Nonuniversal spectral properties of the Luttinger model

K. Schönhammer and V. Meden
Institut für Theoretische Physik, Universität Göttingen, Bunsenstrasse 9, D-3400 Göttingen, Germany
(Received 21 December 1992)

The one-electron spectral functions for the Luttinger model are discussed for large but finite systems. The methods presented allow a simple interpretation of the results. For finite-range interactions, interesting nonuniversal spectral features emerge for momenta that differ from the Fermi points by the order of the inverse interaction range or more. For a simplified model with interactions only within the branches of right- and left-moving electrons, analytical expressions for the spectral function are presented that allow us to perform the thermodynamic limit. As in the general spinless model and the model including spin for which we present mainly numerical results, the spectral functions do not approach the noninteracting limit for large momenta. The implication of our results for recent high-resolution photoemission measurements on quasi-one-dimensional conductors are discussed.

I. INTRODUCTION

The experimental study of quasi-one-dimensional conductors can provide a test of the peculiarities of correlated electrons in one dimension.[1,2] In particular, high-resolution valence photoemission is a very useful tool, as the measured spectra are directly related to the one-particle Green's function of the system. Recent experiments of this type[3] give a strong indication that the interpretation of spectra requires the inclusion of many-body effects. As theoretical approaches usually concentrate on the universal behavior of the spectral functions in the extreme low-energy regime,[4] these experiments provide a stimulus to examine also the *nonuniversal behavior* of the spectral functions for one-dimensional (1D) correlated electrons.

As first discussed by Tomonaga,[5] the problem of 1D electrons with a *long-range* interaction simplifies considerably because it is a good approximation to *linearize* the energy dispersion around the two Fermi points $\pm k_F$. In the Luttinger model[6] an exactly linear dispersion is *assumed*. An exact solution for the Luttinger model was presented by Mattis and Lieb.[7] The original Tomonaga model and the Luttinger model were compared by Gutfreund and Schick,[8] who showed that the low-energy physics in both models is the same for long-range interaction with a rather weak restriction on the interaction strength.

The Luttinger model is often studied with the simplification of a *zero-range* interaction.[9] This is sufficient for the discussion of the low-energy singularities of the spectra. As the interacting ground state of the corresponding model contains holes deep below the Fermi level, a direct comparison with a system of nonrelativistic electrons is doubtful, as the linearization of the energy dispersion is no longer justified for all relevant energies. We therefore study in this paper the spectral properties of the Luttinger model for a *finite-range interaction*.

Even for the spinless case the simplified model with interaction terms only *within* a branch (g_4 interaction in the "g-ology" classification[2]) the nonuniversal behavior of the spectral function is very nontrivial for k values which differ from $\pm k_F$ by the order of the inverse interaction range or more. As the spectral functions of this simplified model can be calculated analytically, this is probably the most simple nontrivial model of interacting electrons for which a complete explicit calculation of spectra can be performed. This solution is presented in Sec. III in two different ways: In a direct approach to calculate the many-electron eigenstates which enter the Lehmann representation for the spectral functions and using the bosonization of the field operators.[4,10,11] In both approaches we first calculate the spectra for systems of finite length L and then perform the limit $L \to \infty$.

For the complete spinless Luttinger model including the g_2-interaction terms between the branches a recursive numerical method is presented in Sec. IV to calculate exact spectra for arbitrarily large systems. This procedure gives more insight into the interesting nonuniversal features of the k and ω dependence of the spectra than the brute-force attempt to perform numerically the double Fourier transform of the Green's functions $G_\alpha^{(<)}(x,t)$ which themselves have to be calculated involving a numerical integration. For the k-integrated spectral function the direct integration procedure is compared with the asymptotic results for very large but finite systems. The approach used for the spinless model is generalized to the model including spin in Sec. V. Our results and the relevance for photoemission spectra are summarized in Sec. VI. Spectral moments and the momentum distribution are discussed in Appendixes.

The present paper with its discussion of spectral properties at *arbitrary* excitation energies is complementary to our recent work[11] in which results for the universal low-energy part of the spectra are presented.

II. SPINLESS LUTTINGER MODEL

As discussed in the Introduction, it is necessary to consider the Luttinger model with a finite-range interaction if one wants the model to work as an approximation to

describe nonrelativistic electrons not only in the asymptotic low-energy region. The original nonrelativistic electrons are assumed to interact by a two-body potential $v(x)$, i.e., the interaction part of the Hamiltonian for a system of finite length L with periodic boundary conditions reads

$$\hat{V} = \frac{1}{2}\int_0^L dx \int_0^L dx' v(x-x')\hat{\rho}(x)\hat{\rho}(x') - \frac{1}{2}v(x=0)\hat{N} = \frac{1}{2L}\sum_{k\neq 0}\hat{\rho}_k\hat{\rho}_{-k}\tilde{v}(k) + \frac{1}{2L}\hat{N}^2\tilde{v}(k=0) - \frac{1}{2}\hat{N}v(x=0), \tag{1}$$

where $\hat{\rho}(x)$ is the operator of the electron density $\hat{\rho}_k$ with $k = 2\pi n/L$, $n \in \mathbb{Z}$ its Fourier components, and $\hat{N} \equiv \int \hat{\rho}(x)dx$ the particle number operator. Here we *assume* that the two-body potential $v(x)$ and its Fourier components $\tilde{v}(k)$ are finite at $x = 0$ and $k = 0$, respectively.

The transition to the interaction term for the Luttinger model occurs by writing $\hat{\rho}_k$ as

$$\hat{\rho}_k = \hat{\rho}_{k,+} + \hat{\rho}_{k,-}, \tag{2}$$

where the $\hat{\rho}_{k,\alpha}$ with $\alpha = +\ (-)$ are the Fourier components of the operators for the densities of right- (left-) moving particles. The first term on the right-hand side (rhs) of Eq. (1) is generalized to[2]

$$\hat{V} = \frac{1}{2L}\sum_{k\neq 0,\alpha=\pm}[g_4(k)\hat{\rho}_{k,\alpha}\hat{\rho}_{-k,\alpha} + g_2(k)\hat{\rho}_{k,\alpha}\hat{\rho}_{-k,-\alpha}] \tag{3}$$

with the original model corresponding to $g_2(k) \equiv g_4(k) \equiv \tilde{v}(k)$. The other two terms in Eq. (1) are usually neglected. This is justified for the calculation of the spectral functions if all frequencies are measured with respect to the chemical potential μ. For the explicit calculation of μ these terms are of importance. One comment on the term involving the interaction $g_4(k)$ should be made, as it is often dropped. This is *only* justified for a *strictly zero-range interaction*, for which the decomposition used in Eq. (1) is not allowed.

In the spinless model the $\hat{\rho}_{q,\alpha}$ are given by

$$\hat{\rho}_{q,\alpha} = \sum_k \hat{a}^\dagger_{k,\alpha}\hat{a}_{k+q,\alpha}, \tag{4}$$

where $\hat{a}^\dagger_{k,\alpha}$ is the creation operator for a particle of type α and momentum k. With a proper normalization

$$\hat{b}_q \equiv \left(\frac{2\pi}{|q|L}\right)^{1/2} \times \begin{cases} \hat{\rho}_{q,+} & \text{for } q > 0 \\ \hat{\rho}_{q,-} & \text{for } q < 0, \end{cases} \tag{5}$$

the density operators obey Bose commutation relations[5–7]

$$[\hat{b}_q\ ,\ \hat{b}^\dagger_{q'}] = \delta_{q,q'},\quad [\hat{b}_q\ ,\ \hat{b}_{q'}] = 0\ . \tag{6}$$

The key to the exact solution of the model lies in the fact that the kinetic energy can also be expressed in terms of the Bose operators[7]

$$\hat{T} = v_F\sum_{k,\alpha}\alpha k\hat{a}^\dagger_{k,\alpha}\hat{a}_{k,\alpha} = v_F\sum_{q\neq 0}|q|\hat{b}^\dagger_q\hat{b}_q + c(\hat{N}_\alpha)\ . \tag{7}$$

The additional term $c(\hat{N}_\alpha)$ involving the particle number operators is irrelevant in the following and will be dropped. A simple unitary transformation brings the total Hamiltonian $\hat{H} = \hat{T} + \hat{V}$ into the form[7]

$$\hat{H} = \sum_{q\neq 0}|q|\tilde{v}_F(q)\hat{\alpha}^\dagger_q\hat{\alpha}_q, \tag{8}$$

where the $\hat{\alpha}_q$ are new boson operators and

$$\tilde{v}_F(q) = v_F\sqrt{[1 + g_4(q)/(2\pi v_F)]^2 - [g_2(q)/(2\pi v_f)]^2}\ . \tag{9}$$

In Eq. (8) a constant and terms involving particle number operators have been dropped in accordance with the discussion of the particle number terms in Eq. (1).

The parts of the one-particle Green's function which lead to the photoemission and inverse photoemission spectra are

$$iG^<_\alpha(x,t) \equiv \langle\, \hat{\psi}^\dagger_\alpha(0,0)\hat{\psi}_\alpha(x,t)\,\rangle, \tag{10}$$

$$iG^>_\alpha(x,t) \equiv \langle\, \hat{\psi}_\alpha(x,t)\hat{\psi}^\dagger_\alpha(0,0)\,\rangle\ . \tag{11}$$

These functions can be calculated exactly, e.g., by bosonizing the field operators $\hat{\psi}_\alpha(x,t)$.[4,10] For finite systems one obtains[11]

$$iG^{(<)}_\alpha(x,t)e^{i\mu t} = \frac{1}{L}e^{i\alpha k_F x}\exp\left\{\frac{2\pi}{L}\sum_{q>0}\frac{1}{q}\{\, e^{(-)^{+}\, i\alpha qx}e^{\ (+)^{-}\, i\omega_q t} + 2\,s^2(q)[\cos(qx)\,e^{(+)^{-}\, i\omega_q t} - 1]\}\right\}, \tag{12}$$

where $\omega_q \equiv |q|\tilde{v}_F(q)$ and $s^2(q) \equiv \sinh^2(\Theta_q)$ with Θ_q the phase in the unitary transformation $\hat{\alpha}_q = \cosh(\Theta_q)\hat{b}_q - \sinh(\Theta_q)\hat{b}^\dagger_{-q}$. From (12) one calculates the relevant spectral function as

$$\rho^<_\alpha(k,\omega) \equiv \langle\, \phi^N_0 \,|\, \hat{a}^\dagger_{k,\alpha}\delta[\omega + (\hat{H} - E^{N-1}_0)]\hat{a}_{k,\alpha}|\, \phi^N_0\,\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty}dt\, e^{i\omega t}\int_{-\infty}^{\infty}dx\, e^{-ikx}e^{i\mu t}iG^<_\alpha(x,t)\ , \tag{13}$$

$$\rho^>_\alpha(k,\omega) \equiv \langle\, \phi^N_0 \,|\, \hat{a}_{k,\alpha}\delta[\omega - (\hat{H} - E^{N+1}_0)]\hat{a}^\dagger_{k,\alpha}|\, \phi^N_0\,\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty}dt\, e^{i\omega t}\int_{-\infty}^{\infty}dx\, e^{-ikx}e^{i\mu t}iG^>_\alpha(x,t)\ . \tag{14}$$

The low-energy singularities of the spectral functions have been known for a long time.[4,12] They are obtained by taking the limit $L \to \infty$, in which the $G^{(<)}_\alpha(x,t)$ can be calculated analytically in the large x and t limit. The

double Fourier transform yields expressions for the critical exponents of the threshold singularities of the spectra.

The aim of our paper is to present results also for the *nonuniversal* frequency range in $\rho_\alpha^{(<)}(k,\omega)$. In principle this can be done by direct numerical integrations after performing the limit $L \to \infty$. As for a finite-range potential the frequencies ω_q have a nontrivial q dependence, the $G_\alpha^{(<)}(x,t)$ have to be calculated involving a numerical integration which has to be followed by a double numerical Fourier integration. It is numerically difficult to obtain the sharp spectral features by this procedure. It also gives little insight into the interpretation of the calculated spectra. We therefore take a different approach and calculate the spectra for *finite* systems in such a way that the double Fourier integral can be performed analytically. For the simplified model with the g_4 interaction only the limit $L \to \infty$ can be directly read off the results for finite L. As this model already shows very interesting nonuniversal behavior of the spectra we start with a detailed discussion of this g_4 model.

III. SPECTRAL FUNCTIONS FOR THE SPINLESS g_4 MODEL

As the Hamiltonian for this special model is a *sum* of Hamiltonians for right- and left-moving electrons it is sufficient to consider, e.g., the right-moving ones only. In the fermion representation the Hamiltonian reads ($\hat{a}_n \equiv \hat{a}_{k_n,+}, \hat{\rho}_n \equiv \hat{\rho}_{q_n,+}$)

$$\hat{H}_+ = \frac{2\pi}{L}\left[v_F \sum_n n\hat{a}_n^\dagger \hat{a}_n + \frac{1}{4\pi}\sum_{n\neq 0} g_4(q_n)\hat{\rho}_n\hat{\rho}_{-n}\right] \tag{15}$$

while the boson representation, apart from an additional term $c_+(\hat{N}_+)$, is given by ($\hat{b}_n \equiv \hat{b}_{q_n}$)

$$\hat{H}_+ = \frac{2\pi}{L}\left[v_F \sum_{n>0} n\hat{b}_n^\dagger \hat{b}_n + \frac{1}{4\pi}\sum_{n>0} n g_4(q_n)(\hat{b}_n^\dagger\hat{b}_n + \hat{b}_n\hat{b}_n^\dagger)\right]. \tag{16}$$

In this representation it is obvious that one can read off the exact energy eigenvalues *without* a canonical transformation. The eigenstates are *identical* to the eigenstates of the *noninteracting system.* Especially the interacting ground state is given by the Fermi sea $|F_+(N)\rangle$, which has the form of a Slater determinant. The general eigenstates have the form

$$|\{m_j\},N\rangle = \prod_{j=1}^{\infty}\left(\frac{1}{m_j!}\right)^{1/2}(\hat{b}_j^\dagger)^{m_j}|F_+(N)\rangle\,. \tag{17}$$

These are *linear combinations* of electron-hole pair excited states which yield the same value of the kinetic energy. It is instructive to write out the states with low excitation energy to see the *high degeneracy* of states with the same kinetic energy.

The spectral functions for the simplified model can be calculated directly *without* using the bosonization of the field operators. The spectral weights of the δ peaks, e.g., in $\rho_+^>(k_n,\omega)$, are given by $|\langle\{m_j\},N+1|\hat{a}_n^\dagger|F_+(N)\rangle|^2$ and can be calculated using $\hat{a}_n^\dagger|F_+(N)\rangle = \hat{a}_n^\dagger\hat{a}_{n_F+1}|F_+(N+1)\rangle$, where $n_F = Lk_F(N)/(2\pi)$, and

$$\hat{b}_n^\dagger = \left(\frac{1}{n}\right)^{1/2}\sum_m \hat{a}_{m+n}^\dagger\hat{a}_m \tag{18}$$

for $n \geq 1$. For $n = n_F+1$ and $n = n_F+2$ the states $\hat{a}_n^\dagger|F_+(N)\rangle$ are eigenstates: $\hat{a}_{n_F+1}^\dagger|F_+(N)\rangle = |F_+(N+1)\rangle$, $\hat{a}_{n_F+2}^\dagger|F_+(N)\rangle = \hat{b}_1^\dagger|F_+(N+1)\rangle$. For $n = n_F+1+\tilde{n}$ with $\tilde{n} \geq 1$ the state $\hat{a}_n^\dagger|F_+(N)\rangle = \hat{a}_{n_F+1+\tilde{n}}^\dagger\hat{a}_{n_F+1}|F_+(N+1)\rangle$ has overlap to the states $\hat{b}_{\tilde{n}}^\dagger|F_+(N+1)\rangle$, $\hat{b}_1^\dagger\hat{b}_{\tilde{n}-1}^\dagger|F_+(N+1)\rangle$, $\hat{b}_2^\dagger\hat{b}_{\tilde{n}-2}^\dagger|F_+(N+1)\rangle$, $(1/2!)^{1/2}(\hat{b}_1^\dagger)^2\hat{b}_{\tilde{n}-2}^\dagger|F_+(N+1)\rangle$ etc., i.e., to states $|\{m_j\},N+1\rangle$ with $\sum_{j\geq 1} jm_j = \tilde{n}$. For large $\tilde{n}$ these eigenstates are rather complicated linear combinations of electron-hole pair excited states, but the expansion coefficient of the component $\hat{a}_{n_F+1+\tilde{n}}^\dagger\hat{a}_{n_F+1}|F_+(N+1)\rangle$ is simple. The square of the overlap is given by

$$|\langle\{m_j\},N+1|\hat{a}_{n_F+1+\tilde{n}}^\dagger|F_+(N)\rangle|^2 = \prod_{j\geq 1}\frac{1}{m_j!}\left(\frac{1}{j}\right)^{m_j} \equiv A(\{m_j\})\,. \tag{19}$$

The factor $1/m_j!$ is due to the corresponding factor in Eq. (17) while the factor $(1/j)^{m_j}$ comes from the prefactor on the rhs of Eq. (18). This yields for the spectral function

$$\rho^>(k_{n_F+1+\tilde{n}},\omega) = \sum_{\{m_j\}}\delta_{\sum_j jm_j,\tilde{n}}A(\{m_j\})\delta\left(\omega - \sum_j m_j\omega_j\right), \tag{20}$$

where $\omega_j = (2\pi/L)[v_F + g_4(q_j)/(2\pi)]$. Alternatively this result can be obtained using Eq. (12) for $s^2(q) \equiv 0$ by formally expanding the exponential function and performing the double Fourier integral in Eq. (14) analytically. We will discuss this procedure in more detail later.

The calculation of $\rho^>(k_n,\omega)$ is therefore reduced to the combinational problem to find all decompositions

$$m_1 + 2m_2 + 3m_3 + \cdots + \tilde{n}m_{\tilde{n}} = \tilde{n} \tag{21}$$

with $m_j \in \mathbb{N}_0$. The solution can be easily produced on a computer, but for large $\tilde{n}$ the number of decompositions increases exponentially. For the special q dependence of g_4 used in the following:

$$g_4(k) = g_4\Theta(k_c^2 - k^2), \tag{22}$$

where $r_c = 1/k_c$ is the effective range of the interaction, we will therefore also use a different technique. The k-dependent Fermi velocity $\tilde{v}_F(k)$ takes only two different values with this assumption

$$\tilde{v}_F(k) = \begin{cases} \tilde{v}_F = v_F\,[1 + g_4/(2\pi v_F)] & \text{for } 0 < k \le k_c \\ v_F & \text{for } k > k_c\,. \end{cases} \quad (23)$$

For a finite system this means that $g_4(k_n) = g_4$ for $1 \le n \le n_c$ where $n_c = Lk_c/(2\pi)$ and $g_4(k_n) = 0$ for $n > n_c$.

From Eq. (20) it follows that $\rho^>(k_n, \omega)$ is *trivial for* $0 < k_n - k_F < k_c$, as m_j can only be different from zero for $j \le \tilde{n}$ and *all* corresponding ω_j are given by $(2\pi/L)\tilde{v}_F$. This yields ($\tilde{k} = k - k_F$)

$$\rho^>(k_F + \tilde{k}, \omega) = \delta(\omega - \tilde{v}_F\tilde{k}) \text{ for } 0 < \tilde{k} < k_c, \quad (24)$$

i.e., Fermi-liquid-like behavior. This is due to the special choice (22) for $g_4(k)$. For $\tilde{k} > k_c$ the spectral functions are nontrivial. One has to distinguish the intervals $mk_c < \tilde{k} < (m+1)k_c$. We discuss in the following small values of m and the limit $m \to \infty$. For $m = 1$ it is still very simple to argue in terms of the decompositions in Eq. (21). For $n_c < \tilde{n} < 2n_c$ one can have at most *one* nonzero m_l ($m_l = 1$) for $n_c < l \le \tilde{n}$. The remaining "momentum" $\tilde{n} - l$ can be decomposed into momenta j which are smaller than n_c. Therefore the energy for all these decompositions is given by $(2\pi/L)\,[lv_F + (\tilde{n} - l)\tilde{v}_F] = (2\pi/L)\,[\tilde{n}v_F + (\tilde{n} - l)(\tilde{v}_F - v_F)]$. The corresponding weight is $1/l$. The remaining weight lies in a δ peak at $(2\pi/L)\tilde{n}\tilde{v}_F$, which corresponds to the decompositions of $\tilde{n}$ with nonzero m_j only for $j \le n_c$. The limit $L \to \infty$ can easily be read off and one obtains

$$\rho^>(k_F + \tilde{k}, \omega) = \frac{\Theta(\omega - v_F\tilde{k})\Theta[v_F\tilde{k} + (\tilde{v}_F - v_F)(\tilde{k} - k_c) - \omega]}{\tilde{v}_F\tilde{k} - \omega} + [1 - \ln(\tilde{k}/k_c)]\delta(\omega - \tilde{v}_F\tilde{k})\,. \quad (25)$$

The weight $z(\tilde{k})$ of the δ peak decreases continuously from 1 to $1 - \ln 2 \approx 0.307$ when $\tilde{k}$ increases from k_c to $2k_c$. The shape of the spectral weight is shown in Fig. 1(b). If one further increases $\tilde{k}$ the calculation of the spectrum using Eq. (20) becomes more and more tedious. We will therefore analyze the spectra by another method.

In the limit $\tilde{k} \to \infty$ one might expect that the spectral function $\rho^>(k_F + \tilde{k}, \omega)$ reduces to a δ function as for noninteracting electrons. That this is *not* the case can be seen quite generally by calculating the first and second moment of $\rho^>(k_F + \tilde{k}, \omega)$. As shown in Appendix A, the result for $\Delta_{\tilde{k}}^2 = \mu_2^>(k_F + \tilde{k}) - [\mu_1^>(k_F + \tilde{k})]^2$ for the special model of Eq. (22) for $\tilde{k} > 2k_c$ is given in the limit $L \to \infty$ by

$$\Delta_{\tilde{k}}^2 = \frac{1}{2}k_c^2(\tilde{v}_F - v_F)^2 = \frac{1}{2}[\mu_1^>(k_F + \tilde{k}) - v_F\tilde{k}]^2, \quad (26)$$

i.e., the effective width of the spectrum is *independent* of $\tilde{k}$ as soon as $\tilde{k}$ is larger than $2k_c$.

We now present a method to calculate the spectra using Eq. (12). As in the following we measure energies with respect to μ and momenta with respect to k_F we have $iG^>(x,t) = \exp[F(x,t)]/L$, where for $g_4(k)$ given by Eq. (22)

$$F(x,t) = \sum_{n=1}^{n_c} \frac{1}{n} \exp\left(i\frac{2\pi}{L}n\,[x - \tilde{v}_F t]\right) + \sum_{n=n_c+1}^{\infty} \frac{1}{n} \exp\left(i\frac{2\pi}{L}n\,[x - v_F t]\right), \quad (27)$$

as $s^2(q) \equiv 0$ for $g_2(k) \equiv 0$. The second sum on the rhs of Eq. (27) is not convergent as it stands. There are two obvious methods to overcome this difficulty. One can add a factor $\exp(-0n)$ or restrict the sum to $n \le M$, where $M(2\pi/L)$ is much larger than the momenta one is interested in. Both procedures give the *same* results for the spectra. Using the first method we can write Eq. (27) as

$$F(x,t) = -\ln\left[1 - \exp\left(i\frac{2\pi}{L}\,[x - v_F t + i0]\right)\right] + \sum_{n=1}^{n_c} \frac{1}{n}\left[\exp\left(i\frac{2\pi}{L}n\,[x - \tilde{v}_F t]\right) - \exp\left(i\frac{2\pi}{L}n\,[x - v_F t]\right)\right]. \quad (28)$$

Unfortunately the finite sums cannot be summed in closed form. We therefore write

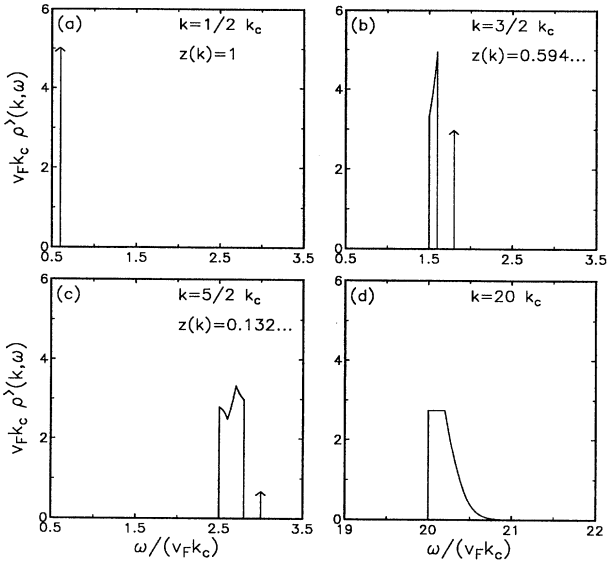

FIG. 1. Spectral function $\rho^>(k,\omega)$ of the spinless g_4 model as a function of $\omega/(v_F k_c)$ for $\tilde{v}_F = 1.2v_F$ and different momenta as indicated in the figures. The arrows represent δ peaks with weight $z(k)$.

$$iG^{>}(x,t) = \frac{1/L}{1-\exp\left(i\frac{2\pi}{L}\left[x-v_F t+i0\right]\right)} \times \prod_{m=1}^{n_c}\left[\sum_{j=1}^{\infty}\frac{(-1/m)^j}{j!}\exp\left(i\frac{2\pi}{L}mj\left[x-v_F t\right]\right)\right]\left[\sum_{l=1}^{\infty}\frac{(1/m)^l}{l!}\exp\left(i\frac{2\pi}{L}ml\left[x-\tilde{v}_F t\right]\right)\right] = \frac{1}{L}\sum_{m=0}^{\infty}a_m^{(n_c)}\exp\left(i\frac{2\pi}{L}m\left[x-v_F t\right]\right)\sum_{l=0}^{\infty}b_l^{(n_c)}\exp\left(i\frac{2\pi}{L}l\left[x-\tilde{v}_F t\right]\right), \tag{29}$$

where the $a_m^{(n_c)}$ and $b_l^{(n_c)}$ can be determined iteratively for $m \geq 1$, $l \in \mathbb{N}_0$, and $i = 0, \ldots, m-1$,

$$a_{lm+i}^{(m+1)} = \sum_{j=0}^{l}\frac{(-1/m)^j}{j!}a_{m(l-j)+i}^{(m)}\,, \qquad b_{lm+i}^{(m+1)} = \sum_{j=0}^{l}\frac{(1/m)^j}{j!}b_{m(l-j)+i}^{(m)}\,. \tag{30}$$

The starting values are $b_m^{(1)} = 1/m!$ and

$$a_m^{(1)} = \sum_{j=0}^{m}(-1)^j/j!\,. \tag{31}$$

Using Eq. (23) the double Fourier transform can be trivially performed ($k_F \equiv 0$)

$$\rho^{>}(k_n,\omega) = \sum_{l=0}^{n}a_{n-l}^{(n_c)}b_l^{(n_c)}\delta\left(\omega-\frac{2\pi}{L}\left[(n-l)\,v_F+l\tilde{v}_F\right]\right)\,. \tag{32}$$

Compared to Eq. (20) this representation avoids the combinatorial problem of Eq. (21) and can be used numerically for much larger values of n_c than Eq. (20). Some important information about the $a_m^{(n_c)}$ and $b_m^{(n_c)}$ can be obtained *analytically*. For the case of the $a_i^{(n_c)}$ it is useful to go back to Eq. (27), where one can directly read off $a_0^{(n_c)} = 1$, $a_i^{(n_c)} = 0$ for $i = 1, \ldots, n_c$, $a_i^{(n_c)} = 1/i$ for $i = n_c+1, \ldots, 2n_c+1$, etc. The limit $m \to \infty$ of $a_m^{(n_c)}$ for fixed n_c follows from Eqs. (30) and (31)

$$a_m^{(n_c)} \to \prod_{n=1}^{n_c}\left(\frac{1}{e}\right)^{1/n}\,. \tag{33}$$

In the large-n_c limit one therefore obtains $a_m^{(n_c)} \to e^{-C}/n_c \approx 0.56/n_c$ where C is Euler's constant. The behavior of the $b_m^{(n_c)}$ for m values of the order of n_c follows from Eq. (27) if one writes the first term on the rhs as a difference of a logarithm similar to the first term on the rhs of Eq. (28) and the sum running from n_c+1 to infinity. This yields $b_i^{(n_c)} = 1$, for $i = 1, \ldots, n_c$,

$$b_i^{(n_c)} = 1 - \left(\frac{1}{n_c+1}+\frac{1}{n_c+2}+\cdots+\frac{1}{i}\right) \tag{34}$$

for $n_c+1 \leq i < 2n_c$, etc.

In this approach it is very simple to obtain the *exact analytical result* for the spectrum in the *large momentum limit* $k \gg k_c$

$$\rho^{>}(k_n,\omega) \to \prod_{m=1}^{n_c}\left(\frac{1}{e}\right)^{1/m}\sum_{l=0}^{\infty}b_l^{(n_c)}\delta\left(\omega-\frac{2\pi}{L}\left[nv_F+l\left(\tilde{v}_F-v_F\right)\right]\right)\,. \tag{35}$$

We discuss this for the special case $n_c = 1$ and $n_c \to \infty$ only. For $n_c = 1$ we have $b_l^{(1)} = 1/l!$ and the spectrum is a *Poisson distribution* with a unit strength parameter. In the thermodynamic limit $L \to \infty$, $n_c \to \infty$ with $k_c = n_c(2\pi/L) = \text{const}$, the spectrum is given by ($\delta v_F \equiv \tilde{v}_F - v_F$)

$$\rho^{>}(k,\omega) = \frac{e^{-C}}{\delta v_F k_c}\times\begin{cases}1 & \text{for } 0 < \omega - v_F k < \delta v_F k_c\\ 1-\ln\left(\frac{\omega-v_F k}{\delta v_F k_c}\right) & \text{for } \delta v_F k_c < \omega - v_F k < 2\delta v_F k_c\,,\end{cases} \tag{36}$$

etc. This behavior is shown in Fig. 1(d). Figure 1 summarizes our results for the g_4 model for $\tilde{v}_F = 1.2v_F$, i.e., a repulsive interaction. In Fig. 1(a) all spectral weight lies in a δ peak at $\omega = \tilde{v}_F k$ ($z = 1$). For $k > k_c$ there continues to be a δ peak at $\omega = \tilde{v}_F k$, but its weight decreases rapidly with increasing momentum as shown in Figs. 1(b) and 1(c). The additional weight lies in the continuous part of the spectrum as discussed analytically in Eqs. (25) and (36). In the limit $k \gg k_c$ one does not recover the limit of noninteracting electrons. The *shape* of the spectrum becomes *independent* of the interaction strength as a function of the scaled variable $(\omega - v_F k_F)/(k_c \delta v_F)$. A calculation of the spectrum using perturbation theory for the self-energy to low order (e.g., second order) completely fails to give the correct shape of the spectrum.

Another quantity of interest is the total spectral density per unit length

$$\rho^{>}(\omega) = \int_0^\infty \frac{dk}{2\pi}\rho^{>}(k,\omega)$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, G^{>}(0,t)\,. \tag{37}$$

For the special g_4 interaction (22) the function $\rho^{>}(k,\omega)$ has been given analytically in Eqs. (24) and (25). The k integration can be simply performed and yields

$$\rho^{>}(\omega) = \frac{1}{2\pi\tilde{v}_F}\times\begin{cases} 1 & \text{for } 0 \le \omega < v_F k_c \\ 1+\ln\left(\frac{\omega}{v_F k_c}\right) & \text{for } v_F k_c < \omega < \tilde{v}_F k_c \\ 1+\ln\left(\frac{\tilde{v}_F}{v_F}\right) & \text{for } \tilde{v}_F k_c < \omega < 2v_F k_c\,, \end{cases} \tag{38}$$

etc. Alternatively the function $F(0,t)$ in Eq. (27) can be calculated numerically in the limit $L \to \infty$ and $G^{>}(0,t)$ is Fourier transformed numerically. The results in both approaches agree and are shown in Fig. 2. The latter method can be performed for *arbitrary* interactions $g_4(k)$. Results for $g_4(k) = g_4 \exp(-|k|/k_c)$ are also shown in Fig. 2 for a positive and a negative value of g_4. For the case of a repulsive interaction there is a *depletion* of the total spectral weight at low frequencies. Due to a sum rule discussed in Ref. 13 the missing weight has to show up in another frequency range. For a model without an upper cutoff in momentum space the missing weight is pushed to *infinity*. If a cutoff is included the missing weight appears at the upper end of the spectrum, as the spectral weight in Fig. 1(d) has to be compared to a δ peak at $v_F k_{\max}$ for noninteracting electrons.

In this section the simple g_4 model has been discussed quite at length, as the generalization to the complete model including g_2 terms has to be performed mainly numerically.

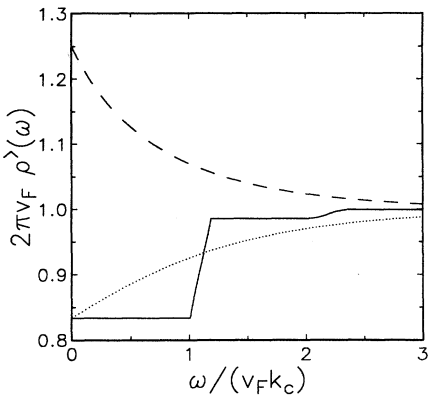

FIG. 2. Total spectral density $\rho^{>}(\omega)$ of the spinless g_4 model as a function of $\omega/(v_F k_c)$. The solid curve shows the result for the step model with $\tilde{v}_F = 1.2 v_F$, the dotted curve the result for the exponential model with $g_4/(2\pi v_F) = 0.2$ (repulsive interaction), and the dashed curve the result for the exponential model with $g_4/(2\pi v_F) = -0.2$ (attractive interaction).

IV. SPECTRAL FUNCTIONS FOR THE GENERAL SPINLESS MODEL

The method to calculate $G^{>}(x,t)$ presented in Eq. (29) is generalized to the full spinless model introduced in Sec. II. We specialize to interactions of the type described in (22), i.e., we also assume $g_2(k) = g_2\Theta(k_c^2 - k^2)$. Then using Eq. (12) the expression for $F(x,t)$ in (27) is generalized to [$s^2 = s^2(q_n)$ for $1 \le n \le n_c$]

$$F(x,t) = \sum_{n=1}^{n_c}\frac{1}{n}\Bigg[(1+s^2)\exp\left(i\frac{2\pi}{L}n\,[x-\tilde{v}_F t]\right) + s^2\exp\left(-i\frac{2\pi}{L}n\,[x+\tilde{v}_F t]\right) - 2s^2 - \exp\left(i\frac{2\pi}{L}n\,[x-v_F t]\right)\Bigg] - \ln\left[1-\exp\left(i\frac{2\pi}{L}n\,[x-v_F t+i0]\right)\right]. \tag{39}$$

Again $iG^{>}(x,t) = \exp[F(x,t)]/L$ is written as a product of power series

$$iG^{>}(x,t) = \frac{1}{L}\left[\sum_{m=0}^{\infty} a_m^{(n_c)}\exp\left(i\frac{2\pi}{L}m\,[x-v_F t]\right)\right] \times\left[\sum_{l=0}^{\infty} b_l^{(n_c)}\exp\left(i\frac{2\pi}{L}l\,[x-\tilde{v}_F t]\right)\right] \times\left[\sum_{r=0}^{\infty} c_r^{(n_c)}\exp\left(-i\frac{2\pi}{L}r\,[x+\tilde{v}_F t]\right)\right] \times\exp\left(-2s^2\sum_{n=1}^{n_c}\frac{1}{n}\right), \tag{40}$$

where the expansion coefficients are determined iteratively as in Sec. III. For $m \ge 1$, $l \in \mathbb{N}_0$, and $i = 0,\ldots,m-1$, one obtains

$$a_{lm+i}^{(m+1)} = \sum_{j=0}^{l}\frac{(-1/m)^j}{j!}a_{m(l-j)+i}^{(m)},$$
$$b_{lm+i}^{(m+1)} = \sum_{j=0}^{l}\frac{([1+s^2]/m)^j}{j!}b_{m(l-j)+i}^{(m)}, \tag{41}$$
$$c_{lm+i}^{(m+1)} = \sum_{j=0}^{l}\frac{(s^2/m)^j}{j!}c_{m(l-j)+i}^{(m)}\,.$$

The starting values are $b_m^{(1)} = (1+s^2)^m/m!$, $c_m^{(1)} = s^{2m}/m!$, and $a_m^{(1)}$ is given by Eq. (31). Using Eq. (40) the double Fourier transform can be simply performed [$A = \exp(\sum_{n=1}^{n_c} 1/n)$]

$$\rho^{>}(k_n,\omega) = A^{-2s^2}\sum_{r=0}^{\infty}\sum_{j=0}^{n+r} c_r^{(n_c)}a_{n+r-j}^{(n_c)}b_j^{(n_c)} \times\delta\left(\omega - \frac{2\pi}{L}[(n+r-j)\,v_F + (r+j)\,\tilde{v}_F]\right). \tag{42}$$

If we write $(n+r-j)v_F+(r+j)\tilde{v}_F = nv_F + j(\tilde{v}_F - v_F) + r(\tilde{v}_F + v_F)$, it is obvious that $\rho^>(k_n,\omega) \equiv 0$ for $\omega < k_n v_F$ is guaranteed only for $\tilde{v}_F - v_F > 0$, i.e., for *repulsive* interactions if $g_4 = g_2$.

The coefficients $a_m^{(n_c)}$ are the same as in Sec. III. The behavior of the $b_m^{(n_c)}$ and $c_m^{(n_c)}$ for $m \le n_c$ follow from the identity

$$\exp\left(\gamma\sum_{n=1}^{n_c}\frac{1}{n}z^n\right) = (1-z)^{-\gamma}\exp\left(-\gamma\sum_{n=n_c+1}^{\infty}\frac{1}{n}z^n\right)$$
$$= \left[1+\sum_{m=1}^{\infty}\left(\prod_{j=1}^{m}\left[1+\frac{\gamma-1}{j}\right]\right)z^m\right]$$
$$\times\exp\left(-\gamma\sum_{n=n_c+1}^{\infty}\frac{1}{n}z^n\right). \quad (43)$$

For $1 \le m \le n_c$ the expansion coefficients are therefore given by the first factor on the rhs of Eq. (43), i.e.,

$$b_m^{(n_c)} = \prod_{j=1}^{m}\left(1+\frac{s^2}{j}\right) \overset{n_c\ge m\gg 1}{\longrightarrow} \text{const}\times m^{s^2},$$
$$c_m^{(n_c)} = \prod_{j=1}^{m}\left(1+\frac{s^2-1}{j}\right) \overset{n_c\ge m\gg 1}{\longrightarrow} \text{const}\times m^{s^2-1}. \quad (44)$$

This power-law behavior of the coefficients $b_m^{(n_c)}$ and $c_n^{(n_c)}$ is responsible for the low-energy power-law singularities of the spectral functions and the power-law behavior of the momentum distribution $n(k)$ discussed in Appendix B. To demonstrate this we consider momenta $1 \ll n \ll n_c$ and frequencies of the order $\tilde{v}_F(2\pi n/L)$ in Eq. (42). As discussed before Eq. (33), $a_m^{(n_c)} = \delta_{m,0}$ for $m \le n_c$. Therefore the spectral function simplifies in this regime to

$$\rho^>(k_n,\omega) = A^{-2s^2}\sum_{r\ge 0} c_r^{(n_c)} b_{n+r}^{(n_c)}\delta\left(\omega - \frac{2\pi}{L}[n+2r]\,\tilde{v}_F\right), \quad (45)$$

i.e., the spectrum consists of δ peaks at $\omega = \tilde{v}_F k_n + \tilde{v}_F(4\pi r/L)$. For $1 \ll r+n \ll n_c$ we can use the asymptotic form of the coefficients in Eq. (44) at $r = (\omega - \tilde{v}_F k_n)/(4\pi\tilde{v}_F/L)$ and $n+r = (\omega+\tilde{v}_F k_n)/(4\pi\tilde{v}_F/L)$ to obtain the weights of the peaks. In the limit $L\to\infty$ this yields

$$\rho^>(k_n,\omega) \sim \Theta(\omega-\tilde{v}_F k)(\omega-\tilde{v}_F k)^{s^2-1}(\omega+\tilde{v}_F k)^{s^2}, \quad (46)$$

i.e., the well-known asymptotic behavior for $k \ll k_c$ and $\omega - \tilde{v}_F k \ll \tilde{v}_F k_c$.[4,11]

In the opposite limit $k \gg k_c$ the coefficients $a_{n+r-j}^{(n_c)}$ in Eq. (42) can be replaced by the constant introduced in Eq. (33). Near the threshold at $\omega = v_F k_n$ Eq. (44) can be used again for the $c_r^{(n_c)}$ and $b_j^{(n_c)}$. Performing the integrations in the limit $L\to\infty$ the finite step at threshold in Eq. (36) is replaced by a power-law behavior $(\omega - v_F k)^{2s^2}$. For arbitrary values of k_n and ω the spectral function $\rho^>(k_n,\omega)$ has to be calculated numerically. Results for the same k values as used for the g_4 model in Sec. III and $g_2(k) \equiv g_4(k)$ are shown in Figs. 3(a)–3(d). The figures show that the δ peak at $\omega = \tilde{v}_F k$ is changed into a power-law behavior and one could analytically show that the critical exponent for $\omega \searrow \tilde{v}_F k$ is given by $s^2 - 1$ as in Eq. (46). The shape of the continuous part of the spectra of the g_4 model is modified quite considerably for the parameters used ($s^2 = 1/8$ corresponding to $\tilde{v}_F = 2v_F$). In contrast to the g_4 model

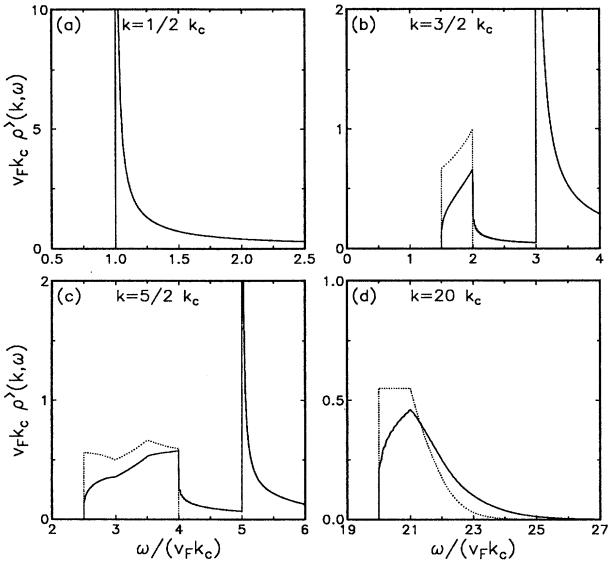

FIG. 3. Same as in Fig. 1, but for the full $g_2 = g_4$ spinless Luttinger model for $\tilde{v}_F = 2v_F$. The small oscillations in the full curves are a finite-size effect. The dotted curves show the continuous part of the related spectral function for the spinless g_4 model with $\tilde{v}_F = 2v_F$.

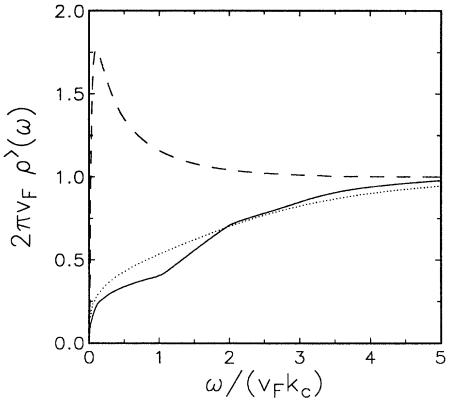

FIG. 4. Same as in Fig. 2, but for the full $g_2 = g_4$ spinless Luttinger model. The solid curve shows the result for the step model with $\tilde{v}_F = 2v_F$, the dotted curve the result for the exponential model with $g/(\pi v_F) = 3$ (repulsive interaction), and the dashed curve the result for the exponential model with $g/(\pi v_F) = -0.98$ (attractive interaction).

$\rho^{>}(k,\omega) \neq 0$ for $k < 0$ and small $|k|$,[11] but the spectral weight for negative k decreases rapidly for increasing $|k|$.

In Fig. 4 we show the integrated spectral weight for $g_4(k) \equiv g_2(k)$ for the step model (22) as well as the exponential model $g_4(k) \equiv g_2(k) = g\exp(-|k|/k_c)$ used also in Fig. 2. At low frequencies the power-law behavior proportional to ω^{2s^2} leads to a suppression of spectral weight.[14] For the case of an *attractive* interaction this leads to a peak in $\rho^{>}(\omega)$.

V. GENERALIZATIONS FOR THE MODEL INCLUDING SPIN

For the Luttinger model including spin the discussion in Sec. II applies up to Eq. (4). If the density operators $\hat{\rho}_{q,\alpha}$ are decomposed into a sum of particle-hole operators an additional spin summation occurs. It is then useful to define charge and spin operators[15]

$$\hat{\rho}_{q,\alpha} \equiv \hat{\rho}_{q,\alpha,\uparrow} + \hat{\rho}_{q,\alpha,\downarrow}, \quad \hat{\sigma}_{q,\alpha} \equiv \hat{\rho}_{q,\alpha,\uparrow} - \hat{\rho}_{q,\alpha,\downarrow} . \tag{47}$$

With a normalization which differs by a factor of $\sqrt{2}$, the analogous definition to Eq. (5) reads

$$\hat{b}_{q,c} \equiv \left(\frac{\pi}{|q|L}\right)^{1/2} \times \begin{cases} \hat{\rho}_{q,+} & \text{for } q > 0 \\ \hat{\rho}_{q,-} & \text{for } q < 0, \end{cases}$$

$$\hat{b}_{q,s} \equiv \left(\frac{\pi}{|q|L}\right)^{1/2} \times \begin{cases} \hat{\sigma}_{q,+} & \text{for } q > 0 \\ \hat{\sigma}_{q,-} & \text{for } q < 0. \end{cases} \tag{48}$$

These operators describe independent boson degrees of freedom. Again the kinetic energy can be expressed in terms of the boson operators and particle number operators

$$\hat{T} = v_F \sum_{q\neq 0} |q| (\hat{b}^{\dagger}_{q,c}\hat{b}_{q,c} + \hat{b}^{\dagger}_{q,s}\hat{b}_{q,s}) + c(\hat{N}) . \tag{49}$$

For the spin-independent interaction (1) the spin degrees are *not* renormalized by including the interaction, i.e., $\tilde{v}_{F,s}(q) \equiv v_F$. For the charge degrees of freedom the problem to find the exact eigenstates is equivalent to the spin-independent problem. The additional factor $\sqrt{2}$ in Eq. (48) modifies Eq. (9) to

$$\tilde{v}_{F,c}(q) = v_F \sqrt{[1 + g_4(q)/(\pi v_F)]^2 - [g_2(q)/(\pi v_f)]^2} . \tag{50}$$

For the propagators the changes are more dramatic. If one denotes the propagator for the spinless model by $G^{(<)}_{\alpha}(x,t;g_2,g_4)$, one obtains, using the bosonization of the fermion field operators

$$iG^{(<)}_{\alpha,\sigma}(x,t;g_2,g_4) = [iG^{(<)}_{\alpha}(x,t;0,0) iG^{(<)}_{\alpha}(x,t;2g_2,2g_4)]^{1/2} . \tag{51}$$

For the g_4 model the more intuitive approach to calculate the spectral functions via the Lehmann representation described for the spinless model in Sec. III can be generalized and leads to a combinatorical problem slightly more complicated than in Eq. (21).

In the following we restrict the discussion to the step model (22). For the g_4 model the spectral function has the same form as (32)

$$\rho^{>}(k_n,\omega) = \sum_{l=0}^{n} \tilde{\alpha}^{(n_c)}_{n-l}\beta^{(n_c)}_{l} \times \delta\left(\omega - \frac{2\pi}{L}[nv_F + l(\tilde{v}_F - v_F)]\right) . \tag{52}$$

The coefficients $\beta^{(n_c)}_l$ are obtained from the power series with the coefficients $b^{(n_c)}_m(2g_4)$ by taking the square root, i.e.,

$$\beta^{(n_c)}_l = \left(b^{(n_c)}_l - \sum_{j=1}^{l-1} \beta^{(n_c)}_{l-j}\beta^{(n_c)}_j\right) \Big/ (2\beta^{(n_c)}_0) . \tag{53}$$

The coefficients $\tilde{\alpha}^{(n_c)}_{n-l}$ are obtained from $\tilde{a}^{(n_c)}_m = \sum_{n=0}^{m} a^{(n_c)}_n(2g_4)$ by the same procedure. For the g_4 model *including* spin the spectral density differs from a simple δ peak already for $k < k_c$. For $l \leq n_c$ the coefficients $\tilde{\alpha}^{(n_c)}_l$ and $\beta^{(n_c)}_l$ can be given explicitly as $\tilde{a}^{(n_c)}_m = 1 = b^{(n_c)}_m$ for $m \leq n_c$. For $1 \ll l \ll n_c$ one has $\tilde{\alpha}^{(n_c)}_l = \beta^{(n_c)}_l \sim l^{-1/2}$. In the limit $L \to \infty$ this yields for $0 < k < k_c$ and $\tilde{v}_F \equiv \tilde{v}_{F,c} > v_F$

$$\rho^{>}(k,\omega) = \text{const} \times ([\omega - v_F k][\tilde{v}_F k - \omega])^{-1/2} \times \Theta(\omega - v_F k)\Theta(\tilde{v}_F k - \omega), \tag{54}$$

as discussed previously.[11,16,17] Note that this non-Fermi-liquid-like behavior occurs, but one still has $n(k) = 0$ for $k > k_F = 0$. The behavior of the spectral function is shown in Fig. 5 (dotted curves). Even for $k \gg k_c$ there remains a square-root singularity at threshold which is "intermediate" between the δ peak expected from the first factor in Eq. (51) and the plateau from the second factor.

For the general model including spin the spectral function follows from Eq. (51) using Eq. (40)

$$\rho^{>}(k_n,\omega) = A^{-s^2} \sum_{r=0}^{\infty}\sum_{j=0}^{n+r} \tilde{\alpha}^{(n_c)}_{n+r-j}\beta^{(n_c)}_j\gamma^{(n_c)}_r \times \delta\left(\omega - \frac{2\pi}{L}[(n+r-j)v_F + (r+j)\tilde{v}_F]\right), \tag{55}$$

where the coefficients $\beta^{(n_c)}_j$ and $\gamma^{(n_c)}_r$ follow from the coefficients $b^{(n_c)}_m(2g_2,2g_4)$ and $c^{(n_c)}_l(2g_2,2g_4)$ defined in Eq. (41) by the procedure (53) to take the square root of a power series, while the $\tilde{\alpha}^{(n_c)}_i$ follow from the $\tilde{a}^{(n_c)}_m = \sum_{i=1}^{m} a^{(n_c)}_i(2g_2,2g_4)$ in the analogous way. In the low-energy regime Eq. (55) leads to the power-law behavior discussed in detail in Refs. 11 and 17. This and the spectral behavior for larger values of k is shown in Fig. 5

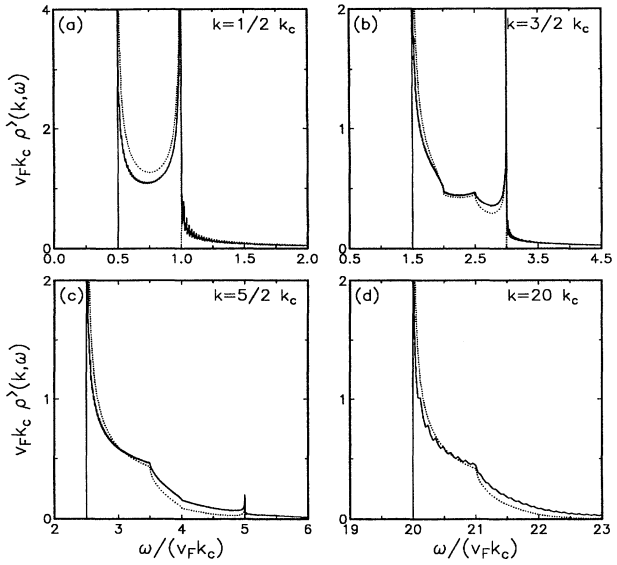

FIG. 5. Same as in Fig. 1, but for the model including spin for $\tilde{v}_F = 2v_F$. The dotted curves represent the results for the g_4 model including spin and the full curves the related results for the full $g_2 = g_4$ Luttinger model including spin. The small oscillations are again a finite-size effect.

(solid curves), where the results of the full model are compared to the g_4 model. For the value of s^2 used the nonuniversal features of the spectra are rather similar, i.e., they can be largely understood examining the much simpler g_4 model.

Results for the integrated spectral density $\rho^>(\omega)$ are qualitatively the same as the corresponding curves for the spinless case shown in Fig. 4.

VI. SUMMARY

In the preceding sections we have presented results for the spectral functions $\rho_+^>(k,\omega)$ and $\rho_+^>(\omega)$ relevant for inverse photoemission. The corresponding functions to describe photoemission are given by the mirror images of the curves presented as $\rho_+^>(k_F+\tilde{k},\omega) = \rho_+^<(k_F-\tilde{k},-\omega)$. In the Luttinger model with its linear energy dispersion the value of the Fermi momentum is irrelevant and has been set to zero in Secs. III–V. If we want to describe nonrelativistic electrons this is no longer the case as k_F is proportional to the electron density. In order to apply results using the Luttinger model one has to keep in mind that the linearization procedure is only allowed for sufficiently *long-range interactions*, i.e., $k_c \ll k_F$.

In Ref. 3 angular-integrated high-resolution photoemission data of quasi-one-dimensional conductors are presented which show a depletion near the Fermi energy and a rather broad peak about 1 eV below the Fermi level. As these spectra are related to $\rho^<(\omega)$ the peak below the Fermi level reminds one of the peak in one of the integrated spectral functions in Fig. 4. But this spectrum corresponds to an *attractive interaction*. Such a peak was first discussed by Suzumura,[13] who obtained it for the case $g_2 > 0$, $g_4 = 0$, i.e., a *repulsive* g_2 interaction but *neglecting* the g_4 interaction. As one can see from Eq. (9) or (50) this leads to $\tilde{v}_F(k \to 0) < v_F$, i.e., an *increase* of spectral weight with decreasing $|\omega|$ until the power-law factor ω^α ($\alpha = 2s^2$ for the spinless model, $\alpha = s^2$ for the model including spin with spin-independent interaction) leads to suppression and the peak emerges. As discussed in Sec. II the *physical model corresponds to* $g_4 = g_2$, i.e., it is *unphysical* to neglect the g_4 interaction for finite-range interactions. Therefore the experimental peak below the Fermi level cannot be explained as a Luttinger liquid feature for repulsive interactions. In order to explain the experimental depletion near the Fermi level a surprisingly large value of the exponent has to be assumed.[14]

In this paper we have presented a detailed study of the nonuniversal spectral properties of the Luttinger model and have shown both analytically and numerically that a suprisingly rich variety of spectral features can emerge.

ACKNOWLEDGMENTS

The authors would like to thank J. Voit for bringing Ref. 13 to their attention.

APPENDIX A

In this appendix we calculate the moments

$$\mu_n^>(k) \equiv \int \omega^n \rho^>(k,\omega) d\omega = i^{n+1} \int \left(\frac{d^n G^>(x,t)}{dt^n} \right)_{t=0} e^{-ikx} dx \qquad \text{(A1)}$$

of the spectral function $\rho^>(k,\omega)$ for $n = 0$, 1, and 2. As $iG^>(x,t) = \exp[F(x,t)]/L$ we have to calculate the derivatives of $F(x,t)$

$$\dot{G}^>(x,0) = \dot{F}(x,0) G^>(x,0),$$
$$\ddot{G}^>(x,0) = [\ddot{F}(x,0) + \dot{F}^2(x,0)] G^>(x,0) . \qquad \text{(A2)}$$

The calculations are straightforward only for the g_4 model. In the spinless case the function $F(x,t)$ takes the form

$$F(x,t) = \sum_{n \geq 1} \frac{1}{n} \exp\left(i \frac{2\pi}{L} n [x - \tilde{v}_F(n) t] \right) . \qquad \text{(A3)}$$

Therefore $F(x,0) = \ln\{1 - \exp[i\frac{2\pi}{L}(x+i0)]\}$, i.e., $G^>(x,0)$ is identical to the noninteracting Green's function

$$iG^>(x,0) = \frac{1}{L} \sum_{n=0}^{\infty} \exp\left(i \frac{2\pi}{L} n [x + i0] \right) . \qquad \text{(A4)}$$

The spatial Fourier transform in (A1) can easily be performed using (A2)–(A4) and yields [$\tilde{v}_F(i) = v_F + g_4(k_i)/(2\pi)$]

$$\mu_1^>(k_n) = \frac{2\pi}{L} \sum_{m=1}^{n} \tilde{v}_F(m), \qquad \text{(A5)}$$

$$\Delta_n^> \equiv \mu_2^>(k_n) - \left[\mu_1^>(k_n)\right]^2 = \left(\frac{2\pi}{L}\right)^2 \frac{1}{2}\sum_{j=1}^{n}\sum_{i=n-j+1}^{n}\left[\tilde{v}_F(i)-\tilde{v}_F(j)\right]^2 . \quad \text{(A6)}$$

For the step model (22) this leads for $n > 2n_c$ to

$$\Delta_n^> = \left(\frac{2\pi}{L}\right)^2 \frac{n_c(n_c+1)}{2}\left(\tilde{v}_F - v_F\right)^2 . \quad \text{(A7)}$$

In the limit $L \to \infty$ this reduces to Eq. (26).

For the general spinless model already the zeroth moment $\mu_0^>(k_n) = 1 - n(k_n)$ is nontrivial. It is discussed in Appendix B. As the expressions for the first and second moment are rather lengthy, we do not present them here.

For the model including spin we restrict ourselves to the g_4 model. If we again write $iG_\sigma^>(x,t) = \exp\left[F(x,t)\right]/L$, the function $F(x,t)$ is given by

$$F(x,t) = \frac{1}{2}\sum_{n\geq 1}\frac{1}{n}\left[\exp\left(i\frac{2\pi}{L}n\left[x-\tilde{v}_F(n)t\right]\right) + \exp\left(i\frac{2\pi}{L}n\left[x-v_F t\right]\right)\right] . \quad \text{(A8)}$$

Therefore $G_\sigma^>(x,0)$ equals the expression on the rhs of Eq. (A1) and the spatial Fourier transform can be performed as in the spinless case. The first and the second moment are given by

$$\mu_{1,\sigma}^>(k_n) = \frac{2\pi}{L}\sum_{m=1}^{n}\left[\tilde{v}_F(m)+v_F\right]/2, \quad \text{(A9)}$$

$$\Delta_n^> \equiv \mu_{2,\sigma}^>(k_n) - \left[\mu_{1,\sigma}^>(k_n)\right]^2 = \left(\frac{2\pi}{L}\right)^2 \frac{1}{2}\sum_{j=1}^{n}\sum_{i=n-j+1}^{n}\left[\tilde{v}_F(i)-\tilde{v}_F(j)\right]^2/4 + \left(\frac{2\pi}{L}\right)^2\sum_{j=1}^{n} j\left[\tilde{v}_F(j)-v_F\right]^2/4 . \quad \text{(A10)}$$

As now $\tilde{v}_F(i) = v_F + g_4(k_i)/\pi$, the first term on the rhs of (A10) is identical to the result for the spinless model. For the step model the additional term is the only contribution to the width of the spectrum in the interval $0 < k < k_c$.

APPENDIX B

In this appendix we present a short discussion of the momentum distribution in the ground state for finite systems and in the limit $L \to \infty$. For the special interaction $g_2(k) \equiv g_4(k) = g\Theta(k_c^2 - k^2)$ the Green's function $G^>(x,0)$ for the spinless model can be written as

$$iG^>(x,0) = \frac{1}{L}\frac{\exp\left(ik_F\left[N+1\right]x\right)}{1-\exp\left(i\frac{2\pi}{L}\left[x+i0\right]\right)}\exp\left(s^2\sum_{n=1}^{n_c}\frac{1}{n}\left[e^{i(2\pi/L)nx}+e^{-i(2\pi/L)nx}-2\right]\right) = \frac{A^{-2s^2}}{L}\frac{\exp\left(ik_F\left[N+1\right]x\right)}{1-\exp\left(i\frac{2\pi}{L}\left[x+i0\right]\right)}\left[\sum_{m=0}^{\infty}c_m^{(n_c)}\exp\left(i\frac{2\pi}{L}mx\right)\right]\left[\sum_{l=0}^{\infty}c_l^{(n_c)}\exp\left(-i\frac{2\pi}{L}lx\right)\right] \quad \text{(B1)}$$

with the coefficients $c_m^{(n_c)}$ given in Eq. (41) and

$$A = \exp\left(\sum_{n=1}^{n_c}\frac{1}{n}\right) \overset{n_c\gg 1}{\to} e^C n_c . \quad \text{(B2)}$$

The momentum distribution $n(k)$ follows from the spatial Fourier transform of $iG^>(x,0)$

$$1 - n\left(\frac{2\pi}{L}\left[n_F+1+\tilde{n}\right]\right) = A^{-2s^2}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty} d_m^{(n_c)}c_l^{(n_c)}\delta_{l+\tilde{n},m}, \quad \text{(B3)}$$

where $d_m^{(n_c)} = \sum_{l=0}^{m} c_l^{(n_c)}$. The asymptotic behavior of $d_m^{(n_c)}$ can be obtained from a comparison of the power series in (B1) for $x = 0$

$$d_m^{(n_c)} \overset{m\to\infty}{\to} \sum_{l=0}^{\infty} c_l^{(n_c)} = A^{s^2} . \quad \text{(B4)}$$

Using (B4) the large momentum behavior $n\left(\frac{2\pi}{L}\left[n_F+1+\tilde{n}\right]\right) \to 0$ for $\tilde{n} \to \infty$ can be read off Eq. (B3). As $1/2 - n(k)$ is symmetric with respect to $(2\pi/L)(n_F+1/2)$, it is sufficient to consider values $\tilde{n} \geq 0$.

In order to obtain the power-law behavior of the momentum distribution for $1 \ll \tilde{n} \ll n_c$ it is useful to calculate the finite differences $\Delta_{\tilde{n}} \equiv n\left(\frac{2\pi}{L}\left[n_F+\tilde{n}\right]\right) - n\left(\frac{2\pi}{L}\left[n_F+1+\tilde{n}\right]\right)$. They are given by

$$\Delta_{\tilde{n}} = A^{-2s^2}\sum_{l=0}^{\infty} c_l^{(n_c)}c_{l+\tilde{n}}^{(n_c)} . \quad \text{(B5)}$$

The singular contribution can already be obtained by restricting the summation in (B5) to values $\tilde{n} + l \leq n_c$. This simplifies the discussion as an analytical expression for the $c_l^{(n_c)}$ (44) is available. If one uses the asymptotic form $c_m^{(n_c)} \sim m^{s^2-1}$, Eq. (B5) involves a summation over $\{1/\left[l\left(l+\tilde{n}\right)\right]\}^{1-s^2}$ which in the thermodynamic limit for $2s^2 < 1$ contains a singular contribution proportional to $(1/\tilde{n})^{1-2s^2}$. We therefore recover the well-known result[7,8]

$$n(k) - 1/2 \sim \mathrm{sgn}\left(k_F - k\right)\left|k-k_F\right|^{2s^2} . \quad \text{(B6)}$$

If one wants to determine the exponent of $\partial n/\partial k$ numerically from Eq. (B5) one has to go to very large values of n_c.

[1]D. Jerome and H.J. Schulz, Adv. Phys. **31**, 299 (1982).
[2]J. Sólyom, Adv. Phys. **28**, 201 (1979).
[3]B. Dardel, D. Malterre, M. Grioni, P. Weibel, Y. Baer, C. Schlenker, and Y. Pétroff, Europhys. Lett. **19**, 525 (1992).
[4]A. Luther and I. Peschel, Phys. Rev. B **9**, 2911 (1974).
[5]S. Tomonaga, Prog. Theor. Phys. **5**, 544 (1950).
[6]J.M. Luttinger, J. Math. Phys. **4**, 1154 (1963).
[7]D.C. Mattis and E.H. Lieb, J. Math. Phys. **6**, 304 (1965).
[8]H. Gutfreund and M. Schick, Phys. Rev. **168**, 418 (1968).
[9]See Ref. 2 for a review.
[10]F.D. Haldane, J. Phys. C **14**, 2585 (1981).
[11]V. Meden and K. Schönhammer, Phys. Rev. B **46**, 15 753 (1992).
[12]A. Theumann, J. Math. Phys. **8**, 2460 (1967).
[13]Y. Suzumura, Prog. Theor. Phys. **63**, 51 (1980).
[14]K. Schönhammer and V. Meden, J. Electron Spectrosc. Relat. Phenom. **62**, 225 (1993).
[15]A. Luther and V.J. Emery, Phys. Rev. Lett. **33**, 2460 (1967).
[16]H.C. Fogedby, J. Phys. C **9**, 3757 (1976).
[17]J. Voit, Phys. Rev. B **47**, 6740 (1993).