# Single-particle motion in a random magnetic flux

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The motion of a quantum particle in a random magnetic flux in two dimensions is investigated. Two situations are distinguished, a "Debye" phase where the fluxes are uncorrelated, and a "Meissner" phase where the fluxes appear as neutral pairs. A geometrical interpretation of effective single-particle action in these phases is emphasized. Results are discussed for (a) a continuum white-noise model where we employ a trial-action method, (b) a continuum model with randomly distributed flux tubes where we obtain the form of the Lifschitz tail, and (c) a lattice model, where numerical results for the density of states and diamagnetic response of Debye and Meissner phases are given. An important conclusion is that the density of states in the Debye phase exhibits a sharp peak at an effective band edge.

## I. INTRODUCTION

The quantum motion of a particle that experiences a random magnetic flux presents a problem of fundamental interest to condensed-matter physics. Aspects of this problem have arisen in several contexts during recent years; notable examples include holon motion in doped Mott insulators,[1,2] vortex liquids in type-II superconductors,[3] and anyons.[4] The correlations or constraints appearing in real systems can often be represented by gauge fields. Thus for example the backflow constraint in a doped Mott insulator is equivalent to a current-current interaction between spins and charges, which can be considered to be mediated by a gauge field.[2,5]

It is clearly important to understand the single-particle static field problem which parallels these many-body examples. Here there are few exact results, although a good deal of understanding has been achieved recently using a variety of techniques.[6–12] The lack of exact results is not surprising; even the conceptually simple situation of a periodically varying magnetic flux (e.g., a periodic array of flux tubes) presents a formidable problem. The reason is that the vector potential associated with a flux line is long ranged.

The problem of particle motion in a random flux $\phi$ can be simply formulated in terms of the imaginary-time path integral

$$Z = \int d[\phi] \exp\left[-\frac{1}{2}\phi D \phi\right] \int d[\mathbf{r}] \exp\left[-S_0 + i \oint \mathbf{a}\cdot d\mathbf{r}\right], \tag{1}$$

where $\phi = \nabla \times \mathbf{a}$, and $D^{-1}(r,r')$ is the flux correlator $\langle \phi_r \phi_{r'} \rangle$. The kinetic term $S_0 = \frac{1}{2}\int_0^\beta d\tau\, \dot{\mathbf{r}}^2$ is the action of a random walker in imaginary time; $\beta$ is the inverse temperature, $\mathbf{r}(\tau) = \mathbf{r}(\tau+\beta)$, and we set $e$, $c$, $\hbar$, and the particle mass equal to unity.

We distinguish two cases which often arise in practice. A "Meissner screened" phase is defined by a static gauge-field correlation $\langle a_q^\alpha a_{-q}^\beta \rangle = (q^2 + \lambda_L^{-2})^{-1} \delta^{\alpha\beta} \langle a_0^2 \rangle$ corresponding to a flux fluctuation

$$\langle \phi(r)\phi(0)\rangle = \langle a_0^2\rangle \left[\delta^{(2)}(\mathbf{r}) - \frac{1}{2\pi\lambda_L^2} K_0(\lambda_L^{-1} r)\right], \tag{2}$$

where $K_0$ is a Bessel function of the second kind (the screened Coulomb potential). The Meissner phase corresponds to the static flux fluctuation in a superconductor. It has an interpretation as a phase in which the fluxes occur only as neutral pairs, with average separation $\lambda_L$. If the screening length is infinite,

$$\langle \phi(r)\phi(0)\rangle = \langle a_0^2\rangle \delta^{(2)}(r), \tag{3}$$

this is called the "Debye" phase because it corresponds to the static part of the magnetic flux fluctuation in a dielectric or metal.

Performing the average over Meissner flux in Eq. (1), the contribution to the single-particle effective action $\frac{1}{2}\oint\oint dr^\alpha dr'^\beta \langle a_r^\alpha a_{-r'}^\beta \rangle$ is

$$\Delta S_{\text{Meissner}} = \frac{\langle a_0^2\rangle}{4\pi} \oint \oint d\mathbf{r}\cdot d\mathbf{r}' K_0(\lambda_L^{-1}|\mathbf{r}-\mathbf{r}'|). \tag{4}$$

In the Debye limit $K_0(\lambda_L^{-1} r) \to -\ln \lambda_L^{-1} r/2$ and this becomes

$$\Delta S_{\text{Debye}} = -\langle a_0^2\rangle \frac{1}{4\pi} \oint\oint d\mathbf{r}\cdot d\mathbf{r}' \ln(|\mathbf{r}-\mathbf{r}'|) \equiv \langle a_0^2\rangle \Omega. \tag{5}$$

Because of the closed-loop boundary condition, the length scale appearing in the logarithm of the Debye action is arbitrary and has been dropped. Note that the effective action [Eqs. (4) and (5)] is independent of the parametrization of the path and so measures a purely geometrical property of the path. The quantity $\Omega$ appearing in the Debye action [Eq. (5)] is the "Amperean area" of the path. Under dilatation of the loop by a scale factor $s$, $\Omega \to s^2 \Omega$.[13]

An equivalent expression for the quantity $\Omega$ is

$$\Omega = \int d^2R\, w^2(R), \tag{6}$$

where $w(R)$ is the winding number of the loop about

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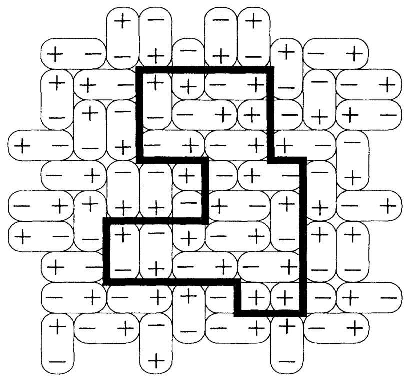

FIG. 1. Pictorial representation of Meissner phase. The flux accumulated in a Meissner phase depends on the perimeter of the loop.

the point $R$. This provides a geometrical interpretation of Eq. (5). The equivalence of the two expressions follows since Eq. (6) can be expressed as $\oint dr^\alpha \oint dr'^\beta \int d^2R a_0^\alpha(r-R) a_0^\beta(r'-R)$, where $a_0(r-R)$ is the vector potential of a unit flux tube located at $R$. The quantity $\int d^2R a_0^\alpha(r-R)\, a_0^\beta(r'-R)$ can be identified with the kernel $\langle a_\tau^\alpha a_{\tau'}^\beta\rangle$ in the Debye phase since its double curl is $\delta^{(2)}(r-r')$, corresponding to the Debye flux fluctuation [Eq. (3)].

A more pictorial description of the Meissner and Debye phases is provided by considering the lattice equivalent of the above models. Figure 1 shows a possible Meissner phase; "molecules" of flux $\phi$ and $-\phi$ are placed randomly on the lattice. If a particle moves about a loop as shown, the flux accumulated in the case of the Meissner phase depends on the number of molecules cut by the loop; the mean-square flux fluctuation therefore follows a perimeter law which corresponds to Eq. (4). In the Debye case, the mean-square accumulated flux follows an area law. This is familiar from lattice-gauge theories where area and perimeter laws of the Wilson loop distinguish confined from deconfined phases.[14]

A lattice version of the model is treated numerically in Sec. IV. The Debye phase is defined by a uniformly distributed *random flux*; the Meissner phase is defined by a uniformly distributed *random phase* on links of the lattice. The numerical results yield the form of the density of states in "typical" regions of the system. A sharp peak in the density of states is clearly resolved in the Debye phase; this feature is entirely absent in the Meissner phase. We associate the peak with the singular long-distance behavior of the Debye action.

First, however, we outline some analytic approaches to the continuum problem.

## II. VARIATIONAL METHOD

A simple but powerful approach to the continuum problem is provided by the trial-action method. A lower bound on $Z$ is obtained using a trial action $S_t = S_0 + \Delta S_t$ and the inequality

$$Z \geq Z_t \exp -\langle \Delta S - \Delta S_t\rangle_{S_t}. \tag{7}$$

For a Gaussian trial action, whose diffusion law we denote as $\langle r^2(\tau)\rangle$, we get for the average of the action [Eq. (4)]:

$$\begin{aligned}\frac{\langle a_0^2\rangle}{4\pi}\left\langle \oint\oint d\mathbf{r}\cdot d\mathbf{r}' K_0(\lambda_L^{-1}|\mathbf{r}-\mathbf{r}'|)\right\rangle_{S_t} &= \frac{\langle a_0^2\rangle}{8\pi^2}\int \frac{d^2q}{q^2+\lambda_L^{-2}}\left\langle \int_0^\beta\int_0^\beta d\tau d\tau' \dot{\mathbf{r}}\cdot\dot{\mathbf{r}}' e^{i\mathbf{q}\cdot\mathbf{r}-i\mathbf{q}\cdot\mathbf{r}'}\right\rangle \\ &= \frac{\langle a_0^2\rangle}{8\pi^2}\int \frac{d^2q}{q^2+\lambda_L^{-2}}\int_0^\beta\int_0^\beta d\tau d\tau' \langle \dot{\mathbf{r}}_\perp\cdot\dot{\mathbf{r}}'_\perp\rangle e^{-\frac{1}{2}q^2\langle(\mathbf{r}_\parallel-\mathbf{r}'_\parallel)^2\rangle} \\ &= \frac{\beta\langle a_0^2\rangle}{8\pi^2}\int \frac{d^2q}{q^2+\lambda_L^{-2}}\int_0^\beta \frac{1}{2}d\tau \frac{d^2}{d\tau^2}\langle r^2\rangle e^{-\frac{1}{2}q^2\langle r^2\rangle}.\end{aligned} \tag{8}$$

$\mathbf{r}_\parallel$ and $\mathbf{r}_\perp$ refer to components parallel and perpendicular to $\mathbf{q}$.

The singular behavior of Eq. (8) at small and high $q$ is controlled by the diffusion law of the trial model. For large $q$ only the short-time diffusion is important; since the short-time diffusion is that of a free particle, $\frac{d^2}{d\tau^2}\langle r_\tau^2\rangle$ can be replaced by $\delta(\tau)$ and the time integration yields a factor independent of $q$. Therefore the integral depends logarithmically on an upper cutoff $q_0$.

For small $q$ the momentum integral is cut off by the screening length $\lambda_L$. A second cutoff at small $q$ is provided by the thermal length $\lambda_{\rm th}$ of the trial model. This follows because for momenta satisfying $q^2\langle r_\tau^2\rangle \ll 1$, the integral $\int_0^\beta d\tau \frac{d^2}{d\tau^2}\langle r_\tau^2\rangle$ vanishes by periodic boundary conditions on the paths. Thus

$$\langle \Delta S\rangle_{S_t} = \begin{cases} \frac{\beta\langle a_0^2\rangle}{4\pi}\ln q_0\lambda_L, & \lambda_L \ll \lambda_{\rm th} \text{ Meissner}, \\ \frac{\beta\langle a_0^2\rangle}{4\pi}\ln q_0\lambda_{\rm th}, & \lambda_L \gg \lambda_{\rm th} \text{ Debye}. \end{cases} \tag{9}$$

In the Debye phase it is advantageous to limit the growth of the thermal length at low temperatures. One Gaussian model which does this is is the Caldeira-Leggett model of dissipative quantum mechanics,[7] which is $S_0 + \pi\eta\int_0^\beta\int_0^\beta d\tau d\tau' \ln|\tau-\tau'|$. The low-temperature partition sum in this model is[8]

$$Z_t = \frac{\eta}{2}e^{-\beta\eta[1+\ln\omega_c/\eta]} \tag{10}$$

and thus $\langle \Delta S_t\rangle_{S_t} = -\eta d\ln Z_t/d\eta = \beta\eta\ln\omega_c/\eta$. $\omega_c$ is a high-frequency cutoff. The thermal length is fixed by the

the dissipation $\eta$: At low temperature $\lambda_{\rm th}^{-2} \sim \eta$. Combining these results, the value of $\eta$ which maximizes the right-hand side of Eq. (7) at low temperature $\beta\frac{\langle a_0^2\rangle}{8\pi} \gg 1$ is

$$\eta = \begin{cases} \frac{1}{\beta}, & \lambda_L^2 \frac{\langle a_0^2\rangle}{8\pi} \ll 1, \text{ Meissner}, \\ \frac{\langle a_0^2\rangle}{8\pi}, & \lambda_L^2 \frac{\langle a_0^2\rangle}{8\pi} \gg 1, \text{ Debye} \end{cases} \quad (11)$$

and

$$Z \gtrsim \begin{cases} \frac{1}{2\beta}\exp\left[-\beta\frac{\langle a_0^2\rangle}{8\pi}\ln q_0^2\lambda_L^2\right], & \frac{\langle a_0^2\rangle}{8\pi}\lambda_L^2 \ll 1, \text{ Meissner}, \\ \frac{\langle a_0^2\rangle}{16\pi}\exp\left[-\beta\frac{\langle a_0^2\rangle}{8\pi}\ln\frac{8\pi q_0^2}{\langle a_0^2\rangle}\right], & \frac{\langle a_0^2\rangle}{8\pi}\lambda_L^2 \gg 1, \text{ Debye}. \end{cases} \quad (12)$$

These expressions can be used to deduce information about densities of states. The exponential decay of the partition sum present in both cases corresponds to an upward shift of the band edge. The behavior of the density of states at the new band edge is markedly different, however; in the Meissner case it corresponds to a weakly renormalized free-particle density of states in two dimensions; in the Debye phase it corresponds to a *$\delta$-function peak*[8] of strength $\frac{\langle a_0^2\rangle}{16\pi}$. The existence of a density of states peak at low energy in the Debye phase is also predicted by perturbation theory[9] and will be confirmed in Sec. IV.

While these results are suggestive, they suffer from the usual limitations of the trial method. A significant defect is that the introduction of a cutoff scale for the flux fluctuation means that a Lifschitz tail[15] is expected in the density of states, starting from zero energy, deriving from exceptional regions of the system with small flux. This effect is completely missed in the analysis leading to Eqs. (12); to capture the Lifschitz tail it is necessary to use a trial action which breaks translational symmetry as described in Sec. III. The presence of a low-energy tail means that the behavior of the density of states at the "effective band edge" can be less singular than that given by Eq. (12) without violating the variational principle.

## III. LIFSCHITZ TAILS

To illustrate the Lifschitz tail in a random magnetic flux we study a system of randomly distributed flux tubes of fixed strength; the model is defined by two parameters: an areal density $\rho$ (equivalent to $q_0^2$ above) and a small flux strength $\mu$. (With flux strengths $\pm\mu$ the Debye model [Eq. (5)] is recovered in the limit of infinite $\rho$ but finite $\rho\mu^2 \equiv \langle a_0^2\rangle$.) We follow the formal technique of Friedberg and Luttinger[16] who studied the Lifschitz tail in a system of randomly distributed, local, repulsive potentials. The key to this approach is once again the use of a trial action and the inequality (7) for the partition sum. The impurity averaged partition sum is

$$Z = \prod_{i=1}^{N}\int_V \frac{d^2R_i}{V}\int d[\mathrm{r}]\exp\left[-S_0 + i\sum_i \oint dr a_\mu(r-R_i)\right], \quad (13)$$

where $a_\mu(r-R)$ is the vector potential of a flux tube of strength $\mu$ located at $R$ and $\nabla\times\mathbf{a}_\mu(r) = \mu\delta^{(2)}(r)$.

Integration over the flux tube positions $R_i$ is easily carried out. Noting that the impurity average $\langle\exp i\oint adr\rangle_{\rm impurity}$ can be written $[1-\rho/N\int d^2R(1-\exp i\oint adr)]^N$, with $\rho = N/V$, and taking the thermodynamic limit, $Z = \int d[r]\exp[-S_0 - W_+]$, where the effective action is

$$W_+\{r\} = \rho\int d^2R\left(1-\exp\left[-i\oint a_\mu(r-R)dr\right]\right). \quad (14)$$

The exponent appearing in Eq. (14) is precisely $\mu$ times the winding number $w(R)$ of the path $r$ about the point $R$. Note that in the limit $\mu\to 0$ but $\rho\mu$ fixed the effective action reduces to $i\rho\mu\int d^2Rw(R)$, which is the action in a uniform field $\rho\mu$ as expected.

We introduce a trial field $\nabla\times a_t$ corresponding to a large hole in an otherwise uniform flux distribution $\rho\mu$ centered at the origin. (The trial flux thus breaks translational symmetry.) The transition amplitudes $(K)$ satisfy the inequality $Z \geq VK_t(0)\exp[-\langle W_+ - W_t\rangle_0]$, where the average is taken with respect to the trial action $S_0 + W_t$, $W_t = \oint dra_t(r)$. It is assumed that $\exp -\beta E_0(a_t) \gg \exp -\beta E_1(a_t)$, where $E_0$ and $E_1$ are the lowest and first excited state in the well, so that for instance we can write $K_t(0) = |\psi_0(0)|^2\exp(-\beta E_0)$, where $\psi_0$ is the ground state of the Schrödinger equation in the trial flux. The validity of this inequality is checked at the end of the calculation.

Retaining leading terms in $\beta$ a simple calculation shows that we must maximize $\exp(-\beta Q[a_t])$ where the functional $Q[a_t]$ is

$$Q[a_t] = E_0[a_t] + \int ja_tdr + \frac{\rho}{\beta}\int d^2R\{1-\exp[-\beta(E_0[a_t+a_\mu(R)] - E_0[a_t])]\}. \quad (15)$$

The first two terms are the leading contributions from $\exp[\ln K_t + \langle W_t\rangle]$ and the last term is the average of the effective action [Eq. (14)]. The energy shift on placing a flux tube of strength $\mu$ at $R$, $E_0[a_t + a_\mu(R)] - E_0[a_t]$, is $-\int\mathbf{j}\cdot\mathbf{a}_\mu d^2r$, where $\mathbf{j}_0 = \psi_0^*(i\nabla - \mathbf{a}_t)\psi_0/2m + \text{c.c.}$ is the physical current flowing in the ground state. Since $\nabla\cdot\mathbf{j} = 0$, $j_0^\alpha$ can be expressed as $\epsilon^{\alpha\beta}\partial_\beta\Lambda$, where the scalar field $\Lambda(\mathbf{r}) = \int_\infty^{\mathbf{r}}\mathbf{j}\times d\mathbf{r}'$. The line integral is taken along an arbitrary path from infinity to $R$. Therefore the energy shift may be written as $\int \epsilon^{\alpha\beta}\partial_\alpha a_\mu^\beta \Lambda d^2r = \mu\Lambda(R)$. The optimum trial flux distribution is given by the solution of $\delta Q/\delta a_t^\alpha(r) = 0$, yielding

$$\frac{|\psi_0^2|}{m} a_t^\alpha(r) - \rho\mu \int d^2R \exp[-\Lambda(R)] \frac{\delta\Lambda(R)}{\delta a_t^\alpha(r)} = 0. \tag{16}$$

Noting that $\delta\Lambda(R)/\delta a_t^\alpha(r) = \int_\infty^R \epsilon^{\beta\gamma}[\delta j_0^\beta/\delta a_t^\alpha(r)]dr'_\gamma = \frac{|\psi_0^2|}{m}\int_\infty^R \delta^{(2)}(r-r')\epsilon^{\alpha\nu}dr'_\nu$ and taking the curl of Eq. (16) yields a relation between the ground-state current and trial flux. The equations to be solved are finally

$$\tfrac{1}{2}[i\nabla_R - a_t(R)]^2\psi_0(R) = E_0\psi_0(R), \tag{17}$$

$$\nabla_R \times a_t(R) = \rho\mu \exp\left[-\beta\mu\int_\infty^R \mathbf{j}_0(r)\times dr\right]. \tag{18}$$

Perhaps, remarkably, the solutions of these nonlinear nonlocal equations are similar to the solution of the corresponding problem for the scalar potential case.[16] Suppose the solution takes the from of a large disk-shaped region of zero field with radius $d$. Outside this region the field takes the average value $\rho\mu$. The ground-state wave function is localized about the origin of the disk, and decays as a Gaussian in the region of finite field with decay length $\lambda$ of order $(\rho\mu)^{-1/2}$ for a large disk radius. For large $d$ where the ground state is expected to be real, the current flow is confined to the edge of the disk; $j_\theta(r) = |\psi_0(r)|^2\Phi/2\pi r$, where $\Phi(r) = \pi\rho\mu(r^2-d^2)$ is the flux enclosed at radius $r > d$. Thus the quantity $\int_\infty^R \mathbf{j}\times\mathbf{dr'}$ in the exponent of Eq. (18) jumps when the point $R$ moves through the edge, forcing the field to decrease for $R < d$. This is a consistent solution provided the edge current is sufficiently large.

To check the self-consistency of this solution we need to know the edge current or the value of the wave function in the boundary region. Since the wave function penetrates a finite distance into the edge, we may apply "soft-wall" boundary conditions $\lambda d\psi/dr + \psi|_d = 0$. The interior wave function is a simple Bessel function $\sim d^{-1}J_0(\sqrt{2E_0}r)$, where for large $d$, $E_0 \approx x_0^2/(2d^2)$, $J_0(x_0) = 0$. At the edge $\psi_0(d) \approx d^{-2}$. The exponent in Eq. (18) is therefore $\sim \mu\beta d^{-4}$ and we obtain a self-consistent solution at low temperature provided $\beta d^{-4}$ is divergent at low temperature. Since this should be satisfied with $d$ as large as possible, this means that $d \sim (\beta)^{1/4}$ is a marginal solution, and we expect logarithmic corrections to this result.[17]

The above analysis assumed that the solution of Eqs. (18) is governed by a single scale, the disk size $d$. This was confirmed by numerical solution of the equations, assuming circular symmetry and real wave functions. The results are shown in Fig. 2. The asymptotic behavior of the partition sum corresponding to the marginal solution is $Z \sim \exp(-\sqrt{2\pi}x_0\rho\beta^{\frac{1}{2}})$ with a corresponding density of states $\exp(-\pi x_0^2\rho/2\epsilon)$. This is the same as the result for repulsive short-ranged potentials in two dimensions.

Finally, we remark that the Debye model discussed in the Introduction can be recovered when there are two species of flux tube each of density $\rho$ and strengths $\pm\mu$. In this case the effective action is

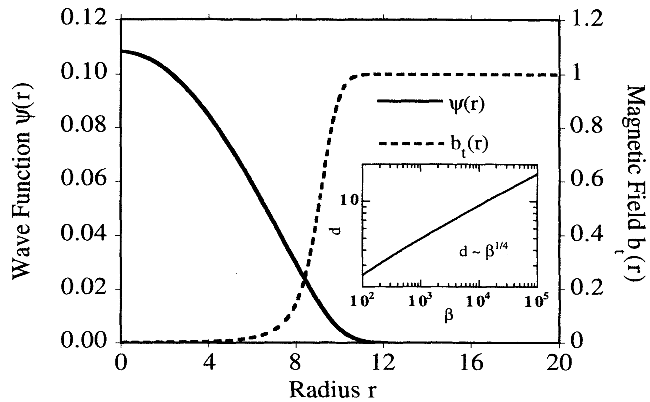


FIG. 2. Self-consistent solution of Eqs. (18) for the trial flux and wave function. Inset: the scaling $d \sim \beta^{1/4}$ which determines the Lifschitz tail is verified. $d$ is defined computing the total flux missing in the hole region and equating this to $\rho\mu\pi d^2$.

$$W_\pm\{r\} = 2\rho\int d^2R\left[1-\cos\left(\oint a_{\text{flux}}(r-R)dr\right)\right]. \tag{19}$$

In the limit $\mu \to 0$ but $\rho\mu^2$ constant, this becomes $W_\pm\{r\} = \rho\mu^2\int d^2Rw^2[R]$. This corresponds to the Debye model described in the Introduction. Moreover, since the Lifschitz tail is infinitely dilute in the limit $\rho\to\infty$, this is consistent with the conclusion of Sec. II that the "band-edge shift" diverges in the limit of large $q_0^2$.

## IV. LATTICE MODEL: DENSITY OF STATES

In this section we study the model

$$H = -\sum_{\langle ij\rangle} e^{ia_{ij}}c_i^\dagger c_j \tag{20}$$

on a square lattice. $i$ and $j$ are nearest-neighbor sites. The flux through a given loop $L$ on the lattice is $\Phi_L = \sum_{ij\in L} a_{ij}$ with real gauge or link fields $a_{ij} = -a_{ji}$. The model is nontrivial even for the well-studied case of uniform flux.[18] For arbitrary $a_{ij}$ the eigenvalues of $H$ lie in the interval $[-4,4]$. There is also an exact symmetry on a bipartite lattice. The Hamiltonian changes sign under the canonical transformation $c_i \to P(i)c_i$, where $P(i) = 1$ on the even and $-1$ on the odd sublattice; if the state $\sum_i \alpha_i c_i^\dagger$ is an eigenfunction with energy $E$, then $\sum_i P(i)\alpha_i c_i^\dagger$ is an eigenfunction with energy $-E$ and so the density of states is symmetric about $E = 0$.

A Meissner phase of the lattice model is readily generated by choosing the link fields to be uniformly, randomly distributed in the interval $[-a_0, a_0]$. To see this recall that from the continuum limit discussed in the Introduction a Meissner phase is defined by the property that the plaquette sum $\sum_{\Box'}\langle\phi_\Box\phi_{\Box'}\rangle$ vanishes. Suppose that $a_0 < \pi/4$ so that the flux is always in the interval $[-\pi,\pi]$. With random link phases, only nearest-neighbor plaquette fluxes are correlated since they share one link. For near neighbors $\langle\phi_1\phi_2\rangle = \langle(a_{12}+a_{23}+a_{34}+a_{41})(a_{43}+a_{35}+a_{56}+a_{64})\rangle = -\langle a^2\rangle$. Thus the sum over nearest neighbors plus the local fluctuation $\langle(a_{12}+a_{23}+a_{34}+a_{41})^2\rangle = 4\langle a^2\rangle$ is zero. The screening length in this phase is clearly of order of a lattice constant. A different construction is required for the Debye phase where we require that there be no correlations between fluxes on neighboring plaquettes. This implies long-range correlations between the link fields $a_{ij}$. We choose fluxes from a uniform distribution $[-\phi_0, \phi_0]$.

For gauge-invariant quantities (such as the local density of states), the relevant object to examine is the loop average

$$\left\langle \prod_{ij\in L} \exp i a_{ij} \right\rangle = \begin{cases} (-)^{nL_N} \exp\left[-\sum_{ij\in L} \ln \frac{\mathcal{N}_{ij}a_0}{\sin \mathcal{N}_{ij}a_0'}\right], & \text{Meissner}, \\ (-)^{nA} \exp\left[-\sum_{\square\in L} \ln \frac{w_\square \phi_0}{\sin w_\square \phi_0'}\right], & \text{Debye}. \end{cases} \tag{21}$$

Here $\mathcal{N} = \mathcal{N}_+ - \mathcal{N}_-$, where $\mathcal{N}_\pm$ are the number of times the link is traversed in each direction, $L_N = \sum_L \mathcal{N}_{ij}$ is the "nonretraced path length" of the loop $L$, $w_\square$ is the winding number of the plaquette, $A = \sum_L w_\square$ is the lattice-oriented area enclosed by $L$, and $\phi_0 \equiv \phi_0' + n\pi$, where $\phi_0' < \pi$, with an equivalent definition for $a_0$. Note that when the amplitude of the random flux or phase is close to an integer multiple of $\pi$ the contribution of nonretraced paths is effectively suppressed. Also, for small amplitude of the fluctuation we recover the continuous form of the effective interaction.

Ground-state energies as a function of the $a_0$ and $\phi_0$ on lattices of size 200×200 with hard-wall boundary conditions were obtained by the Lanczos technique. The results (Fig. 3) clearly illustrate the features anticipated from the loop averages [Eq. (21)]. In the Meissner phase the energy rises to a value close to $-3.5 \sim -2\sqrt{3}$ (the retraced path result; see below) at $a_0 = \pi$ and then remains roughly constant at this level. In the Debye phase the energy reaches the same value at $\phi_0 = \pi$ but continues to rise to a maximum value $\sim -3.35$ at $\phi_0 = 3\pi/2$. For larger fluctuations the ground-state energy oscillates with period $2\pi$. The reason for these contrasting behaviors derives from the sign factors in Eq. (21). Since the nonretraced path length $L_N$ is even for all loops, the sign is positive in the Meissner case. Frustration occurs when the sign is negative in the Debye case; this is analogous to a uniform magnetic field of $\pi$ per plaquette.

The density of states was obtained using the recursion method.[19] This involves construction of the continued-fraction expansion for the local Green's function, and averaging over sites. Representative results are shown in Fig. 4. When $a_0 = 0$ or $\phi_0 = 0$ we obtain the density of states for a square lattice with a "rounded" logarithmic singularity at the band center. The rounding is due in part to the finite number (50) of coefficients retained in the continued-fraction expansion, and in part to the finite size of the lattice. For small values of $a_0$ and $\phi_0$ significant differences between Meissner and Debye phases emerge [Figs. 4(a) and 4(b)]. In the Debye phase a sharp pileup of states near the effective band edge appears; this effect is entirely absent in the Meissner phase. This is qualitatively consistent with the conclusions for the continuum models [Eqs. (4), (5), and (12)] described in the Introduction. Previous numerical studies did not resolve this peak[9,11] which requires use of a large lattice and relatively weak flux fluctuation. Note that while a Lifschitz tail is expected in a random flux, it is very dilute and therefore not present in our numerical results.

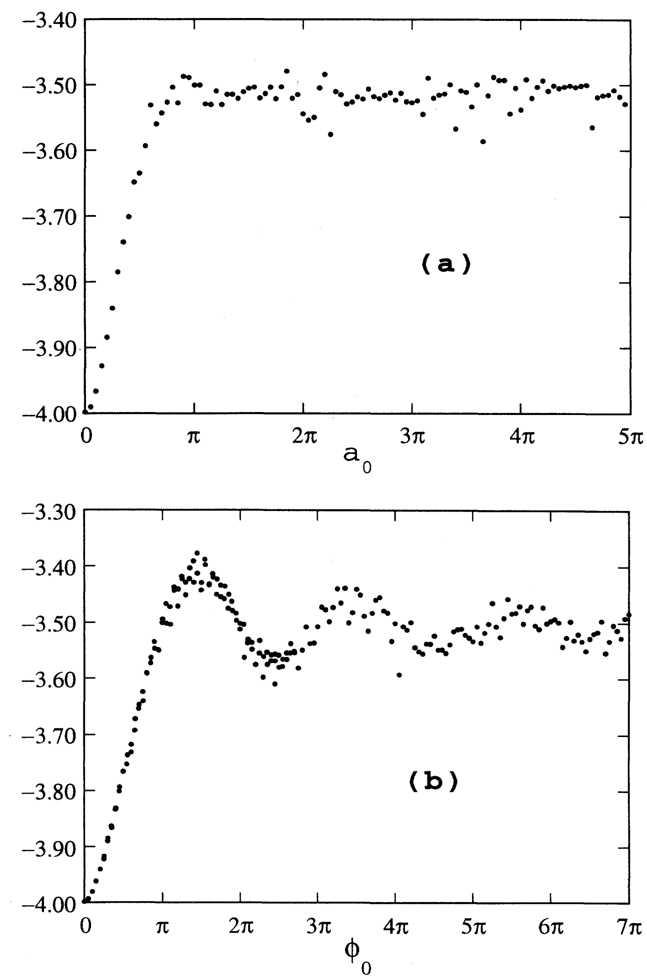


FIG. 3. Dependence of ground-state energies on (a) $a_0$ (Meissner phase) and (b) $\phi_0$ (Debye phase) on a 200×200 square lattice. The structure present in the Debye case is associated with the properties of Eq. (21) as described in the text.

Results for $\phi_0 = \pi$ and $a_0 = \pi$ are compared in Fig. 4(c). Here the numerical results are almost indistinguishable from the retraced path expression $\rho(E) = 1/\pi \, \mathrm{Im} 3/(E+2\sqrt{E^2-12})$ given by Brinkman and Rice[20] in both phases. We conclude that the distinction between the Meissner and Debye phases is completely lost for large flux fluctuations on a lattice. This can be regarded as a topological effect.

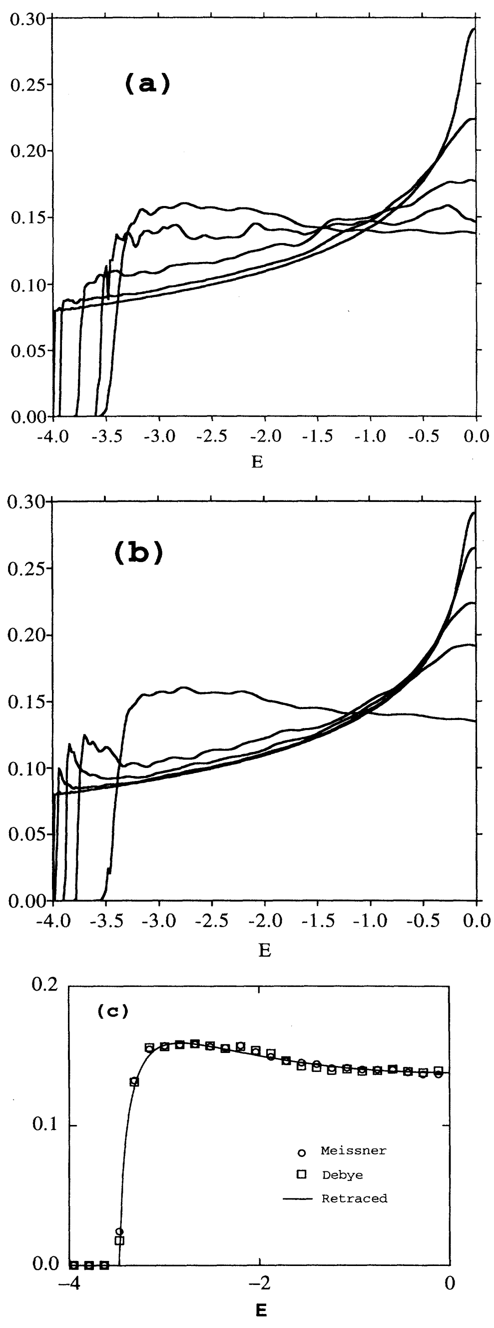


FIG. 4. Density of states obtained by recursion method on a 200×200 lattice using 50 coefficients in the continued-fraction expansion and averaging over sites. (a) Meissner phase $a_0 = 0$, $a_0 = 0.5$, $a_0 = 1.0$, $a_0 = 1.5$ (average over 250 sites), and $a_0 = \pi$ (average over 2500 sites). (b) $\phi_0 = 0$, $\phi_0 = 0.5$, $\phi_0 = 1.0$, $\phi_0 = 1.5$ (250 sites), and $\phi_0 = \pi$ (2500 sites). (c) Comparison of $a_0 = \phi_0 = \pi$ data with the retraced path result.

Finally we make some remarks on the response to a small uniform magnetic field. This is a subtle property which depends on the physics of localization, Lifschitz tails as well the effects of the type of flux correlations present. In the Lifschitz tail regime the behaviors of Meissner and Debye phases are completely different. The reason is that in the Debye phase the applied field can be completely "screened" out since the particle can move to a region where the external field is effectively canceled by the random flux. Since the probability of finding such a region is independent of the external field, the energy shift in an external field vanishes at zero temperature. Conversely in the Meissner phase there is a small probability of finding such a region; the diamagnetic susceptibility is large as for a free particle. Our lattice calculations are not sensitive to this effect since they only yield information about strongly localized states states outside the tail region. In a localized state the effect of an external magnetic field can be treated perturbatively when the magnetic length exceeds the localization length $l_{\mathrm{loc}}$. In this regime the susceptibility is $O(l_{\mathrm{loc}}^{-2})$. The similar behavior of the response of Meissner and Debye phases was confirmed numerically.

## V. CONCLUSION

We have found in this paper that there are profound differences in the quantum motion of a particle in a random magnetic flux in Debye and Meissner phases. This verifies the expectation[7] that the long-range effect in the Debye phase plays a crucial role in the low-energy properties. Variational calculations suggested the existence of a peak in the density of states in the Debye phase, which was confirmed in numerical calculations. We do not know at present whether the Debye phase peak represents a true singularity (pole or cut in the local Green's function) or just a smooth pileup of states. This is an important question which lacks a definitive answer. It seems likely that the single-particle properties described here will nevertheless play a role in the quantitative, if not the qualitative, behavior of Bose systems coupled to gauge fields.

We recently learned of the work of Khveshchenko and Meshkov.[17] These authors performed an elegant calculation of the density of states in the Lifschitz tail using a functional method with conclusions which agree with the results of Sec. III. Moreover, they obtained the diamagnetic response of the Debye phase in the Lifschitz tail regime, and find $T^{\frac{1}{2}}$ behavior. These authors do not report the density of states peak.

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[13]The Amperean area $\Omega$ should be contrasted with the more familiar oriented area $A = \frac{1}{2}\oint d\mathbf{r}\times\mathbf{r}$. For any loop there is an inequality [see Eq. (6)] $\Omega = \int d^2R w^2(R) \geq |A| = |\int d^2R w(R)|$ since the winding number $w(R)$ is an integer. For a simple closed curve (winding number about any interior point unity) the Amperean and oriented areas agree up to a sign. This conclusion could be reached directly from Eq. (5); for example, for the circular path $r = R(\cos\theta, \sin\theta)$, $\Omega = \frac{-R^2}{4}\int_0^{2\pi} d\theta \cos\theta \ln[2-\cos\theta] = \pi R^2$.
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