Hall resistance of a two-dimensional electron gas in the presence of magnetic-flux tubes

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We compute the Hall and diagonal resistivities of a two-dimensional electron gas in the presence of randomly placed magnetic-flux tubes. It is found that there is a suppression of the Hall conductance below its expected value for the corresponding uniform magnetic-field case at low electron densities, in agreement with recent experiments. At densities somewhat lower than achieved thus far experimentally, we find resonances in the transport coefficients due to Landau-level structure in the flux tubes. A classical calculation of the Hall resistance reproduces the broad features of the Hall conductance for integral or half-integral numbers of flux quanta in each flux tube.

In recent years, there has been increasing interest in the behavior of the two-dimensional electron gas (2DEG) in inhomogeneous magnetic fields. Such systems are of interest because they offer a new and unusual way of introducing an external potential in the electron gas, which does not necessarily involve adding random impurities to the system. Much work has focused on the case of a periodic field modulation,[1,2] and its effects on the energy spectrum and transport properties of the system. The effect of the electrostatic analog of this is known to lead to interesting effects, including the Hofstadter butterfly spectrum[3] and quantization of the Hall conductance.[4]

In practice, such inhomogeneous magnetic fields are made possible by depositing a thin superconducting film above the 2DEG, so that the field is broken up into vortex lines just above the electron gas.[5–7] If the ratio of the superconducting flux quantum $hc/2e$ to the magnetic field B is large compared to the area of a typical vortex line, then to a first approximation one may think of the magnetic field as being confined only to small regions of the electron gas. For low enough magnetic fields, the positions of the vortices will be random, due to inhomogeneities in the superconducting film.[5–7] Thus, one is led to consider a system of randomly distributed scatterers, with the scattering potential being due to a strong magnetic field confined to small regions of space. This model has been studied theoretically, with particular attention to the effects of weak localization.[8,9]

In this work, we will investigate the Hall effect in this system using a Boltzmann equation approach. Recent experiments[7] have investigated this quantity, and have found that at high electron densities, the Hall resistivity is essentially the same as in a uniform magnetic field, while at lower densities, it becomes suppressed below this value. Past theoretical work anticipated the former result,[10] although the latter is largely unexplained. Below, we will show explicitly that there is indeed a suppression of the Hall resistivity in a simple model of flux tubes randomly distributed through an electron gas, when $k_F R < 1$, where k_F is the Fermi wavelength and R the radius of a flux tube. This suppression is largely a result of the ineffectiveness of the flux tube as a scatterer if it is too narrow, and can be understood in a purely classical model. We also find that there can be resonances in the Hall resistivity due to the Landau-level structure in the flux tubes, and that there are interesting oscillations in both the Hall resistance and the diagonal resistance as a function of the flux in each vortex, which is a consequence of the Aharanov-Bohm effect.[11]

A typical result is illustrated in Fig. 1, where we plot the Hall factor, defined as $F_H(k_F R) = \rho_{xy} n_s ec/B$, where ρ_{xy} is the Hall resistivity, n_s the sheet density, and B is the spatially averaged magnetic field. The flux parameter α, which is the number of flux quanta contained in a single flux tube, is taken to be 0.5, as is expected for the experiment of Ref. 7. We note also that, throughout this paper, we assume for simplicity that the magnetic field

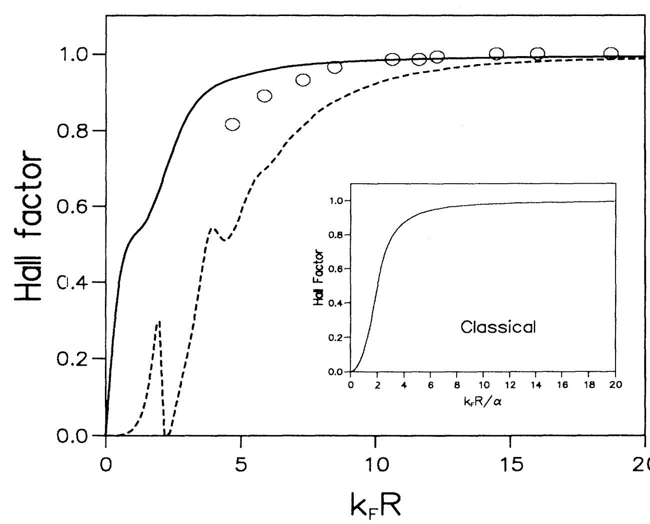

FIG. 1. Hall factor F_H as a function of $k_F R$ for flux tubes containing $\alpha = 0.5$ (solid line) and $\alpha = 2.5$ (dashed line) flux quanta hc/e. Inset: F_H as calculated classically.

is uniform inside the flux tube. One can see a shoulder in the Hall factor near $k_F R = 2$, which may be traced back to a scattering resonance through the first Landau level in the region of the magnetic field. (Such resonances can be quite pronounced for larger values of α, leading to nonmonotonic behavior in F_H; this is also illustrated in Fig. 1.) For comparison, the experimental points, as taken from Ref. 7, are plotted as well, with their assumption that $R = 1000$ Å. One can see good qualitative agreement here. We note that the choice of R is somewhat arbitrary, as the field profile inside the flux tube is not really uniform; we find that the agreement between experiment and theory can be made quantitative if we use an effective radius of $R = 650$ Å.[12]

In Fig. 2, we illustrate (for $\alpha = \frac{1}{2}$ and $\frac{5}{2}$) the quantity $m^*/\hbar n_L \tau(k_F R)$, where m^* is the effective mass, n_L is the number of vortices per unit area, and $\tau(k_F R)$ is the elastic mean lifetime for scattering off the vortices. This quantity is proportional to the diagonal resistivity ρ_{xx}, and so is easily measured experimentally. It is interesting to see that there is a peak in the resistivity near $k_F R = 2$, which once again we interpret as a scattering resonance. Not surprisingly, similar plots for larger values of α reveal a number of such resonances. It is also interesting to look at this quantity for fixed $k_F R = 2$ as a function of α. One finds oscillations of period 1, which are a manifestation of the Aharanov-Bohm effect. Similar oscillations are present in the Hall factor, although they are not as pronounced.

We now discuss the derivation of our results. The magnetic field is applied in the z direction. In the symmetric gauge, and in polar coordinates (r, θ), the vector potential describing a vortex is $\mathbf{A} = (0, A_r)$, with

$$A_r = \begin{cases} \frac{\Phi r}{2\pi R^2} & \text{if} \quad r < R \\ \frac{\phi}{2\pi r} & \text{if} \quad r > R. \end{cases} \tag{1}$$

With this gauge the Hamiltonian describing the electron interacting with a vortex is

$$\frac{\hbar^2}{2m^*}\left(-\frac{\partial^2}{\partial r^2} - \frac{\partial}{r\partial r} + \frac{1}{r^2}\left[-i\frac{\partial}{\partial\theta} - \frac{e}{c\hbar}A_r r\right]^2\right) . \tag{2}$$

In this equation m^* is the effective mass of the host semiconductor. We have neglected the interaction of the electron spin with the magnetic field. This interaction can be relevant for the case of free electrons,[13–15] but in the actual samples the electrons are moving in GaAs where the Zeeman energy is much smaller than the cyclotron energy.

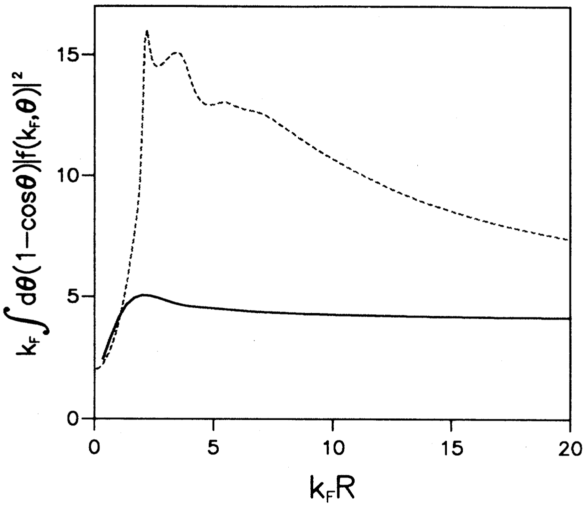

FIG. 2. Unitless measure of the inverse scattering time due to the flux quanta, as given in Eq. (14), for $\alpha = 0.5$ (solid line) and $\alpha = 2.5$ (dashed line).

The eigenstates of the Hamiltonian given by Eq. (3) have the form $\chi(\mathbf{r}) = e^{im\theta}\psi(qr)$, where

$$\psi(qr) = \Delta_1 J_{|m-\alpha|}(qr) + \Delta_2 Y_{|m-\alpha|}(qr) \quad \text{for} \quad r \geq R, \tag{3}$$

and for $r < R$, $\psi(qr)$ is obtained from the equation

$$\left\{ r^2\frac{d^2}{dr^2} + r\frac{d}{dr} + q^2r^2 - \left[m - \alpha\frac{r^2}{R^2}\right]^2 \right\} \psi(qr) = 0 . \tag{4}$$

In these expressions q is related to the electron energy, ε, by the relation $\varepsilon = \hbar^2 q^2/2m^*$, J_m and Y_m are respectively the usual Bessel and Neumann functions, and α is the magnetic flux in units of $\phi_0 = ch/e$; i.e., $\phi = \alpha\phi_0$. Δ_1 and Δ_2 are constants which depend on m^* and q, and are obtained by imposing continuity of the wave function $\psi(qr)$ and its derivative at $r = R$.

For large distances, r, the wave function $\psi(qr)$ can be written as

$$\lim_{r\to\infty} \psi(qr) = e^{iqx} + f(q,\theta)\frac{e^{iqr}}{\sqrt{r}} , \tag{5}$$

where $f(q, \theta)$ is the total scattering amplitude, given by

$$f(q,\theta) = -\frac{1}{\sqrt{2\pi q}}e^{-i\pi/4}\left\{ e^{i\theta/2}\frac{\sin\pi\alpha}{\sin\theta/2} + \sum_{m=-\infty}^{\infty} e^{2i\delta_m}\frac{2i\Delta_2}{\Delta_1 + i\Delta_2}e^{im\theta} \right\} , \tag{6}$$

and

$$\delta_m = \frac{\pi}{2}(m - |m - \alpha|) . \tag{7}$$

In the $R \to 0$ limit, $f(q, \theta)$ is periodic in α and the scattering amplitude reduces to the Aharonov-Bohm amplitude.[16,17]

From the total scattering amplitude it is possible to calculate the T-matrix element between two plane waves of wave vector $\mathbf{k}$ and $\mathbf{k}'$, both with the same magnitude k, and forming an angle θ between them:

$$f(k,\theta) = -\frac{im^*}{2\hbar^2}\sqrt{\frac{2}{\pi k}}e^{-i\pi/4}\langle\mathbf{k}'|T|\mathbf{k}\rangle \quad . \tag{8}$$

In terms of this matrix element, the transition probability from the state $|\mathbf{k}\rangle$ to the state $|\mathbf{k}'\rangle$ is given by the golden rule:

$$W_{\mathbf{k},\mathbf{k}'} = \frac{2\pi}{\hbar}n_L|\langle\mathbf{k}'|T|\mathbf{k}\rangle|^2\,\delta(\varepsilon(\mathbf{k}') - \varepsilon(\mathbf{k})) \quad . \tag{9}$$

Note that since the scattering is due to a magnetic field $f(q,\theta) \neq f(q,-\theta)$, and therefore $W_{\mathbf{k},\mathbf{k}'} \neq W_{\mathbf{k}',\mathbf{k}}$. The Boltzmann equation, in the presence of an electric field $\mathbf{E}$, takes the form

$$\begin{aligned} -e\mathbf{E}\frac{1}{\hbar}\frac{\partial}{\partial\mathbf{k}}g(\mathbf{k}) = &-\sum_{\mathbf{k}'}\{W_{\mathbf{k},\mathbf{k}'}g(\mathbf{k})\,[1-g(\mathbf{k}')] \\ &- W_{\mathbf{k}',\mathbf{k}}g(\mathbf{k}')\,[1-g(\mathbf{k})]\} \\ &- \frac{g(\mathbf{k}) - g_0(\mathbf{k})}{\tau_i} \quad . \end{aligned} \tag{10}$$

Here $\mathbf{E}$ is the applied electric field, $g(\mathbf{k})$ is the number of electrons in the volume element $d\mathbf{k}$ about the point $\mathbf{k}$, and $g_0(\mathbf{k})$ is the equilibrium Fermi distribution. The last term of Eq. (10) describes collisions with the usual impurities, so that τ_i is the relaxation time in the absence of a magnetic field.

It may be shown[10] that this equation is identical to the Boltzmann equation in a uniform magnetic field and a random impurity potential, with a total relaxation time τ of the form

$$\frac{1}{\tau(kR)} = n_L\frac{\hbar k}{m^*}\int_0^{2\pi} d\theta(1-\cos\theta)|f(k,\theta)|^2 + \frac{1}{\tau_i}, \tag{11}$$

and an effective magnetic field $B_{\text{eff}} = BF_H(kR)$, where the Hall factor $F_H(kR)$ is given by the expression

$$F_H(kR) = \frac{k}{2\pi\alpha}\int_0^{2\pi} |f(k,\theta)|^2\sin\theta\, d\theta \quad . \tag{12}$$

Note that both the Hall factor and the relaxation time only depend on the dimensionless quantity kR. From this, the Hall resistance ρ_{xy} is easily shown to be

$$\rho_{xy} = \frac{B}{n_s ec}F_H(k_F R) \quad . \tag{13}$$

The Hall resistance is linear in the applied magnetic field, because the density of flux tubes is proportional to the applied magnetic field, and in the dilute limit, the scattering amplitude is independent of it. At higher magnetic fields the array of vortices is denser and multiple scattering of the electron with the vortices should be included in the calculations. This would modify the dependence of the Hall resistance on B. In any case, at high enough magnetic field the adjacent vortices strongly overlap, and we should obtain the Hall resistance of a 2DEG, $\rho_{xy} = B/n_s ec$.

We illustrate in Fig. 1 the Hall factor in the case of vortices with magnetic flux α=0.5 and 2.5. Note the two following limits in Fig. 1: (a) $F_H \to 0$ when $k_F R \to 0$.[18] This happens because in this limit there is no magnetic field. This behavior is precisely reproduced by our classical model below. (b) $F_H \to 1$ when $k_F R \to \infty$, coinciding with the semiclassical result.[10] However, for 2α not precisely equal to an integer, we find F_H may either be suppressed or elevated from the semiclassical value. Such behavior, we shall see, does not arise in the classical model, and appears to be a purely quantum effect. Physically, this is a result of the Aharanov-Bohm effect, for which there is a relative phase shift of electron paths traversing either side of the flux tube, even without passing through it. Thus, the vector potential *outside* the flux tube is capable of scattering electrons, an effect which never arises classically.

In Fig. 2, we illustrate the variation of the dimensionless quantity

$$k_F\int_0^{2\pi} d\theta(1-\cos\theta)|f(k,\theta)|^2 \tag{14}$$

versus $k_F R$ for different values of the magnetic flux α. This quantity is related with the inverse of the scattering time due to the presence of vortices. From Eq. (11) the value of this function in $k_F R$=0 is $2\sin\pi\alpha^2$. This function develops a maximum at the values of $k_F R$ which correspond to the first Landau levels of the magnetic field in the disk. As α increases more structure is observed because the Landau levels are more separated in energy. In high mobility samples (mean free path around 10^4–10^5 Å), and at magnetic field around 150 G, the scattering time due to the vortices can be shorter than that due to the usual impurities.

Finally, we now show that much of the behavior illustrated in Fig. 1 may be understood from a purely classical approach. We begin by noting

$$|f(k,\theta)|^2 = \frac{d\sigma}{d\theta} \quad , \tag{15}$$

where $d\sigma/d\theta$ is the differential cross section in two dimensions. Noting that the scattering angle θ is a unique function of the impact parameter s, it is not difficult to show that $d\sigma/d\theta = ds/d\theta$, so that

$$F_H(k_F R) = \frac{k_F}{2\pi\alpha}\int_{-R}^{R} ds\,\sin\theta(s). \tag{16}$$

By integrating Newton's equation, one may explicitly compute $\theta(s)$. We find the resulting form of F_H is a universal function of the parameter $k_F R/\alpha$, and plot it in the inset of Fig. 1. One can thus see the suppression of F_H near small $k_F R$, which in the classical case is simply the statement that the cross section of an infinitely thin solenoid is zero. For large $k_F R$, the semiclassical result[10] is recovered. One can show that this arises whenever the velocity of the electron is large compared to R/ω_c, where ω_c is the cyclotron frequency for the magnetic field inside a flux tube, so that all the scattering occurs at small angles. In this situation, to lowest order in $1/k_F R$, one finds

$$\sin\theta(s) = \frac{e}{v_F/m^*c}\int_{x_1}^{x_2} B(x,s)dx \quad , \tag{17}$$

where v_F is the velocity of the electron, $B(x,s)$ is the magnetic-field profile along the incident direction of the electron, and the points x_1 and x_2 lie outside the magnetic-flux tube. Using this in Eq. (16) shows that F_H depends *only* on the total magnetic flux in the vortex, and is independent of its shape or magnetic-field profile.[10] One may thus imagine "spreading out" the flux of all the tubes so that they fill space with a uniform magnetic field; this should not affect the final result. The Hall factor for the uniform magnetic field is trivially 1, and this agrees with our results for integral 2α at large $k_F R$. It is interesting to note that we do not recover the classical result for nonintegral 2α; this is clearly a result of interference effects, due to the vector potential, of electron paths that do not actually traverse the flux tube.

In conclusion, we have studied the Hall effect and the elastic-scattering time for a two-dimensional electron gas scattered by random magnetic-flux tubes using a Boltzmann equation approach. Our results give reasonable agreement with experiment for the suppression of the Hall effect at low electron densities. We find resonances in both the Hall and diagonal resistivities as a function of $k_F R$ that correspond to Landau-level energies in the flux tubes. At large $k_F R$, the Hall resistivity takes the value expected for a uniform magnetic field, provided the number of flux quanta contained in each tube is integral or half-integral. This result may be understood from a purely classical approach. For other values of the flux in a tube, one gets deviations from the classical result even at large $k_F R$, due to the Aharanov-Bohm effect.

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