Skyrmions and the crossover from the integer to fractional quantum Hall effect at small Zeeman energies

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(Received 10 September 1992; revised manuscript received 24 May 1993)

We study the two-dimensional electron gas in a high magnetic field at filling factor $\nu = 1$ for an arbitrary ratio of the Zeeman energy $g\mu_B B$ to the typical interaction energy. We find that the system always has a gap, *even* when the one-particle gap vanishes, i.e., when $g = 0$. When g is sufficiently large, the quasiparticles are perturbatively related to those in the noninteracting limit; we compute their energies to second order in the Coulomb interaction. For g smaller than a critical value g_c the quasiparticles change character; in the limit of $g \to 0$, they are skyrmions—spatially unbounded objects with infinite spin. In GaAs heterojunctions, the gap is, unambiguously, predominantly due to correlation effects; indeed, we tentatively conclude that g is always smaller than g_c, so the relevant quasiparticles are the skyrmions. The generalization to other odd-integer filling factors, and to $\nu = \frac{1}{3}$ and $\frac{1}{5}$, is discussed.

I. INTRODUCTION

In the theory of the quantum Hall effect[1] (QHE) it is customary to distinguish the integer and fractional effects: filling factors ν where the quasiparticle gap arises from a gap in the single-particle spectrum are said to exhibit the integer effect whereas those ν where the gap arises from the electron interactions exhibit the fractional effect. With some exceptions this theoretical classification is supported by comparisons of the experimentally measured gaps, extracted from the temperature dependence of ρ_{xx} in the regime in which it is activated, and the single-particle gaps. Exceptions are the odd integers where the measured gaps exceed the single-particle Zeeman gaps, derived from bulk g factors, by as much as a factor of 20.[2] Clearly, it is not possible to interpret these data without including significant effects of the interactions.

In order to understand the role of the interactions at odd-integer fillings we study the QHE at $\nu = 1$ in two ways. First, we calculate the energies of the quasiparticles in the noninteracting limit to second order in the interactions and confirm that the contribution of the interactions dominates the Zeeman gap in the magnetic fields of interest. We compare our calculation with the data of Usher *et al.* on the $\nu = 1$ activation energies in GaAs heterostructures and find reasonable agreement (Fig. 1, see the discussion following). This work has the merit that we have computed all the terms of a systematic expansion in powers of $1/\sqrt{B}$ that survive in the $B \to \infty$ limit; the physics of the exchange enhancement of the g factor is well known.[3–5] (We have done the same calculation for $\nu = 2$ as well.)

Next we consider a modified problem in which we vary the ratio of the Zeeman energy ($g\mu_B B$; μ_B is the Bohr magneton) to the typical Coulomb interaction energy (e^2/l; l is the magnetic length); this is conveniently accomplished, theoretically, by varying the effective gyromagnetic ratio g. We find that the ground state is independent of g for $g \geq 0$ (at $g = 0$ the ground state is degenerate), and that there is always a gap to creating charged excitations *even* for $g = 0$. Though the gap survives at $g = 0$, the quasiparticles change dramatically as g is lowered. At large g the quasiparticles have the quantum numbers of the single-particle picture, i.e., they have charge $\pm e$ and spin $S_z = 1/2$ (relative to the ground state). As g is reduced there is, beginning at a

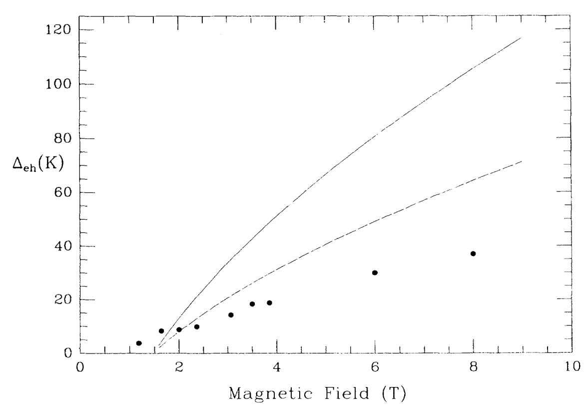

FIG. 1. The quasiparticle gap Δ_{eh} as a function of magnetic field. Solid and dashed lines are, respectively, the theoretical results for an undisordered system with and without the phenomenological 40% finite thickness correction. The points are data taken from Ref. 2. The theoretical gaps vanish at a magnetic field where their computation via low-order perturbation theory is no longer reliable.

critical value g_c, an infinite number of level crossings in which the lowest-lying quasiparticles increase their spin and size while retaining their charge. Near $g = 0$, which is a limit point for these level crossings, the quasiparticles have a divergent size, their spin is macroscopic, and they possess nontrivial spin order—they are skyrmions.

Accordingly, we conclude the following. (a) the QHE at $\nu = 1$ cannot be classified as either fractional or integer; the ground state does not change and the gap persists from the large g limit, where single-particle effects dominate, to the small g limit where correlation effects are paramount. (b) There is a level crossing at $g = g_c$ such that the quasiparticles are perturbatively related to the single-particle excitations for $g > g_c$ whereas for $g < g_c$ they are not. (c) In GaAs at $\nu = 1$, g appears to be less than g_c and the relevant quasiparticles are thus skyrmions. The gap is certainly predominantly due to electron-electron interactions. Later we discuss how the existence of the skyrmion might be detected experimentally as well as the relevance of our work to the QHE in Si devices where the valley SU(2) symmetry, in principle, provides a realization of the small g limit.

A similar scenario holds for the other odd-integer filling factors as well as for $\nu = 1/3$ and 1/5. In the latter cases there is never a correspondence to a noninteracting spectrum but there is a similar sequence of quasiparticle crossings culminating in large skyrmions at small g. Our work builds on earlier work by one of us (E.H.R.).[6,7] More generally it is part of the study of spin-reversed excitations and ground states that was initiated by Halperin's observations on the smallness of the g factor in GaAs.[8]

II. PERTURBATION THEORY FOR THE GAPS

The activation energy Δ for the temperature dependence of ρ_{xx} is related to the excitation energy Δ_{eh} of an infinitely separated quasiparticle and quasihole by the law of mass action, i.e., $\Delta = \Delta_{eh}/2$. A systematic expansion for Δ_{eh} can be generated by perturbing in the interaction around the noninteracting problem. As the latter corresponds to the limit of an infinite magnetic field this is a useful approach for understanding the high-field problem. The noninteracting problem exhibits a degenerate ground-state manifold for filling factors other than the integers. Hence in the remaining cases it is necessary to choose the unperturbed states by degenerate perturbation theory in the interaction. At filling factors that exhibit the fractional QHE this produces a nondegenerate ground state separated by a gap from the quasiparticle states. For Coulomb $(1/r)$ interactions the expansion for Δ_{eh} and other energies of interest is particularly simple in structure:

$$\Delta_{eh} = \hbar\omega_c^* \sum_{k=0}^{\infty} F_k(\nu) \left(\frac{l}{a^*}\right)^k . \tag{1}$$

(Here k is the order of perturbation theory, $\omega_c^* = eB/m^*c$ is the cyclotron frequency of particles with an effective mass m^*, $l = \sqrt{\hbar c/eB}$ is the Landau length, and $a^* = \epsilon\hbar^2/m^*e^2$ is the effective Bohr radius.) Hence perturbation theory in the interactions is also an expansion in powers of $1/\sqrt{B}$.[9] This simplicity is a consequence of the scale invariance of the Coulomb interaction; for a Yukawa interaction the coefficients in the expansion would themselves depend upon l. In Table I and in Appendix A we list some of the known terms in this expansion; here we comment on the first few coefficients. The zeroth-order coefficients are the gaps in the single-particle spectrum in units of $\hbar\omega_c^*$ and therefore are nonvanishing only for integer ν:

TABLE I. Coefficients in the gap expansion.

ν	$F_0(\nu)$	$F_1(\nu)$[a]	$F_2(\nu)$
1	$g\frac{m^*}{2m}$	$\sqrt{\frac{\pi}{2}}$	-0.58
2	$1-g\frac{m^*}{2m}$	$\sqrt{\frac{\pi}{8}}$	-0.47
1/3	0	0.10360	?
1/5	0	0.02440	?

[a]The coefficients for $\nu = 1/3$ and 1/5 are from Ref. 31.

$$\begin{aligned} F_0(\nu) &= 1 - g(m^*/2m), \quad \nu = 2k, \\ &= g(m^*/2m), \quad \nu = 2(k-1), \\ &= 0 \ \text{ otherwise.} \end{aligned} \tag{2}$$

(Here m is the free-electron mass.) The $F_1(\nu)$ multiply $e^2/\epsilon l$ and are nonzero both for integer and for fractional filling factors. They represent the leading contribution of the Coulomb interactions to the gap; for the fractions they signal the existence of the fractional effect. The next set, $F_2(\nu)$, multiply twice the effective Rydberg (Ry$=e^2/\epsilon a^*$) and describe the leading effects of Landau-level mixing. Note that the second-order term is *independent* of the magnetic field; it is therefore important for a quantitative estimate of the gap (even) at high magnetic fields.

We have obtained $F_2(1)$ and $F_2(2)$ by calculating, to second order, the energies of the quasihole and quasielectron and subtracting the ground-state energy calculated to the same order. The unperturbed quasiparticle states consist of states with different particle number at the same magnetic field and area: the quasihole has an emptied orbital in the otherwise filled Landau level, while the quasielectron has an extra electron added to the spin-reversed Landau level ($\nu = 1$) or across the cyclotron gap ($\nu = 2$).[10] The resulting expressions involve multiple sums of matrix elements of the Coulomb interaction; we were able to evaluate the matrix elements in terms of standard functions but were forced to carry out the sums numerically.[11] We find that, in units of 2 Ry, the second-order contributions to the gap are $\Delta_{eh}^{(2)} = \Delta_e^{(2)} + \Delta_h^{(2)} = -0.524(5) - 0.054(2) = -0.58(1)$ at $\nu = 1$ and $\Delta_{eh}^{(2)} = \Delta_e^{(2)} + \Delta_h^{(2)} = -0.352(5) - 0.113(2) = -0.47(1)$ at $\nu = 2$. The corresponding second-order energies for GaAs, with values of $m^* = 0.07m$ and $\epsilon = 13$, are -76 K ($\nu = 1$) and -61 K ($\nu = 2$).

III. INTERACTIONS AND THE $\nu = 1$ GAP

We now make contact with some recent data of Usher *et al.*[2] on activation energies at $\nu = 1$. In calculating

$F_0(1)$, $F_1(1)$, and $F_2(1)$ we have kept all the contributions to the gap that will survive in the high-field limit for an ideal two-dimensional system. Comparison with experiment is, however, complicated by our neglect of disorder, the finite extent of the wave functions orthogonal to the interface, and the mixing of higher interfacial subbands. In Fig. 1 we show the experimental values of the gap (points) and our theoretical result (solid line). We expect a reduction of the gap due to the finite extent of the wave function perpendicular to the interface. If we phenomenologically model this by a 40% reduction of the interaction contribution to the gap the result is the dashed line.[12] Note that $\hbar\omega_c F_0(1) < 3$ K over the range of magnetic fields in Fig. 1 and hence the interactions provide the dominant contribution to the gaps. The agreement with the data in the high-field region is good enough that it is *plausible* that the true quasiparticles are perturbatively related to the noninteracting excitations.[13] However, we will argue below that this is in fact *not* the case.

IV. $\nu = 1$ AT VARYING ZEEMAN ENERGIES

We have established that in GaAs heterojunctions and odd integer ν the observed gap is largely due to the interactions, not to the single-particle (Zeeman) gap. We now explore the possibility that the gap exists even in the *absence* of a single-particle gap and that the quasiparticle properties are qualitatively (in addition to quantitatively) different than those in the noninteracting limit. To this end we investigate the following problem: We imagine that g is a tunable parameter and ask how the physics changes as g varies from zero to some large value at which the Zeeman energy dominates the interactions.

The problem is conveniently studied in the (Chern-Simons) Landau-Ginzburg theory of the Hall effect introduced by Zhang, Hansson, and Kivelson.[14] We use the generalization to electrons with spin that was worked out by Lee and Kane.[15] In this formulation the system is governed by the Lagrangian density

$$\mathcal{L}(\mathbf{r}) = \phi^\dagger(\mathbf{r})\left[i\hbar\partial_t - ea_o\right]\phi(\mathbf{r}) - \frac{1}{2m^*}\left|\left[\frac{\hbar}{i}\nabla - \frac{e}{c}[\mathbf{A}(\mathbf{r}) + \mathbf{a}(\mathbf{r})]\right]\phi(\mathbf{r})\right|^2$$
$$-\tfrac{1}{2}\int d^2r'\, V(\mathbf{r}-\mathbf{r}')[|\phi(\mathbf{r})|^2 - \overline{\rho}][|\phi(\mathbf{r}')|^2 - \overline{\rho}] - \frac{e^2}{4\hbar c\theta}\epsilon^{\mu\nu\sigma}a_\mu(\mathbf{r})\partial_\nu a_\sigma(\mathbf{r}) - \tfrac{1}{2}g\mu_B B\phi^\dagger(\mathbf{r})\sigma^z\phi(\mathbf{r})\ . \quad (3)$$

Here $\phi = (\phi_1, \phi_2)$ is a two-component complex scalar field and $\theta = (2k+1)\pi$ as we are describing bosonized fermions. $V\ (= e^2/\epsilon r)$ is the interparticle potential and we pick $\mathbf{A}$ so that $\mathbf{B} = \nabla\times\mathbf{A} = -B\hat{\mathbf{z}}$. As in the case of spinless electrons[14] we find that the filling factors $\nu = 1/(2k+1)$ (the Laughlin fractions) are special in that at these we can find spatially uniform solutions to the equations of motion. These solutions, which are of the form $\phi = \sqrt{\overline{\rho}}(1,0)$, $\overline{\rho} = \nu/(2\pi l^2)$, minimize the action at *any* value of g. Moreover, to all orders, the fluctuations about these mean-field solutions are independent of g. From this we conclude that the ground state is spin polarized and described by the *same* orbital wave function at all g. Consequently the ground state at all g is obtained by solving the usual problem of fully spin polarized electrons.[16] At $g = 0$ our model is spin rotationally invariant. Correspondingly any solution of the form $\phi = \sqrt{\overline{\rho}}\,(1,0)\,U$, $U \in \mathrm{SU}(2)$ also minimizes the action. A particular choice of U then corresponds to spontaneous symmetry breaking (ferromagnetism).

We now turn to the excitations; we specialize to $\nu = 1$ for now. In the large g limit[17] the low-lying charged excitations have a size of the order of the magnetic length;[18] these are simply the single-particle excitations whose energies we computed above. There are also spin waves that disperse quadratically from $g\mu_B B$. One can obtain wave functions and energies for these excitations if the Hilbert space is restricted to the lowest Landau level of both spin species. There is a branch of low-lying excitations that describes neutral spin waves at small k and a separated quasihole-quasielectron pair as $k \to \infty$. The energy of this branch was computed by Kallin and Halperin[19] and takes the form (I_0 is the modified Bessel function)

$$E(k) = g\mu_B B + \frac{e^2}{\epsilon l}\sqrt{\frac{\pi}{2}}\left[1 - e^{-k^2l^2/4}I_0(k^2l^2/4)\right]\ . \quad (4)$$

The long-wavelength spin waves become gapless as $g \to 0$. While the $k \to \infty$ limit of $E(k)$ still corresponds to the creation of a quasielectron-quasihole pair, the lowest-lying charged excitations are different; they are "skyrmions."[15] These are excitations that are characterized by unusual spin order. At the boundary of the system the local spin takes its value in the ground state ("up") while it is reversed at the center of the skyrmion ("down"). Along any radius it interpolates smoothly between two limits. If we identify the points at the boundary, such spin configurations are described by the simplest homotopically nontrivial maps of the surface of the sphere onto itself. (See Appendix B for explicit examples.) As we shall see the skyrmions carry charge $\pm e$ depending on the sense of their spin twist.

To reveal the properties of these quasiparticles in the small g limit (where they are large), we study the long-wavelength, low-energy dynamics of (3). This is a theory of the long-wavelength spin dynamics. Technically, we can decompose the ϕ field as $\phi_\alpha = \sqrt{\rho}z_\alpha$ where $\sum_\alpha z_\alpha^\dagger z_\alpha = 1$. $\mathcal{L}$ then describes a CP^1 field z_α coupled to a Chern-Simons gauge field and the ρ field. Using the mapping $n^a = z^\dagger\sigma^a z$ we can replace the CP^1 field by an

$O(3)$ sigma-model field. At this point we could obtain the effective sigma-model dynamics by integrating out all the other fields. Though we are unable to do this explicitly we can nevertheless calculate the necessary terms in the effective Lagrangian. We do this by observing that the dynamics is that of a ferromagnet with a long-range interaction arising from the Coulomb interaction between the underlying electrons. This leads to a Lagrangian of the form[20]

$$\mathcal{L}_{\text{eff}} = \alpha\mathcal{A}(\mathbf{n}(\mathbf{r}))\cdot\partial_t\mathbf{n}(\mathbf{r}) + \alpha'(\nabla\mathbf{n}(\mathbf{r}))^2 + g\bar{\rho}\mu_B\mathbf{n}(\mathbf{r})\cdot\mathbf{B} \\ -\tfrac{1}{2}\int d^2r'\,V(\mathbf{r}-\mathbf{r}')q(\mathbf{r})q(\mathbf{r}')\ , \quad (5)$$

where $\mathcal{A}$ is the vector potential of a unit monopole, i.e., $\epsilon^{ijk}\partial_j\mathcal{A}^k = n^i$, and $q(\mathbf{r}) = \epsilon^{ij}\epsilon^{abc}n^a\partial_i n^b\partial_j n^c/8\pi$ is the skyrmion density whose spatial integral is the topological charge (±1). The first three terms would be present for any ferromagnet; however, the last term is specific to our problem and is responsible for the macroscopic character of the skyrmions.

The form of the last term follows from the equality, in the long-wavelength limit, of the skyrmion density and the deviation of the physical density from its uniform value; more generally at $\nu = 1/k$ they are related by $\delta\rho(x) = q(x)/(2k+1)$. To derive the latter note that in terms of the fields in (3) the current is $\mathbf{j} = (e\rho/m^*c)(\mathbf{a}_{\text{sk}} - \mathbf{a} - \mathbf{A})$ where $\mathbf{a}_{\text{sk}} = (\hbar c/e)(z^\dagger\nabla z)$. For finite energy configurations the current must vanish at infinity and hence, at large distances, $\mathbf{a}_{\text{sk}} = \mathbf{a}+\mathbf{A}$. For sufficiently smoothly varying configurations this equality will hold everywhere, asymptotically, as the scale of the variations diverges. From the Chern-Simons relation between the density and the statistical magnetic field, $\delta\rho = (e/2\hbar c\theta)\nabla\times(\mathbf{a}+\mathbf{A})$ we find that $\delta\rho(x) = (1/2\theta)\nabla\times(z^\dagger\nabla z) = q(x)/(2k+1)$. (Also, see Ref 15.)

Due to the singular behavior of the Coulomb interaction at small momenta the interaction term in (5) is *cubic* and not quartic in powers of momentum in the long-wavelength limit. Therefore, by power counting the last term in (5) is the leading irrelevant term; we keep it since it determines the size of the skyrmions in the small g limit. The other terms that will arise are either of higher dimension or will vanish when $g = 0$.[21] The coefficients in $\mathcal{L}_{\text{eff}}$ can be fixed by requiring that (a) it reproduces (4) in the small k limit (i.e., it yields the correct spin-wave dispersion) and (b) it describes correctly the uniform precession of the ferromagnet in a tilted magnetic field. These conditions yield

$$\alpha = \frac{1}{4}\hbar\rho \quad\text{and}\quad \alpha' = \frac{1}{32}\frac{1}{\sqrt{2\pi}}\frac{e^2}{l}\quad . \quad (6)$$

Armed with $\mathcal{L}_{\text{eff}}$ we can now study the skyrmions. In the absence of the Zeeman and interaction terms $\mathcal{L}_{\text{eff}}$ is scale invariant for static configurations and contains solitons (skyrmions) on all length scales.[22] By virtue of the equality of the skyrmion density and the physical density it follows that the skyrmions carry charge $\pm e$ according to the sense of the spin twist. This connection between the (Coulomb) charge and topological charge of the skyrmions can be understood in a more intuitive fashion. In a self-consistent description of the skyrmion state the electrons move in a skyrmionic spin background. As the electron spins attempt to align with the background texture this gives rise to extra phase factors for closed trajectories. The effect of the nontrivial spin background is similar to that of an extra magnetic flux—indeed, it is not hard to see, by recalling the behavior of spin-1/2 particles under rotations, that the presence of a unit of topological charge produces a Berry's phase equal to the change in the Aharonov-Bohm phase produced by the insertion of one quantum of flux. Consequently the charge of the skyrmions is $\pm e/(2k+1)$ at $\nu = 1/(2k+1)$, for much the same reasons as for the usual quasiparticles created by the insertion of one quantum of flux.

For the pure sigma model ($g=0$, $V=0$), analytic expressions are available for the skyrmions (Appendix B) and their energy is $8\pi\alpha'$, independent of their size.[22] This scale invariance is broken by the remaining terms. The interaction favors large skyrmions while the Zeeman term prefers microscopic skyrmions. For $g = 0$, the skyrmions are infinite; consequently the only contribution to their energy comes from the stiffness of the spin waves and equals the sigma model result $8\pi\alpha'$. Away from $g = 0$ the skyrmions acquire a size determined by balancing the Zeeman and Coulomb terms. For $g \ll 1$ the form of the solution in the core region is determined by the scale invariant term alone. We have used the known analytic expressions for these (Appendix B) matched to the solution outside the core (where the equations of motion can be linearized) to determine the global behavior of the solution for $g \ll 1$.[23] These yield expressions for the size and energy of the skyrmions,

$$\left(\frac{\lambda}{l}\right)^3 = \left(\frac{9\pi^2}{2^8}\right)\left(\frac{l}{\epsilon a}\right)(g|\ln g|)^{-1}\ , \quad (7)$$

$$E(g) = \frac{1}{4}\sqrt{\frac{\pi}{2}}\frac{e^2}{\epsilon l}\left[1 + \frac{3\pi}{4}\left(\frac{18}{\pi}\right)^{1/6}\left(\frac{\epsilon a}{l}\right)^{1/3}(g|\ln g|)^{1/3}\right].$$

where $a = \hbar^2/me^2$ is the Bohr radius. These expressions represent the leading asymptotic behavior of λ and $E(g) - E(0)$ at small values of g. At nonzero g there are corrections to these expressions that arise from our not having found the true minima of (5) as well as from our neglect of the remaining terms in the effective Lagrangian. We expect that these corrections become significant when the size of the skyrmions approaches l. Note that the gap to creating a skyrmion-antiskyrmion pair at $g = 0$, $2E(0)$, is *half* the gap to creating a pair of single-particle excitations; hence the skyrmions are the relevant quasiparticles in this limit.[24]

To summarize: There is no change in the ground state of the system as g is varied through nonzero values. This is the familiar behavior of a ferromagnet. However, the excitation spectrum does interesting things: At large g the quasiparticles are single-particle-like—they carry charge $\pm e$ and spin $S_z = \frac{1}{2}$, and have size l. At small g they still carry charge $\pm e$ but diverge in size and have nontrivial spin order with a divergent z component of spin S_z (the number of reversed spins) as well as a divergent total spin S.[25] In between there are an infinite number of level crossings that allow the requisite crossover in the properties of the quasiparticles. There is always a gap in the quasiparticle spectrum; however, the first of the level crossings erases the correspondence to the noninteracting problem.

V. EXACT DIAGONALIZATION STUDIES

Some of this physics is implicit in the previous work by one of us.[6,7] This work consists of a study of up to 10 particles on a finite sphere in the $g = 0$ limit. It was found that for $\nu = 1$ and $\nu = 1/3$, the ground state is ferromagnetic, i.e., has maximal total spin. However, upon changing the flux through the system by one flux quantum, a process that creates a single quasiparticle, the ground state becomes a spin singlet. This state is also an orbital singlet (with respect to rotations of the sphere) and hence describes a quasiparticle with a uniform density distribution. The remaining states are ordered so that their energies and orbital angular momenta increase with their spin. On including the Zeeman term, the S_z eigenstates continue to be eigenstates while their energies are shifted by the Zeeman cost of flipping spins. It follows then that we recover the scenario of level crossings among quasiparticle states that we described above. Variational wave functions for quasielectrons of arbitrary spin were also constructed and it was shown in Ref. 7 (by finite-size scaling from calculations on up to 160 particles), that for $g = 0$ the maximal spin one-polarized-quasielectron state is unstable to decreasing its spin by one unit. By using this variational result we can obtain a lower bound for g_c, $g_c \geq 0.054e^2/(\epsilon l \mu_B B)$. This implies that the value of g for GaAs systems (~ 0.5) is less than g_c for fields below 25 T and therefore the quasiparticles are not fully polarized.

We have extended the finite-size study of the $g = 0$ problem by finite-size scaling. In Fig. 2 we plot the quasiparticle creation energies at fixed number and magnetic field[26] as a function of system size. We find that the energy to create a quasielectron and quasihole extrapolates to within $\frac{1}{2}$% of the calculated energy of the skyrmion-antiskyrmion pair in the Landau-Ginzburg theory, i.e., within the uncertainty of the extrapolation. In fact, as a consequence of particle-hole symmetry the quasihole and quasielectron energies extrapolate to the same limit, which is therefore just the Landau-Ginzburg result for a single skyrmion. We note that this remarkable feature, the calculation of the *exact* quasiparticle energies from the Landau-Ginzburg theory, is a consequence of the divergent size of the quasiparticles. In Fig. 3 we show the correlation functions $\langle\rho(0)\rho(\theta)\rangle$ and $\langle\sigma_z(0)\sigma_z(\theta)\rangle$ where θ is the polar angle on the sphere and σ_z is twice the spin density in units in which ρ_0, the density of the filled Landau level, is 1. The small θ behavior of both correlation functions is dominated by the exchange hole. At larger θ we see behavior characteristic of the skyrmion: the density is uniform and the spin-spin correlation function becomes negative. One may wonder about the spin of the ground state when the system is more than one flux quantum away from commensuration—whether there are any "shell effects" reminiscent of the Nagaoka problem. We found, by exact diagonalization, that the ground state at two flux quanta fewer than $\nu = 1$ is also a singlet. As the removal of enough flux quanta must ultimately produce a singlet $\nu = 2$ state we conjecture that the singlet character of the ground state persists until we have added half a skyrmion per particle.

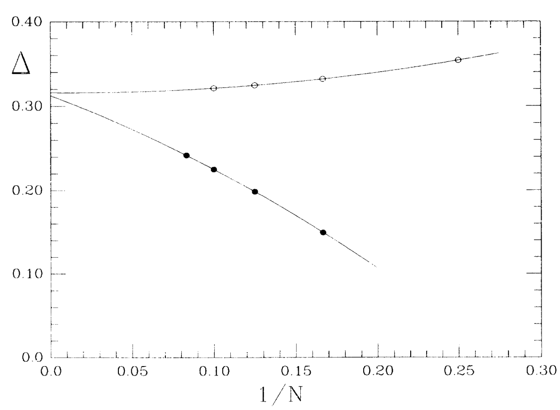

FIG. 2. Singlet quasiparticle energies Δ (in units of $e^2/\epsilon l$) from finite-size calculations. The fits are quadratic polynomials in $1/N$. The quasielectron data (filled circles) extrapolate to $0.3128e^2/\epsilon l$ and the quasihole data (open circles) to $0.3159e^2/\epsilon l$. The Landau-Ginzburg analysis gives $0.3133e^2/\epsilon l$.

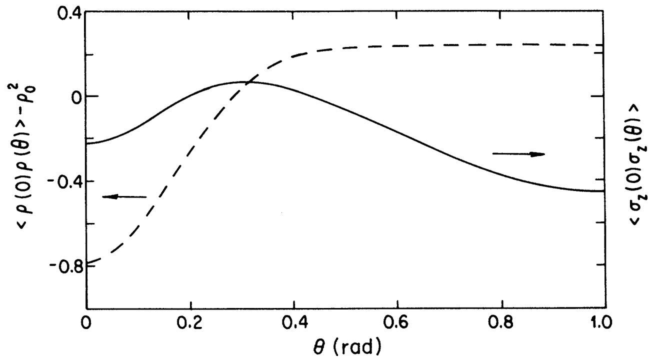

FIG. 3. Correlation functions in the one-quasielectron state. We plot $\langle\rho(0)\rho(\theta)\rangle - \rho_0^2$ (dashed curve) and $\langle\sigma_z(0)\sigma_z(\theta)\rangle$ (solid curve) in units in which the filled Landau level has density $\rho_0 = 1$; θ is the polar angle divided by π. In this case, the quasielectron is the ground state of a system with nine flux quanta and ten electrons ($g = 0$).

VI. EXTENSION TO OTHER FRACTIONS

The physics at the higher odd integer ν is very similar; however, we have not attempted to quantitatively estimate their properties. As we indicated earlier, finite-size studies show that the ground states at $\nu = 1/3$ and $1/5$ are ferromagnetic at $g = 0$ and that the one-quasiparticle states have infinite spin relative to the ground state, (i.e., they are singlets). The Landau-Ginzburg analysis in these cases is identical to that for $\nu = 1$ except that the physical density is now the corresponding fraction of the skyrmion density. Thus, the quasiparticles carry charge $e^* = \pm\nu e$ and again have a spin and size that diverge as g vanishes. In contrast to $\nu = 1$, we do not have access to the exact spin-wave dispersion even in the lowest Landau-level approximation; however, we can use the single-mode approximation[27] to extract an approximate spin-wave stiffness from the structure factor of the Laughlin states at $\nu = 1/3$ and $1/5$.[28] We estimate the skyrmion pair gap for $g = 0$ (in units of $e^2/\epsilon l$) as 0.024 at $\nu = 1/3$ and 0.006 at $\nu = 1/5$. Generalizing (7) to these fractions is straightforward.

VII. DISSIPATION AND EXPERIMENTS

It is essential for the quantum Hall effect that the quasiparticles are localized by disorder. For $g = 0$, the bare quasiparticles are infinite in extent, so the question arises whether they still are localized. We believe that the interaction with an impurity gives the skyrmion a finite size through the balance of the attraction to the impurity and the Coulomb repulsion, and that therefore the skyrmions are localized by disorder. However, the binding is likely to be weaker than for smaller quasiparticles; this may have observable consequences.

Even given the fact that the quasiparticles are pinned by disorder when $g = 0$, the question arises whether the existence of gapless neutral excitations (the spin waves) leads to dissipation and hence destroys the quantum Hall effect. A related problem was studied by Rasolt, Halperin, and Vanderbilt[29] in the context of valley waves in Si devices. They applied a Landau argument and concluded that the quantum Hall effect is not destroyed so long as the condensate velocity is less than the velocity of the neutral mode. In the present case, for $g = 0$, the quadratic dispersion of the spin waves does not permit us to use this argument to deduce dissipationless flow. We think it is likely that dissipationless flow survives the presence of spin waves, since the low-energy spin waves are neutral and hence are irrelevant to charge transport. At the very least, it is clear that for any nonzero g and $T = 0$, there is a quantum Hall effect, so if the $g \to 0$ limit is approached at $T = 0$, the quantum Hall effect survives by continuity. This issue warrants further study.

In GaAs systems, the most interesting confirmation of our analysis would of course be a direct probe of the spin structure of the quasiparticles. In terms of the more usual probes one should look for evidence of level crossings among the quasiparticle states as a function of g. Experimentally, the effective g can be varied by tilting the magnetic field, since only the perpendicular component, B, couples to the orbital motion while the full field couples to the spin. As a result, the effective g is $g/\cos(\theta)$ where θ is the tilting angle. We have estimated the properties of the quasiparticles using the results obtained in (7) above. We find that at 1 T the quasiparticle should extend about 2000 Å and have about 12 reversed spins which would imply that there should be as many level crossings in going down to that field from the high (~ 20 T) field limit. However, our estimate is crude and ignores effects such as Landau-level mixing (important at low fields) and the effects of disorder that may favor smaller quasiparticles. For $\nu = 1/3$ the same procedure leads to the conclusion that for a field of 1 T, the quasiparticle size is about 1400 Å and it involves a couple of reversed spins. This suggests that the smaller Laughlin fractions are worse places to look for evidence of these excitations. Finally, as has been extensively discussed by Rasolt,[30] the valley degeneracy in Si devices behaves like an isospin. Thus, the present considerations apply with small modification with the advantage that the system is automatically in the $g = 0$ limit.

VIII. RELATIONSHIP TO OTHER WORK

In Ref. 1 Haldane suggested that there is charge fractionalization at $\nu = 1$ in the isotropic limit. We do not see any evidence for it and the relationship of our work to his ideas is unclear. Finally, after we had finished this work, we realized that the final footnote in Ref. 29 already refers to the existence of the skyrmions in the context of large-amplitude valley wave configurations.

ACKNOWLEDGMENTS

We would like to thank E. Fradkin, S. Girvin, P. Goldbart, and N. Goldenfeld for useful discussions. We are particularly grateful to K. Yang and K. Moon for catching an error in our original calculation of the size and energy of the skyrmions. This work was supported in part by NSF Grant No. DMR-90-11803 at UCLA (S.A.K and S.L.S) and by NSF Grant No. DMR-91-13876 at CSULA (E.H.R). A.K. was supported by the Swedish Natural Science Research Council.

APPENDIX A

We list here the $F_1(\nu)$ for arbitrary integer ν. With the definition

$$V(l,m) = \frac{1}{\sqrt{2}\,m!}\sum_{r=0}^{l}\binom{l}{r}(-1)^r \times \frac{\Gamma(r+1/2)\Gamma(m-r+\frac{1}{2})}{r!\,\Gamma(\frac{1}{2}-r)}\ , \quad \text{(A1)}$$

they are

$$F_1(2k+1) = V(k,k) \ ,$$
$$F_1(2k) = \sum_{0\le p<k} [V(k-1,p) - V(k,p)]. \tag{A2}$$

APPENDIX B

As we noted in the text the pure sigma model [i.e., Eq. (5) with $g = 0$ and $V = 0$] admits skyrmion solutions on all length scales and explicit analytic expressions are available for them.[22] We record here, for ease of access, the explicit form of the skyrmion with topological charge $Q = \int d^2r q(\mathbf{r}) = +1$ and scale λ and its topological (skyrmion) density:

$$\begin{aligned} n^x(\mathbf{r}) &= \frac{4\lambda x}{r^2+4\lambda^2}, \\ n^y(\mathbf{r}) &= \frac{4\lambda y}{r^2+4\lambda^2}, \\ n^z(\mathbf{r}) &= \frac{r^2-4\lambda^2}{r^2+4\lambda^2}, \\ q(\mathbf{r}) &= \frac{1}{\pi}\frac{4\lambda^2}{(r^2+4\lambda^2)^2}. \end{aligned} \tag{B1}$$

The corresponding antiskyrmion has $Q = -1$ and is of the same form but with $n^y(\mathbf{r}) \to -n^y(\mathbf{r})$ and $q(\mathbf{r}) \to -q(\mathbf{r})$. Since Q is a topological invariant, any continuous deformation of the spin texture in (B1) will also have $Q = 1$.

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[24] We have also calculated the effect of the finite extent (perpendicular to the interface) of the wave functions on the spin-wave velocity, and hence on the skyrmion energy. We find that for GaAs heterojunctions, $E(0)$ is reduced by amounts from 50 % to 70% as the field varies from 2 T to 10 T. The leading g dependence is unchanged; there will of course be quantitative corrections at nonzero g due to the softened interaction. In the same range of fields the energy of the microscopic (i.e., spin-$\frac{1}{2}$) quasiparticle is reduced by 30–50 %. Thus the effect is to further stabilize the skyrmion relative to the perturbative quasiparticle. None of these numbers includes the effects of Landau-level mixing.

[25] By standard arguments the infinite skyrmion represents a quantum state with $S = 0$; relative to the ground state this is an infinite spin.

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