Superfluidity of a hydrogenlike gas in a strong magnetic field

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Superfluidity is studied for a hydrogen gas and a hydrogenlike gas of excitons in a strong magnetic field, such that the distance between the Landau levels, $e\hbar B/m_{\rm eff}c$, is greater than the Rydberg, $m_{\rm eff}e^4/2\hbar^2$. The expression obtained for the energy spectrum demonstrates the existence of a phonon branch of the energy spectrum and the appearance of the superfluid state of the hydrogenlike gas at low temperature. The phase transition into the superfluid state is possible at gas number densities below a certain critical value (which changes with B). The Green's functions, the energy spectrum, and the critical temperature are calculated in a low-density approximation for the condensed state of such atoms in the strong magnetic field.

I. INTRODUCTION

We are interested in a gas of hydrogenlike atoms. This can be a gas of hydrogen atoms or a gas of hydrogenlike excitons in semiconductors. Assuming that the density of the gas is low enough, we can just take into account the pair interaction of the atoms. As is known, the interaction of two hydrogen atoms is caused by van der Waals forces and related to the symmetry of the spin-wave function at large interatomic distances. At small interatomic distances the pair interaction is mainly caused by the Coulomb repulsion of the nuclei. The potential of the pair interaction of two atoms is well known from the solution of the quantum-mechanical problem for the hydrogen molecule.[1] In order to recall the situation, let us first consider two hydrogen atoms in the ground state without a magnetic field. When they approach each other, the potential of the pair interaction depending on the symmetry of the wave function of the electrons forces a final energy state of the molecule to be either a singlet term or a triplet term. The singlet term $({}^1\Sigma_g^+)$ and the triplet term $({}^3\Sigma_u^+)$ correspond to the possible states of two electrons with the antisymmetrical and symmetrical spin-wave function, respectively. In the singlet state, when the total spin of the electrons is equal to zero, the potential energy as a function of the distance R between the nuclei has a deep minimum which lies well below a shallow minimum of the potential energy in the triplet state at all R. This deep minimum located at $R \sim 1$ (in atomic units) is responsible for the formation of the ground state of a stable hydrogen molecule. Now let us take a hydrogen gas and see what happens as we lower the temperature. It is physically clear that a Bose gas of such strongly interacting hydrogen atoms in the ground state will start to solidify if the temperature is decreased. So the transition into a dense molecular phase, and then into the solid state, comes before the Bose condensation (which implies the formation of the superfluid state) takes place. In principle, this phase transition into the superfluid state would readily happen if the bound state of the molecule corresponded to the triplet term. However, it is hardly probable because of a large energy difference between the singlet and the triplet, so that the triplet term may only be a metastable state of the system.

The situation is dramatically changed in the presence of a strong magnetic field B, such that the distance between the Landau levels, $e\hbar B/m_e c$, exceeds the Coulomb unit of energy (Rydberg), $\mathcal{R}=m_e e^4/2\hbar^2$. In the fields $B \gg B_c$ the spins of the electrons are strictly antiparallel to the magnetic-field direction. Under the circumstances, the triplet evidently becomes the lowest state of the system,[2] for the energy difference of the singlet and triplet terms is asymptotically μB (here μ is the magnetic moment of the electron). At the same time, the size of each hydrogen atom becomes smaller by the factor $\ln(B/B_c)$ in comparison with the Bohr radius along the direction of the magnetic field, and smaller by the factor $\sqrt{B/B_c}$ in a plane perpendicular to the magnetic-field direction. The motion of the electrons in this plane is almost suppressed by the magnetic field so that the hydrogen atom looks like a thin needle directed along the magnetic field. Since such atoms have a large quadrupole moment, the pair interaction reduced by the exchange interaction of the spins in the ground triplet state is strongly anisotropic and pretty weak by its nature (see Fig. 1).[2] Therefore, both this sort of pair interaction and the small overlap of the atomic wave functions may lead to a remarkable situation when the collective effects are not able to break the symmetry of the gaseous system by means of the lattice formation even at a large number density. Thus for such a Bose gas of hydrogenlike atoms with the weak anisotropic pair interaction there is the possibility of forming a superfluid state at a low temperature.

The importance of the problem is due to the fact that it arises in various areas of physics, especially in astrophysics and physics of semiconductors. In fact, the magnetic fields, higher than the characteristic atomic value $B_c=m_e^2e^3c/\hbar^3\approx 2.35\times 10^9$ Oe, are observed near pulsars, neutron stars, and white dwarf stars. No doubt, those huge magnetic fields must drastically influence the different physical properties of the compact cosmic objects. That is why researchers have been interested in the behavior of matter under such conditions for a long time.[3–5] On the other hand, in semiconductors (or dielectrics) a high-density exciton gas is the unique object for the laboratory studies of the extreme states of matter in the ultrahigh magnetic field. Since the excitonic Coulomb unit of energy (Rydberg) is $\mathcal{R}_{ex}=m_{eff}e^4/2\varepsilon^2\hbar^2$ (here m_{eff} is the reduced effective mass of the electron and the hole, ε is the dielectric constant), the "atomic" scale of the magnetic field for the hydrogenlike excitons is $B_{ex}=m_{eff}^2e^3c/\varepsilon^2\hbar^3$. This scale may be a few orders of magnitude smaller than for usual hydrogen atoms. For instance, the characteristic atomic field is $B_{ex}\approx 9$ kOe for Ge and $B_{ex}\approx 2$ kOe for InSb, respectively. One should also note the good opportunities to create a high-density exciton gas by a laser pulse. So we see that a high magnetic field may cause so dramatic changes in the hydrogenlike exciton system that, perhaps, no better way to observe the Bose-Einstein condensation, the superfluidity, and related phenomena as well as the behavior of matter in such "cosmic" magnetic fields on the earth could ever be expected.

In this paper the hydrogenlike gas in a strong magnetic field is studied. In Sec. II, we consider the pair interaction of atoms in detail and discuss the possibility of applying a gaseous approximation. Section III is devoted to the thermodynamic properties of the hydrogenlike gas in the strong magnetic field. Using the diagram methods, we obtain the expressions for the normal Green's function, and the anomalous Green's function of the hydrogen gas. The thermodynamic functions, the energy spectrum, the condensate density, and the density of noncondensate particles are calculated in a low-density limit. The existence of the phase transition into the superfluid state is established. At last, we obtain and discuss the dependence of the critical temperature on the total density and the magnetic field B.

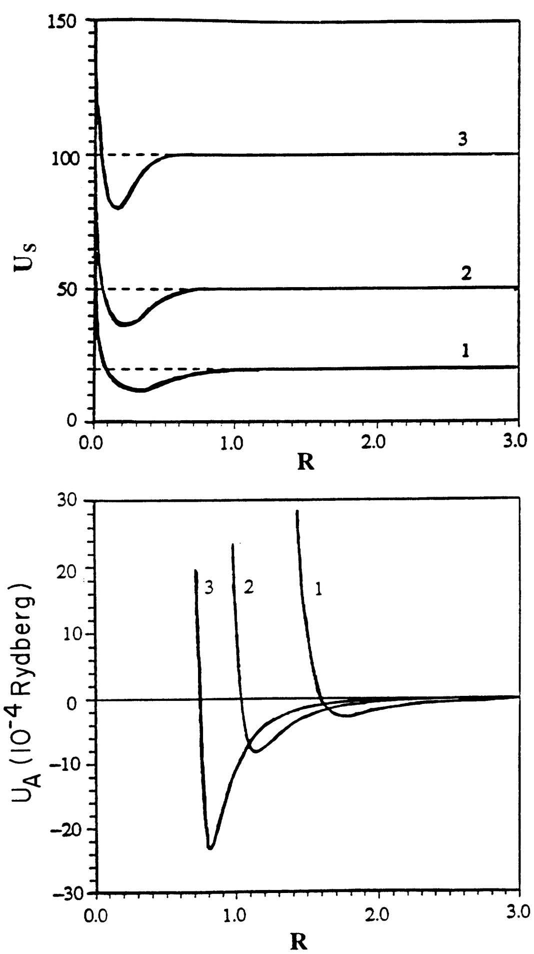

FIG. 1. The potential energy of two interacting hydrogen atoms in the strong magnetic field. The singlet term (top) and triplet term (bottom); lines 1, 2, and 3 correspond to the magnetic fields of 20, 50, and 100, respectively. All values are expressed in atomic units.

II. THE PAIR INTERACTION OF A HYDROGEN GAS IN A STRONG MAGNETIC FIELD

We consider a hydrogen gas in a strong magnetic field, so that $B \gg B_c$. In this paper the Coulomb system of units is used, $e^2/\varepsilon=m_e=\hbar=1$, and the magnetic-field intensity unit is $B_c=m_{eff}^2e^3c/\varepsilon^2\hbar^3$, where m_{eff} is the reduced effective mass of the electron and the nucleus (or the electron and the hole). For the hydrogen atom ($\varepsilon=1$) we have $m_{eff}\approx m_e$. In the case of the hydrogenlike excitons, the reduced effective mass can be one or two orders of magnitude smaller than m_e.

We shall need some facts about the internal structure of the hydrogen atom under these conditions in order to begin our study. To find the wave function of the hydrogen atom in the ground state, one should note that we can, as the first approximation, treat the motion of the electron parallel to the direction of the magnetic field as one-dimensional motion in a Coulomb potential, while considering only the motion in the magnetic field in the perpendicular plane. The wave function of the ground state of the hydrogen atom in a strong magnetic field can be written in accordance with the results of Ref. 6 as

$$\Psi(\mathbf{r})=\phi(\rho)\psi(x)=\frac{1}{\sqrt{2\pi}\lambda}\exp\left[-\frac{\rho^2}{4\lambda^2}\right]\frac{1}{\sqrt{\alpha}}\exp\left[-\frac{|x|}{\alpha}\right], \quad (1)$$

where $\phi(\rho)$ is the wave function of the zeroth Landau level, which corresponds to the motion in the plane perpendicular to the magnetic field ($\rho^2=y^2+z^2$); $\lambda=\sqrt{1/B}$; and $\psi(x)$ is the wave function corresponding to the motion of the electron in the one-dimensional Coulomb field along the magnetic field. The expression which relates the parameter α to the ground-state energy can be written with logarithmic accuracy as follows:

$$E_{atom}=-\frac{1}{2\alpha^2}\approx-\frac{1}{2}\ln^2 B \; . \quad (2)$$

Since the hydrogen atom in the strong magnetic field has a large binding energy ($E_{atom} \gg \mathcal{R}$) and a small characteristic size ($\lambda \ll \alpha \ll 1$), a system of a large number of these atoms may definitely be treated as a low-density gas of structureless particles with the negligible overlap of the wave functions of different atoms at the densities even when the collective effects, such as the formation of either a dense molecular phase for hydrogen atoms or an electron-hole liquid for excitons, dominate in the system without a magnetic field at a low temperature. Of course, the peculiarities in the behavior of the pair interaction in the ground triplet state, and first of all the weakness of the interaction, also play an important role in order that such an approximation becomes legitimate.

An asymptotically exact expression for the energy of the exchange interaction and the binding energy of two hydrogen atoms was obtained,[2] and the potential of the pair interaction is shown in Fig. 1. The hydrogen atoms are stretched out to a great extent parallel to the magnetic field. For that reason, their interaction caused by the quadrupole-quadrupole interaction and the exchange coupling of the spins strongly depends on the angle $\tilde{\theta}$ between the line joining the centers of mass of two atoms and the direction of the magnetic field. The potential energy of the interatomic interaction $U_A(R)$ in the triplet state has a shallow minimum at $R \sim 1$, for $\tilde{\theta} \approx 49°$, and the depth of the well is more than two orders of magnitude smaller than the binding energy of the ordinary hydrogen molecule. In any case, the triplet term lies well below the singlet term; i.e., the interaction of two atoms in the ground state corresponds to the triplet state. The main contribution to the interaction at the small interatomic distance $R \sim \lambda$ is caused by the Coulomb repulsion. Taking into account the results obtained,[2] we can write the expression for the energy of the pair interaction in the ground triplet state with acceptable accuracy in the following form:

$$U=\frac{1}{R}\exp\left[-\frac{R^2}{2\lambda^2}\right]+U_0\exp\left[\frac{R}{R_0}\right]P_4(\cos\tilde{\theta}) , \quad (3)$$

where R is the distance between the centers of mass of two hydrogen atoms, and

$$P_4(\cos\tilde{\theta})=\tfrac{1}{8}(35\cos^4\tilde{\theta}-30\cos^2\tilde{\theta}+3)$$

is the Legendre polynomial, U_0 and R_0 are the effective depth and the size of the potential well in the field $B \gg 1$, respectively. The depth and the size of the potential well depend on the strength of the magnetic field

$$U_0=\left|2.33\ln^2 B\left[10.34\frac{\ln B}{\sqrt{B}}+41.61\right]\exp\left[-10.34\frac{\ln B}{\sqrt{B}}-8.92\right]-3.66\times 10^{-5}\frac{B^{5/2}}{\ln^4 B}\right| , \quad (4)$$

$$R_0 \cong \frac{7.9}{\sqrt{B}}=7.9\lambda . \quad (5)$$

The low-density condition formally means that the characteristic size R_0 of the potential well is much less than the average distance between the atoms

$$R_0 n^{1/3} \ll 1 , \quad (6)$$

where $n=N/V$ is the number density.

Substituting the expression for R_0 into relation (6), we obtain the values of the density for the formal applicability of the gaseous approximation, i.e., when the hydrogenlike atoms can presumably be regarded as bosons, whose internal orbital structure is almost irrelevant: $n \ll 1.3\times 10^{24}$ cm^{-3} at $B=20$ [$R_0=1.76$, $U_0=6.78\times 10^{-4}$ $(2\mathcal{R})=1.8\times 10^{-2}$ eV], and $n \ll 1.5\times 10^{25}$ cm^{-3} at $B=100$ [$R_0=0.79$, $U_0=5.49\times 10^{-3}$ $(2\mathcal{R})=0.15$ eV]. Thus we have made sure that the low-density approximation is quite legitimate at the densities mentioned above, and the hydrogenlike gas in the strong magnetic field may still be looked upon as a gas of weak interacting particles even at such high densities, because of those substantial changes in the structure and the interaction of the atoms.

III. THERMODYNAMIC PROPERTIES OF THE HYDROGEN GAS IN A STRONG MAGNETIC FIELD; GREEN'S FUNCTIONS

The influence of a strong magnetic field on both a single hydrogen atom and the pair interaction immediately leads to the question of how the corresponding ground state of a gas of these atoms is changed at a low temperature. The large binding energy of the atom, the small characteristic size, and at last the very weak pair interaction in the ground triplet state suggest an idea of a phase transition into a superfluid state in a dilute gas of such atoms. In what follows, we shall hold this view, assuming that the ground state of the gas is the Bose condensate at a low temperature. The phase transition into the superfluid state occurs if the temperature is

$$T \sim (1/m)n^{2/3} , \quad (7)$$

where m is the mass of the hydrogenlike atom in atomic units and n is the number density. It is precisely a range of the temperature that will be the matter of our interest.

As is well known,[7] in order to find the important physical properties of a system it is quite sufficient to know the

Green's functions which determine the probability amplitude for the motion of particles in the quantum system. The knowledge of the Green's functions allows to obtain, among other things, the energy spectrum, the gas density, the mean momentum, and the thermodynamic potentials. In the system of interacting Bose particles the normal Green's function $G(p)$ and the anomalous Green's function $G_1(p)$ exist below the transition temperature.[8,9] The Green's functions $G(p)$ and $G_1(p)$ can be expressed through the irreducible self-energy parts A and B, which are the sum of all possible connected Green's function diagrams that cannot be separated into two parts joined by only one continuous line. The corresponding expression is known as the Dyson's equation. For the Bose system its graphical representation is shown in Fig. 2, and we can write it down in an analytical form as

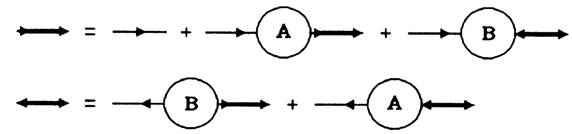

FIG. 2. The graphical representation for the Green's functions.

$$G(p)=\frac{i\omega+(k^2/2m)-\mu+A(-p)}{[i\omega+(k^2/2m)-\mu+A(-p)][i\omega-(k^2/2m)+\mu-A(p)]+|B(p)|^2}\ , \tag{8}$$

$$G_1(p)=-\frac{B(p)}{[i\omega+(k^2/2m)-\mu+A(-p)][i\omega-(k^2/2m)+\mu-A(p)]+|B(p)|^2}\ , \tag{9}$$

where μ is the chemical potential, $\omega\equiv\omega_s=2\pi sT$, s being integer, $p=(\mathbf{k},\omega)$.

The main contribution to the self-energy parts A and B comes from the diagrams[9] shown in Fig. 3. The summation in these diagram expressions reduces to the summation of the t matrix shown in Fig. 4. The result of this summation has the form

$$A(p)=(n_0+n_1)[t(\mathbf{k}/2,\mathbf{k}/2;i\omega-k^2/4m)+t(\mathbf{k}/2,-\mathbf{k}/2;i\omega-k^2/4m)]\ , \tag{10}$$

$$B(p)=n_0 t(\mathbf{k},0;0)\ , \tag{11}$$

where n_0 is the condensate density, n_1 is the density of noncondensate particles ($n=n_0+n_1$). The solution of the equation which gives the expression for the t matrix can be related to the scattering amplitude $f(\mathbf{k},\mathbf{k}')$. Accurate to the terms of the first order in $f(\mathbf{k},\mathbf{k}')$ this relation can be written as follows:

$$t(\mathbf{k}/2,\mathbf{k}'/2;i\omega-k^2/4m)\cong\frac{4\pi}{m}f(\mathbf{k}/2,\mathbf{k}'/2)\ , \tag{12}$$

$$t(\mathbf{k},0;0)\cong\frac{4\pi}{m}f(\mathbf{k},0)\ . \tag{13}$$

The scattering amplitude is obtained as the iterated solution of the integral equation[1] and the final result represented by a Born series is

$$\frac{4\pi}{m}f(\mathbf{k},\mathbf{k}')=U(\mathbf{k}-\mathbf{k}')+\sum_{n=1}^{\infty}\frac{1}{(2\pi)^{3n}}\int dk_1\cdots dk_n\frac{U(\mathbf{k}'-\mathbf{k}_1)\cdots U(\mathbf{k}_n-\mathbf{k})}{[(k^2-k_1^2)/m+i\delta]\cdots[(k^2-k_n^2)/m+i\delta]}\ , \tag{14}$$

where $U(\mathbf{k}-\mathbf{k}')$ is the Fourier transform of the potential of pair interaction.

In the sum over n in Eq. (14) the main contribution comes from the regions of integration where $U\approx U_1$, i.e., when $k\gg 1/R_0$ (here U_1 is the isotropic part of the pair interaction). In the first approximation, neglecting the terms of order λ/R_0, we obtain

$$\frac{4\pi}{m}f(\mathbf{k},\mathbf{k}')=U_2(\mathbf{k}-\mathbf{k}')+t_0\ , \tag{15}$$

where $t_0=(4\pi/m)f_1(\mathbf{k},\mathbf{k}')$, f_1 is the scattering amplitude of the potential U_1, and $U_2(\mathbf{k}-\mathbf{k}')$ is the momentum representation of the anisotropic part of the pair interaction. At the momentum $k\ll 1/\lambda$ the parameter t_0 can

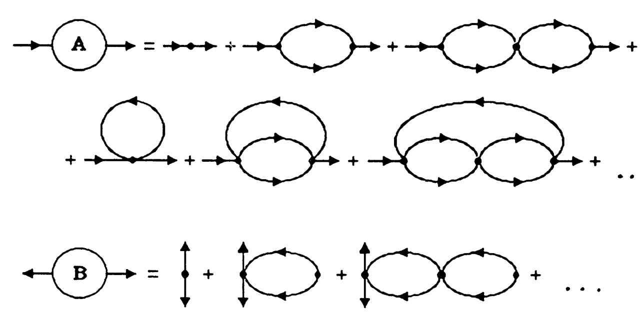

FIG. 3. The diagram expressions for the self-energy parts A and B.

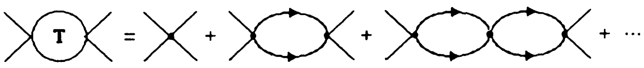

FIG. 4. The diagram representation for the t matrix.

be treated as a constant. One should emphasize now that in obtaining these relations we assumed that two hydrogenlike atoms cannot form a bound state under the conditions considered. The following analysis will show that it means the appearance of the restrictions on a possible magnitude of the magnetic field ($1 \ll B < 1000$) and on the gas number density ($n < 10^{24}$ cm^{-3}).

Thus we can rewrite the expressions for $A(p)$ and $B(p)$ in terms of the scattering amplitude

$$A(p) \cong \Psi_A(\mathbf{k}) = n[2t_0 + U(\mathbf{k})] , \quad (16)$$

$$B(p) \cong \Psi_B(\mathbf{k}) = n_0[t_0 + U(\mathbf{k})] . \quad (17)$$

Here and below, in order not to complicate the notations, $U(\mathbf{k})$ denotes the Fourier transform of the anisotropic part of the pair interaction U_2, i.e.,

$$U(\mathbf{k}) \equiv U_2(\mathbf{k}) = U_0 \int \exp\left\{-i\mathbf{k}\cdot\mathbf{R} - \frac{R}{R_0}\right\} P_4(\cos\tilde{\theta})dV . \quad (18)$$

Using the "addition theorem" for Legendre polynomials

$$P_4(\cos\tilde{\theta}) = \frac{4\pi}{9} \sum_{m=-4}^{m=+4} Y^*_{4m}(\vartheta,\phi) Y_{4m}(\theta,\varphi) , \quad (19)$$

where ϑ,ϕ and θ,φ are the usual latitude and longitude angles which determine the directions of the unit vectors $\mathbf{R}/R$ and $\mathbf{B}/B$, respectively, in the coordinate system with the z axis directed along the vector $\mathbf{k}$, and applying the definition of the spherical Bessel functions[10]

$$j_n(z) = \frac{(-i)^n}{2} \int_0^\pi \exp(iz\cos\theta) P_n(\cos\theta) \sin\theta \, d\theta , \quad (20)$$

we obtain the anisotropic part of the pair interaction in the momentum representation

$$U(\mathbf{k}) = 4\pi U_0 R_0^7 J(R_0 k) k^4 P_4(\cos\theta) = bk^4 Y_{40}(\theta,\varphi) , \quad (21)$$

where θ,φ are the angular coordinates of the vector $\mathbf{k}$ in the coordinate system with the z axis parallel to the magnetic field; $k = |\mathbf{k}|$, and

$$b = \frac{8\pi\sqrt{\pi}}{3} J(R_0 k) U_0 R_0^7 .$$

The function $J(R_0 k)$ is expressed via the spherical Bessel function of the fourth-order $j_4(y)$ as

$$J(R_0 k) = \frac{1}{(R_0 k)^4} \int_0^\infty j_4(R_0 k x) \exp(-x) x^2 dx . \quad (22)$$

The estimate of the above expression shows that the function $J(R_0 k)$ can be replaced with acceptable accuracy by the step function

$$J(R_0 k) = \begin{cases} 16/21, & k < 1/R_0, \\ 0, & k > 1/R_0 . \end{cases}$$

Such an approximation almost has no influence on the results of the following calculations.

Now the formulas obtained allow us to find out the concrete physical properties of the system in the first approximation. The density n_0 of condensate particles is defined by the expression for the chemical potential[9]

$$\mu = A(0) - B(0) \cong \Psi_A(0) - \Psi_B(0) , \quad (23)$$

which is known as Hugenholz-Pines relation. Substituting the first approximation for the self-energy parts into Eq. (23), we find the density of condensate particles

$$n_0 = \mu/t_0 - 2n_1 , \quad (24)$$

where the constant t_0 can be determined in the Born approximation

$$t_0 \approx 4\pi\lambda^2 = 4\pi/B . \quad (25)$$

The density of particles "above the condensate" (with $k \neq 0$) can be written down in the first approximation as ($\varepsilon \to +0$)

$$n_1 = -\frac{T}{V} \sum_p \exp(i\omega\varepsilon) G^0(p) = \frac{\zeta(3/2)}{(2\pi)^{3/2}} (mT)^{3/2} , \quad (26)$$

where $\zeta(x)$ is the Riemann zeta function, $\zeta(3/2) = 2.612$, and

$$G^0(p) = [i\omega - (k^2/2m) + \mu]^{-1} , \quad (27)$$

is the unperturbed Green's function.

At last the Green's functions are given by formulas (8) and (9), where we are to substitute expressions (16) and (17) for the self-energy parts $A(p)$ and $B(p)$ in the first approximation. Below the transition point we get

$$G(p) = -\frac{i\omega + [(k^2/2m) - \mu] + \Psi_A}{\omega^2 + [(k^2/2m) - \mu]^2 + 2[(k^2/2m) - \mu]\Psi_A + \Psi_A^2 - \Psi_B^2} , \quad (28)$$

$$G_1(p) = \frac{\Psi_B}{\omega^2 + [(k^2/2m) - \mu]^2 + 2[(k^2/2m) - \mu]\Psi_A + \Psi_A^2 - \Psi_B^2} . \quad (29)$$

Above a critical temperature, where $n_0=0$, only the normal Green's function $G(p)$ and the corresponding self-energy part $A(p)$ exist. Therefore, the normal Green's function becomes

$$G(p)=\frac{1}{i\omega-[(k^2/2m)-\mu]-\Psi_A} . \quad (30)$$

As was stated earlier, the single-particle Green's functions contain the information about the observable properties of the system. For example, the excitation spectrum of the system is determined by poles of the Green's functions. In that case, a simple calculation yields

$$E=\{[(k^2/2m)-\mu]^2+2[(k^2/2m)-\mu]\Psi_A+\Psi_A^2-\Psi_B^2\}^{1/2} . \quad (31)$$

As a matter of fact, this expression is nothing but Bogoliubov's energy spectrum vanishing linearly as $k\to 0$, with a slope equal to a macroscopic speed of sound, i.e., this spectrum satisfies the Landau criterion for superfluidity. In order to make it clear and consider the main features of the excitation spectrum in detail, we rewrite this expression, using the relation of the chemical potential with the self-energy parts of the Green's functions, as follows:

$$E(\mathbf{k})=\left[\left[\frac{k^2}{2m}+nbk^4Y_{40}(\theta,\varphi)+n_0t_0\right]^2-[n_0bk^4Y_{40}(\theta,\varphi)+n_0t_0]^2\right]^{1/2} . \quad (32)$$

So the energy spectrum of the hydrogenlike gas in an ultrahigh magnetic field has a strong angular dependence on the direction of the magnetic field. Nevertheless, the speed of sound does not depend on the direction of the magnetic field in the first approximation. We can check it out going through some simple manipulations which put the previous expression in the form:

$$E(\mathbf{k})=\left[\left[\frac{k^2}{2m}+\tilde{\Psi}_A\right]^2+\frac{k^2}{m}\Psi_B(0)+2\Psi_B(0)\tilde{\Psi}_A-2\Psi_B(0)\tilde{\Psi}_B-\tilde{\Psi}_B^2\right]^{1/2} , \quad (33)$$

where $\Psi_B(0)=n_0t_0$, $\tilde{\Psi}_B=\Psi_B-\Psi_B(0)=n_0U(\mathbf{k})$, and $\tilde{\Psi}_A=\Psi_A-\Psi_A(0)=nU(\mathbf{k})$. This form of the excitation spectrum proves the existence of a linear phonon branch at small momenta. The phonon branch not having the angular dependence surely satisfies the Landau criterion for superfluidity. In the limit $k\to 0$ we obtain

$$E(\mathbf{k})=k\sqrt{\Psi_B(0)/m}=k\sqrt{4\pi n_0/mB} . \quad (34)$$

The macroscopic speed of sound in the superfluid phase is therefore

$$v_c=\sqrt{4\pi n_0/mB} . \quad (35)$$

The expression obtained for the normal Green's functions gives a chance of calculating the density n_1 of noncondensate particles with a higher accuracy. The density n_1 is directly related to the normal Green's function and below the transition temperature we have ($\varepsilon\to +0$)

$$n_1=-\frac{T}{V}\sum_p \exp(i\varepsilon\omega)G(p)=\frac{1}{2V}\sum_{\mathbf{k}}\left[\frac{(k^2/2m)-\mu+\Psi_A}{E(\mathbf{k})}\coth\left[\frac{E(\mathbf{k})}{2T}\right]-1\right]=\frac{1}{(2\pi)^3}\int d^3k\left[\exp\left[\frac{E(\mathbf{k})}{T}\right]-1\right]^{-1} . \quad (36)$$

The function Ψ_B proportional to n_0 is quite small near the critical temperature. Recalling that this is the region of the temperature we are mainly interested in, we shall use the following expansion of $E(\mathbf{k})$ near the critical temperature in expression (36):

$$E(\mathbf{k})=k^2/2m+\Psi_B(0)+\tilde{\Psi}_A . \quad (37)$$

Taking into account the angular and momentum dependence of the pair interaction according to (21), we can now write out the expression for the density n_1

$$n_1\cong\Gamma_1+\Gamma_2 , \quad (38)$$

where

$$\Gamma_1=\frac{(2m)^{3/2}}{2(2\pi)^3}\int d\Omega\int_0^{\varepsilon_1}\frac{\sqrt{\varepsilon}\,d\varepsilon}{\exp[(\varepsilon+n_0t_0+4m^2nbY_{40}\varepsilon^2)/T]-1}\cong\frac{m^{3/2}T\sqrt{\varepsilon_2}}{\sqrt{2}\pi^2}\ln\left[\frac{1+\sqrt{\varepsilon_1/\varepsilon_2}}{1-\sqrt{\varepsilon_1/\varepsilon_2}}\right] , \quad (39)$$

and

$$\Gamma_2=\frac{(2m)^{3/2}}{2(2\pi)^3}\int d\Omega\int_{\varepsilon_1}^{\infty}\frac{\sqrt{\varepsilon}\,d\varepsilon}{\exp[(\varepsilon+n_0t_0)/T]-1}\cong\frac{\zeta(3/2)}{(2\pi)^{3/2}}(mT)^{3/2}-\frac{\sqrt{2}}{\pi^2}m^{3/2}T[\varepsilon_1+n_0t_0]^{1/2} , \quad (40)$$

here the constants ε_1 and ε_2 are

$$\varepsilon_1=\frac{1}{2mR_0^2},\quad \frac{1}{\varepsilon_2}=\frac{18}{7\sqrt{\pi}}m^2nb=\frac{256\pi}{49}m^2nU_0R_0^7\ . \quad (41)$$

One should emphasize that the condition $1-\sqrt{\varepsilon_1/\varepsilon_2}>0$ has been used to derive these expressions. The final result for the density of the particles above the condensate is

$$n_1=\frac{\zeta(3/2)}{(2\pi)^{3/2}}(mT)^{3/2}+\frac{\sqrt{2}}{\pi^2}m^{3/2}T\times\left[\frac{\sqrt{\varepsilon_2}}{2}\ln\left[\frac{1+\sqrt{\varepsilon_1/\varepsilon_2}}{1-\sqrt{\varepsilon_1/\varepsilon_2}}\right]-[\varepsilon_1+n_0t_0]^{1/2}\right]\ . \quad (42)$$

At the transition point the condensate density is equal to zero, and the density n_1 simply turns into a total density n of the particles. We therefore see that the equation for the critical temperature T_c of the system is

$$n=\frac{\zeta(3/2)}{(2\pi)^{3/2}}(mT_c)^{3/2}+\frac{\sqrt{2}}{\pi^2}m^{3/2}T_c\times\left[\frac{\sqrt{\varepsilon_2}}{2}\ln\left[\frac{1+\sqrt{\varepsilon_1/\varepsilon_2}}{1-\sqrt{\varepsilon_1/\varepsilon_2}}\right]-\sqrt{\varepsilon_1}\right]\ . \quad (43)$$

When either the pair interaction or the total density is quite small ($\sqrt{\varepsilon_1/\varepsilon_2}\ll 1$), the approximate equation for the critical temperature in terms of the interaction parameters is

$$n=\frac{\zeta(3/2)}{(2\pi)^{3/2}}(mT_c)^{3/2}+\frac{128}{147\pi}m^2nU_0R_0^4T_c\ . \quad (44)$$

The results of the calculation of the critical temperature as a function of the density, in accordance with Eq. (43), for various values of the magnetic field are shown in Fig. 5. Lines 1, 2, and 3 in this figure correspond to magnetic fields of 10, 100, and 1000, respectively. The temperature of the Bose condensation of an ideal gas is shown by the dashed line. The critical temperature of the interacting system is nearly the same as for an ideal Bose gas. However, there is the region of the densities ($n<n_c$), where the critical temperature vanishes as $n\rightarrow n_c$. This critical density n_c strongly depends on the magnetic field. For example, the critical density changes from $n_c=1.251\times10^{22}$ cm^{-3} at $B=10$, and $n_c=2.645\times10^{23}$ cm^{-3} at $B=100$ to $n_c=1.047\times10^{24}$ cm^{-3} at $B=1000$.

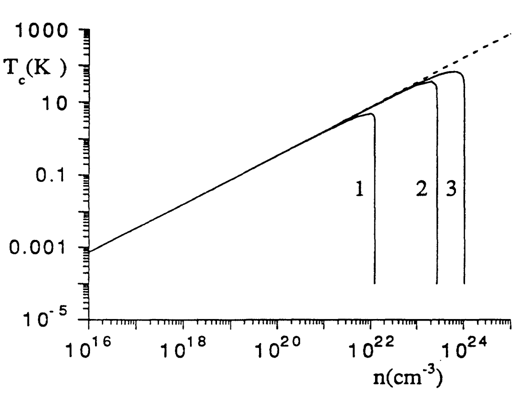

FIG. 5. The critical temperature for the transition of the hydrogen gas into the superfluid state; lines 1, 2, and 3 correspond to magnetic fields of 10, 100, and 1000, respectively, (in atomic units).

Thus the analysis of the properties of the hydrogenlike gas in the strong magnetic field shows that a gaseous phase of the hydrogen atoms can definitely be looked upon as a degenerate almost ideal Bose gas at densities below a certain critical value ($n<n_c$), and the superfluid state arises as a result of the spontaneous symmetry breaking in the system at low temperatures. There are two main points distinguishing this system from the case of an usual dilute Bose gas. First, the hydrogenlike gas in the strong magnetic field has the anisotropic excitation spectrum in the superfluid state. Second, we see the existence of the critical density which is probably related to the fact that the perturbation theory based on the gaseous approximation in this work completely breaks down at the densities $n>n_c$, and a simple picture of a dilute Bose gas of hydrogenlike atoms becomes irrelevant. In that case, when the pair interaction begins to play an important role, other processes may become the most significant. For example, there can be the creation of a new superfluid state made up of weakly bound pairs of the atoms. These pair correlations may lead to the collapse of the previous quantum state which is to be "rebuilt" as the density increases. In principle, there is a real chance that molecular hydrogen would itself be superfluid.[11] Another possibility is the appearance of a new molecular phase. The pair correlations depending on the strength of the magnetic field may cause a "crystallization" of the system. Perhaps, the weakness and the strong anisotropy of the pair interaction in the ground triplet state imply that anisotropic structures such as liquid crystals would arise in the system of these atoms.

IV. OUTLOOK

The consistent treatment of the properties of matter in high magnetic fields opens possibilities of studying and creating a great number of new phenomena having a similar nature in both terrestrial laboratories and space. Maybe, such a number of possibilities of that sort has never appeared in atomic physics before. First of all, an atmosphere around neutron stars or white dwarf stars must be a natural field to observe a drastic influence of a huge magnetic field on the properties of matter. Such huge magnetic fields in the vicinity of compact cosmic objects, which completely change the electronic structure of atoms, have been discovered recently (see, for example, Ref. 5 and references therein). Thus the idea of the superfluidity of the hydrogenlike gas in the atmosphere of the compact cosmic objects seems to be an interesting possibility. The superfluidity probably suggests explaining several peculiarities in the behavior of optical spectra of those objects.

Another matter of considerable interest is mostly related to the terrestrial conditions. Quantum properties of matter are easily seen when one considers so-called quantum liquids at a low temperature. So far, helium has appeared to be the only practical example of a quantum superfluid liquid. Therefore, it is worth looking for another example of a quantum liquid which would have a dissimilar collection of physical phenomena. There may be an exciton liquid in semiconductors (or dielectrics). The idea that an exciton gas can become superfluid has attracted physicists for a long time. Since the various methods of excitation of a semiconductor by lasers easily give a high excitonic concentration, and the lifetime of the excitons is long enough so that the excitonic system reaches quasiequilibrium, one can use a thermodynamic approach to describe possible properties of the system.[12] Then, we could expect that Bose condensation and superfluidity must take place because of a light excitonic mass. Unfortunately, this ideal case is usually destroyed by the collective effects. If the exciton density is high enough, i.e., of $10^{17}-10^{18}$ cm^{-3}, the "interatomic" interaction of the excitons becomes so important that we cannot describe the situation in terms of excitons any longer. The exciton gas turns into an electron-hole liquid.[13] Nevertheless, the first evidence of the Bose-Einstein condensation of the excitonic gas in Cu_2O has been given recently.[14] In this experiment, the results obtained for the luminescence spectrum of a highly degenerate orthoexciton gas created by a laser pulse probably indicate the actual Bose-Einstein condensation. However, the creation of a dense exciton system for observing the effects of quantum statistics still remains to be a very difficult problem. But, as was pointed out, a high magnetic field which completely changes the main interaction parameters of the exciton system may give a real chance of observing these phenomena in a large number of materials. In this case, a large binding energy $[E_{ex} \approx -\mathcal{R}_{ex} \ln^2(B/B_{ex}) \gg \mathcal{R}_{ex}]$ of the exciton, and a small characteristic size, as well as a substantial decrease of the interaction in the ground triplet state of two hydrogenlike excitons allow us to hope that the Bose condensation and the superfluidity will not be suppressed by the collective effects leading to the electron-hole liquid at a high density of the system. The fact of the decrease of the exciton-exciton pair interaction and the absence of the biexcitonic states in the strong magnetic field was already discussed.[15] The possible superfluidity of the high-density exciton liquid may cause a number of new unusual properties of semiconductors such as a superthermal conductivity and supertransparency. Finally, one should point out that the superfluidity and the related phenomena are not all new properties of the excitonic system in the high magnetic field. The pair interaction of the hydrogenlike excitons in the singlet term must be much stronger than the pair interaction of the usual excitons without the magnetic field (see Fig. 1). Therefore, if the gas of the excitons were to exist in the excited singlet state, metastable structures such as long polymeric molecules, regular lattice structures or even liquid crystals would arise due to the extremely strong anisotropic interaction of the excitons. Thus the exciton liquid in the high magnetic field gives another example of a quantum liquid.

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