Single-scattering-path approach to the negative magnetoresistance in the variable-range-hopping regime for two-dimensional electron systems

M. E. Raikh
Department of Physics, University of Utah, Salt Lake City, Utah 84112

G. F. Wessels
Physik-Department E13, Technische Universität München, D-8046 Garching, Federal Republic of Germany
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An alternative approach to the calculation of the scattering-induced magnetoresistance in the regime of variable-range-hopping conductivity is developed. Under the assumption that the direct tunneling and the various single-scattering tunneling paths give the main contribution to the amplitude of a hop, the exact averaging over the positions and energies of scatterers of the logarithm of the resistance is performed. It is shown that in the strong-scattering limit the magnetoresistance is universal in the sense that it does not depend on the scattering strength. The transition from negative magnetoresistance in low magnetic fields, caused by the suppression of the destructive interference, to positive magnetoresistance in strong enough fields, caused by the orbital shrinkage of the impurity wave functions, is traced. The theory developed is applied to the system of two parallel impurity planes in a magnetic field parallel to the planes. The oscillations of the hopping resistance with magnetic field in such a system are studied quantitatively.

I. INTRODUCTION

It is well known that the low-temperature transport in a system with localized electronic states is governed by electron hops over impurity sites with energies close to the Fermi level. The temperature dependence of the resistance is described by Mott's law, which in the two-dimensional (2D) case reads[1]

$$\ln\mathcal{R} = \left[\frac{T_0}{T}\right]^{1/3}, \tag{1.1}$$

where

$$T_0 = \frac{14}{ga^2}, \tag{1.2}$$

g is the density of states at the Fermi level, and a is the decay length of an impurity wave function. The typical hopping length in this variable-range-hopping (VRH) regime is of the order of $r \sim a(T_0/T)^{1/3}$, and exceeds by several times the average distance between the impurities.

The magnetic-field dependence of the resistance in the VRH regime was studied experimentally in several types of 2D systems.[2-6] While experiments performed on a silicon metal-oxide-semiconductor field-effect transistor (MOSFET) (Ref. 2) exhibited an increase in resistance with magnetic field B (positive magnetoresistance) in other systems studied,[3-6] a decrease in resistance with B starting from $B=0$ (negative magnetoresistance) was observed. A similar decrease of the resistance with B was found in experiments performed on bulk samples.[7-10]

Positive magnetoresistance is caused by the orbital shrinkage of the impurity wave functions.[1] The crucial idea for the understanding of negative magnetoresistance was proposed by Nguyen, Spivak, and Shklovskii (NSS) (Refs. 11 and 12). The effect was accounted for by the interference of the different tunneling paths in the process of a single hop. These paths corrrespond to different sequences of scatterings of tunneling electrons by the impurities located within a cigar-shaped area of length r and width $(ra)^{1/2}$. Since the amplitudes of different paths are random, they can cancel each other. Phase factors acquired by each path in a magnetic field destroy this interference, increasing, therefore, the probability of the hop.

Further theoretical study of the NSS mechanism was carried out in Refs. 13–17. The considerations in Refs. 13–16 were based on the assumption that the number of scattering acts corresponding to the typical hop is large. The analytic approach based on independent forward-scattering paths was used in Refs. 13–15. In Refs. 16 and 17, the statistical properties of the interference of different tunneling amplitudes with magnetic-field-dependent phase factors were studied using the computer simulation.

An alternative approach was proposed in Ref. 18. It was assumed that the "interference" area $r^{3/2}a^{1/2}$ contains only one scatterer. The reason for such an assumption was as follows. The average number of scatterers within the interference area can be estimated as

$$m = nr^{3/2}a^{1/2} \sim na^2\left[\frac{T_0}{T}\right]^{1/2}, \tag{1.3}$$

where n is the concentration of impurities. The typical value of the logarithm of the resistance which can be measured experimentally is about 10. At the same time, to provide the insulating regime, the condition $na^2 \lesssim 0.1$ should be satisfied. This leads to the restriction $m \lesssim 3$. Note that the assumption $m=1$ is consistent with the concept of VRH. Indeed, in the VRH regime, the hopping length r should be greater than the average distance between the impurities. The condition $r > n^{-1/2}$ can be rewritten as

$$na^2 \left[\frac{T_0}{T} \right]^{2/3} > 1 \; . \tag{1.4}$$

Since the powers of the ratio T_0/T in (1.3) and (1.4) are different, we can have $m \sim 1$ in the VRH regime for moderate temperatures.

The averaging of the logarithm of the resistance of a hop over the energy-level position and location of a scatterer was performed in Ref. 18, in the framework of the effective-medium approximation.

Both these approaches result in the decrease in resistance with B, thus approving the general character of the initial idea of Ref. 3. Which of these approaches, each implying quite a different picture of scattering, applies in the concrete experimental system depends on the actual type of the disorder in the system.

In crystalline semiconductors, the disorder originates from the randomness in the positions of doping atoms, e.g., donors. The electrons hop over the donors with energies close to the Fermi level, while the donors with energies out of Mott's energy strip $T(T_0/T)^{1/3}$ scatter the tunneling electrons. Therefore, the same type of atom serves both for hopping and for scattering. In this case, the above arguments for a small number of scatterers within the interference area seem to be adequate.

On the contrary, in amorphous or polycrystalline materials, it is natural to expect a large number of irregularities (defects) which do not produce localized states in the vicinity of the Fermi level but can scatter a tunneling electron. Then the approach based upon multiple scattering is relevant.

The approach developed in the present paper is somewhat intermediate between the two cited above. We take into account the interference between the amplitude of the direct tunneling path and *all* the amplitudes of the tunneling paths with a *single* scattering (see Fig. 1). The advantage of such an approach is that it allows us to perform analytically the averaging over the positions and energies of scatterers and to obtain quantitative predictions for $\mathcal{R}(B)$ dependence, which was impossible in the previous works. Indeed, the calculations in Ref. 18 exploiting the effective-medium approximation cannot provide the correct numerical factors and, hence, the correct magnitude of magnetoresistance $\mathcal{R}(B)/\mathcal{R}(0)$. The approximation in Refs. 13–15 of independent direct paths assumes the Gaussian distribution of phase factors in the tunneling amplitudes with some arbitrary width proportional to B, and, therefore, cannot be expected to give correct quantitative predictions.

Our approach also allows us to take into account the orbital shrinkage of the impurity wave functions in a magnetic field. If this effect is neglected, the theory yields $\mathcal{R}(B)$ dependence that decreases with B and saturates for strong enough $B \gg B_0$ at some level $\mathcal{R}(\infty)$. The value B_0 is determined from the condition that the magnetic flux through the interference area $r^{1/2}a$ is on the order of the flux quantum ϕ_0.[11–13] The shape of the curve $\mathcal{R}(B)$ depends on the scattering strenth. If the scattering is strong enough, $\mathcal{R}(B)$ is a universal function of the ratio B/B_0. The total relative decrease of the resistance $[\mathcal{R}(0)-\mathcal{R}(\infty)]/\mathcal{R}(0)$ appears to be 0.42. By including the orbital shrinkage, we trace the transition from the negative magnetoresistance in low fields to the positive magnetoresistance in high fields. Since the typical value of B, relevant for the orbital shrinkage, is of the order of B_0, the $\mathcal{R}(B)$ dependence in the limit of strong scattering is also a universal function of B/B_0, which decreases with B, passes through the minimum, and then increases. The depth of the minimum is $\mathcal{R}_{\min}=0.82\mathcal{R}(0)$.

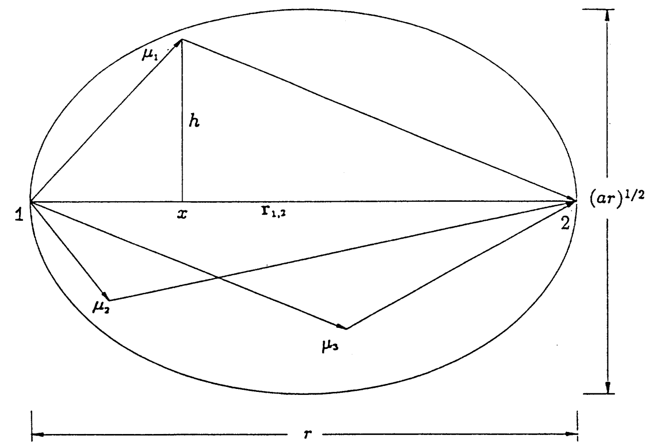

FIG. 1. The cigar-shaped region of the paths of an electron tunneling between impurities 1 and 2. Only the paths with a single scattering are shown.

The calculations in the paper are carried out under the assumption that the hopping length is fixed. The averaging over the various hops, performed according to Refs. 14 and 19, does not change significantly the dependence $\mathcal{R}(B)$.

It should be noted that within our approach, the magnetoresistance in the low-field limit is linear in magnetic field: $[\mathcal{R}(0)-\mathcal{R}(B)] \propto B$. It can be shown, however, that for very low fields $B \ll B_0$, the change of the resistance with B should be quadratic. In the paper, we present the physical reason for B^2 dependence at low fields, which is different from that in Ref. 14, and estimate the range of fields in which the linear dependence applies.

The paper is organized as follows. In Sec. II, we derive the expression for the correction to the resistance of a hop caused by the single-scattering tunneling paths. The analytical averaging of the logarithm of the resistance over the positions and energies of scatterers is performed in Sec. III. In Sec. IV, we present the $\mathcal{R}(B)$ dependences calculated numerically with the use of the expressions derived previously. The results obtained, both without the orbital shrinkage effect and with orbital shrinkage taken into account, are demonstrated. In Sec. V, the theory developed is extended to the case of two parallel impurity planes. The oscillations of $\mathcal{R}(B)$ proposed in Ref. 20 for such a geometry are studied quantitatively. The relative amplitude of the first strongest oscillations is shown to be about several percent. In Sec. VII, we address the limit of very low magnetic fields, in which the magnetoresistance is quadratic. Section VII concludes the paper.

II. INTERFERENCE-INDUCED CORRECTION TO THE RESISTANCE OF THE HOP

Consider the electron hop between two sites 1 and 2 (Fig. 1). The probability of the hop $\mathcal{P}$ is proportional to $|M|^2$, where

$$M=M_0\int d\mathbf{r}\,\Psi_1^*(\mathbf{r})e^{i\mathbf{q}\cdot\mathbf{r}}\Psi_2(\mathbf{r}) \tag{2.1}$$

is the matrix element of the electron-phonon interaction between the wave function $\Psi_1(\mathbf{r}),\Psi_2(\mathbf{r})$ of the initial and the final states; $\mathbf{q}$ is a phonon wave vector and M_0 is a prefactor.

Following Ref. 1, in a calculation of the matrix element, one should take into account the overlap-induced admixture of the wave functions of the other sites to the wave functions of sites 1 and 2:

$$\Psi_1=\Psi_1^{(0)}+\sum_\mu C_{1,\mu}\Psi_\mu^{(0)} , \tag{2.2}$$

$$\Psi_2=\Psi_2^{(0)}+\sum_\mu C_{2,\mu}\Psi_\mu^{(0)} , \tag{2.3}$$

where $\Psi_\mu^{(0)}$ is the wave function of the isolated site μ. As usual,[1,19] we shall assume that the condition

$$qr_{1,2}\gg 1 , \tag{2.4}$$

is fulfilled. Then the matrix element (2.11) takes the form

$$M=M_0I\left[C_{1,2}^*e^{i\mathbf{q}\cdot\mathbf{r}_2}+C_{2,1}e^{i\mathbf{q}\cdot\mathbf{r}_1}+\sum_{\mu\neq 1,2}C_{1,\mu}^*C_{2,\mu}e^{i\mathbf{q}\cdot\mathbf{r}_\mu}\right] , \tag{2.5}$$

where the integral

$$I=\int d\mathbf{r}\,e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}_\nu)}|\Psi_\nu^{(0)}(\mathbf{r})|^2 , \tag{2.6}$$

does not depend on ν. In the square of matrix element (2.5), the terms proportional to $\exp[i\mathbf{q}\cdot(\mathbf{r}_\mu-\mathbf{r}_\nu)]$ vanish after averaging over the directions of $\mathbf{q}$. We then obtain

$$|M|^2=M_0^2\left[|C_{2,1}|^2+|C_{1,2}|^2+\sum_\mu|C_{1,\mu}C_{2,\mu}^*|^2\right] . \tag{2.7}$$

Three terms in Eq. (2.7) correspond to the following scenarios of a hopping act: (i) The electron absorbs (or emits) a phonon at site 1 and then tunnels to site 2; (ii) the electron tunnels to site 2 and absorbs (or emits) a phonon there; and (iii) the electron tunnels to some intermediate site μ, absorbs (or emits) a phonon at that site and then tunnels to site 2. As follows from Eq. (2.7), the probability of the hop is the sum of probabilities for all three above processes.

The perturbation expansions for the coefficients $C_{1,\mu}$ and $C_{2,\mu}$ have the form

$$C_{1,\mu}=\frac{1}{\varepsilon_1-\varepsilon_\mu}\left[V_{1,\mu}+\sum_\nu\frac{V_{1,\nu}V_{\nu,\mu}}{\varepsilon_1-\varepsilon_\nu}+\sum_{\nu,\kappa}\frac{V_{1,\nu}V_{\nu,\kappa}V_{\kappa,\mu}}{(\varepsilon_1-\varepsilon_\nu)(\varepsilon_1-\varepsilon_\kappa)}+\cdots\right] , \tag{2.8}$$

$$C_{2,\mu}=\frac{2}{\varepsilon_2-\varepsilon_\mu}\left[V_{2,\mu}+\sum_\nu\frac{V_{2,\nu}V_{\nu,\mu}}{\varepsilon_2-\varepsilon_\nu}+\sum_{\nu,\kappa}\frac{V_{2,\nu}V_{\nu,\kappa}V_{\kappa,\mu}}{(\varepsilon_2-\varepsilon_\nu)(\varepsilon_2-\varepsilon_\kappa)}+\cdots\right] , \tag{2.9}$$

where ε_μ is the energy of the site μ measured from the Fermi level, and

$$V_{\mu,\nu}=V_0\exp\left[-\frac{r_{\mu,\nu}}{a}\right] \tag{2.10}$$

is the overlap integral between sites μ and ν, with $r_{\mu,\nu}$ being the corresponding distance.

With the use of Eqs. (2.8) and (2.9), the coefficients $C_{1,2}$ and $C_{2,1}$ can be expressed as

$$C_{1,2}=\frac{1}{\varepsilon_1-\varepsilon_2}\left[V_{1,2}+\sum_\nu C_{1,\nu}V_{\nu,2}\right] , \tag{2.11}$$

$$C_{2,1}=\frac{1}{\varepsilon_2-\varepsilon_1}\left[V_{2,1}+\sum_\nu C_{2,\nu}V_{\nu,1}\right] . \tag{2.12}$$

Substituting (2.8)–(2.12) into Eq. (2.7), we obtain

$$|M|^2=M_0^2\frac{|V_{1,2}|^2}{(\varepsilon_1-\varepsilon_2)^2}\left[\left|1+\frac{1}{V_{1,2}}\sum_{\mu\neq 2}C_{1,\mu}V_{\mu,2}\right|^2+\left|1+\frac{1}{V_{2,1}}\sum_{\mu\neq 1}C_{2,\mu}V_{\mu,1}\right|^2+\frac{(\varepsilon_1-\varepsilon_2)^2}{|V_{1,2}|^2}\sum_{\mu\neq 1,2}|C_{1,\mu}C_{2,\mu}^*|^2\right] . \tag{2.13}$$

The terms in the square brackets describe the effect of the scatterers μ on the hopping probability.

The resistance $\mathcal{R}_{1,2}$ of the hop $1\rightarrow 2$ is proportional to $1/|M|^2$. The percolation procedure for the calculation of the hopping conductivity[1] implies the logarithmic averaging of the resistances of different hops. The correction to $\ln\mathcal{R}_{1,2}$ caused by the scattering reads[12]

$$\delta\ln\mathcal{R}=-\ln\left\{\frac{1}{2}\left[\left|1+\frac{1}{V_{1,2}}\sum_{\mu\neq 2}C_{1,\mu}V_{\mu,2}\right|^2+\left|1+\frac{1}{V_{2,1}}\sum_{\mu\neq 1}C_{2,\mu}V_{\mu,1}\right|^2+\frac{(\varepsilon_1-\varepsilon_2)^2}{|V_{1,2}|^2}\sum_{\mu\neq 1,2}|C_{1,\mu}C_{2,\mu}^*|^2\right]\right\} . \tag{2.14}$$

The expansions (2.8) and (2.9) describe all possible paths of the tunneling electron. In the framework of the single-scattering-paths approach, proposed in the present paper, we keep only the first terms in these expansions:

$$C_{1,\mu}=\frac{V_{1\mu}}{\varepsilon_1-\varepsilon_\mu}\ ,\qquad C_{2,\mu}=\frac{V_{2\mu}}{\varepsilon_2-\varepsilon_\mu}\ . \tag{2.15}$$

Then Eq. (2.14) takes the form

$$\delta\ln\mathcal{R}=-\ln\left\{\frac{1}{2}\left[\left|1+\frac{1}{V_{1,2}}\sum_{\mu\neq 1}\frac{V_{1,\mu}V_{\mu,2}}{\varepsilon_1-\varepsilon_\mu}\right|^2+\left|1+\frac{1}{V_{2,1}}\sum_{\mu\neq 2}\frac{V_{2,\mu}V_{\mu,1}}{\varepsilon_2-\varepsilon_\mu}\right|^2+\frac{(\varepsilon_1-\varepsilon_2)^2}{|V_{1,2}|^2}\sum_{\mu\neq 1,2}\frac{|V_{1,\mu}|^2|V_{2,\mu}|^2}{(\varepsilon_1-\varepsilon_\mu)^2(\varepsilon_2-\varepsilon_\mu)^2}\right]\right\}. \tag{2.16}$$

The typical value of the energies ε_1 and ε_2 is the Mott energy strip[1]

$$\Delta=T\left[\frac{T_0}{T}\right]^{1/3}. \tag{2.17}$$

Let us estimate the characteristic energy of scatterer ε_μ in Eq. (2.14). Since the scatterers are located within the interference area $r^{3/2}a^{1/2}$ (Fig. 1), the typical energy E of the site closest to the Fermi level can be found from the condition

$$gr^{3/2}a^{1/2}E\sim 1\ , \tag{2.18}$$

which leads to

$$E\sim T\left[\frac{T_0}{T}\right]^{1/2}. \tag{2.19}$$

Therefore, the energy of scatterer $\varepsilon_\mu\sim E$ exceeds the energies of the initial and final states $\varepsilon_1,\varepsilon_2\sim\Delta$ by the parameter $(T_0/T)^{1/6}\gg 1$.

Making use of inequality $\varepsilon_\mu\gg\varepsilon_1,\varepsilon_2$, we can rewrite Eq. (2.16) as

$$\delta\ln\mathcal{R}=-\ln\left\{\left|1-\frac{1}{V_{1,2}}\sum_\mu\frac{V_{1,\mu}V_{\mu,2}}{\varepsilon_\mu}-\frac{\varepsilon_1+\varepsilon_2}{2V_{1,2}}\sum_\mu\frac{V_{1,\mu}V_{\mu,2}}{\varepsilon_\mu^2}\right|^2+\frac{(\varepsilon_1-\varepsilon_2)^2}{4|V_{1,2}|^2}\left|\sum_\mu\frac{V_{1,\mu}V_{\mu,2}}{\varepsilon_\mu^2}\right|^2+\frac{(\varepsilon_1-\varepsilon_2)^2}{2|V_{1,2}|^2}\sum_\mu\frac{|V_{1,\mu}|^2|V_{\mu,2}|^2}{\varepsilon_\mu^4}\right\}. \tag{2.20}$$

In a magnetic field **B**, each overlap integral (2.10) acquires a phase factor. The combination $V_{1,\mu}V_{\mu,2}/V_{1,2}$ acquires the phase factor $\exp(i\psi_\mu)$, with

$$\psi_\mu=2\pi\frac{(\mathbf{B}\cdot\mathbf{S}_\mu)}{\phi_0}\ , \tag{2.21}$$

where $\mathbf{S}_\mu$ is the vector area of the triangle $1\to\mu\to 2$.

The last two terms in Eq. (2.20) compared to the first one are small in the parameter $[(\varepsilon_1-\varepsilon_2)/\varepsilon_\mu]^2$. The origin of the first of these two terms is that the scattering amplitudes for the electron with energies $\varepsilon=\varepsilon_1$ and ε_2 are slightly different. The second one describes the absorption (emission) of a phonon at the scatterer. These small terms play an important role at very low magnetic fields. Indeed, as it was first pointed out by Nguyen, Spivak, and Shklovski,[11] most sensitive to low magnetic fields are those impurity realizations in which the first term in Eq. (2.20) is 0 for $B=0$. If one neglects the last two terms in Eq. (2.20), these realizations cause the linear magnetoresistance $[\mathcal{R}(0)-\mathcal{R}(B)]\propto B$. If the complex nature of a hop is taken into account, its probability never turns to 0. This leads to the quadratic magnetoresistance at very low fields.[19]

The range of very small B, where the magnetoresistance is quadratic, is discussed in Sec. VI. Out of this range we can neglect the small terms in Eq. (2.20). With the use of (2.21), the scattering-induced correction to the logarithm of a resistance of a hop in the presence of magnetic field can be rewritten as

$$\delta\ln\mathcal{R}=-\ln\left[\left[1-\sum_\mu\frac{V_{1,\mu}V_{\mu,2}}{V_{1,2}\epsilon_\mu}\cos\psi_\mu\right]^2+\left[\sum_\mu\frac{V_{1,\mu}V_{\mu,2}}{V_{1,2}\epsilon_\mu}\sin\psi_\mu\right]^2\right]. \tag{2.22}$$

To find the correction to the total resistance, one should average Eq. (2.22) over the positions and energies of scatterers μ. This averaging is performed in Sec. III.

III. AVERAGING OVER THE REALIZATIONS OF SCATTERERS

To perform the averaging in Eq. (2.22), it is convenient to rewrite it as

$$\delta \ln\mathcal{R} = -\int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv\, L(u,v) \left\langle \delta\left[u - \sum_\mu \frac{V_{1,\mu}V_{\mu,2}}{V_{1,2}\epsilon_\mu}\cos\psi_\mu\right] \delta\left[v - \sum_\mu \frac{V_{1,\mu}V_{\mu,2}}{V_{1,2}\epsilon_\mu}\sin\psi_\mu\right]\right\rangle , \tag{3.1}$$

where

$$L(u,v) = \ln[(1-u)^2 + v^2] , \tag{3.2}$$

and $\langle\ \rangle$ denotes the configurational averaging. Replacing δ functions by their Fourier transforms, we obtain

$$\delta \ln\mathcal{R} = -\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv\, L(u,v) \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} dt\, e^{isu+itv} \left\langle \exp\left[-i\sum_\mu \frac{V_{1,\mu}V_{\mu,2}}{V_{1,2}\epsilon_\mu}(s\cos\psi_\mu + t\sin\psi_\mu)\right]\right\rangle . \tag{3.3}$$

Each exponent in the product can now be averaged independently. We then obtain

$$\left\langle \exp\left[-i\sum_\mu \frac{V_{1,\mu}V_{\mu 2}}{V_{1,2}\epsilon_\mu}(s\cos\psi_\mu + t\sin\psi_\mu)\right]\right\rangle = \exp\left\{-g\int d\epsilon \int d^2r \left[1 - \exp\left[i\frac{V_{1,\mathbf{r}}V_{\mathbf{r},2}}{V_{1,2}\epsilon}[s\cos\psi(\mathbf{r}) + t\sin\psi(\mathbf{r})]\right]\right]\right\} , \tag{3.4}$$

where g is the density of states, $V_{1,\mathbf{r}}$, $V_{2,\mathbf{r}}$, and $\psi(\mathbf{r})$ are the matrix elements and the phase for the scatterer placed at point $\mathbf{r}$. The integration over energy in (3.4) can be performed with the use of the relation

$$\int_{-\infty}^{\infty} d\epsilon \left[1 - \exp\left[-i\frac{C}{\epsilon}\right]\right] = \pi|C| . \tag{3.5}$$

We then have

$$\left\langle \exp\left[-i\sum_\mu \frac{V_{1,\mu}V_{\mu,2}}{V_{1,2}\epsilon_\mu}(s\cos\psi_\mu + t\sin\psi_\mu)\right]\right\rangle = \exp\left[-\pi g\int d\mathbf{r} \frac{V_{1,\mathbf{r}}V_{\mathbf{r},2}}{V_{1,2}}|s\cos\psi(\mathbf{r}) + t\sin\psi(\mathbf{r})|\right] . \tag{3.6}$$

After substitution of (3.6) into (3.3), it is convenient to introduce polar coordinates

$$\begin{aligned} u &= 1 + \rho\sin\varphi_0 , \quad s = R\sin\varphi , \\ v &= \rho\cos\varphi_0 , \quad t = R\cos\varphi . \end{aligned} \tag{3.7}$$

Then expression (3.3) takes the form

$$\delta\ln\mathcal{R} = -\frac{1}{(2\pi)^2}\int_0^{\infty} d\rho\,\rho\ln\rho^2 \int_0^{2\pi} d\varphi_0 \int_0^{\infty} dR\, R\int_0^{2\pi} d\varphi \exp[iR\sin\varphi + i\rho R\cos(\varphi-\varphi_0) - RG(\varphi)] , \tag{3.8}$$

where

$$G(\varphi) = \pi g\int d^2\mathbf{r}\frac{V_{1,\mathbf{r}}V_{2,\mathbf{r}}}{V_{1,2}}|\sin[\psi(\mathbf{r}) - \varphi]| . \tag{3.9}$$

The integration over φ_0 can be performed easily, and we obtain

$$\delta\ln\mathcal{R} = -\frac{1}{2\pi}\int_0^{\infty} d\rho\,\rho\ln\rho^2 \int_0^{\infty} dR\, RJ_0(\rho R)\int_0^{2\pi} d\varphi\exp[iR\sin\varphi - RG(\varphi)] , \tag{3.10}$$

where J_0 is the Bessel function of order zero. For the further calculations, it is convenient to rewrite $\ln\rho^2$ in (3.10) as

$$\ln\rho^2 = \lim_{\nu\to 0}\frac{\partial}{\partial\nu}\rho^{2\nu} . \tag{3.11}$$

We then have

$$\delta\ln\mathcal{R} = -\frac{1}{2\pi}\lim_{\nu\to 0}\frac{\partial}{\partial\nu}\int_0^{\infty} d\rho\,\rho^{2\nu+1}\int_0^{\infty} dR\, RJ_0(\rho R)\int_0^{2\pi} d\varphi\exp[iR\sin\varphi - RG(\varphi)] . \tag{3.12}$$

The integral over ρ can be now expressed through the Γ function,[21]

$$\int_0^{\infty} d\rho\,\rho^{2\nu+1}J_0(\rho R) = \frac{2^{2\nu+1}\Gamma(1+\nu)}{R^{2+2\nu}\Gamma(-\nu)} . \tag{3.13}$$

Then the integration over R results in

$$\int_0^\infty dR \frac{e^{-RG(\varphi)}}{R^{1+2\nu}} \cos(R \sin\varphi) = \Gamma(-2\nu)[G^2(\varphi)+\sin^2\varphi]^\nu \cos\left[2\nu \arctan\left[\frac{\sin\varphi}{G(\varphi)}\right]\right] . \quad (3.14)$$

After both integrations, Eq. (3.12) takes the form

$$\delta \ln\mathcal{R} = -\lim_{\nu\to 0}\frac{\partial}{\partial\nu}\left\{\frac{2^{2\nu}\Gamma(1+\nu)\Gamma(-2\nu)}{\pi\Gamma(-\nu)}\int_0^{2\pi} d\varphi [G^2(\varphi)+\sin^2\varphi]^\nu \cos\left[2\nu \arctan\left[\frac{\sin\varphi}{G(\varphi)}\right]\right]\right\} . \quad (3.15)$$

At small ν, the prefactor of the integral (3.15) behaves as $(1+2\nu\ln 2)/2\pi$. Taking the limit $\nu\to 0$, we obtain

$$\delta \ln\mathcal{R} = -\frac{1}{2\pi}\int_0^{2\pi} d\varphi \{\ln[G^2(\varphi)+\sin^2\varphi]+2\ln 2\} . \quad (3.16)$$

By using the relation

$$\int_0^{2\pi} d\varphi \ln(\sin^2\varphi) = -4\pi \ln 2 , \quad (3.17)$$

we can rewrite the final result in the form

$$\delta \ln\mathcal{R} = -\frac{2}{\pi}\int_0^{\pi/2} d\varphi \ln\left[1+\frac{G^2(\varphi)}{\sin^2\varphi}\right] . \quad (3.18)$$

IV. ASYMPTOTES AND NUMERICAL RESULTS

The magnetic-field dependence of the resistance in Eq. (3.18) comes from the function $G(\varphi)$. The phase $\psi(\mathbf{r})$ in Eq. (3.9) is proportional to B. To demonstrate that the magnetoresistance defined by Eq. (3.18) is negative, we first calculate the interference correction to the logarithm of the resistance for $B=0$. Then we have $\psi(\mathbf{r})\equiv 0$, and the function $G(\varphi)$ takes the form

$$G(\varphi) = \pi A |\sin\varphi| , \quad (4.1)$$

where

$$A = g\int d\mathbf{r} \frac{V_{1,\mathbf{r}} V_{2,\mathbf{r}}}{V_{1,2}} . \quad (4.2)$$

From Eq. (3.18), we immediately obtain

$$\delta \ln\mathcal{R}(0) = -\ln(1+\pi^2 A^2) . \quad (4.3)$$

In the high-field limit ($B\to\infty$), the typical value of the phase $\psi(\mathbf{r}) \gg 1$, and one can replace the fast oscillating factor $|\sin[\psi(\mathbf{r})-\varphi]|$ in Eq. (3.9) by its average value $2/\pi$. We then have

$$G(\varphi) = 2A . \quad (4.4)$$

Substituting this value into (3.18) and performing the integration, we obtain

$$\delta \ln\mathcal{R}(\infty) = -\ln[8A^2+1+4A(4A^2+1)^{1/2}] . \quad (4.5)$$

Equation (4.5) shows that in the strong-field limit, the magnetoresistance saturates. Comparing the values (4.3) and (4.5), we can express the ratio of the resistances in high field and in zero field in the form

$$\frac{\mathcal{R}(\infty)}{\mathcal{R}(0)} = \frac{1+\pi^2 A^2}{1+8A^2+4A(4A^2+1)^{1/2}} . \quad (4.6)$$

Figure 2 shows the ratio $\mathcal{R}(\infty)/\mathcal{R}(0)$ as a function of parameter A. It can be seen that the magnetoresistance is negative, and the total decrease $[\mathcal{R}(\infty)-\mathcal{R}(0)]$ does not exceed 42% of the resistance in zero field. Surprisingly, for $A \geq 0.2$, the ratio $\mathcal{R}(\infty)/\mathcal{R}(0)$ depends very weakly on parameter A. Since parameter A describes the strength of the scattering, this suggests some kind of universal behavior of the magnetoresistance for strong enough scattering.

To calculate the value of parameter A, we substitute the overlap integrals (2.10) into (4.2). We obtain then

$$A = gV_0\int_{-\infty}^{\infty} dh \int_0^{v_{1,2}} dx \exp\left\{-\frac{1}{a}[\sqrt{(r_{1,2}-x)^2+h^2}+\sqrt{x^2+h^2}-r_{1,2}]\right\} , \quad (4.7)$$

where x and h are the coordinates of the point $\mathbf{r}$, as shown in Fig. (1). It is easy to see that the typical values of x and h that provide major contributions to the integral (4.7) are

$$x \simeq r_{1,2} , \qquad h \simeq (ar_{1,2})^{1,2} , \quad (4.8)$$

in agreement with the results of previous considerations.[12,13,18,19] Making use of the fact that $h \ll x, r-x$, we obtain

$$A(r_{1,2}) = \frac{\pi^{3/2}}{2^{5/2}} gV_0(ar_{1,2}^3)^{1/2} . \quad (4.9)$$

Since the value of $r_{1,2}$ is of the order of Mott's hopping length $(a/2)(T_0/T)^{1/3}$, Eq. (4.9) can be presented in the form

$$A \simeq 5\frac{V_0}{(T_0 T)^{1/2}} , \quad (4.10)$$

where the density of states g and the localization radius a enter into Eq. (4.10) through the parameter T_0.

The maximal magnitude of the logarithm of the resis-

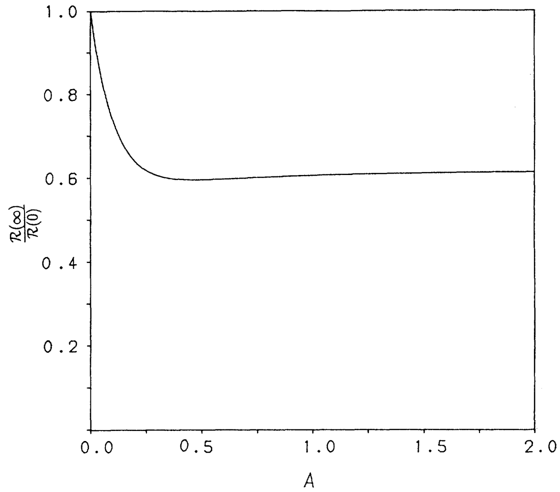

FIG. 2. The saturation value of magnetoresistance as a function of normalized scattering strength A [Eq. (4.6)].

tance, which still can be measured in experiment, is on the order of 10 in absolute value. It then follows from (4.10) that for $V_0 \gtrsim 1.5T$ we already have $A > 0.2$, which corresponds to the saturation of the ratio (4.6). In other words, our theory predicts the saturation value of magnetoresistance $\mathcal{R}(\infty)$ to be equal $0.6\mathcal{R}(0)$ starting from very small values of V_0. The reasonable estimate for parameter V_0 is $V_0 \simeq E_B$—the binding energy of the impurity state.[19] For such V_0, we have $A \gg 1$.

To calculate the dependence $\mathcal{R}(B)$, we substitute

$$\psi(\mathbf{r}) = \frac{\pi B h r_{1,2}}{\phi_0} \tag{4.11}$$

and the matrix elements (2.10) into Eq. (3.9). Using the condition $h \ll r_{1,2}$ we obtain the following expression for the function G:

$$G(\varphi,B) = \pi g V_0 \int_{-\infty}^{\infty} dh \int_0^{r_{1,2}} dx \exp\left[-\frac{h^2 r_{1,2}}{2ax(r-x)}\right] \times \left|\sin\left[\varphi - \frac{\pi B h r_{1,2}}{\phi_0}\right]\right| . \tag{4.12}$$

It is convenient to introduce the new variables

$$x = \frac{r_{1,2}}{2}(1-\sin\alpha) , \quad h = \left[\frac{2ax(r_{1,2}-x)}{r_{1,2}}\right]^{1/2} t , \tag{4.13}$$

in which Eq. (4.12) takes the form

$$G(\varphi,B) = \frac{2}{\pi^{1/2}} A \int_{-\infty}^{\infty} dt\, e^{-t^2} \int_{-\pi/2}^{\pi/2} d\alpha (\cos\alpha)^2 \left|\sin\left[\varphi - t\frac{B}{B_0}\cos\alpha\right]\right| . \tag{4.14}$$

Here parameter A is defined by Eq. (4.9), and the characteristic magnetic field B_0 is defined as

$$B_0 = \frac{2^{1/2}\phi_0}{\pi (a r_{1,2}^3)^{1/2}} . \tag{4.15}$$

Substitution of Eq. (4.14) into (3.18), and numerical calculation of integrals, yields the set of the dependences $\mathcal{R}(B/B_0)$ for various values of parameter A. The results of the calculations are shown in Fig. 3. It can be seen that the resistance decreases with B, for $B \lesssim 2.5 B_0$, and saturates in higher magnetic fields. The saturation value is given by Eq. (4.6). It can also be seen that for small B/B_0, the magnetoresistance is linear. To demonstrate this analytically, we rewrite Eq. (3.18) as

$$\ln\frac{\mathcal{R}(B)}{\mathcal{R}(0)} = -\frac{2}{\pi}\int_0^{\pi/2} d\varphi \ln\left[\frac{\sin^2\varphi + G^2(\varphi,B)}{\sin^2\varphi + G^2(\varphi,0)}\right] . \tag{4.16}$$

For small B, the contribution to the integral comes from small values of φ, so that we can replace $\sin\varphi$ by φ, and extend the integration to ∞. For $\varphi \ll 1$, the function $G(\varphi,B)$ can be written as

$$G(\varphi,B) = \pi A \left[\frac{B}{B_0}\right] F\left[\frac{\varphi B_0}{B}\right] , \tag{4.17}$$

where

$$F(x) = \frac{2}{\pi^{3/2}} \int_{-\infty}^{\infty} dt\, e^{-t^2} \int_{-\pi/2}^{\pi/2} d\alpha (\cos\alpha)^2 |x - t\cos\alpha| \tag{4.18}$$

is the dimensionless function with asymptotes

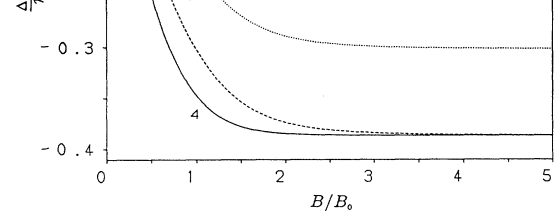

FIG. 3. The behavior of the relative magnetoresistance as a function of the normalized magnetic field for various values of normalized scattering strength A [Eq. (3.18)]. (1) $A = 0.07$, (2) $A = 0.13$, (3) $A = 0.26$, and (4) the strong-scattering limit.

$$F(x)=\begin{cases} 8/3\pi^{3/2}\ , & x \ll 1 \\ x\ , & x \gg 1\ . \end{cases} \tag{4.19}$$

Substituting Eq. (4.17) into Eq. (4.16), and introducing the variable $x=\varphi B_0/B$, we obtain

$$\frac{\mathcal{R}(B)-\mathcal{R}(0)}{\mathcal{R}(0)}=-K(A)\frac{B}{B_0}\ , \tag{4.20}$$

where

$$K(A)=\frac{2}{\pi}\int_0^\infty dx\,\ln\left[1+\frac{\pi^{5/2}A^2[F^2(x)-x^2]}{x^2(2+\pi^{5/2}A^2)}\right]\ . \tag{4.21}$$

The plot of function $K(A)$ is shown in Fig. 4. For $A \gtrsim 1$, the function K saturates at the value $K \approx 1.1$. Therefore, for strong enough scattering, the slope of the magnetoresistance appears to be independent of the scattering strength. The temperature dependence of the slope is then determined by the temperature dependence of B_0:

$$\frac{\Delta\mathcal{R}(B)}{\mathcal{R}(0)}=-1.1\frac{B}{B_0}=-2.4\frac{(ar_{1,2}^3)^{1/2}B}{\phi_0}\propto -T^{-1/2}B\ . \tag{4.22}$$

For $A \ll 1$, the values of x that contribute to the integral (4.21) are small: $x \sim A$. Therefore one can replace $F(x)$ by $F(0)=8/3\pi^{3/2}$ in (4.21). Then the evaluation of the integral yields

$$K(A)=\frac{16}{3\pi^{1/2}}A\ , \quad A \ll 1\ . \tag{4.23}$$

Substituting this into Eq. (4.20), we obtain, in the weak-scattering limit,

$$\frac{\Delta\mathcal{R}(B)}{\mathcal{R}(0)}\propto -r_{1,2}^3B \propto -T^{-1}B\ . \tag{4.24}$$

This result agrees with the result of Schirmacher.[18]

Therefore, the behavior of the magnetoresistance as a function of the normalized scattering strength A is the following. For $A<0.2$, both the slope (4.23) and the saturation value (4.6) increase with A (in the absolute value). For $0.2<A<1$, the saturation value is independent of A [namely, $\Delta\mathcal{R}(\infty)/\mathcal{R}(0)\simeq -0.39$], while the slope still increases with A. Finally, for $A>1$, both the slope and the saturation value are independent of A. Such a behavior is illustrated in Fig. 3.

It was assumed above that the magnetic field reveals itself in the probabilities of the hops only through the

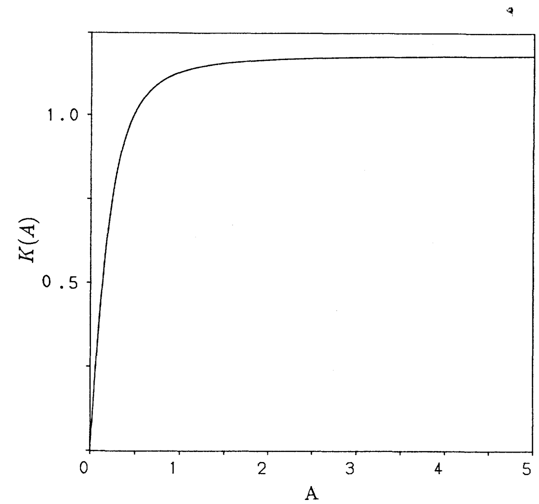

FIG. 4. The zero-field slope of the magnetoresistance as a function of the normalized scattering strength A [Eq. (4.24)].

phase factors (2.10) in the overlap integrals. This assumption is valid only for low enough magnetic fields. With the increase of B, the effect of the orbital shrinkage of the impurity wave functions has to be taken into account. The orbital shrinkage causes the reduction of the overlap integrals $V_{\mu,\nu}$ [Eq. (2.10)] which can be expressed as[1]

$$\tilde{V}_{\mu,\nu}=V_{\mu,\nu}\exp\left[-\frac{r_{\mu,\nu}^3a}{24\lambda^4}\right]\ . \tag{4.25}$$

One obvious consequence of such a reduction is the positive correction to the logarithm of the resistance of the hop caused by the decrease of the probability of the direct tunneling $|V_{1,2}|^2$,[1]

$$(\delta\ln\mathcal{R})_{\rm orb}=\frac{r_{1,2}^3a}{12\lambda^4}=\frac{2}{3}\frac{B^2}{B_0^2}\ , \tag{4.26}$$

where B_0 is defined by Eq. (4.15). We see that the characteristic magnetic field for the orbital shrinkage effect is the same field B_0 at which the interference correction (4.14) saturates.

Another effect results from the modification of the interference contribution (3.18) due to the reduction of the matrix elements $V_{1,r}$, $V_{r,2}$. Indeed, by the substitution of Eq. (4.25) into Eq. (3.9), the function $G(\varphi)$ is replaced by

$$\tilde{G}(\varphi,B)=\frac{2}{\pi^{1/2}}A\int_{-\infty}^{\infty}dt\,e^{-t^2}\int_{-\pi/2}^{\pi/2}d\alpha\,\exp\left[\frac{B^2}{B_0^2}\cos^2\alpha\right](\cos\alpha)^2\left|\sin\left[\varphi-t\frac{B}{B_0}\cos\alpha\right]\right|\ . \tag{4.27}$$

We see that in comparison with Eq. (4.14), $\tilde{G}(\varphi,B)$, which is responsible for the interference, has increased. This is the result of the strong dependence of the factor describing the shrinkage effect on the distance $r_{\mu,\nu}$. Since the distances $1\to r$ and $r\to 2$ are less than $r_{1,2}$, the product $V_{1,r}V_{r,2}$ is less reduced in magnetic field than $V_{1,2}$.

The total magnetoresistance can be presented as a sum of two contributions:

$$\delta\ln\mathcal{R}=(\delta\ln\mathcal{R})_{\rm orb}+(\delta\ln\mathcal{R})_{\rm int}\ . \tag{4.28}$$

The first contribution (4.26) is positive, and increases with B quadratically. The second contribution is described by Eq. (4.16) after substitution of $\tilde{G}(\varphi,B)$ into it. This contribution is negative and is enhanced by the effect of the orbital shrinkage. Since, at low magnetic fields, $(\delta \ln\mathcal{R})_{\rm int}$ is linear in B, it dominates the magnetoresistance for $B \ll B_0$. For $B \gg B_0$, the first term predominates and the magnetoresistance should be positive.

The numerical results for magnetoresistance given by Eq. (4.28) are shown in Fig. 5(a). It can be seen that with increasing B, the resistance passes through a minimum at $B \simeq 0.6 B_0$ and then increases. One can also see that for $A \gtrsim 1$, the curves acquire a shape which is independent of A. The maximal magnitude of the negative magnetoresistance for such A is $\Delta\mathcal{R}/\mathcal{R} \approx -0.18$. Therefore, within the model considered, the magnetoresistance appears to have a universal behavior in the limit of strong enough scattering, even if the shrinkage effect is taken into account.

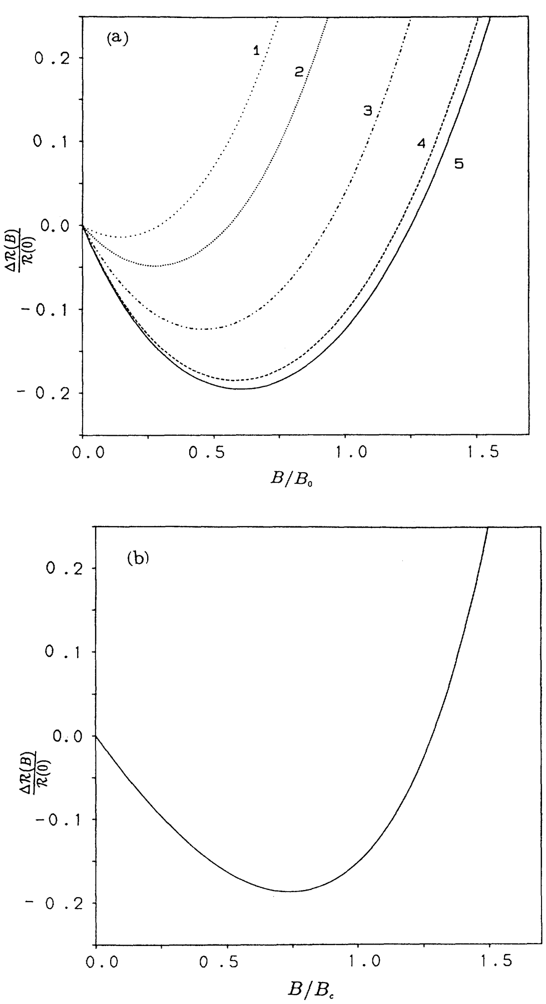

FIG. 5. The behavior of the relative magnetoresistance with orbital shrinkage effect taken into account (a) calculated with the use of Eq. (4.28) for the values of parameter A: (1) $A=0.07$, (2) $A=0.13$, (3) $A=0.23$, (4) $A=1$, and (5) the strong-scattering limit; and (b) after averaging over the hopping lengths in the strong-scattering limit.

Up to now, we have studied the magnetoresistance of the hop with some fixed length $r_{1,2}$. Following the perturbation method in the percolation theory (see Refs. 1, 14, and 19), the correction to the total resistance of the system is determined by Eq. (4.28), averaged over $r_{1,2}$ according to the rule

$$\delta \ln\mathcal{R} = \frac{20}{r_c^5}\int_0^{r_c} dr_{1,2}\, r_{1,2}^3 (r_c - r_{1,2})\, \delta \ln\mathcal{R}(r_{1,2}) , \qquad (4.29)$$

where

$$r_c = \frac{a}{2}\left[\frac{T_0}{T}\right]^{1/3} \qquad (4.30)$$

is the maximal hopping length. For $A \gtrsim 1$ (the strong-scattering limit), the dependence of $\delta \ln\mathcal{R}$, under the integral, on $r_{1,2}$ comes only from $B_0 \propto r_{1,2}^{-3/2}$ [Eq. (4.15)]. The result of averaging in this limit is shown in Fig. 5(b), in which the magnetoresistance is plotted as a function of the ratio B/B_c, with

$$B_c = \frac{2^{1/2}\phi_0}{\pi(a r_c^3)^{1/2}} . \qquad (4.31)$$

It can be seen that, after averaging, the position and the depth of the minimum remain almost unchanged, while the slope at low fields decreases approximately two times.

V. OSCILLATIONS OF THE HOPPING MAGNETORESISTANCE IN THE SYSTEM OF TWO PARALLEL PLANES

In this section, we apply our theory to the quantitative study of the hopping magnetoresistance in a system of two parallel impurity planes. It was suggested in Ref. 20 that in a magnetic field parallel to the planes, the resistance of such a system oscillates with magnetic field. The arguments presented were the following.

If the electron tunnels between two impurities located within one plane, it can be scattered either by impurities from the same plane or by impurities from the neighboring plane. In the first case, the corresponding phase factors (2.10) are strictly unity, since B is parallel to the planes. In the second case, the phase factor ψ_μ can be rewritten as

$$\psi_\mu = \frac{\pi \mathbf{d}\cdot(\mathbf{B}\times\mathbf{r}_{1,2})}{\phi_0} , \qquad (5.1)$$

where $\mathbf{d}$ is the vector distance between the planes (see Fig. 6). Note that in this case ψ_μ does not depend on the position of the scatterer μ. Then the interference correction (2.20) to the resistance of the hop $1\rightarrow 2$ takes the form

$$\delta \ln\mathcal{R}_{1,2} = -\ln\left[1+\left[\sum_\mu \frac{V_{1\mu}V_{\mu 2}}{V_{1,2}\epsilon_\mu}\right]^2 + 2\left[\sum_\mu \frac{V_{1\mu}V_{\mu 2}}{V_{1,2}\epsilon_\mu}\right] \times\cos\left[\frac{\pi B r_{1,2} d \sin\beta}{\phi_0}\right]\right] , \qquad (5.2)$$

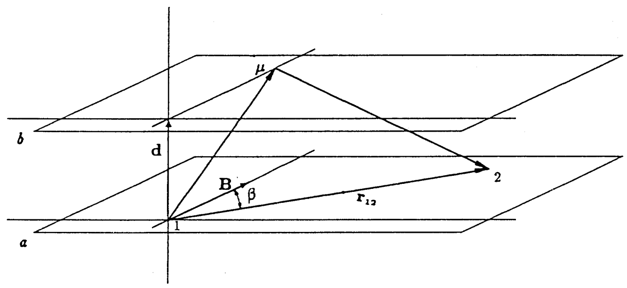

FIG. 6. The schematic illustration of the hopping transport in the system of two impurity planes in a magnetic field parallel to the planes. The electron tunneling between sites 1 and 2 in plane a is scattered by impurity μ in plane b.

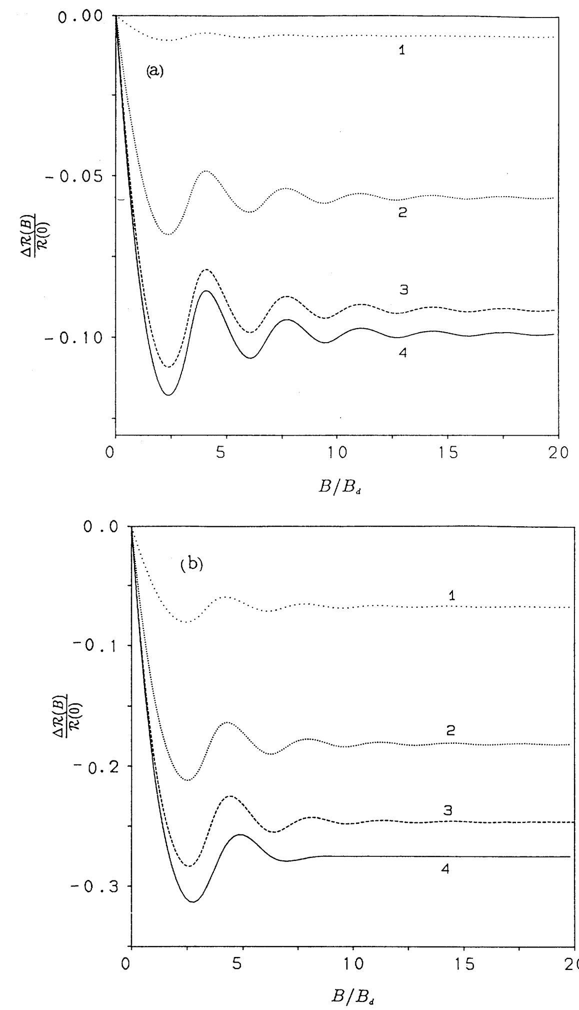

FIG. 7. Oscillations of the magnetoresistance in the system shown in Fig. 6. The curves are calculated with the use of Eq. (5.2) for interplane distance. (a) $d=(2ar_c)^{1/2}$; $A_c=$(1) 1, (2) 10, (3) 18, and (4) 20. (b) $d=(ar_c)^{1/2}$; $A_c=$(1) 0.8, (2) 3, (3) 6, and (4) 20.

where β is the angle between $\mathbf{B}$ and $\mathbf{r}_{1,2}$. We can see that the resistance of the hop oscillates with magnetic field. To find the correction to the total resistance, one should average (5.2) over the energies and positions of the scatterers and also over the directions β and the lengths $r_{1,2}$ of the hop. The latter averaging is performed with the use of Eq. (4.29). It is significant that after such an averaging the oscillations do not disappear. The reason is that the possible values of the hopping length in the integral (4.29) are strictly limited. Indeed, according to the percolation procedure used in calculating the hopping conductivity,[1] only the hops with $r_{1,2}<r_c$ can contribute to the total resistance, the hops with $r_{1,2}>r_c$ being shunted by the shorter ones.

As shown in Sec. III, the averaging of Eq. (5.2) over the positions and energies of scatterers results in Eq. (3.18), with function G defined by Eq. (3.9). For the geometry under study, this function takes the form

$$G(\varphi)=\pi\left|\sin\left[\varphi-\frac{\pi B r_{1,2} d\sin\beta}{\phi_0}\right]\right| A(r_{1,2})\,, \tag{5.3}$$

where the dependence $A(r_{1,2})$ is defined by Eq. (4.2). The integration in Eq. (4.2) is now performed over plane b (see Fig. 6), while points 1 and 2 are located in plane a. Substituting expression (2.10) into Eq. (4.2) for the overlap integrals, and performing the integration, we obtain

$$A(r_{1,2})=A_c Q\left[\frac{r_{1,2}}{r_c}\right], \tag{5.4}$$

where

$$A_c=\frac{\pi^{3/2}}{2^{5/2}} g V_0 (a r_c^3)^{1/2}\simeq 5\frac{V_0}{(T_0 T)^{1/2}}\,, \tag{5.5}$$

and the dimensionless function Q is defined as

$$Q(t)=t^{3/2}\int_0^{\pi/2} d\Theta(\cos\Theta)^2\exp\left[-\frac{2d^2}{r_c a t\cos^2\Theta}\right]. \tag{5.6}$$

The resulting expression for the magnetoresistance can be now written as

$$\delta\ln\mathcal{R}=-\frac{40}{\pi}\int_0^1 dt\, t^3(1-t)\int_0^{2\pi}d\beta\int_0^{\pi/2}d\varphi\ln\left[1+4\frac{A_c^2Q^2(t)\sin^2\left[\varphi-t\frac{B}{B_d}\sin\beta\right]}{\sin^2\varphi}\right], \tag{5.7}$$

where

$$B_d = \frac{\phi_0}{\pi r_c d} \ . \tag{5.8}$$

To derive Eq. (5.7), we have substituted Eq. (5.3) into Eq. (3.18) and introduced the dimensionless variable $t = r_{1,2}/r_c$. The threefold integration in Eq. (5.7) can be reduced to twofold by using integration by parts over β.

It can be seen from Eqs. (5.6) and (5.7) that the behavior of the magnetoresistance as a function of normalized magnetic field B/B_d depends on two dimensionless parameters: d^2/ar_c and the normalized scattering strength A_c. The typical amplitude of oscillations $\Delta\mathcal{R}(B)/\mathcal{R}(0)$ is on the order of 2%. The numerical results for the magnetoresistance at $d^2/ar_c = 1$ and 2, and for various values of A_c, are shown in Fig. 7. It can be seen that the amplitude of the oscillations decreases with B. Since the orbital shrinkage effect was not taken into account, the magnetoresistance saturates at high B. With orbital shrinkage, the dependence $\Delta\mathcal{R}(B)$ should represent the curve having a minimum [as in Fig. 5(a)], with oscillations superimposed on it. As was mentioned in Ref. 20, the analysis of these oscillations allows us to find the value of the hopping length r_c and, thus, the localization radius a. This is significant since the value of a in the vicinity of Mott's transition can exceed the localization radius of an isolated impurity.[19]

VI. MAGNETORESISTANCE AT VERY LOW FIELDS

In the regime of very small B, one should use Eq. (2.20) rather than Eq. (2.22) for the scattering-induced correction to the resistance of a hop.[19] As was mentioned in Sec. II, in this regime the main contribution to the magnetoresistance comes from the realizations of scatterers for which the parameter

$$\tau = 1 + \sum_\mu \frac{V_{1,\mu} V_{\mu,2}}{V_{1,2}\varepsilon_\mu} \tag{6.1}$$

is small: $|\tau| \ll 1$. Then we can present the magnetoresistance of a hop in the following form:

$$\ln\frac{\mathcal{R}(B)}{\mathcal{R}(0)} = \frac{\mathcal{R}(B)-\mathcal{R}(0)}{\mathcal{R}(0)} = -\ln\left[\frac{\tau^2 + 4\pi^2 B^2\omega^2/\phi_0^2 + \lambda}{\tau^2+\lambda}\right] , \tag{6.2}$$

where

$$\omega = \sum_\mu \frac{V_{1,\mu}V_{\mu,2}}{V_{1,2}\varepsilon_\mu} S_\mu \tag{6.3}$$

and

$$\lambda = \frac{(\varepsilon_1-\varepsilon_2)^2}{4|V_{1,2}|^2}\left[\left|\sum_\mu \frac{V_{1,\mu}V_{\mu,2}}{\varepsilon_\mu^2}\right|^2 + 2\sum_\mu \frac{|V_{1,\mu}|^2|V_{\mu,2}|^2}{\varepsilon_\mu^4}\right] . \tag{6.4}$$

Let $W(\tau,\omega,\lambda)$ be the distribution function of the parameters τ, ω, and λ. Then the result of the configurational averaging of Eq. (6.2) can be written as

$$\frac{\Delta\mathcal{R}(B)}{\mathcal{R}(0)} = -\int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\omega \int_0^{\infty} d\lambda \, W(\tau,\omega,\lambda) \times \ln\left[1 + \frac{4\pi^2 B^2\omega^2}{\phi_0^2(\tau^2+\lambda)}\right] . \tag{6.5}$$

Since the relevant values of τ are small, we can replace $W(\tau,\omega,\lambda)$ with $W(0,\omega,\lambda)$. Then the integration over τ can be performed analytically, and yields

$$\frac{\Delta\mathcal{R}(B)}{\mathcal{R}(0)} = -2\pi\int_{-\infty}^{\infty} d\omega \int_0^{\infty} d\lambda \, W(0,\omega,\lambda)\left[\left[\lambda + \frac{4\pi^2B^2\omega^2}{\phi_0^2}\right]^{1/2} - \lambda^{1/2}\right] . \tag{6.6}$$

It can be seen that neglecting the parameter λ (in the square brackets) immediately results in linear magnetoresistance. Retaining this parameter leads to quadratic magnetoresistance at small B:

$$\frac{\Delta\mathcal{R}(B)}{\mathcal{R}(0)} = -\frac{B^2}{B_q^2} , \tag{6.7}$$

with

$$\frac{1}{B_q^2} = \frac{4\pi^3}{\phi_0^2}\int_{-\infty}^{\infty} d\omega \int_0^{\infty} d\lambda \frac{\omega^2}{\lambda^{1/2}} W(0,\omega,\lambda) . \tag{6.8}$$

Let us estimate the characteristic field B_q. This estimate is different in the cases of weak ($A \ll 1$) and strong ($A \gtrsim 1$) scattering. Indeed, as can be seen from Eq. (4.9), parameter A is on the order of V_0/E, where E—the typical energy of the scatterer— is defined by Eq. (2.18).

If $A \ll 1$, we have $V_0 \ll E$. To satisfy the condition $|\tau| \ll 1$, a scatterer with energy $\varepsilon_\mu \sim V_0 \ll E$, located within the interference area is required. The probability of finding such a scatterer is on the order of V_0/E. From Eqs. (4.13) and (4.4), we can see that, for configurations containing such scatterers, the value of parameter ω is on the order of the interference area $(ar_c^3)^{1/2}$, while the value of parameter λ is on the order of $(\varepsilon_1-\varepsilon_2)^2/V_0^2 \sim \Delta^2/V_0^2$, where Δ is the width of Mott's energy strip [see Eq. (2.17)]. Using these estimates, from Eq. (6.8) we obtain

$$\frac{1}{B_q^2} \sim \frac{1}{\phi_0^2}\left[\frac{V_0}{E}\right](ar_c^3)\left[\frac{V_0}{\Delta}\right] \sim \frac{V_0^2}{\Delta E B_c^2} , \tag{6.9}$$

where the characteristic field B_c is defined by Eq. (4.31).

The range of B, where the magnetoresistance is quadratic, can be estimated from Eq. (6.6). One has

$$B < \frac{\phi_0 \lambda^{1/2}}{\omega} \sim \frac{\Delta}{V_0} B_c \sim \left[\frac{\Delta}{E} \right]^{1/2} B_q \ . \tag{6.10}$$

For higher B, the linear dependence (4.20) applies. The temperature dependence of the resistance within the quadratic regime results from the temperature dependencies of parameters Δ, E, and B_c. Using Eqs. (2.17), (2.19), and (4.22), we obtain

$$\frac{\Delta \mathcal{R}(B)}{\mathcal{R}(0)} \propto T^{-13/6} B^2 \ . \tag{6.11}$$

In the limit of strong scattering ($A \gtrsim 1$), we have $V_0 \gtrsim E$, and the condition $|\tau| \ll 1$ can be satisfied for typical realization of scatterers. Substituting $\varepsilon_\mu \sim E$ in Eq. (6.14), we obtain $\lambda \sim \Delta^2/E^2$, while ω is still on the order of $(ar_c^3)^{1/2}$. With these values we obtain, from Eq. (6.8),

$$\frac{1}{B_q^2} \sim \frac{1}{\phi_0^2}(ar_c^3)\frac{E}{\Delta} \sim \frac{E}{\Delta B_c^2} \ . \tag{6.12}$$

Therefore, in the regime of strong scattering, the low-field magnetoresistance behaves as

$$\frac{\Delta \mathcal{R}(B)}{\mathcal{R}(0)} \propto -T^{7/6} B^2 \ . \tag{6.13}$$

This behavior applies within the range

$$B \gtrsim \frac{\Delta}{E} B_c \ . \tag{6.14}$$

For higher B, the magnetoresistance is linear and has the form of (4.22).

VII. CONCLUSION

In the present paper an alternative approach to the calculation of interference-induced magnetoresistance in the variable-range-hopping regime has been developed. The approach is valid for moderate temperatures, when the hops are not very long and the interference area contains only a few scatterers. Note that the regime considered in the paper is the only regime in which the theory is able to provide the explicit quantitative predictions. It also allows us to take into account the effect of orbital shrinkage and, therefore, to trace the transition from negative to positive magnetoresistance with increasing magnetic field.

The condition that the single-scattering tunneling paths make the major contribution to the resistance restricts the possible values of the Mott hopping length to $r \le n^{-2/3} a^{-1/3}$ [see Eq. (1.3)]. Another restriction comes from the condition that the regime of transport is the variable-range hopping, so that the hopping length is larger than the average distance between the impurities $n^{-1/2}$. Therefore, the approach developed applies to the hopping length lying within the interval

$$n^{-2/3} a^{-1/3} \ge r > n^{-1/2} \ . \tag{7.1}$$

For comparison with experiment, it is convenient to reformulate condition (7.1) in terms of the logarithm of the resistance in zero magnetic field $\ln\mathcal{R} = 2r/a$. We then obtain

$$2(na^2)^{-2/3} \ge \ln\mathcal{R} > 2(na^2)^{-1/2} \ . \tag{7.2}$$

For numerical estimates, it seems reasonable to choose the value 0.1 for parameter na^2. Indeed, for higher values of this parameter, the electron gas is metallic. On the other hand, this parameter cannot be much smaller than 0.1, since far from the metal-insulator transition the hopping resistance becomes too high to be measured. With $na^2 = 0.1$, we obtain, from (7.2),

$$9.3 \ge \ln\mathcal{R} > 6.3 \ . \tag{7.3}$$

The most detailed studies of the 2D hopping magnetoresistance in crystalline semiconductor structures were carried out in Refs. 3 and 6. Both works were done on n-type GaAs samples. In Ref. 6, the Si donors with concentration $n = 0.8 \times 10^{11}$ cm^{-2} were arranged in atomically sharp planes (δ doping). Since the localization radius of a donor in GaAs is $a = 1000$ Å, the parameter na^2 was 0.08. The Mott parameter T_0 obtained from the temperature dependence of the resistance in zero field was 340 K. The lowest temperature studied was 1.5 K, which corresponds to $\ln\mathcal{R} = 6.1$–the boundary of the VRH regime. The maximal decrease of the resistance with the magnetic field, $\Delta\mathcal{R}/\mathcal{R}(0)$, was about 0.2. Our theory predicts $\Delta\mathcal{R}/\mathcal{R}(0) = 0.18$. In Ref. 23, the same sample was measured at much lower temperatures (in the interval 0.17–1 K). For values $\ln\mathcal{R} \ge 12$, the decrease of the resistance $\Delta\mathcal{R}/\mathcal{R}(0)$ was about 0.5. This means that the hops at low temperatures ($T < 0.5$ K) were long enough to involve several scattering acts. Another reason for the large values of $\Delta\mathcal{R}/\mathcal{R}(0)$ is that at strong enough B the incoherent mechanism of the negative magnetoresistance proposed in Refs. 22 and 23 may be effective. In Refs. 22 and 23, it was shown that the orbital shrinkage effect, by suppressing the overlap between nearest neighbors, increases the density of states at the Fermi level and, hence, decreases the resistance.

In Ref. 3, the concentration of electrons in the conductive channel could be varied by changing the gate voltage. At $T = 4.2$ K, the value of $\ln\mathcal{R}$ was estimated as 5–6. Increasing the gate voltage (reduction in the number of electrons) varied the maximal decrease of the resistance $\Delta\mathcal{R}/\mathcal{R}(0)$ from 0.1 to 0.35. This is a consequence of the fact that the scattering potential of donors (located mostly outside of the channel) increases as the channel is depopulated, since the small number of electrons is unable to screen the fluctuating potential of charged donors. The maximal value of $\Delta\mathcal{R}/\mathcal{R}(0) \simeq 0.35$ is two times larger than what we predict. However, one should take into account that the thickness of the channel (about 250 Å) was comparable to the hopping length $r \simeq 600$ Å, so that negative magnetoresistance was observed both parallel to and in perpendicular orientation with B. If the direction of the hop is not strictly perpendicular to the magnetic field, the effect of orbital shrinkage is weaker and, hence, the magnetoresistance should be larger.

We have restricted our consideration to the 2D case. However, the theory can be easily extended to the 3D. Indeed, the averaging procedure, described in Sec. III, also applies in the 3D case including the final result (3.18). The only change is in the definition (3.9) of the function $G(\varphi)$ (3D instead of the 2D integration). It is easy to see that in the 3D case, the form of the function $G(\varphi)$ for $B=0$ and for very strong B is the same as in the 2D. Therefore, the result (4.6) for the total decrease of the resistance in the absence of orbital shrinkage is valid in the 3D case as well.

The difference in the energies of sites between which the electron hops, as well as the absorption and emission of phonons at the scatterers, results in the quadratic magnetoresistance at very low fields: $\delta\mathcal{R}(B)/\mathcal{R}(0) \sim -B^2/B_q^2$. We can estimate the characteristic field B_q in the 3D case in the same way as it was done for the 2D in Sec. VI. Consider the limit of the strong scattering; then by analogy with Eq. (6.12), we have

$$\frac{1}{B_q^2} \sim \frac{\widetilde{E}}{\widetilde{\Delta}} \frac{1}{B_c^2} , \tag{7.4}$$

where $\widetilde{\Delta}$ is the Mott energy strip in the 3D case, and $\widetilde{E}$ is the characteristic energy of the scatterer within the "interference volume" ar_c^2. We then have $\widetilde{E} \propto 1/ar_c^2$. Since $B_c \propto r_c^{-3/2}$, from Eq. (7.4) we obtain

$$\frac{1}{B_q^2} \propto \frac{r_c}{\widetilde{\Delta}} . \tag{7.5}$$

Equation (7.5) determines the temperature dependence of the magnetoresistance within the quadratic region. Since, in the 3D case, $r_c \propto T^{-1/4}$ and $\widetilde{\Delta} \propto T^{3/4}$, we have

$$\frac{\Delta\mathcal{R}(B)}{\mathcal{R}(0)} \propto -\frac{1}{T} B^2 . \tag{7.6}$$

The other possible situation is VRH in the regime of the Coulomb gap.[1] In this case, $r_c \propto T^{-1/2}$ and $\widetilde{\Delta} \propto T^{1/2}$. Thus we again arrive at dependence (7.6).

In the experimental paper,[10] the transition from the Coulomb gap regime (at low temperatures) to Mott's VRH (at higher temperatures) was traced by varying the temperature. It was demonstrated that the T dependence of the magnetoresistance within the quadratic region is the same in both regimes: $\Delta\mathcal{R}(B)/\mathcal{R}(0) \propto T^{-\alpha}B^2$, with $\alpha \simeq 1.3$. The value $\alpha \simeq 1.2$ was reported in Ref. 7, where the resistance of the GaAs sample exhibited the Coulomb gap temperature dependence $\ln\mathcal{R}(T) \propto T^{-1/2}$ at zero magnetic field. The difference in the powers of temperature between the theoretical result (7.6) and the experiment[7,10] may be accounted for by the uncertainty in the estimation of the energy $\widetilde{E}$. Indeed, making the estimate, we have assumed that $\widetilde{E}$ lies within the energy range where the density of states is constant. The agreement can be improved if we assume that the density of states increases with ε approximately as $\varepsilon^{1/2}$ within the region $\varepsilon \sim E$.

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