Anholonomy of a moving space curve and applications to classical magnetic chains

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The subject of space curves finds many applicatons in physics such as optical fibers, magnetic spin chains, and vortex filaments in a fluid. We show that the time evolution of a space curve is associated with a geometric phase. Using the concept of Fermi-Walker parallel transport, we show that this phase arises because of the path dependence of the rotation of the natural Frenet-Serret triad as one moves along the curve. We employ Lamb's formalism for space-curve dynamics to derive an expression for the anholonomy density and the geometric phase for a general time evolution. This anholonomy manifests itself as a gauge potential with a monopolelike structure in the space of the tangent vector to the space curve. Our classical approach is amenable to a quantum generalization, which can prove useful in applications. We study the application of our constructive formalism to ferromagnetic and antiferromagnetic (classical) spin chains by first presenting certain classes of exact, physically interesting solutions to these nonlinear dynamical systems and then computing the corresponding geometric phases.

I. INTRODUCTION

Berry[1] has shown that when a quantum system in an eigenstate is evolved by adiabatically varying the parameters in its Hamiltonian around a circuit, it acquires a geometrical phase factor (or anholonomy[2]) in addition to the familiar dynamical one. Subsequently, Aharanov and Anandan[3] have generalized this concept by removing the assumption of adiabaticity and further defining the geometric phase as one associated with a closed circuit in the projective Hilbert space corresponding to the evolution of any normalized state, not necessarily the eigenstate. Thus, the underlying parameter space plays no fundamental role in their discussion, although it can be regarded as a special case when the closed circuit arises from adiabatic evolution of parameters. More recently, a further generalization to noncyclic evolution in the projective Hilbert space has been given by Samuel and Bhandari[4] who show that the natural way to obtain the phase corresponding to an open circuit in this space is to "close" the evolution with a geodesic.

The phenomenon of a geometric phase can arise in a purely classical context,[5] without appealing to quantum-mechanical concepts. Using a single-mode twisted optical fiber, Tomita and Chiao[6] have shown at a classical level that the measured angle of rotation of the linearly polarized light in the fiber is a measure of Berry's phase. Kugler and Shtrikman[7] have observed that the geometric features of this effect can be discussed by considering the optical fiber to be a (static) space curve. The experimental observation can be understood in terms of the parallel transport (along the fiber) of the unit tangent vector characterizing the curve. Geometrical properties of the space curve and its spherical images have been discussed by Dandoloff and Zakrzewski.[8]

The subject of space curves[9] finds other applications in physics such as the description of a vortex filament in a fluid,[10] spin configurations in a magnetic chain,[11] etc. It is therefore of interest to study the geometric phase associated with moving space curves. A brief account of this space-curve formalism has been recently published by us.[12]

In Sec. II, we use space-curve evolution and parallel transport to obtain an expression for the associated anholonomy density. In Sec. III, using Lamb's[13] formalism we derive an expression for the geometric phase. This approach is constructive and applies to a general evolution. Cyclic and adiabatic evolutions in the space of the tangent vector of the space curve can be studied as special cases. The possibility of a quantum analog of the classical result obtained is discussed. In Sec. IV, we derive an expression for the gauge potential and show that it displays a monopolelike character in the tangent vector space. In Sec. V, nonlinear spin-wave solutions and soliton solutions for a ferromagnetic chain are presented and the corresponding anholonomy densities and geometric phases are determined. In Sec. VI, we derive a class of moving domain-wall solutions for the low-energy configurations of an isotropic antiferromagnetic chain and compute the corresponding geometric phase. Instanton solutions are also derived and the phase is shown to be $4\pi n$, where n is an integer. A brief summary is given in Sec. VII.

II. SPACE CURVES, PARALLEL TRANSPORT, AND ANHOLONOMY

A space curve is described either by its parameter equations or by its natural equations: $\kappa=\kappa(s)$ and $\tau=\tau(s)$, where κ, τ, and s are the curvature, the torsion, and the length (treated as the natural parameter) of the space curve. Let us consider a curve γ which, in its parametric form, is described by $\mathbf{r}=\mathbf{r}(s)$ and denote by $\mathbf{t}$ the unit tangent vector to this curve and by $\mathbf{n}$ and $\mathbf{b}$ its principal normal and binormal, respectively. Then $\mathbf{t}$, $\mathbf{n}$, and $\mathbf{b}$ form a moving triad of the curve. They are related by the Frenet-Serret equations:

$$\begin{aligned} \mathbf{t}_s &= \kappa\mathbf{n} , \\ \mathbf{n}_s &= -\kappa\mathbf{t}+\tau\mathbf{b} , \\ \mathbf{b}_s &= -\tau\mathbf{n} , \end{aligned} \tag{2.1}$$

where the subscript s denotes d/ds and κ and τ are given by $\kappa^2=\mathbf{t}_s\cdot\mathbf{t}_s$ and $\tau=\mathbf{t}\cdot(\mathbf{t}_s\times\mathbf{t}_{ss})/\kappa^2$. If we introduce the Darboux vector $\boldsymbol{\xi}=\tau\mathbf{t}+\kappa\mathbf{b}$, the Frenet-Serret equations can be rewritten as follows:

$$\begin{aligned} \mathbf{t}_s &= \boldsymbol{\xi}\times\mathbf{t} , \\ \mathbf{n}_s &= \boldsymbol{\xi}\times\mathbf{n} , \\ \mathbf{b}_s &= \boldsymbol{\xi}\times\mathbf{b} . \end{aligned} \tag{2.2}$$

Thus, $\boldsymbol{\xi}$ plays the role of an angular velocity of the Frenet-Serret triad and has components along the tangent vector $\mathbf{t}$ and the binormal vector $\mathbf{b}$. Let us consider the plane perpendicular to the tangent $\mathbf{t}$, which moves along the curve γ with a constant unit velocity. The vectors $\mathbf{n}$ and $\mathbf{b}$ span this plane. The natural frame ($\mathbf{n}$ and $\mathbf{b}$) rotates around $\mathbf{t}$ with an angular velocity $\tau(s)$. As s increases from s_0 to s_1 the system develops a phase[7,8] $\Phi_1=\int_{s_0}^{s_1}\tau(s)ds$ between $\mathbf{n}$, $\mathbf{b}$, and the corresponding nonrotating frame in this plane. Such a nonrotating frame may be defined by using the usual Fermi-Walker parallel transport along the curve γ (2.2):

$$\frac{DA^i}{ds}=\kappa A^k(t^k n^i - t^i n^k)=\{\kappa\mathbf{b}\times\mathbf{A}\}^i . \tag{2.3}$$

Such a phase Φ_1 appears in the process of propagating light along a twisted waveguide and also in the case of an isolated spin in a constant magnetic field $\mathbf{B}$, where the spin vector plays the role of a tangent $\mathbf{t}$ to a space curve (2.3). The question arises what would happen if the space curve has one more degree of freedom, as, for example, if the twist in the waveguide is time dependent, or if the isolated spin is replaced by a linear chain of interacting spins. This is our main motivation for studying the Fermi-Walker parallel transport for a *moving* space curve.

As time evolves κ and τ, and thus $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$, are, in general, functions of both s and time u, i.e., $\kappa=\kappa(s,u)$, $\tau=\tau(s,u)$. For a fixed s we have

$$\kappa_0^2(u)=\mathbf{t}_u\cdot\mathbf{t}_u$$

and

$$\tau_0(u)=\frac{\mathbf{t}\cdot(\mathbf{t}_u\times\mathbf{t}_{uu})}{\kappa_0^2} ,$$

where the subscript u denotes d/du, and κ_0 and τ_0 represent, respectively, the curvature and the torsion of a new space curve with u as its natural parameter. Analogous to the case of a "spatial" curve as we move from u_0 to u_1 along the "temporal" space curve, a phase $\Phi_0=\int_{u_0}^{u_1}\bar{\tau}_0(u)du$ develops between the natural frame and the corresponding nonrotating frame, where $\bar{\tau}_0$ is to be determined by deriving the corresponding Darboux vector $\boldsymbol{\xi}_0=\bar{\tau}_0\mathbf{t}+B\mathbf{n}+C\mathbf{b}$. The relation between $\bar{\tau}_0$ and τ will be derived in the next section.

Let us now consider the space-time evolution of the tangent to the moving space curve from the point $a=(s,u)$ to the point $d(s+\Delta s,u+\Delta u)$ using paths (a) and (b) as shown in Fig. 1. Path (a) goes first along the "spatial" curve from a to b and then along the "temporal" space curve from b to d. Path (b) goes first along the "temporal" curve $a\rightarrow c$ and then along the "spatial" curve $c\rightarrow d$. The rotation angle Φ is given in the two cases by

$$\begin{aligned} \Phi_1 &= \tau(s,u)\Delta s+\bar{\tau}_0(s+\Delta s,u)\Delta u , \\ \Phi_2 &= \bar{\tau}_0(s,u)\Delta u+\tau(s,u+\Delta u)\Delta s . \end{aligned} \tag{2.4}$$

The phase difference $\delta\Phi=\Phi_1-\Phi_2$ is

$$\begin{aligned} \delta\Phi &= \left[\frac{\partial\bar{\tau}_0}{\partial s}-\frac{\partial\tau}{\partial u}\right]\Delta s\Delta u+O(\Delta^3) \\ &= j(s,u)\Delta s\Delta u+O(\Delta^3) , \end{aligned} \tag{2.5}$$

where $j(s,u)=(\partial\bar{\tau}_0/\partial s-\partial\tau/\partial u)$ can be thought of as a measure of "anholonomy density" of the system. Thus,

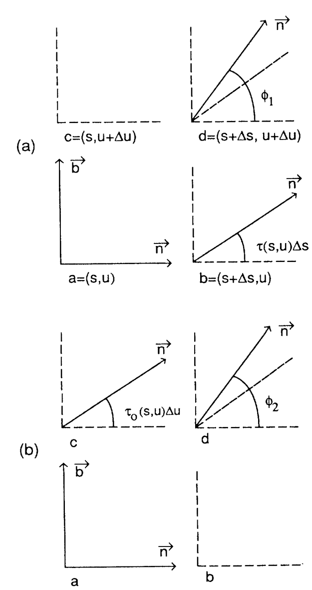

FIG. 1. (a) The route $a\rightarrow b\rightarrow d$: the phase is $\phi_1=\tau(s,u)\Delta s+\tau_0(s+\Delta s,u+\Delta u)\Delta u$. (b) The route $a\rightarrow c\rightarrow d$: the phase is $\phi_2=\tau_0(s,u)\Delta u+\tau(s,u+\Delta u)\Delta s$.

the total "anholonomy" or the "phase" Φ as the system evolves in space from, e.g., $s=-s_0$ to $s=+s_0$ and in time from $u=U_1$ to $u=U_2$, is

$$\Phi=\int_{-s_0}^{+s_0}ds\int_{U_1}^{U_2}du\, j(s,u) \tag{2.6a}$$

$$=\left[\int_{U_1}^{U_2}\bar{\tau}_0(s,u)du\right]_{s=-s_0}^{s=+s_0} - \left[\int_{-s_0}^{+s_0}\tau(s,u)ds\right]_{u=U_1}^{u=U_2} . \tag{2.6b}$$

In order to find Φ we require $\bar{\tau}_0$ and by consequence $\mathbf{t}_u$, $\mathbf{n}_u$, and $\mathbf{b}_u$. These are determined using a procedure suggested by Lamb, which we now discuss.

III. LAMB FORMALISM AND GEOMETRIC PHASE

The Frenet-Serret equations (2.1) combine to give

$$(\mathbf{n}+i\mathbf{b})_s+i\tau(\mathbf{n}+i\mathbf{b})=-\kappa\mathbf{t} . \tag{3.1}$$

Following Lamb, we introduce the quantities

$$\mathbf{N}=(\mathbf{n}+i\mathbf{b})\exp\left[i\int_{-\infty}^{s}ds'(\tau-c_0)\right] \tag{3.2a}$$

and

$$q=\kappa\exp\left[i\int_{-\infty}^{s}ds'(\tau-c_0)\right] , \tag{3.2b}$$

where $\tau(s,u)\rightarrow c_0=\text{const}$ for $|s|\rightarrow\infty$. These definitions lead to $\mathbf{N}_s=-ic_0\mathbf{N}-q\mathbf{t}$ and $\mathbf{t}_s=\frac{1}{2}(q^*\mathbf{N}+q\mathbf{N}^*)$. It is more convenient to describe the temporal evolution of the curve in terms of $\mathbf{t}$, $\mathbf{N}$, and $\mathbf{N}^*$ instead of $\mathbf{t}$, $\mathbf{n}$, and $\mathbf{b}$. $\mathbf{N}$, $\mathbf{N}^*$, and $\mathbf{t}$ satisfy the following conditions:

$$\mathbf{N}\cdot\mathbf{t}=\mathbf{N}^*\cdot\mathbf{t}=\mathbf{N}\cdot\mathbf{N}=0$$

and (3.3)

$$\mathbf{N}\cdot\mathbf{N}^*=2 .$$

The derivatives of $\mathbf{N}$ and $\mathbf{t}$ with respect to u may be written in general as

$$\mathbf{N}_u=\alpha\mathbf{N}+\beta\mathbf{N}^*+\gamma\mathbf{t} ,$$
$$\mathbf{t}_u=\lambda\mathbf{N}+\mu\mathbf{N}^*+\nu\mathbf{t} . \tag{3.4}$$

Multiplying these equations by $\mathbf{N}$ and $\mathbf{t}$ and using Eq. (3.3) gives $\alpha+\alpha^*=0$, $\beta=\nu=0$, and $\gamma=-2\mu$. Further, the compatibility condition $\mathbf{t}_{su}=\mathbf{t}_{us}$ gives $\lambda=\mu^*=(-\gamma^*/2)$. Thus, Eq. (3.4) may be rewritten as

$$\mathbf{N}_u=iR\mathbf{N}+\gamma\mathbf{t} , \tag{3.5a}$$

$$\mathbf{t}_u=-\tfrac{1}{2}(\gamma^*\mathbf{N}+\gamma\mathbf{N}^*) , \tag{3.5b}$$

where $R(s,u)$ is a real function. Using $\mathbf{N}_{su}=\mathbf{N}_{us}$ and $\mathbf{t}_{su}=\mathbf{t}_{us}$ yields

$$q_u+\gamma_s+i(c_0\gamma-Rq)=0 \tag{3.6a}$$

with

$$R_s=i(\gamma q^*-\gamma^*q)/2 . \tag{3.6b}$$

Since $\mathbf{t}\cdot\mathbf{t}_u=0$, a general temporal evolution of the space curve may be written as

$$\mathbf{t}_u=g\mathbf{n}+h\mathbf{b} . \tag{3.7}$$

Using Eqs. (3.7), (3.5), and (3.2), we obtain the following expressions:

$$\gamma=-(g+ih)\exp\left[i\int_{-\infty}^{s}(\tau-c_0)ds'\right] , \tag{3.8a}$$

$$\mathbf{n}_u+i\mathbf{b}_u=-(g+ih)\mathbf{t}+i\left[R-\int_{-\infty}^{s}\tau_u ds'\right](\mathbf{n}+i\mathbf{b}) . \tag{3.8b}$$

Equating real and imaginary parts of the above equation, we get

$$\mathbf{b}_u=\left[R-\int_{-\infty}^{s}\tau_u ds'\right]\mathbf{n}-h\mathbf{t} , \tag{3.9a}$$

$$\mathbf{n}_u=\left[\int_{-\infty}^{s}\tau_u ds'-R\right]\mathbf{b}-g\mathbf{t} . \tag{3.9b}$$

Equations (3.9) together with Eq. (3.7) represent the Frenet-Serret-like equations for the temporal curve. Writing the Darboux vector in the general form $\xi_0=\bar{\tau}_0\mathbf{t}+B\mathbf{n}+C\mathbf{b}$ and requiring that $\mathbf{t}_u=\xi_0\times\mathbf{t}$, etc., Eqs. (3.7)–(3.9) lead to

$$\bar{\tau}_0=\int_{-\infty}^{s}\tau_u ds'-R \tag{3.10a}$$

$B=-h$, and $c=g$:

$$\kappa_0^2=\mathbf{t}_u\cdot\mathbf{t}_u=(g^2+h^2) ,$$
$$\tau_0=\mathbf{t}\cdot(\mathbf{t}_u\times\mathbf{t}_{uu})/\kappa_0^2$$
$$=\int_{-\infty}^{s}ds'\tau_u-R-(gh_u-g_hh)/(g^2+h^2)$$
$$=\bar{\tau}_0+\frac{gh_u-g_uh}{g^2+h^2} . \tag{3.10b}$$

Thus, the anholonomy density obtained in Eq. (2.5) can be directly computed from Eq. (3.10) as

$$j(s,u)=\left[\frac{\partial\bar{\tau}_0}{\partial s}-\frac{\partial\tau}{\partial u}\right]=-\frac{\partial R}{\partial s} . \tag{3.11a}$$

$(\partial R/\partial s)$ can be calculated from Eq. (3.6) by using the definition for q given in Eq. (3.2) and the expression for γ obtained in Eq. (3.8a). We obtain

$$R_s=\kappa h .$$

Hence,

$$j(s,u)=-\kappa h . \tag{3.11b}$$

Thus, if the space curve evolves such that $h=0$ [see Eq. (3.7)], then $j(s,u)=0$. Further, since $\mathbf{t}_s=\kappa\mathbf{n}$ and $\mathbf{t}_u=g\mathbf{n}+h\mathbf{b}$, we get $(\mathbf{t}_s\times\mathbf{t}_u)=\kappa h\mathbf{t}$.

The total phase Φ can therefore be written in one of the following forms:

$$\Phi=-\int_{U_1}^{U_2}du\int_{-s_0}^{s_0}ds\, R_s \tag{3.12a}$$

$$=-\int_{U_1}^{U_2}du\int_{-s_0}^{s_0}ds\,\kappa h \tag{3.12b}$$

$$=-\int_{U_1}^{U_2}du\int_{-s_0}^{s_0}ds\,\mathbf{t}\cdot(\mathbf{t}_s\times\mathbf{t}_u) . \tag{3.12c}$$

Using Eqs. (2.1), (3.7), and (3.9) we can show that the compatibility condition $\mathbf{t}_{su}=\mathbf{t}_{us}$ leads to the following coupled equations for g and h:

$$g_s=\kappa_u+h\tau \ , \tag{3.13}$$

$$h_s=\kappa\left[\int_{-\infty}^{s}(\tau_u-R)ds'\right]-g\tau \ . \tag{3.14}$$

The expressions for Φ given in Eqs. (3.12) are valid for a general evolution. The special case of cyclic evolution in $\mathbf{t}$ space corresponds to boundary conditions on $\mathbf{t}(s,u)$ such that $\mathbf{t}(s,u)$ attains the same value $\mathbf{t}_0$ on the unit sphere at the two space-time end points. Furthermore, if the end points correspond to space-time infinity, Eq. (3.12c) shows that the first term can be written as $4\pi n$, where n is the Pontryagin index. This suggests its topological significance and will be illustrated using an example in Sec. VI.

The expression for Φ given by Eq. (3.12a) shows that it is possible to interpret the first term as the classical analog of the *quantum* geometric phase introduced by Berry. Using Eq. (3.3) in Eq. (3.5a) gives $R=(-i/2)\mathbf{N}^*\cdot\mathbf{N}_u$. Hence,

$$\Phi=\frac{i}{2}\int_{-s_0}^{s_0}ds\frac{\partial}{\partial s}\int_{U_1}^{U_2}\mathbf{N}^*\cdot\mathbf{N}_u du \ . \tag{3.15}$$

Replacing the complex unit vector $\mathbf{N}/\sqrt{2}$ by a quantum state $|N(u)\rangle$, we see that $i\int_{U_1}^{U_2}\langle N|\partial N/\partial u\rangle du$ plays the role, in our approach, of a *local* Berry phase at the point s. Equation (3.15) shows that in applications such as interacting many-body systems, the geometric phase is not the sum of the above local Berry phases, but rather the sum of their "differences" (i.e., gradients).

Now let us consider a space curve such that its torsion does not depend on time: $\partial\tau/\partial u=0$. Then

$$\delta\Phi=\frac{\partial\bar{\tau}_0}{\partial s}\Delta s\Delta u=(-\partial R/\partial s)\Delta s\Delta u \ .$$

Thus,

$$\Phi=\int_{U_1}^{U_2}\tau_0(s,u)du\,\bigg|_{s=s_0}-\int_{U_1}^{U_2}\tau_0(s,u)du\,\bigg|_{s=-s_0} .$$

Hence this particular time evolution is a subclass of the general case discussed earlier [Eq. (2.6b)].

IV. GAUGE POTENTIAL

In this section we determine the corresponding gauge potential related to the phase Φ. Consider the following construction. We transport, by Euclidean parallel transport, all the tangent vectors to our "spatial" and "temporal" curves to the center of a unit sphere. The tips of the tangent vectors trace out the spherical images of these space curves on the unit sphere. Now, consider a small plaquette $abcd$ on the surface of the unit sphere, where point a corresponds to point (s,u) of Fig. 1, b to $(s+\Delta s,u)$, c to $(s,u+\Delta u)$, and d to $(s+\Delta s,u+\Delta u)$. We note that the vector $d\mathbf{t}$ is tangent to the spherical images of the space curves: $\mathbf{t}$ is a unit vector along the radius of the unit sphere $\mathbf{t}^2=1$, $\mathbf{t}\cdot d\mathbf{t}=0$. We now consider the following expression for the phase difference $\delta\Phi$ in terms of the gauge potential:

$$\begin{aligned}\delta\Phi=\oint \mathbf{A}\cdot d\mathbf{t}=&A_i(s,u)\frac{\partial t_i}{\partial s}\Delta s+A_i(s+\Delta s,u)\frac{\partial t_i}{\partial u}\Delta u\\&-A_i(s+\Delta s,u+\Delta u)\frac{\partial t_i}{\partial s}\Delta s\\&-A_i(s,u+\Delta u)\frac{\partial t_i}{\partial u}\Delta u \ ,\end{aligned} \tag{4.1}$$

where $\mathbf{A}=\mathbf{A}(\mathbf{t})$ is the vector potential and $\int$ is the closed integral over the plaquette $abcd$. After developing A_i in power series of Δs and Δu and keeping only terms up to second order in Δs and Δu we obtain the following expression for $\delta\Phi$:

$$\begin{aligned}\delta\Phi&=\oint \mathbf{A}\cdot d\mathbf{t}\\&=\left[\frac{\partial}{\partial s}\left[\mathbf{A}\cdot\frac{\partial\mathbf{t}}{\partial u}\right]-\frac{\partial}{\partial u}\left[\mathbf{A}\cdot\frac{\partial\mathbf{t}}{\partial s}\right]\right]\Delta s\Delta u \ .\end{aligned} \tag{4.2a}$$

A short calculation shows that

$$\delta\Phi=(\nabla_{\mathbf{t}}\times\mathbf{A})\cdot(\mathbf{t}_s\times\mathbf{t}_u)\Delta s\Delta u \ . \tag{4.2b}$$

If $\nabla_{\mathbf{t}}\times\mathbf{A}=\mathbf{t}$, we recover our previous expression Eq. (3.12c) for the phase Φ. This identifies $\mathbf{A}(\mathbf{t})$ as the vector potential of a unit monopole at the center of the unit sphere. In Sec. VI, we will show that the continuum version of a Heisenberg antiferromagnetic chain can be discussed within the framework of the moving space-curve formalism. Note that our constructive formalism provides a justification for the Berry phase expression

$$\int\int\left\{\frac{\partial}{\partial s}\left[\mathbf{A}\cdot\frac{\partial\mathbf{t}}{\partial u}\right]-(s\leftrightarrow u)\right\}ds\,du$$

constructed by Shankar [Eq. (2.3) and following discussion of Ref. 14] in the context of the $(1+1)d$ antiferromagnetic chain. A comparison of Eqs. (4.2a) and (2.5) leads to $\mathbf{A}\cdot\mathbf{t}_u=\bar{\tau}_0$ and $\mathbf{A}\cdot\mathbf{t}_s=\tau$. Using Eqs. (2.1) and (3.7) in these relations gives

$$\mathbf{A}=\left[\frac{\tau}{\kappa}\right]\mathbf{n}+\left\{\frac{1}{h}\left[\bar{\tau}_0-\frac{g\tau}{\kappa}\right]\right\}\mathbf{b} \ . \tag{4.3}$$

In the next two sections we show that the continuum versions of the classical one-dimensional isotropic Heisenberg ferromagnet and antiferromagnet provide examples of interacting spin systems in which the formalism of space curves can be applied to determine the geometric phase associated with these systems.

V. THE FERROMAGNETIC CHAIN

Consider a one-dimensional magnetic system described by the Heisenberg exchange Hamiltonian

$$H=-J\sum_n\mathbf{S}_n\cdot\mathbf{S}_{n+1} \ , \tag{5.1}$$

where $\mathbf{S}_n$ denotes the classical spin vector at the nth lattice site, and $(\mathbf{S}_n)^2=S^2=\text{const}$ for all n. J represents the nearest-neighbor exchange interaction. $J>0$ corresponds to the ferromagnetic chain. It is well known[15] that the time evolution of $\mathbf{S}_n(t)$ obtained from Eq. (5.1) is given by

$$\frac{d\mathbf{S}_n}{dt}=J[\mathbf{S}_n\times(\mathbf{S}_{n+1}+\mathbf{S}_{n-1})] . \tag{5.2}$$

This is studied in the continuum approximation by assuming a slow variation of the spin vectors along the chain, i.e.,

$$\mathbf{S}_n(t)\rightarrow\mathbf{S}(x,t)$$

and

$$\mathbf{S}_{n+1}(t)\rightarrow\mathbf{S}(x,t)+a\,\partial\mathbf{S}/\partial x+\frac{1}{2}a^2\frac{\partial^2\mathbf{S}}{\partial x^2} ,$$

where a is the nearest-neighbor separation. Equation (5.2) becomes

$$\frac{\partial\mathbf{S}}{\partial t}=Ja^2(\mathbf{S}\times\mathbf{S}_{xx}) . \tag{5.3}$$

Defining the unit vector $\mathbf{t}=\mathbf{S}/S$ and the dimensionless variables $u=JSt$ and $s=(x/a)$, we get

$$\mathbf{t}_u=\mathbf{t}\times\mathbf{t}_{ss} , \tag{5.4}$$

where $\mathbf{t}$ can be identified with the tangent to a space curve (with natural parameter s) evolving in time. For the general evolution $\mathbf{t}_u=g\mathbf{n}+h\mathbf{b}$ there is a dependence of g and h on κ and τ [see Eqs. (3.13) and (3.14)]. Further, since $\mathbf{t}$ satisfies the partial differential equation (5.4), we will see that this implies coupled differential equations for κ and τ.

Using Frenet-Serret equations (3.1) in Eq. (5.4), we get

$$\mathbf{t}_u=\mathbf{t}\times(\kappa\mathbf{n}_s+\kappa_s\mathbf{n})=(-\kappa\tau\mathbf{n}+\kappa_s\mathbf{b}) .$$

Thus,

$$g=-\kappa\tau$$

and (5.5)

$$h=\kappa_s .$$

Using Eq. (5.5) in Eq. (3.8a) gives

$$\gamma=-i(\kappa_s+i\kappa\tau)\exp\left[i\int_{-\infty}^{s}(\tau-c_0)ds'\right]=-iq_s$$

and from Eq. (3.6b) we get

$$R=\int_{-\infty}^{s}\kappa\kappa_s ds'=\tfrac{1}{2}|q|^2 ,$$

where $q=\kappa\exp\{i\int_{-\infty}^{s}\tau ds'\}$ [see Eq. (2.6b)]. Substituting these expressions for γ and R in Eq. (3.6) shows that the ferromagnetic chain can be mapped[16] to the following nonlinear Schrodinger equation[17] for q, for $c_0=0$:

$$iq_u+q_{ss}+\tfrac{1}{2}|q|^2q=0 . \tag{5.6}$$

This is well known to be a completely integrable[18] equation. It is clear that this equation leads to coupled differential equations for κ and τ referred to earlier.

Let us compute the anholonomy density for this system. Using Eq. (5.5) in Eq. (3.11b) we get

$$j(s,u)=-\kappa\kappa_s . \tag{5.7}$$

Hence, knowing a solution of the nonlinear equation (5.3), we can determine the anholonomy density corresponding to it. As explained in Sec. III, the phase is

$$\phi_F=\int\int\mathbf{t}\cdot(\mathbf{t}_s\times\mathbf{t}_u)ds\,du=\frac{1}{2}\int\int(\kappa^2)_s ds\,du . \tag{5.8}$$

As illustrative examples, let us consider two classes of solutions, viz., *nonlinear* spin waves and solitons in a ferromagnetic chain.

(i) Nonlinear spin waves. Equation (5.4) supports[19] the following periodic solutions:

$$\mathbf{t}(s,u)=\{\hat{e}_x\cos(ps-\omega u)+\hat{e}_y\sin(ps-\omega u)\}\sin\alpha_0 +\hat{e}_z\cos\alpha_0 . \tag{5.9}$$

Here α_0 is a constant and $(\hat{e}_x,\hat{e}_y,\hat{e}_z)$ denotes the Cartesian coordinate system. Further, $\omega=p^2\cos\alpha_0$. (Note that for $\alpha_0\rightarrow0$, we have $\omega\sim p^2$ corresponding to the usual "linear" spin waves in a ferromagnetic chain). For all α_0, we can verify that

$$\kappa^2(s,u)=(\partial\mathbf{t}/\partial s)^2=p^2\sin^2\alpha_0=\text{const} . \tag{5.10}$$

Hence $(\kappa^2)_s=0$, and from Eq. (5.8) the anholonomy density $j(s,u)=0$. Thus, the phase ϕ_F also vanishes.

(ii) Soliton solutions. It is well known[20] that Eq. (5.3) represents a completely integrable equation with strict soliton solutions. Writing

$$\mathbf{t}(s,u)=(\sin\theta\cos\varphi,\sin\theta\sin\varphi,\cos\theta) , \tag{5.11}$$

these solutions correspond to[21]

$$\cos\theta=[1-2(1-\alpha^2)\text{sech}^2(s-vu)/\Gamma] , \tag{5.12}$$

where α,v,Γ are dimensionless velocity parameters. α is related to the soliton velocity v:

$$\alpha=v/(4\omega)^{1/2},\quad 0\le\alpha\le1 , \tag{5.13}$$

and Γ is the soliton width:

$$\Gamma=\{\omega(1-\alpha^2)\}^{-1/2} . \tag{5.14}$$

ω is an angular frequency parameter and φ is given by

$$\varphi=\varphi_0+\omega u+\tfrac{1}{2}v(s-vu-s_0) +\tan^{-1}\left[\frac{2}{v\Gamma}\tanh[s-vu-s_0]/\Gamma\right] . \tag{5.15}$$

Further, $(\partial\varphi/\partial s)=v/(1+\cos\theta)$.

A short calculation yields

$$\kappa^2=(\partial\mathbf{t}/\partial s)^2=\left\{\left[\frac{\partial\theta}{\partial s}\right]^2+\sin^2\theta\left[\frac{\partial\varphi}{\partial s}\right]^2\right\} =(4/\Gamma^2)\text{sech}^2(s-vu-s_0)/\Gamma . \tag{5.16}$$

Using Eq. (5.16) in Eq. (5.8) shows that the effective anholonomy density is nonvanishing for a soliton solution:

$$(\kappa^2)_s=(8/\Gamma^3)\mathrm{sech}^2\{(s-vu-s_0)/\Gamma\}\times\tanh\{(s-vu-s_0)/\Gamma\}\ . \tag{5.17}$$

From Eq. (5.8) for $-L/a<s<L/a$ and $U_1<u<U_2$, we obtain the geometric phase as

$$\phi_F=(2/\Gamma v)\left[\tanh\left[\frac{L}{a}-vU_2\right]\Big/\Gamma+\tanh\left[\frac{L}{a}+vU_2\right]\Big/\Gamma-\tanh\left[\frac{L}{a}-vU_1\right]\Big/\Gamma-\tanh\left[\frac{L}{a}+vU_1\right]\Big/\Gamma\right], \tag{5.18}$$

where $U_i=JSt_i$, $i=1,2$. Hence the phase depends on the soliton parameters, i.e., its width and velocity as well as the magnitude S of the spin.

For $L\to\infty$ and T_1,T_2 finite, this phase vanishes. However, for $L\to\infty$, $T_1\to 0$, $T_2\to L/avJS\to\infty$ (v finite), we get

$$\phi_F=(-4/\Gamma_v)\ . \tag{5.19}$$

Comparing ϕ_F with the expression for φ in Eq. (5.15), we observe that $\phi_F=-\frac{1}{2}\tan(\Delta\varphi/2)$, where $\Delta\varphi$ is the phase shift[21] associated with φ as one moves across a region of width Γ about the center of the solitary wave.

VI. THE ANTIFERROMAGNETIC CHAIN

To study the antiferromagnetic chain described by $H=-J\sum_n \mathbf{S}_n\cdot\mathbf{S}_{n+1}$, $J<0$, we proceed as follows. Since the neighboring spin vectors on the chains will have a tendency to be antiparallel to each other due to energetic considerations, it is convenient to study the problem by dividing the lattice sites on the chain into "odd" and "even" sublattices, so that subsequently the continuum approximation can be used within a given sublattice.

The equations of motion of the spin vectors $\mathbf{S}_{i,0}$ and $\mathbf{S}_{i-1,e}$ on "odd" and "even" sites (as the subscripts denote) are found by using Eq. (5.2):

$$\frac{d\mathbf{S}_{i,0}}{dt}=J\mathbf{S}_{i,0}\times(\mathbf{S}_{i+1,e}+\mathbf{S}_{i-1,e})\ , \tag{6.1}$$

$$\frac{d\mathbf{S}_{i-1,e}}{dt}=J\mathbf{S}_{i-1,e}\times(\mathbf{S}_{i,0}+\mathbf{S}_{i-2,0})\ . \tag{6.2}$$

Since the continuum approximation is justified for neighboring spin vectors in the same sublattice, we have

$$\mathbf{S}_{i+1,e}+\mathbf{S}_{i-1,e}\to 2\mathbf{S}_e(x-a)+2a\,\partial\mathbf{S}_e/\partial x \tag{6.3}$$

and

$$(\mathbf{S}_{i,0}+\mathbf{S}_{i-2,0})\to 2\mathbf{S}_o(x)-2a\,\partial\mathbf{S}_o/\partial x\ . \tag{6.4}$$

Using Eqs. (6.3) and (6.4) in Eqs. (6.1) and (6.2) we get

$$\partial\mathbf{S}_o(x)/\partial t=2J\{\mathbf{S}_o(x)\times\mathbf{S}_e(x-a)+a\mathbf{S}_o(x)\times\partial\mathbf{S}_e/\partial x\} \tag{6.5}$$

and

$$\partial\mathbf{S}_e(x-a)/\partial t=2J\{\mathbf{S}_e(x-a)\times\mathbf{S}_o(x)-a\mathbf{S}_e(x-a)\times\partial\mathbf{S}_o/\partial x\}\ . \tag{6.6}$$

(Explicit time dependence is omitted to avoid cumbersome notation.) Adding Eqs. (6.5) and (6.6) yields

$$\frac{\partial}{\partial t}[\mathbf{S}_o(x)+\mathbf{S}_e(x-a)]=2Ja\frac{\partial}{\partial x}[\mathbf{S}_o(x)\times\mathbf{S}_e(x-a)]\ . \tag{6.7}$$

Subtracting Eq. (6.6) from Eq. (6.5)

$$\frac{\partial}{\partial t}[\mathbf{S}_o(x)-\mathbf{S}_e(x-a)]=4J[\mathbf{S}_o(x)\times\mathbf{S}_e(x-a)]+2Ja\left[\mathbf{S}_o(x)\times\frac{\partial\mathbf{S}_e}{\partial x}-\frac{\partial\mathbf{S}_o}{\partial x}\times\mathbf{S}_e\right]. \tag{6.8}$$

We define

$$(\mathbf{S}_o-\mathbf{S}_e)=2S\sqrt{1-\epsilon^2}\boldsymbol{\eta}$$

and

$$(\mathbf{S}_o+\mathbf{S}_e)=2S\epsilon\boldsymbol{\zeta}\ , \tag{6.9}$$

where $\boldsymbol{\eta}$ and $\boldsymbol{\zeta}$ are unit vectors, $\boldsymbol{\eta}\cdot\boldsymbol{\zeta}=0$, and $\epsilon^2=\frac{1}{2}\{1+(\mathbf{S}_o\cdot\mathbf{S}_e/S^2)\}$. Substituting Eq. (6.9) in Eqs. (6.7) and (6.8) yields

$$\partial\boldsymbol{\zeta}/\partial t=2JaS\sqrt{1-\epsilon^2}\frac{\partial}{\partial x}(\boldsymbol{\eta}\times\boldsymbol{\zeta}) \tag{6.10}$$

and

$$\partial\boldsymbol{\eta}/\partial t=4JS\epsilon(\mathbf{n}\times\boldsymbol{\zeta})+2JaS\frac{\epsilon^2}{\sqrt{1-\epsilon^2}}\left[\boldsymbol{\zeta}\times\frac{\partial\boldsymbol{\zeta}}{\partial x}\right]-2JaS\sqrt{1-\epsilon^2}\left[\boldsymbol{\eta}\times\frac{\partial\boldsymbol{\eta}}{\partial x}\right]. \tag{6.11}$$

It is clear that the low-energy configurations would correspond to $|\mathbf{S}_o-\mathbf{S}_e|\simeq 2S$ and $|\mathbf{S}_o+\mathbf{S}_e|\simeq 0$ in Eq. (6.11), corresponding to $\epsilon\ll 1$. In this case, the dynamics is dominated by

$$\frac{\partial\boldsymbol{\eta}}{\partial t}=2JaS\left[\frac{\partial\boldsymbol{\eta}}{\partial x}\times\boldsymbol{\eta}\right]. \tag{6.12}$$

It is interesting to contrast this with the dynamical equation for the ferromagnetic chain given in Eq. (5.3).

Defining dimensionless variables $u=JSt$ and $s=(x/2a)$ and identifying $\boldsymbol{\eta}$ with the tangent to a space curve, we get

$$\mathbf{t}_u=(\mathbf{t}_s\times\mathbf{t}) \ . \qquad (6.13)$$

From Eq. (3.1), since $\mathbf{t}_s=\kappa\mathbf{n}$, Eq. (6.13) yields

$$\mathbf{t}_u=-\kappa\mathbf{b} \ .$$

Comparison with $\mathbf{t}_u=g\mathbf{n}+h\mathbf{b}$ gives

$$g=0$$

and (6.14)

$$h=-\kappa \ .$$

Hence, γ for this problem can be found from Eq. (3.8a) as

$$\gamma=-i\kappa\exp\left[i\int_{-\infty}^{s}(\tau-c_o)ds'\right]=-iq \ . \qquad (6.15)$$

Also,

$$R=\int_{-\infty}^{s}\kappa h\, ds'=-\int_{-\infty}^{s}\kappa^2 ds'=-\int_{-\infty}^{s}|q|^2 ds \ . \qquad (6.16)$$

Using these expressions for γ and R in Eq. (3.6), we see that the antiferromagnetic chain can be mapped on to the following nonlinear evolution equation for the complex function q:

$$iq_u+q_s-q\int_{-\infty}^{s}|q|^2 ds'=0 \ . \qquad (6.17)$$

This is the analog of the nonlinear Schrödinger equation [Eq. (5.6)] obtained for the ferromagnetic chain.

Next we compute the anholonomy density $\mathbf{j}(s,u)$. Since $g=0$ and $h=-\kappa$, Eq. (3.11b) yields

$$j(s,u)=\kappa^2(s,u) \ . \qquad (6.18)$$

Thus, knowing a solution of the nonlinear equation of motion (6.13), the anholonomy density corresponding to it can be calculated. The geometric phase is

$$\begin{aligned}\phi_{\mathrm{AF}}&=\int\int \mathbf{t}\cdot(\mathbf{t}_s\times\mathbf{t}_u)ds\,du\\&=-\int\int\kappa^2(s,u)ds\,du \ . \end{aligned}\qquad (6.19)$$

Before proceeding to find the solutions, let us notice certain general properties of Eq. (6.13) which are not satisfied by its analog Eq. (5.4) for the ferromagnetic chain.

Since $\mathbf{t}\cdot\mathbf{t}_u=\mathbf{t}\cdot\mathbf{t}_s=0$, Eq. (6.13) also implies

$$\mathbf{t}_s=-(\mathbf{t}_u\times\mathbf{t}) \ . \qquad (6.20)$$

Combining Eqs. (6.13) and (6.20), we may write

$$\begin{aligned}&\partial t^\alpha/\partial x_\mu=\epsilon_{\mu\nu}\epsilon_{\alpha\beta\gamma}t^\beta\partial t^\gamma/\partial x_\nu \ ,\\&(x_\mu,x_\nu)=(s,u) \ .\end{aligned}\qquad (6.21)$$

These equations were studied by Belavin and Polyakov[22] in the case of the time-independent, two-dimensional, isotopic ferromagnet, where in contrast to our case, s and u were spatial indices.

Equations (6.13) and (6.20) lead to

$$\mathbf{t}\cdot(\mathbf{t}_u\times\mathbf{t}_s)=\tfrac{1}{2}\{(\mathbf{t}_u)^2+(\mathbf{t}_s)^2\} \ . \qquad (6.22)$$

Hence

$$\phi_{\mathrm{AF}}=-\frac{1}{2}\int\int\{(\mathbf{t}_u)^2+(\mathbf{t}_s)^2\}ds\,du \ . \qquad (6.23)$$

It is interesting to note that the nonlinear equation (6.12) cannot support a pure traveling wave-type solution $\mathbf{t}=\mathbf{t}(z)$, where $z=s-vu$, since it leads to $-v\mathbf{t}_z=\mathbf{t}\times\mathbf{t}_z$ which have no nontrivial solutions. Thus, the nonlinear spin-wave type of solutions discussed for the ferromagnet are not possible. Spin waves require that we allow fluctuations in ξ also [see Eq. (6.10)]. Furthermore, it is not possible to have purely static solutions with $\mathbf{t}_u=0$ or purely dynamic solutions with $\mathbf{t}_s=0$. We will study two classes of solutions of (6.12), the twist solutions and the instantons, and the geometric phase corresponding to them.

Since $\mathbf{t}(s,u)$ is a unit vector, we consider a solution as in Eq. (5.11). Substituting this in Eq. (6.13) leads to

$$\partial\theta/\partial u=-(\partial\varphi/\partial s)\sin\theta \ , \qquad (6.24)$$

$$\partial\varphi/\partial u=(\sin\theta)^{-1}\partial\theta/\partial s \ . \qquad (6.25)$$

One of the compatibility conditions, viz., $\theta_{su}=\theta_{us}$, gives

$$\partial^2\varphi/\partial s^2+\partial^2\varphi/\partial u^2=0 \ . \qquad (6.26)$$

(i) Twist solutions. Consider the following simplest solution to Eq. (6.26):

$$\varphi(s,u)=(k_0 s+\omega_0 u) \ . \qquad (6.27a)$$

A brief calculation yields

$$\cos\theta(s,u)=-\tanh(\omega_0 s-k_0 u) \ . \qquad (6.27b)$$

Note the interchanged roles of ω_0 and k_0 in φ and θ. Again, it is clear that it is impossible, in general, to have purely static or purely time-dependent solutions of (6.24) and (6.25). Substituting Eqs. (6.27a) and (6.27b) in Eq. (5.11), we get

$$\begin{aligned}\mathbf{t}(s,u)=\operatorname{sech}(\omega_0 s-k_0 u)\{&\hat{e}_x\cos(k_0 s+\omega_0 u)\\&+\hat{e}_y\sin(k_0 s+\omega_0 u)\}\\&-\hat{e}_z\tanh(\omega_0 s-k_0 u) \ .\end{aligned}\qquad (6.28)$$

Thus,

$$\mathbf{t}(s,u)\rightarrow(0,0,\pm1) \quad \text{for } s\rightarrow\mp\infty \ .$$

This represents a domain-wall configuration, with a type of spin wave existing essentially within the wall thickness. At a given instant of time as s increases from $-\infty$, the spin vector $(\mathbf{S}_o-\mathbf{S}_e/2)$ pointing along $\hat{e}_z$ starts precessing around the $\hat{z}$ axis with the precession angle increasing, until at $s\rightarrow+\infty$ it points along $-\hat{e}_z$. In other words, if one transports the spin vectors at all the (paired) lattice sites at a given instant of time to the center of a unit sphere, then one may imagine the tip of the spin vector lying at the north pole at $s\rightarrow-\infty$ to execute a spiral motion on the surface of the sphere (the axis of the spiral being the polar axis) in such a way that the tip reaches the south pole at $s\rightarrow+\infty$.[23] From Eq. (6.28), we find the curvature $\kappa(s,u)$:

$$\kappa^2(s,u)=(\partial \mathbf{t}/\partial s)^2=(\omega_0^2+k_0^2)\operatorname{sech}^2(\omega_0 s-k_0 u) . \quad (6.29)$$

The geometric phase is found from Eq. (6.19) to be

$$\phi_{AF}=-(\omega_0^2+k_0^2)\int_{-L/2a}^{L/2a} ds \int_{U_1}^{U_2} du \operatorname{sech}^2(\omega_0 s-k_0 u) , \quad (6.30)$$

where $U_i=JST_i$, $i=1,2$. Hence

$$\phi_{AF}=\frac{(\omega_0^2+k_0^2)}{\omega_0 k_0}\ln\left[\frac{\cosh(\omega_0 L/2a-k_0U_2)\cosh(\omega_0 L/2a+k_0U_1)}{\cosh(\omega_0 L/2a+k_0U_2)\cosh(\omega_0 L/2a-k_0U_1)}\right] . \quad (6.31)$$

In the limit $L\to\infty$, with (T_1-T_2) finite, we get

$$\phi_{AF}=2(\omega_0^2+k_0^2)(U_1-U_2)/\omega_0 . \quad (6.32)$$

Hence by mutually adjusting the domain-wall parameters and the time of evolution, different values of ϕ_{AF} can be obtained.

(ii) Instanton solutions. Let us consider another class of solutions of Eqs. (6.24) and (6.25). As already mentioned, such equations were studied by Belavin and Polyakov[22] as the minimum energy configurations of the static two-dimensional *ferromagnet* (with both s and u being spatial coordinates).[24] In the present case, they were derived as dynamical equations for low-energy configurations of the one-dimensional antiferromagnetic chain. Define

$$\begin{aligned} M(s,u)&=\cot\tfrac{1}{2}\theta(s,u)\exp i\varphi(s,u)\\ &=M_1(s,u)+iM_2(s,u) . \end{aligned} \quad (6.33)$$

We can easily verify that the Cauchy-Reimann conditions

$$\partial M_1/\partial s=\mp\partial M_2/\partial u$$

and (6.34)

$$\partial M_1/\partial u=\pm\partial M_2/\partial s$$

are implied by Eqs. (6.24) and (6.25). This shows that if $z=s+iu$, then $M(z)$ corresponds to any analytic function of z (or z^*) for the upper (or lower) sign in Eqs. (6.34). Choosing the former, we consider the simplest solution

$$M(z)=\{(z-z_0)/\lambda\}^n , \quad (6.35)$$

where n is a positive integer. z_0 is a complex number, λ is a real number. The geometric phase corresponding to this solution is obtained by using the definition given in Eq. (6.23). We find

$$\begin{aligned}\left[\frac{\partial t}{\partial s}\right]^2+\left[\frac{\partial \mathbf{t}}{\partial u}\right]^2&=\left\{\left[\frac{\partial\theta}{\partial s}\right]^2+\sin^2\theta\left[\frac{\partial\varphi}{\partial s}\right]^2+\left[\frac{\partial\theta}{\partial u}\right]^2+\sin^2\theta\left[\frac{\partial\varphi}{\partial u}\right]^2\right\}\\ &=4(1+|M|^2)^{-2}\left\{\frac{\partial M}{\partial s}\frac{\partial M^*}{\partial s}+\frac{\partial M}{\partial u}\frac{\partial M^*}{\partial u}\right\} ,\end{aligned} \quad (6.36)$$

where Eq. (6.33) has been used. Since $(\partial M/\partial z^*)=0$, we get

$$\frac{1}{2}\left\{\left[\frac{\partial \mathbf{t}}{\partial s}\right]^2+\left[\frac{\partial \mathbf{t}}{\partial u}\right]^2\right\}=4\left|\frac{\partial M}{\partial z}\right|^2\Big/\{1+|M|^2\}^2 . \quad (6.37)$$

Substituting Eq. (6.37) in Eq. (6.23) and using the form Eq. (6.35) for $M(z)$ yields[25]

$$\phi_{AF}=4\pi n, \quad n>0 .$$

Choosing M to be an analytic function of z^* would give

$$\phi_{AF}=4\pi n, \quad n<0 .$$

Note that in this example $M(z)\to\infty$ as $z\to\infty$, i.e., $\theta=0$. Since the field $\mathbf{t}(s,u)$ points towards the north pole at all points at infinity, the s-u plane can be compactified into a spherical surface, S^2. Furthermore, the space of fields $\mathbf{t}(s,u)$ is also a spherical surface of radius unity, S^2. Thus, the configuration $\mathbf{t}(s,u)$ is just a mapping of S^2 into S^2, which can be classified into homotopy sectors characterized by the set of integers n.

It is clear that, in general, for any choice of the analytic function $M(z)$ such that $\mathbf{t}(s,u)\to\mathbf{t}_0$, a constant vector at space-time infinity, as, for example.

$$M(z)=\prod_i\left[\frac{z-z_i}{\lambda}\right]^{m_i}\prod_j\left[\frac{\lambda}{z-z_j}\right]^{n_j} \quad \text{with } \sum_i m_i\geq\sum_j n_j , \quad (6.38)$$

one can show[22]

$$\phi_{AF}=4\pi n , \quad (6.39)$$

where $n=\sum_j m_j$, since there are n roots of z for a given M.

For the one-instanton solution with $n=1$ it is easily seen that the spin configurations are given by

$$\cos\theta(s,u)=1-2\lambda^2/\{(s-s_0)^2+(u-u_0)^2+\lambda^2\} \tag{6.40a}$$

and

$$\varphi(s,u)=\tan^{-1}[(u-u_0)/(s-s_0)] . \tag{6.40b}$$

Thus (s_0,u_0) is the position of the instanton and λ is its "size," i.e., it is localized in time as well, unlike the moving twist solutions given in Eq. (6.27). Finally, we remark that knowing explicit solutions of θ and φ [as in Eqs. (6.27) and (6.40b)], it is possible to find κ and τ and therefore corresponding solutions of Eq. (6.17), which is a nonlinear equation for q.

VII. SUMMARY

We have studied the spatial and time evolution of a space curve described by spatial and temporal curvature and torsion κ,τ and κ_0,τ_0, respectively. We have shown that the evolution is associated with an anholonomy density $j(s,u)=(\partial\bar{\tau}_0/\partial_s-\partial\tau/\partial u)$, where s and u are the natural parameters corresponding to spatial and temporal evolution, respectively. The relation between the "geodesic torsion," $\bar{\tau}_0$, and τ_0 has been derived in Sec. IV [Eq. (3.10b)]. The total phase $\Phi=\int ds\int du\; j(s,u)$.

Starting with a general time evolution for the tangent $\mathbf{t}(s,u)$ to the space curve, $\mathbf{t}_u=g\mathbf{n}+h\mathbf{b}$, where g and h are functions of (s,u) and $\mathbf{n}$ and $\mathbf{b}$ are the normal and binormal vectors, we have derived $j(s,u)=-\kappa h$. The coupled equations for g and h are given in Eq. (3.14). It is clear that for the class of evolving space curves for which h or k vanish, there is no associated anholonomy. The identity $\kappa h=\mathbf{t}\cdot(\mathbf{t}_s\times\mathbf{t}_u)$ shows that $j(s,u)$ has a topological origin. A second identity $\kappa h=(-i/2)(\partial/\partial s)(\mathbf{N}^*\cdot\mathbf{N}_u)$, where $\mathbf{N}=(\mathbf{n}+i\mathbf{b})\exp\{i\int_{-\infty}^{s}\tau ds'\}$ helps us to write this topological phase as $(i/2)\int ds(\partial/\partial s)\int du(\mathbf{N}^*\cdot\mathbf{N}_u)$. Replacing the complex unit vector $\mathbf{N}/\sqrt{2}$ by a quantum state $|N(u)\rangle$, we see that our expression is the classical analog of the quantum geometric phase introduced by Berry when the system evolves with time and with one more parameter s. This is of relevance when one deals with a continuum description of interacting many-body systems in one dimension, and shows that the total Berry phase is the sum of "differences" of local Berry phases [using the natural definition of "local" Berry phase in our formalism Eq. (3.15)].

We have applied this formalism to the dynamics of classical magnetic chains in the continuum limit. The spin evolution equation of motion for the ferromagnetic chain is known to be integrable. It has (localized) soliton solutions and (extended) nonlinear spin-wave solutions. We have shown by explicit calculation that the anholonomy density is nonzero for the one-soliton solution and vanishes for the latter. The geometric phase for the former depends on the soliton parameters. The question of whether the total phase for an N-soliton solution is the sum of the phases of each of the N solitons is worth further investigation.

The dynamics of the antiferromagnetic chain differs from the ferromagnetic chain in that one must now define the sublattice and solve the coupled equations (6.10) and (6.11) in general. We have considered the dynamics of the low-energy configurations, and shown that the evolution equations obtained are self-dual and are identical in form to those studied by Belavin and Polyakov,[22] with the difference that the independent variables were both spatial in their discussion. The equations support instanton solutions. We show that a new type of "twist" solution (which is not a pure unidirectional traveling wave) also exists. The anholonomy density is *nonvanishing* in both cases. For the former the geometric phase in $4\pi n$ where n is the number of instantons. For the latter, the phase depends on the twist parameters.

It would be interesting to carry out experiments to measure the phase and to verify Eq. (3.12c) of our paper. It may be more practical to consider the case when the two independent variables are both spatial, rather than one spatial and the other temporal as in Eq. (3.12c). Also, optical systems may be more practical than magnetic ones. A fine mesh made of twisted optical fibers which forms a twisted surface could be fabricated, and a method to observe the angle of rotation of the polarization vector for the paths (abd) and (acd) in Fig. 1 measured to see if there is a phase difference. It is clear that the fibers must have variable torsion (or pitch) since their derivatives occur in the anholonomy density. A simple case is when the torsions are linear functions leading to the phase difference which is proportional to the area of the twisted surface. Experimental implications of our formalism for a two-dimensional (2D) magnetic system require appropriate NMR methods capable of studying "local" dynamics of the derivatives of spin vectors (tangents) which are constrained to lie on the n-b plane. The aim should be to probe the twisting of the derivative vectors as one moves from point to point on the surface. Generalization of our formalism to $(2+1)$-dimensional cases will be presented elsewhere.[26]

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