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Mutual friction in superfluid ^{3}He. II. Continuous vortices in ^{3}He-A at low temperatures

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The mutual-friction parameters for axisymmetric continuous vortices in the A phase at low temperatures are calculated on the basis of microscopic BCS theory for nonstationary processes. The parameters depend on the ratio of the distance between the energy levels of quasiparticles localized in the vortex, ω_0, and the relaxation rate of these quasiparticles, $1/\tau$. The transition from viscous to dissipationless flow of vortices occurs when $\omega_0\tau \sim 1$, the corresponding temperatures being well below T_c. From the viscosity coefficient, we estimate the effective cross section of the continuous vortex and its mass. We show that, with the continuous vortex, one can associate a normal region with a radius of order $R(T/T_c)$, R being the size of the vortex. The effect of mutual friction on the vortex eigenmodes is discussed.

I. INTRODUCTION

Mutual friction in superfluids is an interaction between the normal and superfluid components provided by vortices. They couple the Magnus force produced by the superfluid part of the liquid and the force exerted by the normal excitations. Since the latter contains a drag term, the mutual friction manifests itself in experiment as a dissipation present in superfluid state.

The Magnus force appears as a result of the relative motion of the superfluid (with the velocity $\mathbf{v}_s$) and the vortex (with the velocity $\mathbf{v}_L$). For an isotropic superfluid,

$$\mathbf{F}^{(M)}=\rho_s(\mathbf{v}_s-\mathbf{v}_L)\times\boldsymbol{\kappa} \ . \tag{1}$$

Here $\boldsymbol{\kappa}$ is the circulation vector with the magnitude $\kappa=\kappa_0 N$ for the vortex having N circulation quanta $\kappa_0=\pi/m$, where m is the mass of a ^{3}He atom.

The force produced by the normal excitations can be written as

$$\mathbf{F}^{(\mathrm{exc})}=D(\mathbf{v}_n-\mathbf{v}_L)+D'\hat{\boldsymbol{\kappa}}\times(\mathbf{v}_n-\mathbf{v}_L) \ , \tag{2}$$

where $\mathbf{v}_n$ is the velocity of the normal component and $\hat{\boldsymbol{\kappa}}$ is the unit vector along $\boldsymbol{\kappa}$.

The equation of the force balance for a rectilinear vortex is

$$\mathbf{F}^{(M)}+\mathbf{F}^{(\mathrm{exc})}=\mathbf{0} \ . \tag{3}$$

The vortex velocity can be expressed through the superfluid and normal velocities using Eqs. (1)–(3):[1]

$$\mathbf{v}_L=\mathbf{v}_s+\frac{\rho_n}{2\rho_{\mathrm{He}}}[B'(\mathbf{v}_n-\mathbf{v}_s)+B\hat{\boldsymbol{\kappa}}\times(\mathbf{v}_n-\mathbf{v}_s)] \ . \tag{4}$$

Here ρ_{He} is the density of ^{3}He. Experimentally, it is more easy to measure the mutual friction parameters B and B' which are connected with the coefficients D and D' through Eqs. (1)–(3):

$$B-iB'=\frac{2\rho_{\mathrm{He}}}{\kappa\rho_n\rho_s}\left[\frac{1}{D+iD'}-\frac{1}{i\rho_s\kappa}\right]^{-1} \ . \tag{5}$$

The mutual friction parameter B has been measured both in ^{3}He-B (Refs. 2 and 3) and in ^{3}He-A (Refs. 4–6) for temperatures of order of T_c. For both singular and continuous vortices, experiments yield B of order of unity. There are no measurements in the A phase at low temperatures yet.

Theoretical calculations of the mutual friction parameters have been done for continuous vortices in the A phase close to the critical temperature[7,8] and for singular vortices in the B phase at low temperatures.[9] The calculations give $B\sim 1$ and agree with the experiment by the order of magnitude. However, the physical mechanisms responsible for the mutual friction for continuous and singular vortices are different in the situations considered. In Ref. 7, the hydrodynamic approximation has been applied for continuous vortices in phase A assuming that the mean free path of quasiparticles is shorter than the vortex size. In this limit, the mutual friction is due to the Cross-Anderson viscosity.[10]

The opposite limit of a very long quasiparticle mean free path at low temperatures has been considered in Ref. 9. It has been shown that, for singular vortices in phase B, the most important contribution comes from the interaction between bound quasiparticles in the vortex core and the normal excitations outside the vortex. This interaction provides a mechanism of mutual friction in addition to the scattering of excitations by the vortex potential, which is believed to be the main reason for the mutual friction in superfluid ^{4}He (Ref. 1) (see also the reviews in Refs. 11 and 12, and references therein).

The estimates made in Ref. 9 show that the ballistic regime for which one can consider the quasiparticle scattering by the vortex potential, is realized for the quasiparticle mean free path l such that $l \gg p_F L^2$, where L is the characteristic size of the vortex (not necessarily equal to the vortex radius R, see Sec. II B). For continuous vortices, L is much longer than the coherence length ξ, and this condition is fulfilled at very low temperatures when there is practically no normal component in the superfluid. Moreover, the transport cross section produced by the vortex potential in superfluid ^{3}He can be estimated as[9]

$$\sigma_{\rm tr} \sim \frac{1}{p_F^2 L} \ . \tag{6}$$

Even for a singular vortex with L of the order of ξ, this cross section is too small to account for the observed values of $B \sim 1$, to say nothing of continuous vortices having $L \gg \xi$. This shows that the scattering of excitations by the vortex potential is irrelevant for the mutual friction in ^{3}He.

According to the results of Ref. 9, the mutual friction coefficients D and D' for singular vortices depend on the parameter $\omega_0\tau$, where $\omega_0 \sim T_c^2/E_F$ is the distance between the bound-state levels of the quasiparticles with neighboring angular momentum projections on the vortex axis, and τ is the mean free time of quasiparticles. The transition from viscous to dissipationless flow of singular vortices occurs at temperatures of the order of T_c, when $\omega_0\tau \sim 1$.

To summarize, the interaction of the localized quasiparticles with excitations outside the vortex core can explain the general dynamics of singular vortices in phase B. However, the behavior of continuous vortices is not yet well understood. The major questions are (1) what is the mechanism behind the dynamics of continuous vortices at low temperatures when the mean free path of quasiparticles is longer than the vortex size, and (2) at what temperatures does the transition from viscous to dissipationless flow occur? In the present paper we address these problems for axisymmetric continuous vortices in phase A. This paper continues, after the publication,[9] the consideration of the mutual friction in ^{3}He on the basis of the microscopic BCS theory of nonstationary processes.

We calculate the mutual friction parameters using the approach developed in Ref. 9. The scattering of quasiparticles is considered within the relaxation-time approximation. We show that, as for singular vortices, the bound states localized inside a continuous vortex play a decisive role in the vortex dynamics at low temperatures. The transition from viscous to dissipationless flow of vortices now occurs when the scattering rate of the localized quasiparticles by excitations coming from outside the vortex becomes of the order of $\omega_0 \sim (\Delta_0^2/E_F)(\xi/R)$, where R is the vortex radius. Such scattering rates correspond to temperatures well below T_c.

We estimate the cross section for the scattering of incident quasiparticles by those localized in the vortex. The effective vortex cross section found for this process is quite big; it corresponds to the scattering by normal quasiparticles that would occupy a region of a macroscopic radius $R^* \sim R\,(T/\Delta_0)$.

We calculate also the mass of a continuous vortex as the coefficient in the frequency expansion of D. By the order of magnitude, it is equal to the mass of the liquid confined within the region of the radius R^*. This picture suggests that, with a continuous vortex, one can associate a region of the radius R^* filled with normal excitations, i.e., a continuous vortex has a big "normal core," even though the gap does not vanish there for all directions of the quasiparticle momentum. The radius of the normal region decreases with temperature following the decrease in the occupation of the bound states inside the vortex. The big cross section of the continuous vortex can, probably, be measured by vibrating wire radiators, which emit quasiparticles as described in Ref. 13. We discuss also the effect of mutual friction on the vortex eigenmodes.

In Sec. II, we consider bound states in continuous vortices and calculate the mutual-friction parameters. In Sec. III, we discuss the obtained results, estimate the vortex cross section, and calculate the vortex mass. Results are summarized in Sec. IV. The relaxation-time approximation is discussed in detail in the Appendix.

II. BOUND STATES IN CONTINUOUS VORTICES AND THE MUTUAL-FRICTION PARAMETERS

A. General expressions

The general microscopic approach to the problem of mutual friction in superfluid ^{3}He based on the microscopic BCS theory of nonstationary processes has been developed in Ref. 9 for axisymmetric vortices. The approach uses the scheme derived for superconductors (see reviews in Refs. 14 and 15), first implemented in Ref. 16 for materials with a very long relaxation time. The microscopic calculations confirm the phenomenological picture of Eq. (3) and provide the expressions for D and D'. For low temperatures, $T \ll T_c$, the behavior of the mutual-friction parameters is determined by the bound states of the Bogoliubov quasiparticles located within the vortex.[9]

In terms of the bound-state energies $E_{k,n,r,s}$ of quasiparticles, the mutual-friction parameters are[9] (see also the Appendix)

$$D = m\kappa_0 \int_0^{p_F} \frac{p_F^2 - k^2}{16\pi^2 T} dk \sum_{n,r,s} \cosh^{-2}\left[\frac{E_{k,n,r,s}}{2T}\right] \frac{\tau(\partial E_{k,n,r,s}/\partial n)^2}{\tau^2(\partial E_{k,n,r,s}/\partial n)^2 + 1} \ , \tag{7}$$

$$D' = -m\kappa_0 \int_0^{p_F} \frac{p_F^2 - k^2}{16\pi^2 T} dk \sum_{n,r,s} \cosh^{-2}\left[\frac{E_{k,n,r,s}}{2T}\right] \frac{\partial E_{k,n,r,s}/\partial n}{\tau^2(\partial E_{k,n,r,s}/\partial n)^2 + 1} \ . \tag{8}$$

Here k is the quasiparticle momentum along the vortex axis (the z axis), n is the "azimuthal" quantum number defined later (for a general definition of n see, for example, Ref. 17), r is the radial quantum number, and s is the quasiparticle spin.

To calculate D and D' one needs to know the structure of the bound states for a continuous vortex. The energy $E_{k,n,r,s}$ can be found from the equation for the Bogoliubov wave function $\mathcal{U}$ which is a vector in the Nambu space:

$$\mathcal{U}=\begin{bmatrix}\hat{U}\\ -\hat{V}\end{bmatrix},\quad \mathcal{U}^{\dagger}=(\hat{U}^{\dagger},\hat{V}^{\dagger})\ . \tag{9}$$

The eigenvalue equation is

$$\left[-\left[\frac{\nabla^2}{2m}+E_F\right]\check{\tau}_0+\mathcal{H}\right]\mathcal{U}_{k,n,r,s}=E_{k,n,r,s}\check{\tau}_3\mathcal{U}_{k,n,r,s}\ . \tag{10}$$

Here

$$\mathcal{H}=\begin{bmatrix}0 & -\hat{\Delta}_{\mathbf{p}}\\ \hat{\Delta}_{\mathbf{p}} & 0\end{bmatrix}, \tag{11}$$

$\check{\tau}_3$ is the Pauli matrix, and $\check{\tau}_0$ is the unit matrix in the Nambu space. The wave functions obey the orthogonality conditions

$$\sum_{n,r,s}\int\frac{dk}{2\pi}\check{\tau}_3\mathcal{U}_{k,n,r,s}(\mathbf{r}_1)\mathcal{U}^{\dagger}_{k,n,r,s}(\mathbf{r}_2)=\check{\tau}_0\delta(\mathbf{r}_1-\mathbf{r}_2)\ , \tag{12}$$

and

$$\mathrm{Tr}\int d^3\mathbf{r}\,\mathcal{U}^{\dagger}_{k,n,r,s}(\mathbf{r})\check{\tau}_3\mathcal{U}_{k',n',r',s'}(\mathbf{r}) =2\pi\delta(k-k')\delta_{n,n'}\delta_{r,r'}\delta_{s,s'}\ . \tag{13}$$

The quasiparticle relaxation enters Eqs. (7) and (8) through the relaxation time τ. As in Ref. 9, the quasiparticle collisions are considered within the "relaxation-time approximation." This approximation is used to simplify the calculations. However, as it has been shown in Ref. 16 for the scattering of quasiparticles by impurities, the exact calculations give very similar results. We discuss the relaxation time approximation in more detail in the Appendix.

The localized quasiparticles have momenta close to $\mathbf{p}=\pm p_F\mathbf{l}$ for the local orientation of the anisotropy vector $\mathbf{l}$ in the continuous vortex. Their scattering rate is proportional to their density, i.e., to the fraction $(T/T_c)^2$ of the total Fermi surface area times another factor of T/T_c due to their energies of the order of T. The relaxation of the localized quasiparticles proceeds through collisions with the quasiparticles whose energies are of the order of Δ_0, which can overcome the energy barrier produced by the variation of $\mathbf{l}$ in the vortex and escape to infinity where they relax on the walls. Thus, the relaxation rate $1/\tau$ contains also the exponential factor $\exp(-a\Delta_0/T)$. The function $a(\mathbf{r})<1$ is determined by the optimal trajectory of a quasiparticle escaping from the given point in the vortex; it depends on the vortex structure, on the surrounding $\mathbf{l}$ texture, and on the experimental geometry. We will ignore these details using the simple estimate

$$\frac{1}{\tau}\sim\frac{1}{\tau_n(T_c)}\left[\frac{T}{T_c}\right]^3\exp\left[-a\frac{\Delta_0}{T}\right] \tag{14}$$

with a constant $a<1$. Here $\tau_n(T_c)\sim E_F/T_c^2$ is the relaxation time in the normal state at $T=T_c$. The mean free path $l(T_c)$ is of the order of 10 μm; it increases rapidly with lowering the temperature. We will see, however, that the quasiparticle scattering is very essential even for temperatures when l is much longer than the vortex size.

B. Bound states

The A phase can be stabilized at low temperatures in a magnetic field of the order of several kG. If the magnetic field is applied along the rotation axis of the container the spin vector $\mathbf{d}$ will be locked in the plane perpendicular to the rotation axis; therefore, only nonaxisymmetric continuous vortices can appear in such a geometry. (For the description of various types of vortices in superfluid ^{3}He see the review in Ref. 18.) The axisymmetric vortices can be formed if the magnetic field is applied perpendicular to the rotation axis.

In this section we consider the doubly quantized axisymmetric continuous Anderson-Toulouse-Chechetkin (ATC) vortex. The A-phase order parameter is

$$\hat{\Delta}_{\mathbf{p}}=i\hat{\sigma}_\alpha\hat{\sigma}_2 d_\alpha\Delta_0(\mathbf{\Delta}'+i\mathbf{\Delta}'')\cdot(\mathbf{p}/p_F)\ . \tag{15}$$

Here σ_α is the Pauli matrix in the spin space, and $\mathbf{\Delta}'$ and $\mathbf{\Delta}''$ are mutually orthogonal unit vectors which are functions of coordinates. The unit vectors $\mathbf{\Delta}'$ and $\mathbf{\Delta}''$ and the anisotropy vector $\boldsymbol{l}=\mathbf{\Delta}'\times\mathbf{\Delta}''$ can be parametrized with the Euler angles (α,β,γ):

$$\boldsymbol{l}=(-\sin\alpha\sin\beta,\cos\alpha\sin\beta,\cos\beta)\ , \tag{16}$$

$$(\mathbf{\Delta}'+i\mathbf{\Delta}'')\cdot(\mathbf{p}/p_F)=-i\frac{e^{-i\gamma}}{p_F}\left[\left[\cos\alpha\frac{\partial}{\partial x}+\sin\alpha\frac{\partial}{\partial y}\right]+i\cos\beta\left[-\sin\alpha\frac{\partial}{\partial x}+\cos\alpha\frac{\partial}{\partial y}\right]-i\sin\beta\frac{\partial}{\partial z}\right]\ . \tag{17}$$

The superflow velocity produced by the vortex is

$$\mathbf{v}_s=-\frac{1}{2m}(\nabla\gamma+\cos\beta\nabla\alpha)\ . \tag{18}$$

If the circulation is along the positive z direction of the cylindrical coordinate frame (ρ,ϕ,z), one can put $\gamma=-\phi$ and $\alpha=\phi-\eta$. In these coordinates, $l_\rho=\sin\beta\sin\eta$, $l_\phi=\sin\beta\cos\eta$, $l_z=\cos\beta$. The constant angle η parametrizes various types of the ATC vortices; for example, $\eta=\pi/2$ for the so-called v vortex, and $\eta=0$ for the w vortex. The angle $\beta(\rho)$ between the vortex axis and the anisotropy vector varies from $\beta=0$ for $\rho=0$ to $\beta=\pi$ for $\rho/R\to\infty$. The size R of the ATC vortex in a high magnetic field is of the order of the dipole length ξ_d.

The orbital part of the order parameter becomes

$$\Delta_0(\boldsymbol{\Delta}'+i\boldsymbol{\Delta}'')\cdot(\mathbf{p}/p_F)=-i\frac{\Delta_0 e^{i\phi}}{p_F}\left[[\cos\eta+i\cos\beta\sin\eta]\frac{\partial}{\partial\rho}+[-\sin\eta+i\cos\beta\cos\eta]\frac{1}{\rho}\frac{\partial}{\partial\phi}-i\sin\beta\frac{\partial}{\partial z}\right] . \quad (19)$$

Since the spin vector d_α of the order parameter is locked in the plane perpendicular to the magnetic field, one can separate the spin part of the wave functions:

$$\hat{U}_{k,n,r,s}=\hat{s}U_{k,n,r},\quad \hat{V}_{k,n,r,s}=-i\hat{\sigma}^*_\alpha\hat{\sigma}_2\hat{s}d_\alpha V_{k,n,r} , \quad (20)$$

considering $\hat{s}$ as a constant normalized spinor. According to Eq. (19), the orbital wave functions have integer azimuthal quantum numbers n:

$$U_{k,n,r}=e^{ikz}e^{in\phi}u_{k,n,r}(\rho),\quad V_{k,n,r}=e^{ikz}e^{i(n-1)\phi}v_{k,n,r}(\rho) . \quad (21)$$

The Bogoliubov equations, Eq. (10), are

$$\left[\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}-\frac{n^2}{\rho^2}+q^2+2mE\right]u(\rho) -\frac{2m\Delta_0}{p_F}\left[-i[\cos\eta+i\cos\beta\sin\eta]\frac{\partial}{\partial\rho}-[\sin\eta-i\cos\beta\cos\eta]\frac{n}{\rho}-ik\sin\beta\right]v(\rho)=0 , \quad (22)$$

$$\left[\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}-\frac{(n-1)^2}{\rho^2}+q^2-2mE\right]v(\rho) -\frac{2m\Delta_0}{p_F}\left[i[\cos\eta-i\cos\beta\sin\eta]\frac{\partial}{\partial\rho}+[\sin\eta+i\cos\beta\cos\eta]\frac{n-1}{\rho}-ik\sin\beta\right]u(\rho)=0 , \quad (23)$$

where $q^2=p_F^2-k^2$.

Consider the wave function with a positive radial momentum:

$$u=H^{(1)}_{n-1/2}(q\rho)a(\rho),\quad v=H^{(1)}_{n-1/2}(q\rho)b(\rho) , \quad (24)$$

where $H_n^{(1)}$ is the Hankel function of the first kind. We introduce the notations $k=p_F\cos\theta$, where θ is the angle between the momentum and the z axis; $q_\rho=\sqrt{q^2-(n-1/2)^2/\rho^2}$, and $(n-1/2)/\rho=q\cos\lambda$, so that $q_\rho=q\sin\lambda$, where the angle λ defines the orientation of the quasiparticle momentum in the (ρ,ϕ) plane.

Applying the asymptotic expressions for $H^{(1)}_{n-1/2}$, one has

$$iq_\rho\frac{\partial a}{\partial\rho}+m\left[E-\frac{n}{2m\rho^2}\right]a-m\Delta_0(i[\sin\theta\cos\beta\cos\chi-\cos\theta\sin\beta]+\sin\theta\sin\chi)b=0 , \quad (25)$$

$$iq_\rho\frac{\partial b}{\partial\rho}-m\left[E-\frac{n}{2m\rho^2}\right]b-m\Delta_0(i[\sin\theta\cos\beta\cos\chi-\cos\theta\sin\beta]-\sin\theta\sin\chi)a=0 , \quad (26)$$

where $\chi=\lambda-\eta$ is the angle between the projection on the (ρ,ϕ) plane of the quasiparticle momentum and that of the vector $\mathbf{l}$.

Equation (26) can be solved within the WKB approximation by putting $a,b\propto\exp(\pm i\int p_\rho d\rho)$. One obtains

$$p_\rho=\frac{m}{q_\rho}\left\{\left[\left[E-\frac{n}{2m\rho^2}\right]^2-\Delta_0^2\sin^2\theta\sin^2\chi\right]-\Delta_0^2[\cos\beta\sin\theta\cos\chi-\sin\beta\cos\theta]^2\right\}^{1/2} . \quad (27)$$

The Bohr-Sommerfeld quantization rule for the momentum p_ρ is

$$\int_{\rho_1}^{\rho_2}p_\rho d\rho=\pi r , \quad (28)$$

where r is the radial quantum number assuming integer values, and $\rho_{1,2}$ are the turning points, i.e., the zeros of the square root in Eq. (27).

For $0<\eta<\pi$, the low-energy bound states correspond to the small angles $\chi\ll 1$ and are localized near the point $\rho_0(\theta)$ such that $\beta(\rho_0)=\theta$. Expanding $\sin(\theta-\beta)=-[\partial\beta(\rho_0)/\partial\rho](\rho-\rho_0)$, we obtain

$$E_{k,n,r}=\pm\left[\Delta_0^2\sin^2\theta\sin^2\chi+\frac{2q_0\Delta_0}{m}\left|\frac{\partial\beta(\rho_0)}{\partial\rho}\right|r\right]^{1/2} . \quad (29)$$

Here we omitted the small term $n/2m\rho^2$. The notation q_0 is used for the radial momentum q_ρ taken at $\rho=\rho_0$. In this case

$$\sin\chi=\frac{1}{q}\left[q_0\cos\eta-\frac{n}{\rho_0}\sin\eta\right] . \quad (30)$$

Strictly speaking, this semiclassical result holds for large r, however, it gives the correct answer for $r=0$ with the plus sign in Eq. (29). To prove this we consider the level with $r=0$ separately using the approach developed in Ref. 19 for a similar problem in superconductors. Let $a=\exp(i\varphi-\psi)$, $b=\exp(-i\varphi-\psi)$, then

$$q_\rho \frac{\partial\psi}{\partial\rho}+m\Delta_0[\sin(\theta-\beta)\cos2\varphi-\sin\theta\sin\chi\sin2\varphi]=0\ , \tag{31}$$

$$q_\rho \frac{\partial\varphi}{\partial\rho}+m\Delta_0[\sin(\theta-\beta)\sin2\varphi+\sin\theta\sin\chi\cos2\varphi] -mE=0\ . \tag{32}$$

We assume that $\sin2\varphi$ is small. To choose the proper branch of φ one has to require that ψ is positive and increases away from the point $\rho=\rho_0$. Since $\partial\beta/\partial\rho>0$, the function φ should be close to zero, and Eq. (31) gives

$$\psi=\frac{m\Delta_0}{2q_0}\left|\frac{\partial\beta(\rho_0)}{\partial\rho}\right|(\rho-\rho_0)^2\ . \tag{33}$$

Equation (33) defines the localization radius of the bound states, $L\sim\sqrt{\xi R}$, where ξ is the coherence length; L is much larger than ξ but smaller than the vortex radius. It is this localization radius which has been used as the characteristic vortex size L for estimates made in the Introduction. One can neglect effects of the quasiparticle relaxation on the low-energy bound states if the mean free path l is much longer than L. This condition is satisfied at all temperatures for vortices with R of the order of the dipole length.

Equation (32) now gives

$$\varphi=e^{2\psi(\rho)}\int_{-\infty}^{\rho}\frac{m}{q_{\rho'}}[E-\Delta_0\sin\theta\sin\chi]e^{-2\psi(\rho')}d\rho'\ . \tag{34}$$

The integration constant here has been chosen to give a finite φ for $\rho-\rho_0\to-\infty$. Requiring a finite φ also for $\rho-\rho_0\to+\infty$, and keeping in mind that the localization radius $L\ll R$, one obtains $E_{k,n,0}=\Delta_0\sin\theta\sin\chi$, or

$$E_{k,n,0}=\frac{\Delta_0}{p_F}\left[q_0\cos\eta-\frac{n}{\rho_0}\sin\eta\right]\ . \tag{35}$$

This coincides with the semiclassical result. E turns to zero for $n=n_0=q\rho_0\cos\eta$, i.e., when $\mathbf{p}=p_F\mathbf{l}$.

To obtain Eq. (35) we assumed that φ remains small. This holds if $(n-n_0)/p_FR\ll1$, or $E\ll\Delta_0$. The distance between the energy levels with neighboring values of n is

$$-\omega_0(\theta)=\left[\frac{\partial E_{n,k,0}}{\partial n}\right]_{n=n_0}=-\frac{\Delta_0}{p_F\rho_0\sin\eta}\ . \tag{36}$$

The derivative $\partial E_{k,n,0}/\partial n$ is negative, ω_0 being of the order of $(\Delta_0^2/E_F)(\xi/R)$, which is by the factor ξ/R less than that for a singular vortex with the core size of the order of the coherence length.

The two branches (with plus and minus signs) in Eq. (29) for levels with $r\neq0$ are even functions of $(n-n_0)$, while the level $E_{k,n,0}$ is odd: as a function of n, it crosses zero and runs over the energies within the interval of the order of $(-\Delta_0,\Delta_0)$. The levels $E_{k,n,0}$ taken as functions of k for various n, constitute a set of the so-called anomalous branches discussed in Refs. 17 and 20. The specific role of this level for the mutual friction will be clarified in Sec. III B.

If $-\pi<\eta<0$, one has to choose $\chi=\lambda-\eta\approx\pi$. The low-energy levels are localized near the point $\rho_0(\theta)$ such that $\theta+\beta(\rho_0)=\pi$, and we obtain Eq. (29) where now

$$\sin\chi=\frac{1}{q}\left[q_0\cos\eta+\frac{n}{\rho_0}|\sin\eta|\right]\ . \tag{37}$$

For the odd energy level we have

$$E_{k,n,0}=-\frac{\Delta_0}{p_F}\left[q_0\cos\eta+\frac{n}{\rho_0}|\sin\eta|\right]\ . \tag{38}$$

The energy turns to zero for $n_0=-q\rho_0\cos\eta$, so that $\mathbf{p}=-p_F\mathbf{l}$. The distance between the levels is

$$\omega_0(\theta)=\frac{\Delta_0}{p_F\rho_0|\sin\eta|}\ . \tag{39}$$

This expression holds for all η. The case of negative radial momenta of the quasiparticles corresponds to the choice of $H_{n-1/2}^{(2)}$ in Eq. (24), and can be considered in a similar way.

For both positive and negative radial momenta and various values of the parameter η, the bound-state energy levels can be written in a simple form:

$$E_{k,n,0}=-\omega_0(\theta)(n-n_0) \tag{40}$$

for $r=0$, and

$$E_{k,n,r}=\pm\sqrt{[\omega_0(\theta)(n-n_0)]^2+\zeta^2r} \tag{41}$$

for $r\neq0$. Here $\omega_0(\theta)$ is defined by Eq. (39),

$$\zeta^2=\frac{2q|\sin\eta|\Delta_0}{m}\left|\frac{\partial\beta(\rho_0)}{\partial\rho}\right|\ , \tag{42}$$

and ρ_0 is determined either by the equation $\beta(\rho_0)=\theta$ for positive radial momenta or by $\beta(\rho_0)=\pi-\theta$ for negative radial momenta in the case $0<\eta<\pi$, and vice versa in the case $-\pi<\eta<0$.

C. Mutual-friction parameters

Given the bound-state energy spectrum, one can calculate the mutual-friction parameters. In our case, the energies do not depend on spin; therefore, the summation over the spin quantum number s in Eqs. (7) and (8) simply provides the additional factor 2. Moreover, for the two-quantum ATC vortex, there exist two potential wells filled by localized quasiparticles for each momentum direction $k=p_F\cos\theta$ with $0<\theta<\pi/2$. The wells are located at the points $\rho_{0\pm}$ such that $\beta(\rho_{0+})=\theta$ for a quasiparticle with the positive radial momentum, and $\beta(\rho_{0-})=\pi-\theta$ for the negative radial momentum in the case $0<\eta<\pi$ (or vice versa, for $-\pi<\eta<0$). In other words, these two points are where $\mathbf{l}=\pm\mathbf{p}/p_F$ for a given quasiparticle momentum. Both wells provide the energy spectra of the type of Eqs. (40) and (41) which have to be taken into account while calculating D and D'.

First, we calculate D'. Inspection of Eq. (8) shows that only the odd branches of the energy spectrum, $E_{k,n,0}$, contribute to Eq. (8) since all the other branches have odd derivatives $\partial E_{k,n,r}/\partial n$ and give zero after summation over $\nu = n - n_0$. Since the distance between the energy levels with neighboring n is much smaller than temperature one can replace the sum over ν with the integral from $-\infty$ to $+\infty$. With help of Eq. (40), we have

$$D' = \kappa \rho_{\mathrm{He}} \int_0^{\pi/2} \tfrac{3}{4} \sin^3\theta \, d\theta \sum_i \frac{1}{\omega_{0i}(\theta)^2 \tau^2 + 1} . \tag{43}$$

Here we put $\omega_{0i}(\theta) = \Delta_0 / (p_F \rho_{0i} |\sin\eta|)$, where i is either $+$ or $-$. The circulation for the two-quantum vortex is $\kappa = 2\kappa_0$.

Using Eqs. (41) and (40), we obtain for the viscosity coefficient D:

$$D = m\kappa \int_0^{p_F} \frac{q^2 dk}{4\pi^2} \sum_i \left[\frac{\omega_{0i}\tau}{\omega_{0i}^2 \tau^2 + 1} + \frac{1}{T} \int_0^{\infty} d\nu \sum_{r=1}^{\infty} \cosh^{-2} \left[\frac{\sqrt{\omega_{0i}^2 \nu^2 + \zeta_i^2 r}}{2T} \right] \frac{\tau \omega_{0i} 4\nu^2}{\tau^2 \omega_{0i}^4 \nu^2 + \omega_{0i}^2 \nu^2 + \zeta_i^2 r} \right] . \tag{44}$$

Here ζ_i is defined by Eq. (42) with $\rho_0 = \rho_{0i}$. The extra factor 2 in the second term of Eq. (44) appears due to two branches of Eq. (41).

The levels with $r \gg 1$ are excited when $\zeta \ll T$. In this limit, one can replace the sum over r with the integral and obtain

$$D = \kappa \rho_{\mathrm{He}} \int_0^{\pi/2} \tfrac{3}{4} \sin^3\theta \, d\theta \sum_i \left[\frac{\omega_{0i}(\theta)\tau}{\omega_{0i}^2(\theta)\tau^2 + 1} + \frac{4\pi^2}{3} \left[\frac{T}{\zeta_i} \right]^2 f[\omega_{0i}(\theta)\tau] \right] . \tag{45}$$

Here

$$f(\omega_0\tau) = \frac{\omega_0\tau - \arctan(\omega_0\tau)}{\omega_0^2\tau^2} . \tag{46}$$

The function $f(\omega_0\tau) = (\omega_0\tau)^{-1}$ for $\omega_0\tau \gg 1$, and $f(\omega_0\tau) = (\omega_0\tau)/3$ for $\omega_0\tau \ll 1$.

If many levels of $r \neq 0$ are excited, i.e., when $T \gg \zeta$, or

$$T \gg \Delta_0 \sqrt{\xi/R} , \tag{47}$$

the second term in Eq. (45) gives the main contribution. On the contrary, if the inverse inequality of Eq. (47) holds, only the first term exists. Therefore, one can consider Eq. (45) as an interpolation between these two limits.

III. DISCUSSION

A. Transition from the viscous flow to the dissipationless regime

The mutual-friction parameters define, through the equation of the force balance, Eq. (3), the dissipation in the liquid and the angle Θ at which the vortex moves with respect to the superflow velocity $\mathbf{v}_s$. Equations (43) and (45) show that it is the parameter $\omega_0\tau$ that determines the mutual friction. The characteristic parameter $\omega_0\tau$ becomes of the order of unity at the temperature T^* such that

$$\tau(T^*)/\tau_n(T_c) \sim (R/\xi) . \tag{48}$$

For high magnetic fields, the vortex size R is of the order of the dipole length ξ_d, the ratio ξ_d/ξ being of the order of 10^3. With the estimate of Eq. (14) for the relaxation rate, Eq. (48) is fulfilled for T^* well below T_c. For temperature $T \sim T^*$, one should expect a transition from the viscous flow of continuous vortices (perpendicular to the superflow velocity) to the small-viscosity regime when they more essentially with the superflow.

Indeed, for temperatures $T^* \ll T \ll T_c$, when the quasiparticle relaxation time is short, $\omega_0\tau \ll 1$, the reactive mutual-friction parameter is $D' \approx \kappa\rho_{\mathrm{He}}$. If $v_n = 0$, the angle Θ is

$$\tan\Theta = \frac{D}{\kappa\rho_s - D'} . \tag{49}$$

As a result, the vortex moves almost at the right angle to the superflow ($\rho_s = \rho_{\mathrm{He}}$ for $T \ll T_c$). The fact that D' is almost exactly equal to $\kappa\rho$ for $\omega_0\tau \ll 1$ has been first established in Ref. 16 for superconductors, and in Refs. 9 and 21 for ^{3}He. It is a consequence of the general topological structure of the anomalous energy branches of the excitation spectra for quasiparticles localized in the vortex, and does not depend on the specific type of the vortex.[22]

The temperatures above T^* satisfy Eq. (47), therefore, the viscosity coefficient in Eq. (45) is

$$D \sim \kappa\rho_{\mathrm{He}} (T/\zeta)^2 (\omega_0\tau) . \tag{50}$$

Since $\zeta \sim \Delta_0\sqrt{\xi/R}$ and $\omega_0 \sim (\Delta_0^2/E_F)(\xi/R)$ we have

$$D \sim \kappa\rho_{\mathrm{He}} \frac{T^2\tau}{E_F} \sim \kappa\rho_{\mathrm{He}} (T/T_c)^2 \frac{\tau}{\tau_n(T_c)} . \tag{51}$$

This result agrees, by the order of magnitude, with the viscosity coefficient calculated by Ref. 7 for temperatures close to T_c and with the experimental results of Refs. 4–6 in the temperature range $T \sim T_c$.

In the limit of very long relaxation times, i.e., for temperatures below T^*, when $\omega_0\tau \gg 1$, the reactive coefficient is small: $D' \sim \kappa\rho_{\mathrm{He}}/(\omega_0\tau)^2$. The viscosity is

$$D \sim \kappa\rho_{\mathrm{He}} (T/\zeta)^2 (\omega_0\tau)^{-1}$$
$$\sim \kappa\rho_{\mathrm{He}} (T/\Delta_0)^2 (R/\xi)^2 [\tau_n(T_c)/\tau] , \tag{52}$$

for temperatures satisfying Eq. (47), and it is

$$D \sim \kappa \rho_{\mathrm{He}} (\omega_0 \tau)^{-1} \sim \kappa \rho_{\mathrm{He}} (R/\xi)[\tau_n(T_c)/\tau] , \tag{53}$$

in the opposite limit of very low temperatures.

Consider now the mutual friction parameters B as a function of temperature. We have from Eq. (5)

$$\rho_n B/\rho_{\mathrm{He}} = \frac{2D/\kappa\rho_s}{(D/\kappa\rho_s)^2 + [1-(D'/\kappa\rho_s)]^2} . \tag{54}$$

In the temperature range $T^* \ll T \ll T_c$, where $\omega_0 \tau \ll 1$ and $D' \approx \kappa \rho_{\mathrm{He}}$, Eq. (54) and (51) give

$$\rho_n B/\rho_{\mathrm{He}} = 2\kappa\rho_{\mathrm{He}}/D \sim (T_c/T)^2 [\tau_n(T_c)/\tau] . \tag{55}$$

The combination $\rho_n B/\rho_{\mathrm{He}}$ decreases with temperature down to the values of the order of $2(\zeta/T)^2$.

When the temperature is decreased below T^*, one has $\omega_0\tau \gg 1$ and $D' \ll \kappa\rho_{\mathrm{He}}$. Now the combination $\rho_n B/\rho_{\mathrm{He}}$ increases with lowering the temperature as long as $D/\kappa\rho_{\mathrm{He}}$ remains much larger than 1. Indeed, in this limit, $\rho_n B/\rho_{\mathrm{He}} = 2\kappa\rho_{\mathrm{He}}/D$. It is of the order of $\omega_0\tau(\zeta/T)^2$ according to Eq. (52). The combination $B\rho_n/\rho_{\mathrm{He}} = 1$ when $D = \kappa\rho_{\mathrm{He}}$, i.e., for the temperature $T^{**} < T^*$ such that $(T^{**}/\zeta)^2 \sim \omega_0\tau(T^{**})$. Below this temperature, $\rho_n B/\rho_{\mathrm{He}} \propto D$ and starts to decrease again. Therefore, this combination has a deep minimum around the temperature $T = T^*$, followed by the maximum at $T = T^{**}$.

An interesting observation can be made for B-phase vortices which, in addition to the hard core of the order of ξ, have a large soft core determined by the dipole-dipole interaction. The soft-core size decreases with magnetic field and reaches ultimately the dipole length ξ_d. If there are bound states in the soft core, they can also give a contribution to the last term in Eq. (45) for not very low temperatures. It will result in the magnetic field dependence of D. Indeed, the bound states in the soft core resemble those in the continuous vortex and are distinct from the bound states in the hard core: they have a small level separation ζ with respect to the radial quantum number. Since ζ grows as the size of the soft core decreases, the friction coefficient D will decrease with increasing magnetic field. This behavior is in agreement with the experimental results of Ref. 3 and the corresponding discussion therein.

B. The normal-region model for a continuous vortex

We discuss now another aspect of the results obtained for the viscosity D in the limit of long relaxation times. The viscosity can be expressed through the transport cross section for the scattering of incident quasiparticles by the vortex. According to Ref. 23

$$D = -\frac{1}{2}\int \left[\frac{\partial n_F}{\partial E}\right] q^2 v_G \sigma_{\mathrm{tr}}(q) \frac{d^3\mathbf{p}}{(2\pi)^3} . \tag{56}$$

Here n_F is the Fermi distribution function, and v_G is the group velocity. Comparing Eq. (56) with Eqs. (52) and (53), we obtain that the effective vortex cross section per unit length is

$$\sigma_{\mathrm{tr}} \sim (R^2/l)(T/\Delta_0)^2 \tag{57}$$

if temperatures satisfy Eq. (47) and

$$\sigma_{\mathrm{tr}} \sim R\xi/l \tag{58}$$

for very low temperatures.

The scattering potential provided by order-parameter variations in a continuous vortex produces very small cross section as we already discussed in the Introduction. In addition to the scattering by the vortex potential, incident quasiparticles are scattered by the quasiparticles localized inside the vortex. One can easily estimate the cross section of a vortex due to the scattering by the localized quasiparticles under the assumption that a continuous vortex is equivalent to a normal region with the radius R^*. The cross section per unit length of a vortex is then $\sigma \sim \sigma_0 \cdot nR^{*2}$, where $\sigma_0 = 1/(nl)$ is the cross section of one quasiparticle, and n is the density of quasiparticles. As a result, we have $\sigma \sim R^{*2}/l$. Comparing this with Eqs. (57) and (58), we obtain the radius of the normal region associated with the continuous vortex:

$$R^* \sim R\frac{T}{\Delta_0} \tag{59}$$

for temperatures $\Delta_0 \gg T \gg \Delta_0\sqrt{\xi/R}$, and

$$R^* \sim \sqrt{R\xi} \tag{60}$$

in the limit $T \ll \Delta_0\sqrt{\xi/R}$. In this limit, R^* coincides with the localization radius of the bound state of Eq. (35). For higher temperatures, when more levels become excited, the radius of the effective normal region increases in accordance with Eq. (59).

The scattering cross section of a continuous vortex, Eq. (57) or (58), is quite big. One may expect that this cross section can be measured, for example, by deflection of a quasiparticle beam emitted by vibrating wire radiators described in Ref. 14.

C. The frequency dependence, the vortex mass

If the vortex velocity is not constant in time the mutual-friction parameters will depend on the frequency of the vortex motion ω. According to Ref. 24, this dependence can be taken into account by the substitution $\epsilon \to \epsilon \pm \omega/2$ in Eq. (72) for the retarder (advanced) Green's function. We obtain, as in Ref. 9,

$$D(\omega) = m\kappa_0 \int_0^{p_F} \frac{q^2 dk}{16\pi^2 T} \sum_{n,r} \cosh^{-2}\left[\frac{E_{k,n,r}}{2T}\right] \left[\frac{\partial E_{k,n,r}}{\partial n}\right]^2 \left[\frac{i}{\partial E_{k,n,r}/\partial n + \omega + i/\tau} - \frac{i}{\partial E_{k,n,r}/\partial n - \omega - i/\tau}\right] , \tag{61}$$

$$D'(\omega) = -m\kappa_0 \int_0^{p_F} \frac{q^2 dk}{16\pi^2 T} \sum_{n,r} \cosh^{-2}\left[\frac{E_{k,n,r}}{2T}\right] \frac{\partial E_{k,n,r}}{\partial n} \left[2 - \frac{\partial E_{k,n,r}/\partial n}{\partial E_{k,n,r}/\partial n + \omega + i/\tau} - \frac{\partial E_{k,n,r}/\partial n}{\partial E_{k,n,r}/\partial n - \omega - i/\tau}\right] . \tag{62}$$

Equations (61) and (62) show that the dispersion of the mutual friction occurs at frequencies $\omega \sim \max(\omega_0, 1/\tau)$ which can be several tens of Hz. In this range of frequencies, the mutual-friction parameters D and D' can essentially modify the spectrum of the vortex eigenmodes. The spectrum can be obtained from the equation of the force balance, Eq. (3), supplemented by the force due to the vortex bending, $\mathbf{F}^{(\text{tens})} = \mu(\partial^2 \mathbf{u}_L / \partial z^2)$. Here $\mu \sim \kappa^2 \rho_s$ is the linear tension of the continuous vortex, and $\mathbf{u}_L$ is the vortex displacement. For $\mathbf{u}_L = \mathbf{u}_0 \exp(-i\omega t + ik_z z)$ one obtains

$$\mu k_z^2 = \{[\kappa \rho_s - D'(\omega)] + iD(\omega)\}\omega \ . \tag{63}$$

As we pointed out in Refs. 24 and 9, dissipation exists even in the limit of a long mean free path if the frequency is such that $\omega = \omega_0(\theta)$ for some direction of the quasiparticle momentum. The dissipation is due to a resonant absorption by the quasiparticles localized in the vortex.

For small $\omega \ll \omega_0$ in the limit of long τ, one obtains $D = -i\omega M$, where M is the "vortex mass." It is defined as the coefficient in front of the vortex acceleration in the force balance equation Eq. (3):

$$\mathbf{F}^{(M)} + \mathbf{F}^{(\text{tens})} = M \frac{\partial \mathbf{v}_L}{\partial t} \ . \tag{64}$$

The mass is

$$M = m\kappa_0 \int_0^{p_F} \frac{q^2 dk}{8\pi^2 T} \sum_{n,r} \cosh^{-2}\left[\frac{E_{k,n,r}}{2T}\right] . \tag{65}$$

Performing the summation in Eq. (65), one obtains

$$M = \kappa \rho_{\text{He}} \int_0^{\pi/2} \tfrac{3}{4} \sin^3\theta \, d\theta \times \sum_i \left\{ \left[1 + \frac{4\pi^2}{3} \left[\frac{T}{\zeta_i} \right]^2 \right] \Big/ \omega_{0i}(\theta) \right\} . \tag{66}$$

The vortex mass is $M \sim \rho_{\text{He}} R^{*2}$; it is of the order of the liquid mass confined within the normal region of the radius R^* defined by Eqs. (59) and (60). This agrees with the normal-region model for a continuous vortex suggested in the previous section.

IV. CONCLUSIONS

In the present paper we have calculated the mutual-friction parameters for axisymmetric continuous vortices in phase A at low temperatures. We show that the dynamics of the continuous vortices is governed by the interaction between the quasiparticles localized in the vortex and excitations outside the vortex. The interaction is taken into account through the relaxation-time approximation. The mutual-friction parameters are determined by the ratio of the distance between the energy levels of quasiparticles localized in the vortex, ω_0, and the relaxation rate of these quasiparticles, $1/\tau$. The transition from viscous to dissipationless flow of vortices occurs when $\omega_0 \tau \sim 1$, the corresponding temperature T^* being below T_c. With lowering the temperature, the combination $\rho_n B / \rho_{\text{He}}$ exhibits a minimum at T^* followed by a maximum at the temperature $T^{**} < T^*$. It further decreases to zero as $T \to 0$. In the limit of a long mean free path, the viscosity of a continuous vortex is considerably higher than that of a singular vortex.

From the viscosity coefficient D, we estimate the effective cross section of a continuous vortex and its mass. These estimates show that, with a continuous vortex, one can associate a normal region having the radius of order of $R(T/\Delta)$, where R is the size of the vortex. We believe that large effective cross sections of continuous vortices can be measured in experiments with a quasiparticle beam emitted by vibrating wire radiators. The effect of mutual friction on vortex eigenmodes is discussed.

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APPENDIX: THE RELAXATION-TIME APPROXIMATION

In this appendix we discuss the relaxation-time approximation for the quasiparticle scattering and outline the calculations of the mutual-friction parameters. According to Ref. 9 the force produced by normal excitations is

$$\mathbf{F}^{(\text{exc})} \cdot \mathbf{d} = \frac{1}{L} \int d^3\mathbf{r} \int \frac{d\epsilon}{8\pi i} \text{Tr}\{\mathcal{H}_d(\mathbf{r})[\mathcal{G}_\epsilon^{(n)}(\mathbf{r},\mathbf{r}) - \mathcal{G}_\epsilon^{(a)}(\mathbf{r},\mathbf{r})]\} \ . \tag{A1}$$

Here $\mathbf{d}$ is an arbitrary constant vector, and L is the length of the vortex.

The Green's function $\mathcal{G}$ are matrices both in Nambu and spin spaces:

$$\mathcal{G}_\epsilon = \begin{bmatrix} \hat{G}_\epsilon & \hat{F} \\ -\hat{F}^\dagger & \hat{\bar{G}} \end{bmatrix} . \tag{A2}$$

The function $\mathcal{G}^{(n)}$ is defined as follows:

$$\mathcal{G}_\epsilon^{(n)}(\mathbf{r}_1,\mathbf{r}_2) = -\frac{i}{2T} \cosh^{-2}\left[\frac{\epsilon}{2T}\right] [\mathbf{v}_L \cdot \boldsymbol{\nabla}_2 \mathcal{G}_\epsilon^R(\mathbf{r}_1,\mathbf{r}_2) + \mathbf{v}_L \cdot \boldsymbol{\nabla}_1 \mathcal{G}_\epsilon^A(\mathbf{r}_1,\mathbf{r}_2)] \ , \tag{A3}$$

where $\mathcal{G}^R$ ($\mathcal{G}^A$) is the retarded (advanced) Green's function. The operator ∇_i acts on the coordinate $\mathbf{r}_i$. The "anomalous" Green's function is[25]

$$\mathcal{G}_\epsilon^{(a)}(\mathbf{r}_1,\mathbf{r}_2) = \frac{i}{2T} \cosh^{-2}\left[\frac{\epsilon}{2T}\right] \int \mathcal{G}_\epsilon^R(\mathbf{r}_1,\mathbf{r}) \mathcal{H}_v(\mathbf{r}) \mathcal{G}_\epsilon^A(\mathbf{r},\mathbf{r}_2) d^3\mathbf{r} + \int \int \mathcal{G}_\epsilon^R(\mathbf{r}_1,\mathbf{r}') \check{\Sigma}_\epsilon^{(a)}(\mathbf{r}',\mathbf{r}'') \mathcal{G}_\epsilon^A(\mathbf{r}'',\mathbf{r}_2) d^3\mathbf{r}' d^3\mathbf{r}'' \ . \tag{A4}$$

Here the self-energy due to quasiparticle collisions, $\check{\Sigma}_\epsilon^{(a)}$, is proportional to $\mathcal{G}^{(a)}$ (for its definition see Ref. 25); and

$$\mathcal{H}_d = \mathbf{d}\cdot\nabla\mathcal{H}, \quad \mathcal{H}_v = \mathbf{v}_L\cdot\nabla\mathcal{H} \ . \tag{A5}$$

The retarded and advanced Green's functions can be expanded in terms of the Bogoliubov quasiparticle wave functions. Within the τ approximation for the quasiparticle scattering, we can write

$$\mathcal{G}_\epsilon^{R(A)}(\mathbf{r}_1,\mathbf{r}_2) = \sum_{n,r,s}\int \frac{dk}{2\pi}\frac{\mathcal{U}_N(\mathbf{r}_1)\mathcal{U}_N^\dagger(\mathbf{r}_2)}{E_N-\epsilon\mp i/2\tau_N} \ . \tag{A6}$$

Here we put N to be the set of the quantum numbers $\{k,n,r,s\}$, and denoted

$$\frac{i}{\tau_N} = \langle \mathcal{U}_N^\dagger(\check{\Sigma}^R-\check{\Sigma}^A)\mathcal{U}_N\rangle \ , \tag{A7}$$

where

$$\langle \cdots \rangle = \mathrm{Tr}\int d^3\mathbf{r}(\cdots) \ .$$

The τ approximation in Eq. (A6) assumes that the self-energies

$$\check{\Sigma}^R-\check{\Sigma}^A = \sum_{n,r,s;n',r',s'}\int \frac{dk\,dk'}{(2\pi)^2}\mathcal{U}_N(\mathbf{r}_1)\Sigma_{N,N'}\mathcal{U}_{N'}^\dagger(\mathbf{r}_2)$$

have diagonal matrix elements:

$$\Sigma_{N,N'} = \check{\tau}_3\frac{i}{\tau_N}2\pi\delta(k-k')\delta_{n,n'}\delta_{r,r'}\delta_{s,s'} \ .$$

In general, this is not the case. However, for very long relaxation times, this approximation seems to be reasonable for our problem.

Using Eqs. (A4) and (A6), one can write

$$\mathcal{G}_\epsilon^{(a)}(\mathbf{r}_1,\mathbf{r}_2) = \sum_{n,n';r,r';s,s'}\int \frac{dk\,dk'}{(2\pi)^2}\mathcal{U}_N(\mathbf{r}_1)X_{N,N'}(\epsilon)\mathcal{U}_{N'}^\dagger(\mathbf{r}_2) \ , \tag{A8}$$

where

$$X_{N,N'}(\epsilon) = \frac{1}{(E_N-\epsilon-i/\tau_N)(E_{N'}-\epsilon+i/\tau_{N'})} \times \left[\frac{i}{2T}\cosh^{-2}\left[\frac{\epsilon}{2T}\right] \times \langle \mathcal{U}_N^\dagger\mathcal{H}_v\mathcal{U}_{N'}\rangle + \langle \mathcal{U}_N^\dagger\check{\Sigma}^{(a)}\mathcal{U}_{N'}\rangle\right] \ . \tag{A9}$$

Equation (A1) for the force $\mathbf{F}^{(\mathrm{exc})}$ contains the integral of $\mathcal{G}^{(a)}$ over ϵ. If we define

$$Y = \int \frac{d\epsilon}{2\pi i}X(\epsilon) \ ,$$

then, for energies $E_N - E_{N'} \ll E_N$,

$$(E_{N'}-E_N)Y_{N,N'} + \left[\frac{i}{\tau_N}Y_{N,N'} - \langle \mathcal{U}_N^\dagger\check{\Sigma}^{(a)}\mathcal{U}_{N'}\rangle\right]_{\epsilon=E_N} = \frac{i}{2T}\cosh^{-2}\left[\frac{E_N}{2T}\right]\langle \mathcal{U}_N^\dagger\mathcal{H}_v\mathcal{U}_{N'}\rangle \ . \tag{A10}$$

We retain only the residue term due to the denominator in Eq. (A9) since it gives the main contribution to the integral.

One can calculate the force $\mathbf{F}^{(\mathrm{exc})}$ using the relations, derived in Ref. 9,

$$\langle \mathcal{U}_{N'}^\dagger\mathcal{H}_v\mathcal{U}_N\rangle = (E_N-E_{N'})\langle \mathcal{U}_{N'}^\dagger\check{\tau}_3(\mathbf{d}\cdot\nabla)\mathcal{U}_N\rangle \tag{A11}$$

and

$$\mathbf{d}\cdot\nabla\mathcal{U}_{k,n,r,s} = qd_+\mathcal{U}_{k,n-1,r,s} - qd_-\mathcal{U}_{k,n+1,r,s} \ , \tag{A12}$$

where $d_\pm = d_x \pm i d_y$. Here we specified all the quantum numbers explicitly. One obtains

$$\frac{1}{L}\int d^3\mathbf{r}\int \frac{d\epsilon}{2\pi i}\mathrm{Tr}[\mathcal{H}_d(\mathbf{r})\mathcal{G}_\epsilon^{(a)}(\mathbf{r},\mathbf{r})] = \sum_{n,n';r,s}\int \frac{dk}{2\pi}Y_{N,N'}(E_{k,n,r,s}-E_{k,n',r,s})(qd_+\delta_{n',n-1}-qd_-\delta_{n',n+1}) \ , \tag{A13}$$

In this equation, the matrix element $Y_{N,N'}$ has $N=\{k,n,r,s\}$ and $N'=\{k,n',r,s\}$, the difference being only in the azimuthal quantum numbers n and n'. Therefore, Eq. (A13) involves the transitions only between the states with neighboring azimuthal quantum numbers $n \to n\pm 1$ which makes the difference $(E_{k,n,r,s}-E_{k,n',r,s})$ very small.

The matrix elements $Y_{N,N'}$ in Eq. (A13) are to be found by solving Eq. (A10). Using Eqs. (A11) and (A12) as in Ref. 9, one would obtain Eqs. (7) and (8) with τ_N instead of τ, if $\check{\Sigma}^{(a)}$ were absent. However, the self-energy $\check{\Sigma}^{(a)}$ results in a renormalization of the relaxation time. One can easily check that the expression

$$\left[\frac{i}{\tau_N}Y_{N,N'} - \langle \mathcal{U}_N^\dagger\check{\Sigma}^{(a)}\mathcal{U}_{N'}\rangle\right] \tag{A14}$$

is the matrix element of the "collision integral" $J_1 = (i/4)\mathrm{Tr}[\mathcal{J}^{(a)}]$, where (see Refs. 7 and 15)

$$\mathcal{J}^{(a)} = \check{\Sigma}^R\mathcal{G}^{(a)} - \mathcal{G}^{(a)}\check{\Sigma}^A + \check{\Sigma}^{(a)}\mathcal{G}^A - \mathcal{G}^R\check{\Sigma}^{(a)} \ . \tag{A15}$$

The collision integral enters the kinetic equation for the distribution function of excitations. There are two kinds of excitations: those localized in the vortex, and the excitations which can escape to infinity. The latter are in equilibrium with the container walls. All the excitations interact with each other, and the resulting distribution function has to be found from the full kinetic equation. The analytical solution of the kinetic equation is hardly possible. Instead, we use the "relaxation-time approximation" according to which Eq. (A14) can be reduced to

$$\left[\frac{i}{\tau_N} Y_{N,N'} - \langle \mathcal{U}_N^\dagger \check{\Sigma}^{(a)} \mathcal{U}_{N'} \rangle \right] = \frac{i}{\tau} Y_{N,N'} , \qquad \text{(A16)}$$

where τ is an effective relaxation time. We assume a constant τ for simplicity. With the help of Eqs. (A16), (A11), and (A12) we return now to the mutual friction coefficients D and D' from Eqs. (7) and (8).

To estimate τ, one can argue as follows. The collision integral vanishes if all the excitations are in equilibrium with each other. Therefore, if the delocalized excitations (which are in equilibrium with the walls) were absent (for example, at very low temperatures), the quasiparticles localized in the vortex would form a closed subsystem which is in equilibrium with itself and with the moving vortex. This would result in zero relaxation rate $1/\tau$. Therefore, the relaxation rate $1/\tau$ is proportional to the density of delocalized quasiparticles which can relax at the container walls. To escape to infinity where the anisotropy vector **l** is oriented differently from its local direction in the vortex, the quasiparticles need energies of the order of Δ_0 and, thus, we obtain Eq. (14) for the estimate of $1/\tau$.

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