Conformal theory of the two-dimensional Ising model with homogeneous boundary conditions and with disordered boundary fields

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The critical Ising model defined on the upper half plane is studied with conformal-invariance methods. For homogeneous free and fixed-spin boundary conditions we derive all the many-point correlation functions $\langle\sigma_1\sigma_2\ldots\sigma_n\rangle$ and $\langle\epsilon_1\epsilon_2\ldots\epsilon_n\rangle$ of the spin σ and the energy density ϵ. We also treat the case of inhomogeneous boundary fields that orient the spins on the x axis up for $\zeta_1 < x < \infty$, down for $\zeta_2 < x < \zeta_1$, up for $\zeta_3 < x < \zeta_2$, etc. Exact expressions for the correlation functions $\langle\sigma\rangle_{\zeta_1\zeta_2\ldots\zeta_m}$, $\langle\sigma_1\sigma_2\rangle_{\zeta_1\zeta_2\ldots\zeta_m}$, and $\langle\epsilon_1\epsilon_2\ldots\epsilon_n\rangle_{\zeta_1\zeta_2\ldots\zeta_m}$ are obtained. Examples of droplet shapes for random ζ_i are shown.

I. INTRODUCTION

The conformal invariance approach of Belavin, Polyakov, and Zamolodchikov[1,2] determines the bulk critical indices and many-point correlation functions of an infinite class of two-dimensional critical systems.

Conformal invariance also yields information about systems with surfaces.[3] Cardy[2,4] has shown how to apply the conformal theory to the semi-infinite geometry with a uniform boundary condition, such as free or fixed boundary spins. Cardy[5] and Burkhardt and Xue[6] have made a further extension to systems with mixed, piecewise-constant boundary conditions.

This paper considers the critical Ising model, defined on the upper half plane. Using methods of conformal invariance, we study the many-point correlations of the spin variables $\sigma_i = \sigma(z_i, \bar{z}_i)$ and the energy density $\epsilon_i = \epsilon(z_i, \bar{z}_i)$ for several different boundary conditions on the x axis. Here $z = x + iy$ and $\bar{z} = x - iy$ are complex position coordinates.

The *bulk* many-point correlations of the critical Ising model have been worked out in detail.[7] However, in the semi-infinite geometry with uniform fixed-spin or free-spin boundary conditions, only the two-[2,4] and three-point functions [6,8] appear to have been calculated previously at all points in the plane. We obtain $\langle\sigma_1\sigma_2\ldots\sigma_n\rangle$ and $\langle\epsilon_1\epsilon_2\ldots\epsilon_n\rangle$ with these two boundary conditions for arbitrary n.

The two-dimensional Ising model is of direct physical relevance in the phase transitions of adsorbates on crystalline substrates. The melting transition of an ordered adsorbate phase existing on either of two equivalent sublattices generally belongs to the Ising universality class. An irregularly shaped boundary that passes through sites on both sublattices corresponds to quenched disordered boundary magnetic fields in the equivalent Ising model.[9]

We have also considered the semi-infinite critical Ising model with disordered boundary fields. The semi-infinite geometry can, of course, be conformally mapped[2] onto fully finite geometries more appropriate to adsorbed systems. Specifically we apply spin-up boundary conditions on the x axis for $\zeta_1 < x < \infty$, spin-down boundary conditions for $\zeta_2 < x < \zeta_1$, spin-up boundary conditions for $\zeta_3 < x < \zeta_2$, etc., where $\zeta_1, \zeta_2, \ldots, \zeta_m$ and m may be chosen arbitrarily. Simple exact expressions for the correlation functions $\langle\sigma\rangle_{\zeta_1\zeta_2\ldots\zeta_m}$, $\langle\sigma_1\sigma_2\rangle_{\zeta_1\zeta_2\ldots\zeta_m}$, and $\langle\epsilon_1\epsilon_2\ldots\epsilon_n\rangle_{\zeta_1\zeta_2\ldots\zeta_m}$ are derived. Examples of droplet shapes for random $\zeta_1, \ldots, \zeta_m$, calculated from the result for $\langle\sigma\rangle_{\zeta_1\ldots\zeta_m}$, are shown.

In the conformal classification[1,2] the two Ising operators σ and ϵ are degenerate at level 2. The bulk $n+m$ point correlation function $\langle\sigma_1\ldots\sigma_n\epsilon_{n+1}\ldots\epsilon_{n+m}\rangle_{\text{bulk}}$ satisfies $n+m$ partial differential equations[1,2]

$$\left[\frac{\partial^2}{\partial z_i^2} - \frac{2}{3}(1+2\Delta_i)\sum_{\substack{j=1\\ j\neq i}}^{n+m}\left(\frac{1}{z_{ij}}\frac{\partial}{\partial z_j} + \frac{\Delta_j}{z_{ij}^2}\right)\right]G^{(n,m)}(z_1,\ldots,z_{n+m}) = 0 \;, \tag{1a}$$

where

$$\Delta_i = \begin{cases} \frac{1}{16} \;, i = 1,\ldots,n \\ \frac{1}{2} \;, i = n+1,\ldots,n+m \end{cases} \tag{1b}$$

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and $z_{ij} = z_i - z_j$. The form invariance of bulk correlation functions under regular conformal mappings $z' = (az + b)(cz+d)^{-1}$ of the entire complex plane onto itself restricts the physically relevant solutions of Eq. (1) for even n and m to functions of the form[1,2]

$$G^{(n,m)}(z_1,\ldots,z_{n+m}) = F(X_1, X_2, \ldots) \prod_{\substack{i=1 \\ i \text{ odd}}}^{n+m-1} (z_i - z_{i+1})^{\Delta_i + \Delta_{i+1}} , \tag{2}$$

where $X_1, X_2, \ldots$ are independent cross ratios, for example, $z_{13}z_{24}/z_{14}z_{23}$, that can be formed from $z_1, \ldots, z_{n+m}$.

Cardy[4] has shown that the n-point correlation function in the half-space with a uniform, conformally invariant boundary satisfies the same differential equations in the variables $z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n$ as the bulk $2n$-point correlation function in the variables $z_1, z_2, \ldots, z_{2n}$. Thus both bulk and half-space correlation functions can be constructed from the solutions to Eqs. (1) and (2).

II. HOMOGENEOUS BOUNDARY CONDITIONS

We first consider correlation functions $\langle \sigma_1 \sigma_2 \ldots \sigma_n \rangle$, i.e., $m = 0$ in Eq. (1). For $n = 2$ Eqs. (1) and (2) have a single solution

$$G^{(2,0)}(z_1, z_2) = z_{12}^{-1/8}. \tag{3}$$

For $n = 4$ there are two linearly independent solutions[1,2,4]

$$G^{(4,0)}_{1,2}(z_1, z_2, z_3, z_4) = (z_{12}z_{34})^{-1/8}(\xi_{13} \pm \xi_{13}^{-1})^{1/2} , \tag{4a}$$

$$\xi_{13} = \left(\frac{z_{13}z_{24}}{z_{14}z_{23}} \right)^{1/4} . \tag{4b}$$

The case $n = 6$ has also been studied in detail.[8] Equations (1) and (2) have four linearly independent solutions,

$$G_1^{(6,0)}(z_1, \ldots, z_6) = (z_{12}z_{34}z_{56})^{-1/8} \left(\xi_{13}\xi_{15}\xi_{35} + \frac{\xi_{35}}{\xi_{13}\xi_{15}} + \frac{\xi_{15}}{\xi_{13}\xi_{35}} + \frac{\xi_{13}}{\xi_{15}\xi_{35}} \right)^{1/2} , \tag{5a}$$

$$\xi_{13} = \left(\frac{z_{13}z_{24}}{z_{14}z_{23}} \right)^{1/4}, \quad \xi_{15} = \left(\frac{z_{15}z_{26}}{z_{16}z_{25}} \right)^{1/4}, \quad \xi_{35} = \left(\frac{z_{35}z_{46}}{z_{36}z_{45}} \right)^{1/4} , \tag{5b}$$

and solutions $G_2^{(6,0)}, G_3^{(6,0)}, G_4^{(6,0)}$ in which the four terms under the square root in Eq. (5a) have signs $+--+$, $+-+-$, and $++--$, respectively, instead of $++++$.

With Eqs. (4) and (5) in mind we have guessed the form of $G_\alpha^{(n,0)}(z_1, \ldots, z_n)$ for arbitrary even n and confirmed, by lengthy but straightforward calculations, that the differential equations (1) are indeed satisfied by the guess. In this way we obtain the $2^{n/2-1}$ linearly independent solutions

$$G_\alpha^{(n,0)}(z_1, \ldots, z_n) = (z_{12}z_{34} \ldots z_{n-1,n})^{-1/8} \left\{ \frac{1}{2} \sum_{\mu_1=\pm 1} \sum_{\mu_3=\pm 1} \cdots \sum_{\mu_{n-1}=\pm 1} S_\alpha(\mu_1, \mu_3, \ldots, \mu_{n-1}) \prod_{\substack{i<j \\ i,j \text{ odd}}} \xi_{ij}^{\mu_i \mu_j} \right\}^{1/2} , \tag{6a}$$

$$\xi_{ij} = \left(\frac{z_{i,j} z_{i+1,j+1}}{z_{i,j+1} z_{i+1,j}} \right)^{1/4} . \tag{6b}$$

The quantities $S_\alpha(\mu_1, \mu_3, \ldots, \mu_{n-1})$, $\alpha = 1, 2, \ldots, 2^{n/2-1}$, in Eq. (6) are the even operators 1, $\mu_k\mu_l$ with $k < l$, $\mu_k\mu_l\mu_m\mu_n$ with $k < l < m < n$, etc., where $k, l, \ldots$ take the values $1, 3, 5, \ldots, n-1$. Note that the functions $G_\alpha^{(n,0)}$ are invariant (apart from a multiplicative constant) under the interchange $z_i \leftrightarrow z_{i+1}$ of coordinates, where i is odd.

For odd n the bulk many-spin correlation functions $\langle \sigma_1 \sigma_2 \ldots \sigma_n \rangle_{\text{bulk}}$ vanish identically at criticality. For even n the requirements that the correlation functions be real and single valued, factor properly for large separations consistent with the normalization $\langle \sigma_1 \sigma_2 \rangle_{\text{bulk}} = |z_{12}|^{-1/8}$, and satisfy Eq. (1) and similar equations in the $\bar{z}_i$ lead to the expression

$$\langle \sigma_1 \sigma_2 \ldots \sigma_n \rangle_{\text{bulk}} = \frac{1}{2^{n/2-1}} \sum_{\alpha=1}^{2^{n/2-1}} |G_\alpha^{(n,0)}(z_1, \ldots, z_n)|^2 \tag{7}$$

in terms of the $G_\alpha^{(n,0)}$ of Eq. (6). We have checked that Eqs. (6) and (7) reproduce the known exact bulk correlation functions[7]

$$\langle\sigma_1\sigma_2\ldots\sigma_n\rangle_{\rm bulk} = \left\{\frac{1}{2^{n/2}}\sum_{\substack{\nu_1=\pm1 \\ \nu_1+\cdots+\nu_n=0}}\cdots\sum_{\nu_n=\pm1}\prod_{i<j}|z_{ij}|^{\nu_i\nu_j/2}\right\}^{1/2} . \tag{8}$$

As mentioned above, the n-spin correlation function in the half-space geometry with free- or fixed-spin boundary conditions follows[2,4] from the replacement $(z_1, z_2, \ldots, z_{2n}) \to (z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n)$ in the proper linear combination of the functions $G_\alpha^{(2n,0)}$ of Eq. (6). The proper linear combination follows from the requirements that the correlation factor correctly at large separations and reproduce exact results for the two-[2,4] and three-spin[8] correlation functions in the half-space. In this way we obtain

$$\langle\sigma_1\sigma_2\ldots\sigma_n\rangle_{\rm fixed} = (2^n y_1 y_2\ldots y_n)^{-1/8}\left\{\frac{1}{2^{n/2}}\sum_{\mu_1=\pm1}\cdots\sum_{\mu_n=\pm1}\prod_{i<j}\chi_{ij}^{\mu_i\mu_j}\right\}^{1/2} , \tag{9a}$$

$$\langle\sigma_1\sigma_2\ldots\sigma_n\rangle_{\rm free} = (2^n y_1 y_2\ldots y_n)^{-1/8}\left\{\left(-\frac{1}{2}\right)^{n/2}\sum_{\mu_1=\pm1}\cdots\sum_{\mu_n=\pm1}\mu_1\mu_2\ldots\mu_n\prod_{i<j}\chi_{ij}^{\mu_i\mu_j}\right\}^{1/2} , \tag{9b}$$

$$\chi_{ij} = \left|\frac{z_i - z_j}{z_i - \bar{z}_j}\right|^{1/2} = (1 + 4y_i y_j/r_{ij}^2)^{-1/4} . \tag{9c}$$

Equations (9a)–(9c) hold for even and odd n, with $\langle\sigma_1\sigma_2\ldots\sigma_n\rangle_{\rm free}$ in Eq. (9b) vanishing identically for odd n.

We now consider the correlation functions $\langle\epsilon_1\epsilon_2\ldots\epsilon_m\rangle$, which satisfy Eqs. (1) and (2) with $n = 0$. For $n = 0$, $m = 2$ the equations have a single solution

$$G^{(0,2)}(z_1, z_2) = z_{12}^{-1} . \tag{10}$$

For $n = 0$, $m = 4$ there are two linearly independent solutions[1,5]

$$G^{(0,4)}(z_1, z_2, z_3, z_4) = (z_{12}z_{34})^{-1} - (z_{13}z_{24})^{-1} + (z_{14}z_{23})^{-1} \tag{11}$$

and a second solution, which, as explained by Mattis,[10] is unphysical.

For $n = 0$ and arbitrary even m there seems to be only one physical solution to Eqs. (1) and (2),

$$G^{(0,m)}(z_1, z_2, \ldots, z_m) = {\rm Pf}^{(m)}\frac{1}{z_{ij}} . \tag{12}$$

Here and below ${\rm Pf}^{(m)}A_{ij}$ denotes the Pfaffian[11] of the $m \times m$ antisymmetric matrix A_{ij}. Equation (12) is compatible with our detailed study of the case $m = 6$ and the known result[7]

$$\langle\epsilon_1\epsilon_2\ldots\epsilon_n\rangle_{\rm bulk} = \left|{\rm Pf}^{(n)}\frac{1}{z_{ij}}\right|^2 \tag{13}$$

for arbitrary even n. For odd n $\langle\epsilon_1\epsilon_2\ldots\epsilon_n\rangle_{\rm bulk}$ vanishes identically, as implied by duality.

To obtain results for the semi-infinite geometry with uniform free or fixed boundary spins, we make the substitution $(z_1, z_2, \ldots, z_{2n}) \to (z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n)$ in Eq. (12). This yields

$$\langle\epsilon_1\epsilon_2\ldots\epsilon_n\rangle_{\text{free or fixed}} = i^n\ {\rm Pf}^{(2n)}\frac{1}{\mathcal{Z}_{\alpha\beta}} , \tag{14a}$$

$$(\mathcal{Z}_1, \ldots, \mathcal{Z}_{2n}) = (z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n) \tag{14b}$$

for even or odd n.

III. DISORDERED BOUNDARY MAGNETIC FIELDS

Thus far we have only considered *uniform* boundary conditions. Now we turn to the case of piecewise-constant boundary conditions that change at points $\zeta_1, \zeta_2, \ldots, \zeta_m$ on the x axis. The n-point correlation function $\langle\phi_1\phi_2\ldots\phi_n\rangle_{\zeta_1\ldots\zeta_m}$ can, in general, be written in the form[5,6]

$$\langle\phi_1\phi_2\ldots\phi_n\rangle_{\zeta_1\zeta_2\ldots\zeta_m} = \frac{N(z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n; \zeta_1, \ldots, \zeta_m)}{D(\zeta_1, \ldots, \zeta_m)} , \tag{15}$$

where the numerator N and the denominator D satisfy the same differential equations as particular *bulk* $2n + m$ point and m point correlation functions, respectively.

We specialize to the Ising model with edge fields that orient the boundary spins up for $\zeta_1 < x < \infty$, down for $\zeta_2 < x < \zeta_1$, up for $\zeta_3 < x < \zeta_2$, etc. The Ising boundary operator that reverses the boundary spins at ζ_i corresponds[5,6] to the energy density in the related bulk correlation functions. Thus the numerator N in Eq. (15) satisfies the same differential equations in the $2n + m$ variables $z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n, \zeta_1, \ldots, \zeta_m$ as $\langle\phi_1\ldots\phi_{2n}\epsilon_{2n+1}\ldots\epsilon_{2n+m}\rangle_{\rm bulk}$ in the variables $z_1, \ldots, z_{2n+m}$. The denominator D satisfies the same differential equations in the variables $\zeta_1, \ldots, \zeta_m$ as $\langle\epsilon_1\ldots\epsilon_m\rangle_{\rm bulk}$ in the variables $z_1, \ldots, z_m$.

The relevant physical solution to the differential equations for the bulk correlation functions of the energy density is given in Eq. (12). Thus in the presence of disordered boundary fields on the x axis, the many-point correlations of the energy density are given by

$$\langle\epsilon_1\epsilon_2\dots\epsilon_n\rangle_{\zeta_1\zeta_2\dots\zeta_m} = i^n \frac{\mathrm{Pf}^{(2n+m)}\dfrac{1}{\mathcal{W}_{\alpha\beta}}}{\mathrm{Pf}^{(m)}\dfrac{1}{\zeta_{ij}}}, \tag{16a}$$

$$(\mathcal{W}_1,\dots,\mathcal{W}_{2n+m}) = (z_1,\bar{z}_1,\dots,z_n,\bar{z}_n,\zeta_1,\dots,\zeta_m) . \tag{16b}$$

Equation (16) holds for even or odd n and even m. It is simple to obtain the general result for an odd number of ζ's by taking the limit $\zeta_m \to -\infty$ in Eq. (16).

The many-spin correlation functions $\langle\sigma_1\sigma_2\dots\sigma_n\rangle_{\zeta_1\dots\zeta_m}$ with disordered boundary fields also have the form (15) with $D(\zeta_1,\dots,\zeta_m) = \mathrm{Pf}^{(m)}\zeta_{ij}^{-1}$. The numerator N is determined by the same differential equations as $\langle\sigma_1\dots\sigma_{2n}\epsilon_{2n+1}\dots\epsilon_{2n+m}\rangle_{\text{bulk}}$; i.e., N is a linear combination of the solutions $G_\alpha^{(2n,m)}(z_1,\bar{z}_1,\dots,z_n,\bar{z}_n,\zeta_1,\dots,\zeta_m)$ to Eq. (1).

We have made guesses for $\langle\sigma\rangle_{\zeta_1\dots\zeta_m}$ and $\langle\sigma_1\sigma_2\rangle_{\zeta_1\dots\zeta_m}$ consistent with the bulk operator-product expansion[1,2] and the expansion in surface operators of a bulk operator near the boundary[5] and then confirmed, by lengthy but straightforward calculations, that the guesses do indeed satisfy the set of differential equations (1). In this way we find that the one- and two-point functions $\langle\sigma\rangle_{\zeta_1\dots\zeta_m}$ and $\langle\sigma_1\sigma_2\rangle_{\zeta_1\dots\zeta_m}$ with arbitrary even m can be expressed in terms of the simpler correlation functions $\langle\sigma\rangle_{\zeta_i\zeta_j}$ and $\langle\sigma_1\sigma_2\rangle_{\zeta_i\zeta_j}$, respectively, with only two ζ's, according to

$$\langle\sigma_1\dots\sigma_n\rangle_{\zeta_1\zeta_2\dots\zeta_m} = \langle\sigma_1\dots\sigma_n\rangle_{\text{fixed}} \frac{\mathrm{Pf}^{(m)}\left[\dfrac{1}{\zeta_{ij}}\dfrac{\langle\sigma_1\dots\sigma_n\rangle_{\zeta_i\zeta_j}}{\langle\sigma_1\dots\sigma_n\rangle_{\text{fixed}}}\right]}{\mathrm{Pf}^{(m)}\dfrac{1}{\zeta_{ij}}}, \tag{17}$$

with $n = 1$ and 2. The correlation functions appearing on the right side of Eq. (17) for $n = 1, 2$ are given by[6]

$$\langle\sigma\rangle_{\zeta_1\zeta_2} = \langle\sigma\rangle_{\text{fixed}}|(z-\zeta_1)(z-\zeta_2)|^{-1}\mathrm{Re}[(z-\zeta_1)(\bar{z}-\zeta_2)] , \tag{18a}$$

$$\begin{aligned}\langle\sigma_1\sigma_2\rangle_{\zeta_1\zeta_2} = {}& \langle\sigma_1\sigma_2\rangle_{\text{fixed}}|(z_1-\zeta_1)(z_1-\zeta_2)(z_2-\zeta_1)(z_2-\zeta_2)|^{-1} \\ &\times\{\mathrm{Re}[(z_1-\zeta_1)(\bar{z}_1-\zeta_2)(z_2-\zeta_1)(\bar{z}_2-\zeta_2)] + \tfrac{1}{2}\zeta_{12}^2|z_1-\bar{z}_2|(|z_1-\bar{z}_2|-|z_1-z_2|)\}.\end{aligned} \tag{18b}$$

The quantities $\langle\sigma\rangle_{\text{fixed}}$, $\langle\sigma_1\sigma_2\rangle_{\text{fixed}}$ in Eqs. (17) and (18), which denote the one- and two-spin functions in the half-space with uniform spin-up boundary conditions, have the form [see Refs. 2 and 4 and Eq. (9) above]

$$\langle\sigma\rangle_{\text{fixed}} = 2^{1/4}(2y)^{-1/8} , \tag{19a}$$

$$\langle\sigma_1\sigma_2\rangle_{\text{fixed}} = (4y_1y_2)^{-1/8}\left\{\left|\frac{z_1-z_2}{z_1-\bar{z}_2}\right|^{1/2} + \left|\frac{z_1-\bar{z}_2}{z_1-z_2}\right|^{1/2}\right\}^{1/2} . \tag{19b}$$

Equations (17)–(19) completely specify the one- and two-spin functions for an arbitrary even number of points $\zeta_1,\dots,\zeta_m$ at which the boundary spins change sign. It is simple to obtain the general result for an odd number of ζ's by taking the limit $\zeta_m \to -\infty$ in Eqs. (17) and (18).

We emphasize that we have only *proved* Eq. (17) for $n = 1$ and 2. Of course, it is quite likely that the relation holds for general n. Various checks support this view. For example, we have derived Eq. (16) for $\langle\epsilon_1\dots\epsilon_{n/2}\rangle_{\zeta_1\dots\zeta_m}$ from Eq. (17) for $\langle\sigma_1\dots\sigma_n\rangle_{\zeta_1\dots\zeta_m}$ for several values of $n > 2$ on letting pairs of points approach each other and using the first two terms in the operator product expansion[1,2]

$$\sigma_1\sigma_2 \longrightarrow |z_{12}|^{-1/4}[1 + \tfrac{1}{2}|z_{12}|\epsilon(z_1,\bar{z}_1) + \dots] \tag{20}$$

as $z_2 \to z_1$. Note that expressions for all of the mixed correlation functions $\langle\sigma_1\dots\sigma_i\epsilon_{i+1}\dots\epsilon_{i+j}\rangle_{\zeta_1\dots\zeta_m}$ can be derived from Eq. (17) in this way.

IV. DROPLET SHAPES

The disordered boundary fields nucleate "droplets" with nonzero magnetization in the critical Ising model. Each droplet is bounded by the edge of the system and a line on which the magnetization $\langle\sigma\rangle_{\zeta_1\dots\zeta_m}$ vanishes. Examples of droplets are shown in Figs. 1–3. To obtain the droplet shapes, we calculated $\langle\sigma\rangle_{\zeta_1\dots\zeta_m}$ as a function of position for given values of $\zeta_1\dots\zeta_m$ numerically using Eqs. (17)–(19) and looked for the lines where it vanishes. These lines are shown in the figures.

The case of two ζ's or $+-+$ boundary conditions is considered in Fig. 1(a). The negative droplet corresponding to the region $\langle\sigma\rangle_{\zeta_1\zeta_2} < 0$ is semicircular, as has been discussed previously.[12,13]

The case of three ζ's or $+-+-$ boundary conditions is shown in Figs. 1(b)–1(f). As the base $\zeta_2 - \zeta_3$ of the small positive droplet in Fig. 1(b) is lengthened with its midpoint at $\frac{1}{2}(\zeta_2+\zeta_3)$ held fixed, it coagulates with the infinite positive droplet, and a finite negative droplet is formed.

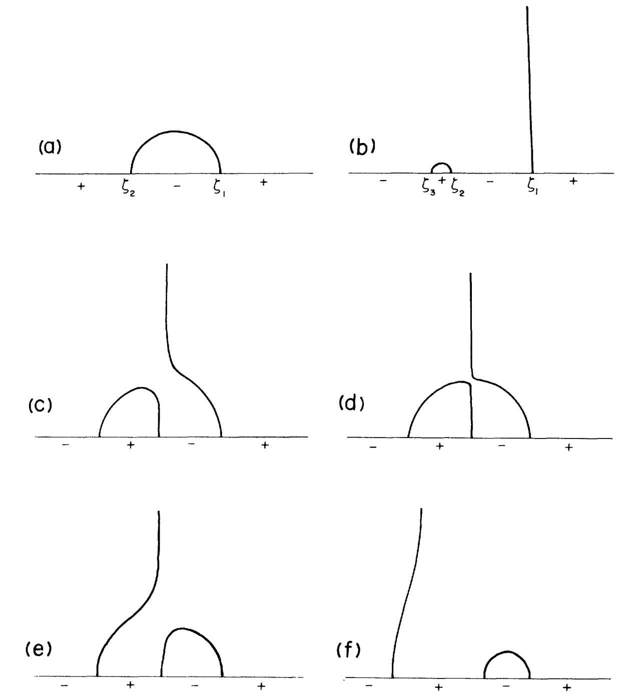

FIG. 1. (a) Semicircular droplet corresponding to $\langle\sigma\rangle_{\zeta_1\zeta_2} < 0$ in the semi-infinite geometry with $+ - +$ boundary conditions. (b)-(f) Coagulation of droplets for $+ - + -$ boundary conditions as $\zeta_2 - \zeta_3$ increases with the midpoint $\frac{1}{2}(\zeta_2 + \zeta_3)$ and ζ_1 fixed.

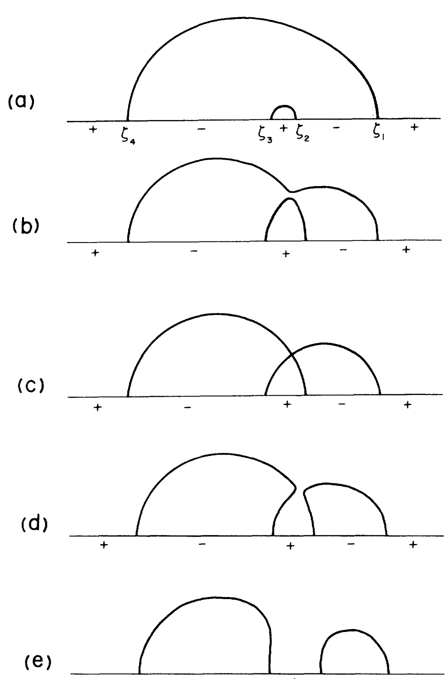

FIG. 2. (a)–(e) Coagulation of droplets for $+ - + - +$ boundary conditions as $\zeta_2 - \zeta_3$ increases with the midpoint $\frac{1}{2}(\zeta_2 + \zeta_3)$, ζ_1, and ζ_4 fixed.

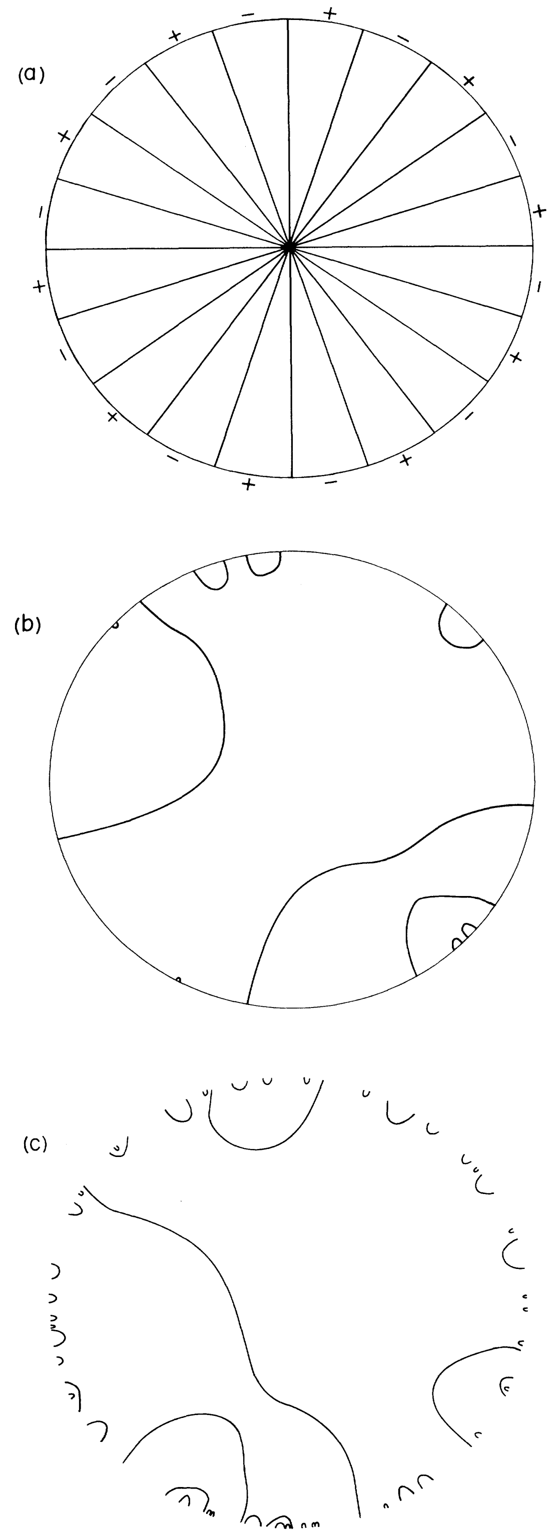

FIG. 3. Droplets in the critical Ising model defined on a disk with m points on the perimeter at which the boundary spins change sign. (a) 20 equally spaced points. (b) 20 randomly distributed points. (c) 100 randomly distributed points.

The case of four ζ's is shown in Figs. 2(a)–2(e). As the base of the small positive drop is lengthened with its midpoint held fixed, it coagulates with the infinite positive droplet, leaving two disconnected negative droplets.

Correlation functions in fully finite geometries more appropriate to adsorbed systems can be obtained from correlation functions in the half-space using the covariance under conformal mappings.[1,2] Transforming our result for $\langle\sigma\rangle_{\zeta_1\ldots\zeta_m}$ in the half-space with the mapping $w = (z - i)(z + i)^{-1}$, we obtained the droplets in the finite Ising model with a circular boundary shown in Fig. 3. In Fig. 3(a) there are 20 evenly spaced points at which the fixed boundary spins change sign. The droplets look like identical slices of a pie. In Figs. 3(b) and 3(c) the m points on the circumference at which the spins change sign were chosen randomly, with $m = 20$ and 100, respectively.

V. CONCLUDING REMARKS

It would be interesting to extend this work in the following two directions.

(1) The semi-infinite Ising model with quenched random magnetic fields acting on the boundary spins has been analyzed, using the replica formalism, by Cardy.[9] In principle one can treat the case of random boundary fields by averaging our results for $\langle\sigma_1\ldots\sigma_n\rangle_{\zeta_1\ldots\zeta_m}$ and $\langle\epsilon_1\ldots\epsilon_n\rangle_{\zeta_1\ldots\zeta_m}$ over the ζ's and avoid the replica method.

(2) The structure factor or Fourier transform of the spin-spin correlation function has been calculated in fully finite geometries with free boundary conditions[14,15] for comparison with experimental results on adsorbed systems. As discussed above and in Ref. 9, the disordered-field boundary condition may be more appropriate in some experimental situations. One could calculate the structure factor with this type of boundary condition beginning with Eq. (17).

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