading to the scaling value $2/l^2$.

As a result, Eq. (65) becomes

$$\sigma_{yx}^{(0)}=-f_N e^2/h\ ,$$

which, upon combination with (59) in Eq. (56), yields the quantized plateaus of the Hall conductivity

$$\sigma_{yx}=(N+f_N)e^2/h-f_N e^2/h=Ne^2/h\ .$$

The behavior of σ_{yx} in the intermediate region will be qualitatively discussed in the next section, in connection with the diagonal conductivity.

VI. DIAGONAL CONDUCTIVITY

The diagonal conductivity is calculated from Eq. (47) with the choice of iA_- instead of $-A_+$. The analysis at the beginning of Sec. V still holds but in the bracket on the rhs of (53), the terms in u_y now add up, whereas those in q_x cancel. The behavior of the resulting expression of

σ_{xx} turns out to be similar to the ideal case, every fraction in Eq. (47) taking on a purely imaginary value in the limit of infinite relaxation rate, thus leading to zero on adding up the two conjugate quantities.

To show this we first notice that in Eq. (47) each fraction is determined by the states b, c, and z. Due to the disorder of the x_I's, the $V_{Nk}^{N'k'}$ matrix elements are very loosely dependent on $k+k'$ involved in $k+k'-2x_I/l^2$ (see Appendix A) and are, therefore, very nearly a function of $k'-k$ only. Further $\mathbb{d}_{zb}$ can be ignored in comparison with the divergent relaxation bracket in zb. Then given the initial state b, it is easy to see that if the overall factor i is discarded, the fractions (b,c,z) and (b,c',z') for which c and c', z and z' belong to the same band with symmetrical momentum with respect to b ($k_{c'b}=-k_{cb}$, $k_{z'b}=-k_{zb}$), are always conjugate to each other. It follows that all fractions can be associated by pairs so as to get a real quantity, or a purely imaginary one on restoring the i factor, leading indeed to zero in summing with the c.c.

This property essentially rests on the neglect of $\mathbb{d}_{zb}$ which, unlike the terms in angle brackets in Eq. (47), remains unchanged in passing from (b,c,z) to (b,c',z')($\mathbb{d}_{z'b}=\mathbb{d}_{zb}=i\Omega$). Thus the vanishing of the diagonal conductivity results from the neglect of the cyclotron frequency relative to divergent relaxation brackets. This is certainly true inside the magnetic-field range defined in (27), where the density of states of the relevant final states is very large. Instead, in the intermediate range additional transitions occur between N and $N-1$, or N and $N+1$, as already considered in the ideal case (see Sec. IV A). The related density of states is overwhelmingly reduced in a ratio up to the order of $\exp(-\hbar\Omega/2k_BT)$, owing to the exclusion principle. The relaxation rate is reduced accordingly and the cyclotron frequency (i.e., the $\mathbb{d}_{zb}$'s) may no longer be negligible, so that for these transitions the two conjugate quantities in σ_{xx} now combine into a real and finite result. Of course, the same arguments invalidate the demonstration of Sec. V B leading to Eq. (65), showing therefore that the Hall conductivity also takes unquantized values in the intermediate range.

In summary, one can say that along the quantized plateaus of the Hall conductivity, *the diagonal conductivity is zero essentially because of the infinite relaxation rate* in the unfilled Landau level. This is the very argument which is responsible for the plateaus, at once. The complete quantitative analysis of both conductivities in the intermediate range should require the calculation of the effective density of states as a function of filling. This should lead to some threshold value for the departure from quantization, depending on both the magnetic field and the density of scatterers.

VII. CONCLUSION

Thanks to its ability to overcome quasi-infinite densities of states, the determinantal method provides us with new means of investigation into the theory of the quantum Hall effect. In the standard approach, the density of states in the collision-broadened Landau spectrum is calculated using the widespread self-consistent Born approximation.[27] In that technique, the approximation made in using the lowest order is, to some extent, counterbalanced by the requirement of self-consistency. This, however, seems questionable on account of the drastic role played by high collision orders, as stressed above. In addition, even if the broadened spectrum is known, the problem remains of calculating the related wave functions of the localized-extended states, in order to proceed further.

Instead, the present method mainly consists in calculating the statistical average of the current density in the *unperturbed* basis of Landau's eigenstates. In the limit of vanishing bandwidth which renders infinite the relaxation parameters in the unfilled band, finite limiting values have been extracted for the contributions of the orbital motion and the center-of-orbit migration to the Hall conductivity. The vanishing of the diagonal conductivity is found to be strongly correlated to the occurrence of the Hall plateaus, in accordance with experiment. The point is that the knowledge of the eigenstates of the collision-perturbed Hamiltonian is no longer required.

Even though the treatment looks completely new, it finally amounts to a change of basis with regard to previous (mainly qualitative) attempts and so there is no essential reason for the underlying physical ground to be inconsistent with existing ideas. The decisive contribution of configurational disorder is emphasized in some places, particularly in the calculation of the center-of-orbit conductivity. The broadening effect of collisions is embodied in the initial matrix density (Sec. III). Also, it would be quite attractive to clear up the connection between the present treatment based on Landau's eigenstates of divergent densities of states, with the localization picture based on broadened eigenstates. Very large relaxation parameters in the unfilled band, which are responsible for quantum effects, can be thought of, in some sense, as the equivalent of the localization resulting from suitable combinations of Landau's eigenstates. It is also worth emphasizing that the emergence of these singular parameters is strongly dependent on using the Landau-state basis. In fact, they behave more like intermediate mathematical entities than realistic relaxation frequencies, since they are finally deleted from the result. In particular, there is no discrepancy with the finite relaxation frequency associated with collision-broadened levels. Only some formal question might arise about whether or not such quasi-infinite transition rates are consistent with the time-energy uncertainty relationship, since the $\Delta E\Delta t$ product goes seemingly to zero. The answer is that in the case of transitions toward a very narrow band containing a large number of states, the relevant transition time entering the uncertainty relationship is no longer connected with the relaxation rate but instead, with the very inverse of the bandwidth. This could be shown in the framework of the determinantal theory as well.

Possibly, disorder effects could be dealt with in a more rigorous way, with the help of scaling considerations. Because of the physical simplicity stressed above, we are not sure to gain much more insight regarding the emergence of the Hall plateaus and the vanishing of the diagonal conductivity. Instead, more refined statistical

methods would likely be quite useful for a quantitative treatment of the plateau widths and the finite diagonal conductivity in the intermediate ranges, where we have to take into account the strong reduction of the density of states as a function of filling in the N band, together with the contribution of additional interband transitions.

We have restricted ourselves to the case of independent electrons. Including the effect of electron-electron interactions, e.g., in the form of Wigner crystallization, would likely permit the extension of the theory to the fractional quantum Hall effect.

In summary, even if improvements and developments are still required for a complete quantitative treatment and comparison with experiment, the reliability of the determinantal theory seems, now and henceforth, well established to warrant further investigations. Perhaps the overall simplicity of the present formalism, using elementary mathematics, is the most attractive feature associated with the use of the Landau-state basis.

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APPENDIX A: MATRIX ELEMENTS OF THE SCATTERING POTENTIAL

We shall consider for definiteness the collision potential

$$V(r)=\sum_I v(\mathbf{r}-\mathbf{r}_I)\ , \tag{A1}$$

where the sum runs over random scatterers located at the $\mathbf{r}_I$'s. Matrix elements of $V(\mathbf{r})$ between Landau states are currently expressed in terms of the Fourier transform

$$V(r)=\frac{L_xL_y}{(2\pi)^2}\int\left[\sum_I \exp(-i\mathbf{q}\cdot\mathbf{r}_I)v(\mathbf{q})\right]\exp(i\mathbf{q}\cdot\mathbf{r})d^2\mathbf{q}\ ,$$

with

$$v(\mathbf{q})=\frac{1}{L_xL_y}\int_{L_xL_y} v(\mathbf{r})\exp(-i\mathbf{q}\cdot\mathbf{r})d^2\mathbf{r}\ .$$

Using the well-known eigenfunctions of Landau states,[28]

$$\langle \mathbf{r}|Nk\rangle=(2^N N!\sqrt{\pi}lL_y)^{-1/2}\exp[-\tfrac{1}{2}(x/l+lk)^2]\times H_N(x/l+lk)\exp(iky)\ , \tag{A2}$$

where H_N is the Hermite polynomial of order N, we obtain

$$V_{Nk}^{N'k'}=\frac{L_x}{2\pi}\sum_I\int_{-\infty}^{+\infty}v(q_x,u_y)J_{N'N}(k',q_x,k)\times\exp[-(iq_xx_I+u_yy_I)]dq_x \tag{A3}$$

with $u_y=k'-k$. $J_{N'N}$ denotes the matrix elements of the plane wave $\exp(iq_xx)$ between normalized Hermite functions φ_{Nk} [the factor of $\exp(iky)$ in (A2)]:

$$J_{N'N}=\langle\varphi_{N'k'}|\exp(iq_xx)|\varphi_{Nk}\rangle=i^{(N'+N)}\exp[-\tfrac{1}{2}\chi^2+\tfrac{1}{2}il^2q_x(k'+k)]\begin{cases}(-)^N[l(q_x-iu_y)/\sqrt{2}]^{(N'-N)}\sqrt{N!/N'!}L_N^{(N'-N)}(\frac{1}{2}\chi^2) & (N'\geq N)\\ (-)^{N'}[l(q_x+iu_y)/\sqrt{2}]^{(N-N')}\sqrt{N'!/N!}L_{N'}^{(N-N')}(\frac{1}{2}\chi^2) & (N'\leq N)\ ,\end{cases} \tag{A4}$$

where $\chi=(q_x^2+u_y^2)^{1/2}l/\sqrt{2}$, and $L_N^{(Q)}$ denotes the Laguerre polynomial of order N and parameter Q. In the case of intraband transitions ($N'=N$), the matrix elements (A3) become

$$V_{Nk}^{Nk'}=\frac{L_x}{2\pi}\sum_I\int_{-\infty}^{+\infty}v(q_x,u_y)\exp\{-\tfrac{1}{2}\chi^2+iq_x[\tfrac{1}{2}l^2(k'+k)-x_I]-iu_yy_I\}L_N^{(0)}(\tfrac{1}{2}\chi^2)dq_x\ . \tag{A5}$$

For transitions between nearest bands, we have

$$V_{Nk}^{N+1k'}=i\frac{L_x}{2\pi}\sum_I\int_{-\infty}^{+\infty}v(q_x,u_y)\exp\{-\tfrac{1}{2}\chi^2+iq_x[\tfrac{1}{2}l^2(k'+k)-x_I]-iu_yy_I\}\frac{l(q_x-iu_y)L_N^{(1)}(\frac{1}{2}\chi^2)}{\sqrt{2(N+1)}}dq_x\ , \tag{A6}$$

$$V_{N+1k}^{Nk'}=i\frac{L_x}{2\pi}\sum_I\int_{-\infty}^{+\infty}v(q_x,u_y)\exp\{-\tfrac{1}{2}\chi^2+iq_x[\tfrac{1}{2}l^2(k'+k)-x_I]-iu_yy_I\}\frac{l(q_x+iu_y)L_N^{(1)}(\frac{1}{2}\chi^2)}{\sqrt{2(N+1)}}dq_x\ . \tag{A7}$$

Hence, using Eq. (A6) twice together with the recurrence formula[29]

$$L_N^{(1)}(x)-L_{N-1}^{(1)}(x)=L_N^{(0)}(x)\ ,$$

the combination arising in Eq. (61) becomes

$$\sqrt{N+1}V_{Nk'}^{N+1k}-\sqrt{N}V_{N-1k'}^{Nk}=-lu_yV_{Nk'}^{Nk}/\sqrt{2}+i\frac{L_xl}{2\pi\sqrt{2}}\sum_I\int_{-\infty}^{+\infty}v(q_x,u_y)\exp\{-\tfrac{1}{2}\chi^2+iq_x[\tfrac{1}{2}l^2(k'+k)-x_I]+iu_yy_I\}\times L_N^{(0)}(\tfrac{1}{2}\chi^2)q_xdq_x\ . \tag{A8}$$

APPENDIX B: LANDAU-LEVEL BROADENING IN A PERIODIC POTENTIAL

The preceding expressions of the scattering potential matrix elements can also be used to set up the secular equation leading to the broadening of Landau states by the crystal periodic potential. In that case $U(\mathbf{r})$ is first written in terms of its Fourier series as

$$U(\mathbf{r})=\sum_{K} U(\mathbf{K})\exp(-i\mathbf{K}\cdot\mathbf{r}) ,$$

where the $\mathbf{K}$'s represent the fundamental vectors of the two-dimensional reciprocal lattice. The perfect crystal Hamiltonian now connects any $|N,k\rangle$ to all $|N,k+K_y\rangle$'s, so that the general state becomes

$$|\psi\rangle=\sum_{NK_y} \alpha_{NK_y}|N,k+K_y\rangle ,$$

with the α_{NK_y}'s being the solution of the secular equation

$$(\varepsilon_N-\varepsilon)\alpha_{NK_y}+\sum_{N'K_y'}\langle N,k+K_y|U(\mathbf{r})|N',k+K_y'\rangle\alpha_{N'K_y'}=0 . \tag{B1}$$

It will be sufficient for the present purpose to restrict ourselves to intraband contributions. The U-matrix elements between $N,k+K_y$ and $N,k+K_y'$ are readily found using Eq. (A5) to be

$$\langle N,k+K_y'|U(\mathbf{r})|N,k+K_y\rangle=\sum_{K_x'K_x} U(\mathbf{K}-\mathbf{K}')\exp[-\tfrac{1}{2}l^2(\mathbf{K}-\mathbf{K}')^2+\tfrac{1}{2}il^2(K_x-K_x')(2k+K_y+K_y')]$$
$$\times L_N^{(0)}[\tfrac{1}{2}l^2(\mathbf{K}-\mathbf{K}')^2] .$$

Upon the Rauh substitution[20]

$$\alpha_{NK_y}=\exp[\tfrac{1}{2}il^2(K_x-K_x')(k+K_y)]\beta_{NK} , \tag{B2}$$

Eq. (B1) becomes

$$(\varepsilon_N-\varepsilon)\beta_{NK}+\sum_{K_y'} U(\mathbf{K}-\mathbf{K}')\exp[-\tfrac{1}{2}l^2(\mathbf{K}-\mathbf{K}')^2]$$
$$\times L_N^{(0)}[\tfrac{1}{2}l^2(\mathbf{K}-\mathbf{K}')^2]\beta_{NK'}=0 .$$

The k dependence is washed out, showing that the related degeneracy of the N level is not lifted by the periodic potential, as stated by Rauh with a similar equation.[20] The bandwidth is the same for any k, and of very small magnitude. An upper limit can be found in the literature.[16,17,20] Difficulties occurring for particular values and orientations of the magnetic field were discussed by Wannier and co-workers who extended the above conclusions to almost all cases.[19,21,22]

APPENDIX C: LIFTING OF THE FINAL-STATE DIVERGENCE IN THE DENSITY MATRIX

The $\mathbb{d}_{cb}^{-1}$ divergence appearing in Eq. (19) is due to the degeneracy of the relevant final states. This is similar to the well-known zero-frequency divergence in the dc conductivity, already considered elsewhere and properly lifted.[14(d)]

The method consists of extracting, first, all terms in $\mathbb{d}_{cb}^{-1}$ from the determinant $\mathbb{D}_{zb}^{cb}$, in Eq. (18), taken with $ml=cb$, through expansion by the cb row, before performing the division by the diagonal minor $\mathbb{D}_{zb}^{zb}$. The divergence is lifted on combining these terms with the similar ones extracted from the denominator. But since we also have to extract terms in $\mathbb{d}_{zb}^{-1}$, both operations can be accumulated from the beginning by taking the second-order minor $\mathbb{D}_{zbcb}^{zbcb}$ instead of $\mathbb{D}_{zb}^{zb}$, which gives

$$\mathbb{X}^c{}_b=\frac{\mathbb{D}_{zb}^{cb}(\mathbb{D}_{zbcb}^{zbcb})^{-1}}{\mathbb{d}_{zb}\mathbb{D}(\mathbb{D}_{zbcb}^{zbcb})^{-1}}K_{Fc}^z+p^*(cb)$$

$$=\frac{\mathbb{D}_{zb}^{cb}(\mathbb{D}_{zbcb}^{zbcb})^{-1}}{\mathbb{d}_{zb}\mathbb{D}(\mathbb{D}_{zb}^{zb})^{-1}\mathbb{D}_{zb}^{zb}(\mathbb{D}_{zbcb}^{zbcb})^{-1}}K_{Fc}^z+p^*(cb) .$$

On carrying out the determinant divisions, we obtain

$$\mathbb{X}^c{}_b=-\frac{\langle cb|\mathbb{K}(\mathbb{I}+\mathbb{Q}_{zb}\mathbb{Q}_{cb}\mathbb{d}^{-1}\mathbb{K})^{-1}|zb\rangle K_{Fc}^z}{[\mathbb{d}_{zb}+\langle zb|\mathbb{K}(\mathbb{I}+\mathbb{Q}_{zb}\mathbb{d}^{-1}\mathbb{K})^{-1}|zb\rangle][\mathbb{d}_{cb}+\langle cb|\mathbb{K}(\mathbb{I}+\mathbb{Q}_{zb}\mathbb{Q}_{cb}\mathbb{d}^{-1}\mathbb{K})^{-1}|cb\rangle]}+p^*(cb) .$$

In a compact form, this reads

$$\mathbb{X}^c{}_b\cong-\frac{\Gamma_{zb/cb}K_{Fc}^z}{(\mathbb{d}_{zb}+\Gamma_{zb})(\mathbb{d}_{cb}+\Gamma_{cb})} .$$

The vanishing of $\mathbb{X}^c{}_b$ in the zero bandwidth limit is due to the divergence of the relaxation parameter Γ's, all close to one another.

APPENDIX D: STATISTICAL AVERAGE OF $(k'-k)^2$

The calculation of the average value of the squared momentum transfer by the Boltzmann factor, as defined

in Eq. (65), reduces to the simpler average value by any power $q+1$ of the intraband collision potential, namely,

$$\overline{(k-k')^2}=\frac{\sum_{k'}(k-k')^2 V_{Nk'}^{Nk}\langle Nk'|(V^{NN})^q|Nk\rangle}{\sum_{k'}V_{Nk'}^{Nk}\langle Nk'|(V^{NN})^q|Nk\rangle}\ . \tag{D1}$$

The reason shown hereafter is that this latter quantity does not depend on the order $q+1$. We simply notice at the moment that, due to values significantly larger than 1 of the ratio V^{NN}/k_BT at low temperature, the predominant orders of the exponential expansions in Eq. (65) are expected to be very high. We thus have to investigate the behavior of diagonal products of large order of the type

$$V_{Nk'}^{Nk}(V_{Nk_{q-1}}^{Nk'}\cdots V_{Nk_j}^{Nk_{j+1}}\cdots V_{Nk_1}^{Nk_2}V_{Nk}^{Nk_1})\ . \tag{D2}$$

The $Nk-Nk'$ element is singled out for later convenience. From Eq. (A5) the current matrix element $V_{Nk_j}^{Nk_{j+1}}$ is given by

$$V_{Nk_j}^{Nk_{j+1}}=\frac{L_x}{2\pi}\sum_J\int_{-\infty}^{+\infty}v_J(q_x,u_{jy})\times\exp[-\tfrac{1}{2}\chi_j^2+iq_x(\zeta_j-x_J)-iu_{jy}y_J]L_N^{(0)}(\tfrac{1}{2}\chi_j^2)dq_x\ , \tag{D3}$$

with $\zeta_j=k_{j+1}+k_j$, $u_{jy}=k_{j+1}-k_j$, $\chi_j^2=q_x^2+u_{jy}^2$, and $l/\sqrt{2}$ used as the unit of length. The scatterer ordinates y_J come into the diagonal products (D2) through the sums $\sum_j u_{jy}y_J$ for different choices of y_J. Due to the evident relationship $\sum_j u_{jy}=0$, all y_J's can be referred to one of them without change, e.g., the y_I involved in the terminal $k'\to k$ transition on the left. This of course is a consequence of the translational invariance of the system in the y direction. Moreover, even though the variables ζ_j and u_{jy} are not independent, strictly speaking, they can be regarded as quasifree on account of the very large number of transitions involved in (D2).

In Eq. (D3) the integrand extends over l^{-1} at most, because of the Gaussian factor $\exp(-\frac{1}{2}\chi_j^2)$. On the other hand, current scattering potentials have a range of the same order or smaller than the cyclotron radius l (of the order of a few hundred Å). For example, screening radii fall noticeably below l in partly filled Landau levels.[14(e),30,31] We shall simply assume, hereafter, the potential $v_J(q_x,u_y)$ of the scatterer at $\mathbf{r}_J$ to be sufficiently well represented, in that range, by a Taylor expansion in even powers of q_x,u_y,

$$v_J(q_x,u_{jy})=\sum_{r,s=0}^{\infty}v_{Jrs}q_x^{2r}u_{jy}^{2s}\ . \tag{D4}$$

Consistently, the variables q_x,u_{jy} can be separated in the Laguerre polynomials $L_N^{(0)}(\frac{1}{2}\chi_j^2)$, by making use of the addition theorem[29]

$$L_N^{(0)}[\tfrac{1}{2}(q_x^2+u_{jy}^2)]=\exp(\tfrac{1}{2}u_{jy}^2)\times\sum_{P=0}^{\infty}\frac{(-)^P}{P!}(\tfrac{1}{2}u_{jy}^2)^P L_N^{(P)}(\tfrac{1}{2}q_x^2)\ . \tag{D5}$$

From Eqs. (D4) and (D5), the matrix elements (D3) then become

$$V_{Nk_j}^{Nk_{j+1}}=\frac{L_x}{2\pi}\sum_J\sum_{P=0}^{\infty}\frac{(-)^P}{P!}(\tfrac{1}{2}u_{jy}^2)^P\exp(-iu_{jy}y_{JI})\sum_{r,s=0}^{\infty}v_{Jrs}u_{jy}^{2s}\int_{-\infty}^{+\infty}q_x^{2r}\exp(-\tfrac{1}{2}q_x^2)L_N^{(P)}(\tfrac{1}{2}q_x^2)\cos[q_x(\zeta_j-x_J)]dq_x \tag{D6}$$

($y_{JI}=y_J-y_I$). Let us carry out the integration over q_x for the current term in q_x^m of $L_N^{(P)}(\frac{1}{2}q_x^2)$. We obtain

$$\int_{-\infty}^{+\infty}q_x^{2(r+m)}\exp(-\tfrac{1}{2}q_x^2)\cos[q_x(\zeta_j-x_J)]dq_x=(-\tfrac{1}{2})^{r+m}\sqrt{2\pi}\exp[-\tfrac{1}{2}(\zeta_j-x_J)^2]H_{2(r+m)}[(\zeta_j-x_J)/\sqrt{2}]\ , \tag{D7}$$

where $H_{2(r+m)}$ is the Hermite polynomial of order $2(r+m)$. Except for $r=m=0$, this expression represents an oscillating function of ζ_j around the abscissa x_J of the scatterers, damped over a range of order l^{-1}. Due to randomness, these incoherent oscillations cancel out on summing over all scattering events in products (D2) of very large order. This property can be given a more quantitative proof by taking into account that the dependence on the x_J's of any $V_{Nk_j}^{Nk_{j+1}}$ in (D2) is completely uncorrelated to that of the product of all other factors. Then, assuming a suitable average value of the v_J's, the summation over the x_J's taken with equal statistical weight gives indeed zero for $r+m\neq 0$, because of the orthogonality relationship[29]

$$\int_{-\infty}^{+\infty}\exp[-\tfrac{1}{2}(\zeta_j-x_J)^2]H_{2(r+m)}\times[(\zeta_j-x_J)/\sqrt{2}]dx_J=\sqrt{2\pi}\delta_{0,r+m}\ . \tag{D8}$$

One could consider, equivalently, that all values of x_J are of equal probability for any given scatterer, and the same result would still be obtained through integration by the quasifree variables ζ_j in Eq. (D7). Besides these complementary arguments, it must be borne in mind that it will be sufficient for the present purpose that the vanishing of the $r+m\neq 0$ contributions occurs in any product (D2) as a whole.

It follows that nonoscillating contributions in $r=m=0$ will be overwhelmingly predominant. They are obtained by taking $q_x=0$ in Eqs. (D4) and (D5). Using the corre-

sponding value $\sqrt{2\pi}$ of the integral (D8), Eq. (D6) then gives

$$V_{Nk_j}^{Nk_{j+1}}=\frac{L_x}{\sqrt{2\pi}}\sum_J \exp[-\tfrac{1}{2}(\xi_j-x_J)^2] \times\sum_{s=0}^{\infty} v_{J0s}u_{jy}^{2s}\exp(-\tfrac{1}{2}u_{jy}^2) \times\cos(u_{jy}y_{JI})L_N^{(0)}(\tfrac{1}{2}u_{jy}^2)\ . \quad \text{(D9)}$$

Of course, similar arguments are expected to work in the y direction, as well. Through integration by u_{jy} for the current term in u_{jy}^m of the Laguerre polynomial, in Eq. (D9), we obtain

$$\int_{-\infty}^{+\infty} u_{jy}^{(s+m)}\exp(-\tfrac{1}{2}u_{jy}^2)\cos(u_{jy}y_{JI})du_{jy} =(-\tfrac{1}{2})^{s+m}\sqrt{2\pi}\exp(-\tfrac{1}{2}y_{JI}^2)H_{2(s+m)}(y_{JI}/\sqrt{2})\ .$$

For $s+m\neq 0$ we now have a damped oscillatory function of y_{JI} which will again lead to cancellation in summing over scatterers, making thereby the $s=m=0$ contributions strongly predominant.

The above properties can now be applied to the terminal matrix element $V_{Nk'}^{Nk}$ in the sequence (D2), which becomes, from Eq. (D9),

$$V_{Nk'}^{Nk}=\frac{L_x}{\sqrt{2\pi}}\sum_I \exp[-\tfrac{1}{2}(\xi-x_I)^2]\exp(-\tfrac{1}{2}u^2)\ , \quad \text{(D10)}$$

with the variables $\xi=k+k'$ and $u=k-k'$ still being very nearly free. At this stage, it will be of interest to get some more insight into that point. The number $q+1$ of collisions involved in (D2) being very large, k and k' are completely uncorrelated in the bracketed product, at least if $|k'-k|$ is much smaller than the sum of all q potential ranges (in the reciprocal space). In other words, given k, the term in brackets is independent of k', especially if $|k'-k|$ is of the order of one range only, as imposed by $V_{Nk'}^{Nk}$ on the left. An additional argument is provided by noticing that the derivation of the term in brackets with respect to k' introduces odd dependences with respect to intermediate k's, leading to zero in subsequent summations. This was already invoked in Eq. (54) and is certainly true insofar as the intermediate summations on the k's are free enough, which again amounts to assuming $|k'-k|$ to be much smaller than the overall range in the brackets.

As a result, one can write from Eq. (D10) the following equation relative to each terminal scatterer I:

$$\frac{\int\int_{-\infty}^{+\infty} u^2\exp[-\tfrac{1}{2}(\xi-x_I)^2]\exp(-\tfrac{1}{2}u^2)\langle Nk'|(V^{NN})^q|Nk\rangle d\xi\,du}{\int\int_{-\infty}^{+\infty}\exp[-\tfrac{1}{2}(\xi-x_I)^2]\exp(-\tfrac{1}{2}u^2)\langle Nk'|(V^{NN})^q|Nk\rangle d\xi\,du}=\frac{\int_{-\infty}^{+\infty}u^2\exp(-\tfrac{1}{2}u^2)du}{\int_{-\infty}^{+\infty}\exp(-\tfrac{1}{2}u^2)du}=1\ .$$

The angular brackets and the integral by ξ have been simplified out. Using this property for every scatterer in $V_{Nk'}^{Nk}$, we straightforwardly obtain

$$\overline{(k-k')^2}=\frac{\sum_{k'}(k-k')^2V_{Nk'}^{Nk}\langle Nk'|(V^{NN})^q|Nk\rangle}{\sum_{k'}V_{Nk'}^{Nk}\langle Nk'|(V^{NN})^q|Nk\rangle}=1\ . \quad \text{(D11)}$$

Obviously, the same conclusion holds for the Boltzmann-like mean value defined in Eq. (65).

We shall make two final remarks. First, the oscillating contributions that have been neglected need not be very small by themselves alone but only in comparison with nonoscillating ones which add up coherently. This is due to the fractional form (D1) of the average which is to be calculated, and explains the very great accuracy which can be reached. Second, no severe restriction is made on the range of scatterers since the above arguments are, in principle, valid at any order of the Taylor expansion (D4). In the case of very long range where the method may be questionable, the Fourier transform $v(q_x,u_y)$ approaches a δ function and it is easy to see that the corresponding average of $(k-k')^2$ then tends to zero. This means that very long-range scatterers do not contribute significantly, in complete agreement with the localization description. Ando has shown that short-range potentials are indeed dominant in determining localization.[4(b)]

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